Several Complex Variables
MSRI Publications
Volume **37**, 1999

# Nevanlinna Theory and Diophantine Approximation

PAUL VOJTA

Abstract. As observed originally by C. Osgood, certain statements in value distribution theory bear a strong resemblance to certain statements in diophantine approximation, and their corollaries for holomorphic curves likewise resemble statements for integral and rational points on algebraic varieties. For example, if $X$ is a compact Riemann surface of genus $> 1$, then there are no non-constant holomorphic maps $f : \mathbb{C} \to X$; on the other hand, if $X$ is a smooth projective curve of genus $> 1$ over a number field $k$, then it does not admit an infinite set of $k$-rational points. Thus non-constant holomorphic maps correspond to *infinite* sets of $k$-rational points.

This article describes the above analogy, and describes the various extensions and generalizations that have been carried out (or at least conjectured) in recent years.

When looked at a certain way, certain statements in value distribution theory bear a strong resemblance to certain statements in diophantine approximation, and their corollaries for holomorphic curves likewise resemble statements for integral and rational points on algebraic varieties. The first observation in this direction is due to C. Osgood [1981]; subsequent work has been done by the author, S. Lang, P.-M. Wong, M. Ru, and others.

To begin describing this analogy, we consider two questions. On the analytic side, let $X$ be a connected Riemann surface. Then we ask:

Question 1. Does there exist a non-constant holomorphic map $f : \mathbb{C} \to X$?

The answer, as is well known, depends only on the genus $g$ of the compactification $\overline{X}$ of $X$, and on the number of points $s$ in $\overline{X} \setminus X$. See Table 1.

On the algebraic side, let $k$ be a number field with ring of integers $R$, and let $X$ be either an affine or projective curve over $k$. Let $S$ be a finite set of places of $k$ containing the archimedean places. For such sets $S$ let $R_S$ denote the localization of $R$ away from places in $S$ (that is, the subring of $k$ consisting of elements that can be written in such a way that only primes in $S$ occur in

Supported by NSF grant DMS95-32018 and the Institute for Advanced Study.

the denominator). We assume again that $X$ is nonsingular. If $X$ is affine, then fix an affine embedding; we then define an $S$**-integral point** of $X$ (or just an **integral point**, if it is clear from the context) to be a point whose coordinates are elements of $R_S$. If $X$ is projective, then we define an integral point on $X$ to be any $k$-rational point (that is, any point that can be written with homogeneous coordinates in $k$). In this case, we ask:

Question 2. Do there exist infinitely many integral points on $X$?

Again, let $\overline{X}$ be a nonsingular projective completion of $X$, let $g$ be the genus of $\overline{X}$ (which is the same as the genus of the corresponding Riemann surface), and let $s$ be the degree of the divisor $\overline{X} \setminus X$ (the sum of the degrees of the fields of definition of the points; over $\mathbb{C}$ this is just the number of points).

The answers to both of the above questions are summarized in the following table:

| $g$ | $s$ | Holo. curve? | $\infty$ many integral points? |
|---|---|---|---|
| 0 | 0 | Yes | Maybe |
| | 1 | Yes | Maybe |
| | 2 | Yes | Maybe |
| | $> 2$ | No | No |
| 1 | 0 | Yes | Maybe |
| | $> 0$ | No | No |
| $> 1$ | | No | No |

Table 1

The entries "Maybe" in the right-hand column require a little explanation. In each case there exists a curve with the given values of $g$ and $s$ with *no* integral points; but, for any curve with the given $g$ and $s$, over a large enough number field $k$ and with a large enough set $S$, there are infinitely many integral points. In that spirit, the two columns on the right have exactly the same answers.

This table could be summarized more succinctly by noting that the answer is "No" if and only if $2g - 2 + s > 0$. This condition holds if and only if $X$ is of "logarithmic general type." On the analytic side, there is a single proof of the non-existence of these holomorphic curves, relying on a Second Main Theorem for curves. For integral points, the corresponding finiteness statements were proved separately for $g = 0$, $s > 2$ and $g > 0$, $s > 0$ by Siegel in 1921; and for $g > 1$, $s = 0$ by Faltings in 1983 (the Mordell conjecture). One of the first major applications of the analogy with Nevanlinna theory was to find a finiteness proof that unified these various proofs. This proof consisted of proving an inequality in diophantine approximation that closely parallels the Second Main Theorem.

The analogy goes into more detail on how the statements of Nevanlinna theory and diophantine approximation correspond; this will be described more fully in the first section. It allows the statements of theorems such as the First and Second Main Theorems to be translated into statements of theorems in number theory (and vice versa), but it is not as useful for translating proofs. In particular, the proofs of the first and second main theorems do not translate in this analogy, but some of the derivations of other results from these theorems can be translated. Thus, the analogy is largely formal.

In addition, it is important to note that the analogue of *one* (non-constant) holomorphic map is an *infinite* set of integral points. It is not the same analogy as one would obtain by first looking at diophantine problems over function fields, and then treating the corresponding polynomials or algebraic functions as holomorphic functions.

I thank William Cherry for many thoughtful comments on this paper.

## 1. The Dictionary

The analogy mentioned above is quite precise, at least as far as the statements of theorems is concerned. Before describing it, though, we briefly review some of the basics of number theory; for more details see any of the standard texts, such as [Lang 1970].

Let $k$ be a number field; that is, a finite field extension of the rational number field $\mathbb{Q}$. Let $R$ be its ring of integers; that is, the integral closure of the rational integers $\mathbb{Z}$ in $k$. We have a standard set $M_k$ of **places** of $k$; it consists of **real places**, **complex places**, and **non-archimedean places**. The real places are defined by embeddings $\sigma : k \hookrightarrow \mathbb{R}$; the complex places, by complex conjugate (unordered) pairs $\sigma, \bar{\sigma} : k \hookrightarrow \mathbb{C}$; the non-archimedean places, by non-zero prime ideals $\mathfrak{p} \subseteq R$. The real and complex places are referred to collectively as the **archimedean** places.

Each place has an associated absolute value $\|\cdot\|_v : k \to \mathbb{R}_{\geq 0}$. If $v$ is a real or complex place, corresponding to $\sigma : k \hookrightarrow \mathbb{R}$ or $\sigma : k \hookrightarrow \mathbb{C}$, respectively, then this absolute value is defined by $\|x\|_v = |\sigma(x)|_v$ or $\|x\|_v = |\sigma(x)|^2$, respectively. If $v$ is non-archimedean, corresponding to a prime ideal $\mathfrak{p} \subseteq R$, then we define $\|x\|_v = (R : \mathfrak{p})^{-\operatorname{ord}_{\mathfrak{p}}(x)}$ if $x \neq 0$; here $\operatorname{ord}_{\mathfrak{p}}(x)$ denotes the exponent of $\mathfrak{p}$ occurring in the prime factorization of the fractional ideal $(x)$. We will also write $\operatorname{ord}_v(x) = \operatorname{ord}_{\mathfrak{p}}(x)$. (Of course, we also define $\|0\|_v = 0$.) Here we use a little abuse of terminology when referring to "absolute values," since $\|\cdot\|_v$ does not obey the triangle inequality when $v$ is a complex place.

The simplest example of all of this is $k = \mathbb{Q}$; in that case we have $R = \mathbb{Z}$ and $M_k = \{\infty, 2, 3, 5, 7, \ldots\}$. Here $\|x\|_\infty$ is just the usual absolute value of a rational number, and $\|x\|_p = p^{-m}$ if $x$ can be written as $p^m a/b$ with $a$ and $b$ integers not divisible by $p$.

We can now describe the most fundamental ingredients of the analogy between Nevanlinna theory and diophantine approximation. This takes the form of a dictionary for translating various concepts between the two fields. This dictionary starts out with just a few ideas, which seem to come from nowhere. These ideas allow one to translate the basic definitions of Nevanlinna theory, and consequently the statements of many of the theorems.

On the complex analytic side of this dictionary, let $X$ be a complex projective variety and let $f : \mathbb{C} \to X$ be a (non-constant) holomorphic curve. On the algebraic side, let $k$ be a number field, let $X$ be a projective variety over $k$ (that is, an irreducible projective scheme over $k$), and let $S$ be a finite set of places of $k$, containing all the archimedean places.

We are comparing $f$ to an infinite set of rational points, so it is useful to split $f$ into infinitely many pieces, each of which can be compared to one of the rational points. This is done as follows: for each $r > 0$, let $f_r$ denote the restriction $f_r := f\big|_{\overline{\mathbb{D}}_r}$. Assume for the moment that $X = \mathbb{P}^1$, so $f$ is a meromorphic map and the rational points are just rational numbers (or $\infty$).

In this dictionary, the domain $\overline{\mathbb{D}}_r$ of $f_r$ is compared to $M_k$. Points on the boundary are compared to places $v \in S$, and $|f_r(re^{i\theta})|$ is translated into $\|x\|_v$ (for the rational point $x$ being compared to $f_r$). Interior points $w \in \mathbb{D}_r$ are compared to places $v \notin S$: we translate $r/|w|$ to $(R : \mathfrak{p})$, where $\mathfrak{p}$ is the prime ideal in $R$ corresponding to $v$. We also translate $\operatorname{ord}_w(f_r) = \operatorname{ord}_w(f)$ to $\operatorname{ord}_{\mathfrak{p}}(x)$. Then the counterpart of $-\log\|x\|_v$ is $\operatorname{ord}_w(f) \cdot \log(r/|w|)$. Dividing by $|w|$ requires that we rule out $w = 0$ in the above translations; this is an imperfection in the analogy, but a minor one.

Thus, the ring of meromorphic functions on $\overline{\mathbb{D}}_r$ has something close to archimedean absolute values on the boundary, and non-archimedean absolute values on the interior of the domain.

The following table summarizes the dictionary, so far.

| **Nevanlinna Theory** | **Number Theory** |
|---:|:---|
| $f$ meromorphic on $\mathbb{C}$ | $\mathscr{S} \subseteq k$ |
| $f\big\|_{\overline{\mathbb{D}}_r}$ $(r > 0)$ | $x \in \mathscr{S}$ |
| $\overline{\mathbb{D}}_r$ | $M_k$ |
| $\theta$ | $v \in S$ |
| $w \in \mathbb{D}_r^\times$ | $v \notin S$ |
| $\|f(re^{i\theta})\|$, $0 \le \theta < 2\pi$ | $\\|x\\|_v$, $v \in S$ |
| $\operatorname{ord}_w f$ | $\operatorname{ord}_v x$ |
| $\frac{r}{\|w\|}$ | $(R : \mathfrak{p})$ |
| $\operatorname{ord}_w f \cdot \log \frac{r}{\|w\|}$ | $\operatorname{ord}_v x \cdot \log(R : \mathfrak{p}) = -\log \\|x\\|_v$ |

Table 2. Fundamental part of the dictionary.

Let $X$ now be an arbitrary projective variety over $\mathbb{C}$ or $k$, and let $\phi$ be a rational function on $X$ whose zero or pole set does not contain the image of $f$ or does not contain infinitely many of the rational points under consideration. Then one can apply the same dictionary above to $\phi \circ f_r$ in the analytic case and to $\phi(x)$ in the number field case.

More generally still, in the analytic case we can let $s$ be a rational section of a metrized line sheaf $\mathscr{L}$ on $X$. Then we again have $|s(f_r(re^{i\theta}))|$ on the boundary of $\overline{\mathbb{D}}_r$ and $\operatorname{ord}_w(s \circ f_r)$ on the interior. These can be translated into the number field case as follows. Let $X$ be a projective variety over $k$, let $\mathscr{L}$ be a line sheaf on $X$, and let $s$ be a section of $\mathscr{L}$. For archimedean places $v$, let $\|\cdot\|_v$ be a metric on the lifting $\mathscr{L}_{(v)}$ of $\mathscr{L}$ to $X \times_\sigma \mathbb{C}$, where $\sigma : k \hookrightarrow \mathbb{C}$ is an embedding corresponding to $v$. Then, for a rational point $x \in X(k)$, $\|s(x)\|_v$ is defined via the metric on $\mathscr{L}_{(v)}$. If $v$ is non-archimedean, one can define something similar to a metric, via the absolute values on the completions $k_v$. These must be done consistently, though, so that infinite sums in these absolute values converge. This can be done either via Weil functions [Lang 1983, Chapter 10] or Arakelov theory. As an example of such a consistent choice of metrics, if $X = \mathbb{P}^n$, if $\mathscr{L} = \mathscr{O}(1)$, if $s$ is the standard section of $\mathscr{O}(1)$ vanishing at infinity, and if $x$ has homogeneous coordinates $[x_0 : \cdots : x_n]$, then

$$\|s(x)\|_v = \frac{\|x_0\|_v}{\max\{\|x_0\|_v, \dots, \|x_n\|_v\}} \tag{1.1}$$

is one possible choice. By tensoring and pulling back, this example can be used to construct such systems of metrics in general. One can then define $\operatorname{ord}_v(s(x))$ in terms of $\|s(x)\|_v$.

Applying these more general definitions to the dictionary in Table 2 gives the following table translating the proximity, counting, and height (characteristic) functions of Nevanlinna theory into the arithmetic setting. Here $s$ is the canonical section of $\mathscr{O}(D)$, for a divisor $D$ on $X$.

| **Nevanlinna Theory** | **Number Theory** |
|---|---|
| Proximity function $m(D,r) = \int_0^{2\pi} -\log\left|s(f(re^{i\theta}))\right| \frac{d\theta}{2\pi}$ | $m(D,x) = \frac{1}{[k:\mathbb{Q}]} \sum_{v \in S} -\log \|s(x)\|_v$ |
| Counting function $N(D,r) = \sum_{w \in \mathbb{D}_r^\times} \operatorname{ord}_w f^*s \cdot \log \frac{r}{\lvert w\rvert}$ | $N(D,x) = \frac{1}{[k:\mathbb{Q}]} \sum_{v \notin S} -\log \|s(x)\|_v$ |
| Height (characteristic function) $T_D(r) = m(D,r) + N(D,r)$ | $h_D(x) = \frac{1}{[k:\mathbb{Q}]} \sum_{v \in M_k} -\log \|s(x)\|_v$ |

Table 3. Higher-level entries in the dictionary.

Note that the integrals over the set (of finite measure) of values of $\theta$ translate into sums over the finite set $S$.

Also note that the above functions are additive in $D$ (up to $O(1)$, or assuming compatible choices of metrics). They are also functorial; for example, if $\phi : X \to Y$ is a morphism of varieties and $D$ is a divisor on $Y$ whose support does not contain the image of $\phi$, then $m_f(\phi^* D, r) = m_{\phi\circ f}(D, r)$ for holomorphic curves $f : \mathbb{C} \to X$ and $m(\phi^* D, x) = m(D, \phi(x))$ for $x \in X(k)$, again assuming that the metrics on $\mathscr{O}(D)$ and $\phi^*\mathscr{O}(D)$ are compatible.

We conclude this section by considering the case of affine varieties $X$. Since everything is functorial, we may assume that $X = \mathbb{A}^n$. Regard it as embedded into $\mathbb{P}^n$, and let $D$ be the divisor at infinity. A holomorphic curve in $\mathbb{A}^n$ does not meet $D$; therefore $N(D, r) = 0$. Likewise, a rational point $x = [1 : x_1 : \cdots : x_n]$ is an $S$-integral point if and only if $x_1, \dots, x_n$ lie in $R_S$. If so, then $\|x_i\|_v \le 1$ for all $i$ and all $v \notin S$ (corresponding to the coordinates of $f_r$ not having poles); hence by (1.1), we have $N(D, x) = 0$ again.

Thus, it is also true on affine varieties that a non-constant holomorphic curve corresponds to an infinite set of integral points. More generally, let $X$ be any quasi-projective variety, and write $X = \overline{X} \setminus D$, where $D$ is a divisor; then $f$ is a holomorphic curve in $X$ if and only if $N(D, r) = 0$; likewise an infinite collection of rational points $x \in X(k)$ is a set of integral points if $N(D, x) = O(1)$. (The different choices of metrics may lead to $N(D, x)$ varying by a bounded amount; by the same token, different affine embeddings may introduce bounded denominators. Also, the situation is more complicated in function fields, since in that case $N(D, x)$ may be bounded, but the denominators may come from an infinite set of primes.)

The first indication that this dictionary is useful comes from the translation of Jensen's formula

$$\log|f(0)| = \int_0^{2\pi} \log|f(re^{i\theta})| \frac{d\theta}{2\pi} + N(\infty, r) - N(0, r)$$

into the number field case (here we assume that $f$ does not have a zero or pole at the origin). The right-hand side translates (up to a factor $1/[k : \mathbb{Q}]$) into

$$\sum_{v\in S} \log\|x\|_v + \sum_{v\notin S} \log^+ \|x\|_v - \sum_{v\notin S} \log^+ \|1/x\|_v = \sum_{v\in M_k} \log\|x\|_v,$$

which is zero by the product formula [Lang 1970, Chapter V, § 1]. Consequently, the First Main Theorem (which, with the above definitions, asserts that the height $T_D(r)$ depends up to $O(1)$ only on the linear equivalence class of $D$, and hence we may write $T_{\mathscr{L}}(r)$ for a line sheaf $\mathscr{L}$) translates into the same assertion for the height $h_D(x)$, which is again a standard fact.

Likewise, consider the following weak version of the Second Main Theorem: if $X$ is a smooth complex projective curve, if $D$ is an effective divisor on $X$ with no multiple points, if $K$ is a canonical divisor on $X$, if $A$ is an ample divisor on

$X$, and if $\varepsilon > 0$, then any holomorphic curve $f : \mathbb{C} \to X$ satisfies

$$m(D, r) + T_K(r) \leq_{\text{exc}} \varepsilon\, T_A(r) + O(1). \tag{1.2}$$

Here the subscript "exc" means that the inequality holds for all $r > 0$ outside a subset of finite Lebesgue measure. This inequality implies all the "No" entries in the middle column of Table 1.

This translates into the number field case as follows [Vojta 1992]: if $X$ is a smooth projective curve over a number field $k$, if $D$ is an effective divisor on $X$ with no multiple points, if $K$ is a canonical divisor on $X$, if $A$ is an ample divisor on $X$, if $\varepsilon > 0$, and if $C$ is a constant, then

$$m(D, x) + h_K(x) \leq \varepsilon\, h_A(x) + C \tag{1.3}$$

for all but finitely many $x \in X(k)$. Again, this implies all the "No" entries in the right-hand column of Table 1.

If $\dim X > 1$, note that the above dictionary still refers to holomorphic curves, so that equidimensional results do not play a role here (although they motivate conjectures for holomorphic curves). In this case one may restrict $f$ to be a Zariski-dense holomorphic curve, and correspondingly restrict the set of rational points to be such that every infinite subset is Zariski-dense.

## 2. Holomorphic Curves in Varieties of Dimension Greater Than 1

In arbitrary dimension, non-existence results for non-constant holomorphic curves mainly concern subvarieties of semiabelian varieties, and quotients of bounded symmetric domains. We first consider the former.

Recall that a **semiabelian variety** over $\mathbb{C}$ is a complex group variety $A$ such that there exists an exact sequence of group varieties,

$$0 \to \mathbb{G}_m^\mu \to A \to A_0 \to 0,$$

where $A_0$ is an abelian variety. A semiabelian variety over a number field is a group variety over that number field that becomes a semiabelian variety over $\mathbb{C}$ after base change. In the context of this section, it is useful to regard semiabelian varieties as the generalization of Albanese varieties to the case of quasi-projective varieties; hence it is also common to refer to them as **quasi-abelian varieties**.

The main theorem for holomorphic curves in semiabelian varieties is the following.

THEOREM 2.1. *Let $A$ be a semiabelian variety defined over $\mathbb{C}$, let $X$ be a closed subvariety of $A$, and let $D$ be an effective divisor on $X$. Then the Zariski closure of the image of any holomorphic curve $f : \mathbb{C} \to X \setminus D$ is the translate of a subgroup of $A$ contained in $X \setminus D$.*

It is useful to think of two special cases of this theorem:

(a) $D = 0$ (Bloch's theorem)
(b) $X = A$

The general case follows from these special cases by an easy argument.

THEOREM 2.2. *Let $k$, $S$, and $R_S$ be as usual. Let $A$ be a semiabelian variety defined over $k$, let $X$ be a closed subvariety of $A$, and let $D$ be an effective divisor on $X$. Let $\mathscr{Y}$ be a model for $X \setminus D$ over* $\operatorname{Spec} R_S$. *Then the set $\mathscr{Y}(R_S)$ of $R_S$-valued points in $\mathscr{Y}$ is a finite union*

$$\mathscr{Y}(R_S) = \bigcup_i \mathscr{B}_i(R_S),$$

*where each $\mathscr{B}_i$ is a subscheme of $\mathscr{Y}$ whose generic fiber $B_i$ is a translated subgroup of $A$.*

Here the idea of a **model** for a variety comes from Arakelov theory; see [Soulé 1992, § O.2]. This more general notion is necessary because, in general, a semiabelian variety is neither projective nor affine.

Another way to view Theorem 2.2 is that, if $Z$ is the Zariski closure of the set of integral points of $X \setminus D$ (defined relative to some fixed model), then any irreducible component of $Z$ must be the translate of a subgroup of $A$ contained in $X \setminus D$. (In the case of holomorphic curves, the Zariski closure of the image of the curve is already irreducible.)

In the case of holomorphic curves, the special case $D = 0$ was proved by Bloch [1926] if $A$ is an abelian variety; see also [Siu 1995] for a history of the other contributors to this theory, including Green-Griffiths, Kawamata, and Ochiai. The more general case when $D = 0$ and $A$ is semiabelian was proved by Noguchi [1981]. The case $X = A$ was proved by Siu and Yeung [1996a] when $A$ is an abelian variety and by Noguchi [1998] when $A$ is semiabelian. This is one of the few cases in which something was proved in the number field case before the complex analytic case.

In the number field case, Faltings proved the special case in which $A$ is an abelian variety (if $X = A$ is an abelian variety, then $D$ is generally assumed to be ample, and then one obtains finiteness of integral points; this implies the result for general $D$). The general case was proved by the author. See [Faltings 1991; 1994; Vojta 1996a; 1999].

**Bounded symmetric domains.** Let $D$ be a bounded symmetric domain. Recall that the underlying real manifold can be realized as a quotient $G/K$, where $G$ is a semisimple Lie group and $K$ is a maximal compact subgroup. The group $G$ can be identified with the connected component of the group of holomorphic automorphisms of $D$. A subgroup $H$ of $G$ is called an **arithmetic subgroup** if there exists a map $i : G \to \mathrm{GL}_n(\mathbb{R})$ of Lie groups inducing an isomorphism of $G$ with a closed subvariety of $\mathrm{GL}_n(\mathbb{R})$ defined over $\mathbb{Q}$, such that $H$ is commensurable with $i^{-1}(\mathrm{GL}_n(\mathbb{Z}))$. Here two subgroups $H_1$ and $H_2$ of a group $G$

are **commensurable** if $H_1 \cap H_2$ is of finite index in $H_1$ and $H_2$. See [Baily and Borel 1966, 3.3]. Finally, an **arithmetic quotient** of $D$ is a quotient of $D$ by an arithmetic subgroup of $G$.

The quotient of a bounded symmetric domain $D = G/K$ by an arithmetic subgroup $\Gamma$ of $G$ does not contain a nontrivial holomorphic curve. Indeed, this follows by lifting the curve to $D$ and applying Liouville's theorem. For a proof in the spirit of Nevanlinna theory, see [Griffiths and King 1973, Corollary 9.22]. Also, for a slightly stronger result, see [Vojta 1987, 5.7.7], using Theorem 5.7.2 instead of Conjecture 5.7.5 of the same reference.

Of course, nothing in the above paragraph made essential use of the fact that $\Gamma$ is an arithmetic subgroup. The interest in arithmetic subgroups stems from the result of Baily and Borel [1966], showing that an arithmetic quotient of a bounded symmetric domain is a complex quasi-projective variety.

In general there is a wide choice of immersions $i$, leading to a wide choice of commensurability classes of arithmetic subgroups. Therefore, an arithmetic quotient is not necessarily defined over a number field. When it is, however, the philosophy of Section 1 suggests that the set of integral points on any given model of the quotient would be finite:

CONJECTURE 2.3. *Let $X$ be a quasi-projective variety over a number field $k$, whose set of complex points is isomorphic to an arithmetic quotient of a bounded symmetric domain. Then, for any $S$ and model $\mathscr{X}$ for $X$ over $R_S$ (where $S$ and $R_S$ are as in the introduction of this chapter), the set $\mathscr{X}(R_S)$ of integral points of $\mathscr{X}$ is finite.*

This is unknown except for one special case. Let $\mathscr{A}_{g,n}$ denote the moduli space of principally polarized abelian varieties of dimension $g$ with level-$n$ structure. For $n$ sufficiently large, $\mathscr{A}_{g,n}$ is a quasi-projective variety defined over a number field, and its set of complex points is isomorphic to an arithmetic quotient of a bounded symmetric domain (in fact, the Siegel upper half plane). By Conjecture 2.3, $\mathscr{A}_{g,n}$ should have only finitely many integral points over $R_S$, for any number field over which this variety is defined. In fact, this is true, since $S$-integral points correspond to abelian varieties with good reduction outside $S$ with given level-$n$ structure, and there are only finitely many such varieties for given $g$, $n$, $k$, and $S$. This was conjectured by Shafarevich and proved by Faltings [1991]. By an extension of the Chevalley–Weil theorem [Vojta 1987, Theorem 5.1.6], this then extends to the quotient by any subgroup commensurable with $\mathrm{Sp}_{2g}(\mathbb{Z})$.

Faltings' proof of the Shafarevich conjecture, however, does not correspond to the proof of Griffiths-King. Of course it is difficult to compare proofs between Nevanlinna theory and number theory, especially for the fundamental results. However, essentially all other results of Second Main Theorem type in the number field case are proved by constructing an auxiliary polynomial. Faltings' proof, on the other hand, uses Hodge theory. Therefore a proof of the Shafarevich conjecture via construction of an auxiliary polynomial would be good to have. Indeed, the proof of this result for holomorphic curves has some moderate differences from other proofs for holomorphic curves, so this may shed more light on the analogy of Section 1.

**Conjectures on general varieties.** Concerning the qualitative question of existence of non-constant holomorphic curves or infinite sets of integral points, a general conjecture has been formulated by S. Lang [1991, Chapter VIII, Conjecture 1.4]. Let $X$ be a variety defined over a subfield of $\mathbb{C}$. We begin by describing various **special sets** in $X$.

DEFINITION 2.4. The **algebraic special set** $\mathrm{Sp}_{\mathrm{alg}}(X)$ is the Zariski closure of the union of all images of non-constant rational maps $f : G \to X$ of group varieties into $X$.

DEFINITION 2.5. If $X$ is defined over $\mathbb{C}$, the **holomorphic special set** $\mathrm{Sp}_{\mathrm{hol}}(X)$ is the Zariski closure of the union of all images of non-constant holomorphic maps $f : \mathbb{C} \to X$.

We have, trivially, $\mathrm{Sp}_{\mathrm{alg}}(X) \subseteq \mathrm{Sp}_{\mathrm{hol}}(X)$.

A general conjecture for rational points on projective varieties is:

CONJECTURE 2.6 [Lang 1991, Chapter VIII, Conjectures 1.3 and 1.4]. *Let $X$ be a projective variety defined over a subfield $K$ of $\mathbb{C}$ finitely generated over $\mathbb{Q}$. Then*

$$\mathrm{Sp}_{\mathrm{alg}}(X) \times_K \mathbb{C} = \mathrm{Sp}_{\mathrm{hol}}(X \times_K \mathbb{C});$$

*that is, the algebraic and holomorphic special sets are the same. Moreover, the following are equivalent:*

(i) *$X$ is of general type.*
(ii) *$X$ is* **pseudo-Brody hyperbolic**; *that is,* $\mathrm{Sp}_{\mathrm{hol}}(X \times_K \mathbb{C}) \subsetneqq X \times_K \mathbb{C}$.
(iii) *$X$ is* **pseudo Mordellic**; *that is,* $\mathrm{Sp}_{\mathrm{alg}}(X) \subsetneqq X$ *and for any finitely generated extension field $K'$ of $K$, $(X \setminus \mathrm{Sp}_{\mathrm{alg}}(X))(K')$ is finite.*

Here we are primarily interested in the case in which $K$ is a number field. However, the above also contains the Green–Griffiths conjecture (implicit in [Green and Griffiths 1980]; see also [Lang 1991, Chapter VIII, § 1]), which says that if $X$ is a complex projective variety of general type, then the image of a holomorphic curve cannot be Zariski dense. Indeed, if $X$ is defined over $\mathbb{C}$, then it can be obtained from a variety $X$ as above; then use the implication (i) $\Longrightarrow$ (ii).

For integral points, the case is not so clear, since the boundary may affect things in a number of different ways. So fewer implications are conjectured here. See [Lang 1991, Chapter IX, § 5] for explanations.

CONJECTURE 2.7. *Let $R$ be a subring of $\mathbb{C}$, finitely generated over $\mathbb{Z}$, let $K$ be its field of fractions, and let $X$ be a quasi-projective variety over $K$. Consider the following conditions.*

(i) *$X$ is of logarithmic general type.*

(ii) *$X$ is* **pseudo-Brody hyperbolic**.

(iii) *$X$ is* **pseudo Mordellic**; *that is*, $\mathrm{Sp}_{\mathrm{alg}}(X) \subsetneqq X$, *and for every scheme* $\mathscr{X}$ *over* Spec $R$ *with generic fiber isomorphic to* $X$, *and for any finitely generated extension ring* $R'$ *of* $R$, *all but finitely many points of* $\mathscr{X}(R')$ *lie in* $\mathrm{Sp}_{\mathrm{alg}}(X)$.

*Then* (i) $\Longrightarrow$ (ii) *and* (i) $\Longrightarrow$ (iii).

## 3. Conjectural Second Main Theorems

Motivated by the situation in the equidimensional case, it is generally believed that the Second Main Theorem should hold for holomorphic curves in arbitrary (nonsingular) varieties:

Definition. Let $X$ be a nonsingular complex variety and let $D$ be a divisor on $X$. We say that $D$ is a **normal crossings divisor** (or that $D$ **has normal crossings**) if each point $P \in X$ has a open neighborhood (in the classical topology) with local coordinates $z_1, \dots, z_n$ such that $D$ is locally equal to the principal divisor $(z_1 \cdots z_r)$ for some $r \in \{0, \dots, n\}$. Note that this implies that $D$ is effective and has no multiple components. If $X$ is a variety over a number field $k$, then a divisor $D$ on $X$ has normal crossings if the corresponding divisor $X \times_k \mathbb{C}$ does, for some embedding $k \hookrightarrow \mathbb{C}$.

Conjecture 3.1. *Let $X$ be a nonsingular complex projective variety, let $D$ be a normal crossings divisor on $X$, let $K$ be a canonical divisor on $X$, let $A$ be an ample divisor on $X$, and let $\varepsilon > 0$. Then there exists a proper Zariski-closed subset $Z \subseteq X$, depending only on $X$, $D$, $A$, and $\varepsilon$, such that for any holomorphic curve $f : \mathbb{C} \to X$ whose image is not contained in $Z$,*

$$m(D, r) + T_K(r) \le_{\mathrm{exc}} \varepsilon\, T_A(r) + O(1).$$

Here the notation $\le_{\mathrm{exc}}$ means that the inequality holds for all $r$ outside a set of finite Lebesgue measure.

The corresponding statement in the number field case is also highly conjectural:

Conjecture 3.2. *Let $X$ be a nonsingular projective variety over a number field $k$, let $D$ be a normal crossings divisor on $X$, let $K$ be a canonical divisor on $X$, let $A$ be an ample divisor on $X$, and let $\varepsilon > 0$. Then there exists a proper Zariski-closed subset $Z \subseteq X$, depending only on $X$, $D$, $A$, and $\varepsilon$, such that for all rational points $x \in X(k)$ with $x \notin Z$,*

$$m(D, x) + T_K(x) \le \varepsilon\, h_A(x) + O(1).$$

Moreover, the set $Z$ should be the same in both these conjectures.

Conjectures 3.1 and 3.2 give the implications (i) $\Longrightarrow$ (ii) and (i) $\Longrightarrow$ (iii) of Conjecture 2.7, respectively.

The set $Z$ must depend on $\varepsilon$: see [Vojta 1989b, Example 8.15].

## 4. Approximation to Hyperplanes in Projective Space

In dimension $> 1$, the only case in which Conjectures 3.1 and 3.2 are known in their full strength is when $X = \mathbb{P}^n$ and $D$ is a union of hyperplanes. This was proved for holomorphic curves by Cartan [1933] and for rational points by W. M. Schmidt [1980, Chapter VI, Theorem 1F]. See [Vojta 1987, Chapter 2] for a description of how to formulate Schmidt's theorem in a form similar to Cartan's.

We recall the statement of Cartan's theorem; except for the stronger error term, this is proved in [Cartan 1933]:

THEOREM 4.1. *Let $n > 0$, let $H_1, \dots, H_q$ be hyperplanes in $\mathbb{P}^n_{\mathbb{C}}$ lying in general position* (*that is, so that $H_1 + \cdots + H_q$ is a normal crossings divisor*), *and let $f : \mathbb{C} \to \mathbb{P}^n_{\mathbb{C}}$ be a holomorphic curve not lying in any hyperplane* (*linearly nondegenerate*). *Then*

$$\sum_{i=1}^{q} m(H_i, r) \le_{\text{exc}} (n+1)T(r) + O(\log^+ T(r)) + o(\log r). \tag{4.1.1}$$

This is a special case of Conjecture 3.1, except for the stronger error term and the weaker condition concerning $Z$.

Actually, a straightforward translation of Theorem 4.1 into the number field case gives something that is not quite as strong as Schmidt's subspace theorem, due to the fact that Schmidt's theorem allows a different collection of hyperplanes for each $v \in S$, so that the aggregate collection is not necessarily in general position.

To describe this further, let $x_0, \dots, x_n$ be homogeneous coordinates on $\mathbb{P}^n$. Write $D = H_1 + \cdots + H_q$ and for each $i$ let $a_{i0}x_0 + \cdots + a_{in}x_n$ be a nonzero linear form vanishing on $H_i$. Let $k$ be a number field and $S$ a finite set of places of $k$. Then, for $v \in S$ and $x \in \mathbb{P}^n(k) \setminus H_i$, we define the **Weil function** for $H_i$ as

$$\lambda_{H_i,v}(x) = -\log \frac{\|a_{i0}x_0 + \cdots + a_{in}x_n\|_v}{\max\{\|x_0\|_v, \dots, \|x_n\|_v\}}, \tag{4.2}$$

These Weil functions depend on the choice of $a_{i0}, \dots, a_{in}$ only up to $O(1)$; the choice of a linear form $a_{i0}x_0 + \cdots + a_{in}x_n$ amounts to choosing a section $s$ of $\mathscr{O}(1)$; the fraction in (4.2) can then be regarded as a metric of that section. Thus, as in Table 3, we may take

$$m(H_i, x) = \frac{1}{[k : \mathbb{Q}]} \sum_{v \in S} \lambda_{H_i,v}(x)$$

Then Schmidt's Subspace Theorem can be stated as follows.

THEOREM 4.3. *Let $k$ be a number field, let $S$ be a finite set of places of $k$, let $n > 0$, let $H_1, \dots, H_q$ be hyperplanes in $\mathbb{P}^n_k$, and let $\varepsilon > 0$. Then*

$$\frac{1}{[k : \mathbb{Q}]} \sum_{v \in S} \max_L \sum_{i \in L} \lambda_{H_i,v}(x) \le (n+1+\varepsilon)h(x) + O(1) \tag{4.3.1}$$

*for all* $x \in \mathbb{P}^n(k)$ *outside of a finite union of proper linear subspaces depending only on* $k$, $S$, $H_1, \dots, H_q$, *and* $\varepsilon$. *Here* $L$ *varies over all subsets of* $\{1, \dots, q\}$ *for which the set* $\{H_i\}_{i \in L}$ *lies in general position.*

Its translation into the case of holomorphic curves is also true:

THEOREM 4.4 [Vojta 1997]. *Let* $n > 0$, *let* $H_1, \dots, H_q$ *be hyperplanes in* $\mathbb{P}^n_{\mathbb{C}}$, *and let* $f : \mathbb{C} \to \mathbb{P}^n_{\mathbb{C}}$ *be a holomorphic curve not lying in any hyperplane* (*linearly nondegenerate*). *Then*

$$\int_0^{2\pi} \max_L \sum_{j \in L} \lambda_{H_j}(f(re^{i\theta}))\frac{d\theta}{2\pi} \leq_{\text{exc}} (n+1)T(r) + O(\log^+ T(r)) + o(\log r). \quad (4.4.1)$$

Note that, if $H_1, \dots, H_q$ lie in general position (that is, the divisor $H_1 + \dots + H_q$ has normal crossings), then we may take $L = \{1, \dots, q\}$, and the maximum occurs up to $O(1)$ at that value of $L$. Thus, the left-hand side of (4.4.1) is

$$\int_0^{2\pi} \sum_{j=1}^{q} \lambda_{H_j}(f(re^{i\theta}))\frac{d\theta}{2\pi} + O(1) = \sum_{j=1}^{q} m(H_j, r) + O(1),$$

which coincides with the left-hand side of (4.1.1).

Similar comments apply to Theorem 4.3.

**Exceptional sets.** In [Vojta 1989b; 1997], the conditions on exceptional sets or nondegeneracy were sharpened as follows.

THEOREM 4.5. *In the notation of Theorem* 4.3, *there exists a proper Zariski-closed subset* $Z \subsetneq \mathbb{P}^n_k$, *depending only on* $H_1, \dots, H_q$, *such that* (4.3.1) *holds for all* $x \in \mathbb{P}^n(k) \setminus Z$.

THEOREM 4.6. *In the notation of Theorem* 4.4, *there exists a proper Zariski-closed subset* $Z \subsetneq \mathbb{P}^n_{\mathbb{C}}$, *depending only on* $H_1, \dots, H_q$, *such that* (4.4.1) *holds for all holomorphic curves* $f$ *whose image does not lie in* $Z$.

Thus, for hyperplanes in projective space, the condition regarding $Z$ is sharper than in Conjectures 3.1 and 3.2, because it does not depend on $\varepsilon$.

**Cartan's conjecture.** A conjecture of Cartan concerns the question of what happens in Theorem 4.1 if the holomorphic curve is linearly degenerate. If it is degenerate, say if the linear span of its image has codimension $t$, then Cartan conjectured that the $n + 1$ factor in front of $T(r)$ should increase to $n + t + 1$. This was finally proved by Nochka [1982; 1983]; it was subsequently improved by Chen, and converted to the number field case by Ru and Wong [1991]. See also [Vojta 1997].

THEOREM 4.7. *Let* $n > 0$, *let* $H_1, \dots, H_q$ *be hyperplanes in* $\mathbb{P}^n_{\mathbb{C}}$ *in general position*, *let* $f : \mathbb{C} \to \mathbb{P}^n_{\mathbb{C}}$ *be a holomorphic curve*, *and let* $t$ *be the codimension* *of the linear span of $f$. Then*

$$\sum_{i=1}^{q} m(H_i, r) \le_{\text{exc}} (n+t+1)T(r) + O(\log^+ T(r)) + o(\log r).$$

This was proved by assigning a **Nochka weight** $\omega_i$ to each hyperplane $H_i$. Shiffman observed that one of the conditions on these weights can be interpreted as the $\mathbb{Q}$-divisor $\omega_1 H_1 + \cdots + \omega_q H_q$ being **log canonical**; see [Shiffman 1996].

**Additional refinements.** The fact that one is working with hyperplanes in projective space means that one can work largely in linear algebra. The proof of Cartan's conjecture, for example, is largely a question of some (very difficult) linear algebra. Some other, simpler results of this nature have also been known.

THEOREM 4.8 [Dufresnoy 1944, Theorem XVI]. *Let $n > 0$ and $k > 0$, let $H_1, \dots, H_{n+k}$ be hyperplanes in $\mathbb{P}^n_{\mathbb{C}}$ in general position, and let $f : \mathbb{C} \to \mathbb{P}^n_{\mathbb{C}}$ be a holomorphic curve that does not meet $H_1, \dots, H_{n+k}$. Then the image of $f$ is contained in a linear subspace of dimension $\le [n/k]$, where the brackets denote greatest integer.*

This result has also been independently rediscovered by other authors.

COROLLARY 4.9. *A holomorphic curve that misses $2n+1$ hyperplanes in general position must be constant.*

Consequently, the complement of those hyperplanes is **Brody hyperbolic**.

These results hold also in the number field case; see [Ru and Wong 1991].

**A diophantine inequality and semistability.** Faltings and Wüstholz prove a finiteness result involving the notion of semistability of a filtration on a vector space. This requires a few definitions to state.

Let $K$ be a number field, let $L$ be a finite extension of $K$, and let $S$ be a finite set of places of $K$. For each $w \in S$ let $I_w$ be a finite index set. For each $\alpha \in I_w$ let $s_{w,\alpha}$ be a nonzero section of $\Gamma(\mathbb{P}^n_K, \mathscr{O}(1))$; that is, a nonzero linear form in $X_0, \dots, X_n$. Also choose a real number $c_{w,\alpha} \ge 0$ for each $w$ and $\alpha$.

Let $V = \Gamma(\mathbb{P}^n_K, \mathscr{O}(1))$ and $V_L = V \otimes_K L$. For each $w \in S$ the choices of $s_{w,\alpha}$ and $c_{w,\alpha}$ define a filtration

$$V_L = W_w^0 \supseteq W_w^1 \supseteq \cdots \supseteq W_w^{e+1} = 0$$

Indeed, for $p \in \mathbb{R}$ let $W_{w,p}$ be the subspace spanned by $\{s_{w,\alpha} : c_{w,\alpha} \ge p\}$. For $j = 0, \dots, e$ let $p_{w,e}$ be the smallest value of $p$ for which $W_w^j = W_{w,p}$, and let $p_{w,e+1} = p_{w,e} + 1$.

DEFINITION 4.10. With the above notation, and for all nonzero linear subspaces $W$ of $V_L$, let

$$\mu_w(W) = \frac{1}{\dim W} \sum_{j=1}^{e} p_j \dim((W \cap W_w^j)/(W \cap W_w^{j+1})).$$

DEFINITION 4.11. With the above notation, we say that the data $s_{w,\alpha}$ and $c_{w,\alpha}$ define a **jointly semistable** fibration on $V_L$ if, for all nonzero linear subspaces $W \subseteq V_L$, we have

$$\sum_{w \in S} \mu_w(W) \le \sum_{w \in S} \mu_w(V_L).$$

The main theorem of Faltings and Wüstholz is then the following:

THEOREM 4.12 [Faltings and Wüstholz 1994, Theorem 8.1]. *With the notation above, assume that the data $s_{w,\alpha}$ and $c_{w,\alpha}$ define a jointly semistable fibration on $V_L$. Assume furthermore that*

$$\sum_{w \in S} \mu_w(V_L) > [L : K].$$

*Then the set of all points $x \in \mathbb{P}^n(K)$ satisfying*

$$\|s_{w,\alpha}(x)\|_w \le H_K(x)^{-c_{w,\alpha}} \qquad \text{for all } w \in S,\ \alpha \in I_w$$

*is finite.*

We remark that this result proves finiteness, not just that the set of points $x$ lies in a finite union of proper linear subspaces. M. McQuillan and R. Ferretti (unpublished) have translated it into a statement for holomorphic curves.

**Approximation to other divisors on $\mathbb{P}^n$.** Approximation to divisors of higher degree on $\mathbb{P}^n$ is a trickier question. At present, no results approaching the bounds of Conjecture 3.1 are known, but weaker bounds can be obtained either from the methods of Faltings and Wüstholz [1994] (over number fields), or by using a $d$-uple embedding as noted in [Shiffman 1979] (for holomorphic curves, but the translation to number fields is immediate).

## 5. The Complement of Curves in $\mathbb{P}^2$

Conjecture 3.1 implies that, if $D$ is a normal crossings divisor on $\mathbb{P}^2$ of degree at least 4, then any holomorphic curve in $\mathbb{P}^2 \setminus D$ must lie in a fixed divisor $E$ depending only on $D$. Moreover, it is conjectured that if $\deg D \ge 5$, then we may take $E = 0$ for a suitably generic choice of $D$. More specifically, let $d_1, \dots, d_k$ be positive integers with $d_1 + \cdots + d_k \ge 5$. Then it is conjectured that there exists a dense Zariski-open subset of the space of all divisors with irreducible components of degrees $d_1, \dots, d_k$, respectively, such that if $D$ is a divisor corresponding to a point in that open subset, then we may take $E = 0$. Partial results for the latter conjecture are as follows:

If $D$ consists of five or more lines in general position, then the conjecture follows from Corollary 4.9. If $D$ consists of any five components such that no three intersect, the conjecture was proved by Babets [1984] and by Eremenko and Sodin [1991]; more generally, their proof applies to any divisor $D$ on $\mathbb{P}^n$ with at least $2n+1$ irreducible components, such that any $n+1$ of them have empty intersection.

The case of a quadric and four lines was proved by M. Green [1975]. If $D$ is composed of at least three irreducible components, none of which is a line, then the conjecture was proved by Grauert [1989] and by Dethloff, Schumacher and Wong [Dethloff et al. 1995a; 1995b].

The case in which $D$ is irreducible and smooth is much harder; in this case the only known general results are due Zaĭdenberg and to Siu and Yeung:

THEOREM 5.1. *For positive integers $d$, let $\Sigma_d$ denote the set of complex curves in $\mathbb{P}^2$ of degree $d$.*

(a) [Zaĭdenberg 1988] *If $d \geq 5$ then the set of points in $\Sigma_d$ corresponding to smooth curves whose complement is* (*Kobayashi*) *hyperbolic and hyperbolically embedded*, *is nonempty and open in the classical topology*.
(b) [Siu and Yeung 1996b] *If $d \geq 5 \times 10^{13}$ then there exists a Zariski-dense open subset of $\Sigma_d$ such that*, *if $C$ is a curve corresponding to a point in that subset*, *then $\mathbb{P}^2 \setminus C$ is Brody hyperbolic.*

Recall that Kobayashi hyperbolicity implies Brody hyperbolicity, but that the converse holds only on compact manifolds. Thus the above theorem provides partial answers to a question posed by S. Kobayashi [1970, p. 132]: Is the complement in $\mathbb{P}^n$ of a generic hypersurface of high degree hyperbolic?

See also [Masuda and Noguchi 1996], for hypersurfaces defined by polynomials with few terms.

Of these results, only those relying on the Borel lemma translate to the number field case. See [Ru and Wong 1991] for the result concerning five lines in general position, and [Ru 1993] for the result of Babets and Eremenko and Sodin.

In addition, Ru [1995] has shown that the complement of a collection of hyperplanes in $\mathbb{P}^n$ has no nontrivial holomorphic curves if and only if it has only finitely many integral points.

## 6. Refinements of the Error Term

Motivated by Khinchin's theorem on approximation to arbitrary real numbers by rational numbers, Lang [1971] conjectured that the error term in Roth's theorem could be strengthened considerably. See also [Lang and Cherry 1990, page 10, including the footnote], for references to earlier, weaker conjectures.

No progress has been made on this, but corresponding questions in Nevanlinna theory have been solved; these questions were motivated by the conjecture in number theory and the dictionary with Nevanlinna theory. For example, the lemma on the logarithmic derivative has been strengthened by Miles as follows:

THEOREM 6.1 [Miles 1992]. *Let $f$ be a meromorphic function on $\mathbb{C}$, and let $\phi : [1, \infty) \to [1, \infty)$ be a continuous function such that $\phi(x)/x$ is nondecreasing* *and*

$$\int_1^\infty \frac{dx}{\phi(x)} < \infty.$$

*Then*

$$m(r, f'/f) <_{\text{exc}} \log^+ \left( \frac{\phi(T_f(r))}{r} \right) + O(1),$$

*where in this case the notation* $<_{\text{exc}}$ *means that the inequality holds for all* $r$ *outside a set* $E$ *with*

$$\int_E \frac{dt}{t} < \infty.$$

Corresponding versions of the Second Main Theorem for meromorphic functions and for hyperplanes in $\mathbb{P}^n$ were proved by Hinkkanen [1992] and by Ye [1995], respectively (although the details of the error terms vary somewhat).

## 7. Slowly Moving Targets

Early on, Nevanlinna conjectured that the Second Main Theorem should remain valid if the constants $a_i$ being approximated were replaced by meromorphic functions $a_i(z)$, provided that these functions move *slowly;* that is, $T_{a_i}(r) = o(T_f(r))$, where $f$ is the meromorphic function that is doing the approximating.

THEOREM 7.1. *Let $f$ be a meromorphic function and let $a_1, \dots, a_q$ be meromorphic functions with $T_{a_j}(r) = o(T_f(r))$ for all $j$. Then, for all $\varepsilon > 0$,*

$$\sum_{j=1}^{q} m_f(a_j, r) \le_{\text{exc}} (2 + \varepsilon) T_f(r).$$

Nevanlinna proved this when $q \le 3$. The general case was proved by Osgood [1981] (motivated, surprisingly, by the proof of Roth's theorem, and Osgood's own analogy with Nevanlinna theory). Soon after that, Steinmetz [1986] found a simple, elegant proof. It was generalized to the case of moving hyperplanes in $\mathbb{P}^n$ by Ru and Stoll [1991a]. In that case, extra care is necessary: since the hyperplanes are moving, the diagonal hyperplanes are also moving, and one needs to make sure that the holomorphic curve does not stay within such a diagonal (or other linear subspace, as in Theorem 4.6), because then the inequality would no longer hold. Therefore a stronger condition than linear nondegeneracy is needed.

Describing this stronger condition requires some additional notation. Let $n > 0$ and let $H_1, \dots, H_q : \mathbb{C} \to (\mathbb{P}^n)^*$ be moving hyperplanes. For each $j$ choose holomorphic functions $a_{j0}, \dots, a_{jn}$ such that $H_j$ is the hyperplane determined by the vanishing of the linear form $a_{j0}x_0 + \dots + a_{jn}x_n$. For such a collection $\mathscr{H} := \{H_1, \dots, H_q\}$, let $\mathscr{R}_{\mathscr{H}}$ denote the field of meromorphic functions generated over $\mathbb{C}$ by all ratios $a_{jk}/a_{jl}$ such that $a_{jl} \ne 0$, where $j = 1, \dots, q$, $k = 0, \dots, n$, $l = 0, \dots, n$.

Definition 7.2. Let $f : \mathbb{C} \to \mathbb{P}^n$ be a holomorphic curve, written in homogeneous coordinates as $f = [f_0 : \cdots : f_n]$, where $f_0, \ldots, f_n$ are holomorphic functions with no common zero. Then we say that $f$ is **linearly nondegenerate** over $\mathscr{R}_{\mathscr{H}}$ if the functions $f_0, \ldots, f_n$ are linearly independent over the field $\mathscr{R}_{\mathscr{H}}$.

Theorem 7.3 [Ru and Stoll 1991a]. *Let $n$, $H_1, \ldots, H_q$ and $f$ be as above. Assume that*:

(i) *for at least one value of $z$ (and hence for almost all $z$), $H_1(z), \ldots, H_q(z)$ are in general position*;
(ii) *$T_{H_j}(r) = o(T_f(r))$ for all $j$ (where $T_{H_j}(r)$ is defined via the isomorphism $(\mathbb{P}^n)^* \cong \mathbb{P}^n$); and*
(iii) *$f$ is linearly nondegenerate over $\mathscr{R}_{\mathscr{H}}$.*

*Then, for all $\varepsilon > 0$,*

$$\sum_{j=1}^{q} m_f(H_j, r) \le_{\text{exc}} (n + 1 + \varepsilon) T_f(r).$$

In [Ru and Stoll 1991b] this theorem was generalized to the case of Cartan's conjecture (Theorem 4.7).

These results were carried over to the number field case by Bombieri and van der Poorten [1988] and by the author [Vojta 1996b] for Roth's theorem; by Ru and Vojta [1997] for Schmidt's theorem and Cartan's conjecture; and by Tucker [1997] for approximation to moving divisors on an elliptic curve.

A representative sample of such a statement is that of Schmidt's theorem with moving targets. We begin with some definitions.

Definition 7.4. Let $I$ be an infinite index set.

(i) A **moving hyperplane** indexed by $I$ is a function $H : I \to (\mathbb{P}^n)^*(k)$, denoted $i \mapsto H(i)$.
(ii) Let $H_1, \ldots, H_q$ be moving hyperplanes. For each $j = 1, \ldots, q$ and each $i \in I$ choose $a_{j,0}(i), \ldots, a_{j,n}(i) \in k$ such that $H_j(i)$ is cut out by the linear form $a_{j,0}(i)X_0 + \cdots + a_{j,n}(i)X_n$. Then a subset $J \subseteq I$ is **coherent** with respect to $H_1, \ldots, H_q$ if, for every polynomial

$$P \in k[X_{1,0}, \ldots, X_{1,n}, \ldots, X_{q,0}, \ldots, X_{q,n}]$$

that is homogeneous in $X_{j,0}, \ldots, X_{j,n}$ for each $j = 1, \ldots, q$, either

$$P(a_{1,0}(i), \ldots, a_{1,n}(i), \ldots, a_{q,0}(i), \ldots, a_{q,n}(i))$$

vanishes for all $i \in J$, or it vanishes for only finitely many $i \in J$.
(iii) We define $\mathscr{R}_I^0$ to be the set of equivalence classes of pairs $(J, a)$, where $J \subseteq I$ is a subset with finite complement; $a : J \to k$ is a map; and the equivalence relation is defined by $(J, a) \sim (J', a')$ if there exists $J'' \subseteq J \cap J'$ such that $J''$ has finite complement in $I$ and $a\big|_{J''} = a'\big|_{J''}$. This is a ring containing $k$ as a subring.

(iv) Let $H_1, \dots, H_q$ be moving hyperplanes, denoted collectively by $\mathscr{H}$. If $J$ is coherent with respect to $\mathscr{H}$, and if $a_{j,\alpha}(i) \neq 0$ for some $i \in J$, then $a_{j,\beta}/a_{j,\alpha}$ defines an element of $\mathscr{R}_J^0$. Moreover, by coherence, the subring of $\mathscr{R}_J^0$ generated by all such elements is *entire*. We define $\mathscr{R}_{J,\mathscr{H}}$ to be the field of fractions of that entire ring.

Thus, a little additional work is needed in order to define something having a property that comes automatically with meromorphic functions: that is, a meromorphic function either vanishes identically, or it is nonzero almost everywhere.

Given a hyperplane $H$ defined by the linear form $a_0X_0 + \dots + a_nX_n$ and a point $\boldsymbol{x} \notin H$ with homogeneous coordinates $[x_0 : \dots : x_n]$, we define a more precise Weil function at a place $v \in M_k$ by

$$\lambda_{H,v}(\boldsymbol{x}) = -\log \frac{\|a_0x_0 + \dots + a_nx_n\|_v}{\max_{0\le\alpha\le n}\|a_\alpha\|_v \cdot \max_{0\le\alpha\le n}\|x_\alpha\|_v}. \tag{7.5}$$

The extra term $\max_{0\le\alpha\le n}\|a_\alpha\|_v$ ensures that $\lambda_{H,v}(\boldsymbol{x})$ depends only on $H$ and $\boldsymbol{x}$, and not on $a_0, \dots, a_n$ or on the choice of homogeneous coordinates $[x_0 : \dots : x_n]$.

Schmidt's subspace theorem with moving targets can now be stated as follows.

THEOREM 7.6. *Let $k$ be a number field, let $S$ be a finite set of places of $k$, let $n > 0$, let $I$ be an index set, and let $H_1, \dots, H_q$ be moving hyperplanes in $\mathbb{P}_k^n$, denoted collectively by $\mathscr{H}$. Also let $\boldsymbol{x} : I \to \mathbb{P}^n(k)$ be a sequence of points, and let $[x_0 : \dots : x_n]$ be homogeneous coordinates for $\boldsymbol{x}$. Suppose that*

(i) *for all $i \in I$, the hyperplanes $H_1(i), \dots, H_q(i)$ are in general position*;
(ii) *for each infinite coherent subset $J \subseteq I$, $x_0\big|_J, \dots, x_n\big|_J$ are linearly independent over $\mathscr{R}_{J,\mathscr{H}}$; and*
(iii) *$h_k(H_j(i)) = o(h_k(\boldsymbol{x}(i)))$ for all $j = 1, \dots, q$ (that is, for all $\delta > 0$,*

$$h_k(H_j(i)) \le \delta h_k(\boldsymbol{x}(i))$$

*for all but finitely many $i \in I$).*

*Then for all $\varepsilon > 0$ and all $C \in \mathbb{R}$,*

$$\frac{1}{[k:\mathbb{Q}]}\sum_{v\in S}\sum_{j=1}^{q}\lambda_{H_j(i),v}(\boldsymbol{x}) \le (n+1+\varepsilon)h(\boldsymbol{x}(i)) + C$$

*for all but finitely many $i \in I$.*

Theorem 7.6 is proved using an extension of Steinmetz's method; in the end it reduces to reducing the problem to Schmidt's subspace theorem with fixed targets, but in a space of much higher dimension. As M. Ru points out [1997], it is more convenient to use the variant Theorems 4.3 and 4.4 instead of the formulation of Theorem 4.1. By the same token, it would be better to phrase Theorem 7.6 in these terms as well; in fact, the proof of [Ru and Vojta 1997] actually gives the following stronger result.

THEOREM 7.7. *Let $k$, $S$, $I$, $\mathscr{H}$, $H_1, \dots, H_q$, $\boldsymbol{x}$, and $[x_0 : \cdots : x_n]$ be as in the first two sentences of Theorem* 7.6. *Also let $\mathscr{L}$ be a collection of subsets of $\{1, \dots, q\}$. Suppose that*

(i) *for all $i \in I$ and all $L \in \mathscr{L}$, the hyperplanes $H_j(i)$, $j \in L$ are in general position*;
(ii) *for each infinite coherent subset $J \subseteq I$, $x_0\big|_J, \dots, x_n\big|_J$ are linearly independent over $\mathscr{R}_{J,\mathscr{H}}$*; *and*
(iii) *$h_k(H_j(i)) = o(h_k(\boldsymbol{x}(i)))$ for all $j = 1, \dots, q$.*

*Then for all $\varepsilon > 0$ and all $C \in \mathbb{R}$,*

$$\frac{1}{[k:\mathbb{Q}]} \sum_{v \in S} \max_{L \in \mathscr{L}} \sum_{j \in L} \lambda_{H_j(i),v}(\boldsymbol{x}) \le (n+1+\varepsilon) h(\boldsymbol{x}(i)) + C$$

*for all but finitely many $i \in I$.*

Similarly, we can define a more precise Weil function in the context of holomorphic curves as

$$\lambda_H(\boldsymbol{x}) = -\log \frac{|a_0 x_0 + \cdots + a_n x_n|}{\max_{0 \le \alpha \le n} |a_\alpha| \cdot \max_{0 \le \alpha \le n} |x_\alpha|}$$

(using the notation of (7.5)). Then the methods of [Ru and Stoll 1991a] immediately give:

THEOREM 7.8. *Let $n > 0$ be an integer, let $H_1, \dots, H_q$ be moving hyperplanes in $\mathbb{P}^n$, and let $f$ be a holomorphic curve in $\mathbb{P}^n$. Assume that*:

(i) *$T_{H_j}(r) = o(T_f(r))$ for all $j$ (where $T_{H_j}(r)$ is defined via the isomorphism $(\mathbb{P}^n)^* \cong \mathbb{P}^n$); and*
(ii) *$f$ is linearly nondegenerate over $\mathscr{R}_{\mathscr{H}}$.*

*Then, for all $\varepsilon > 0$,*

$$\int_0^{2\pi} \max_L \sum_{j \in L} \lambda_{H_j}(f(re^{i\theta})) \frac{d\theta}{2\pi} \le_{\text{exc}} (n+1+\varepsilon) T_f(r),$$

*where $L$ varies over all subsets of $\{1, \dots, q\}$ for which $(H_j)_{j \in L}$ lie in general position (for at least one value of $z$).*

## 8. Discriminant Terms

Instead of working with rational points in inequalities such as (1.3) and Conjecture 3.2, one may conjecture more generally that the inequalities hold for *algebraic* points, provided that the inequalities are modified appropriately. This modification involves the discriminant of the number field generated by the algebraic point in question. To justify this suggestion, we begin with Nevanlinna theory.

Inequality (1.2) may be rewritten, via the definition $T_D(r) = m(D,r) + N(D,r)$, as

$$N(D,r) \geq_{\text{exc}} T_{K+D}(r) - \varepsilon\, T_A(r) - O(1).$$

This may be sharpened to

$$N^{(1)}(D,r) \geq_{\text{exc}} T_{K+D}(r) - \varepsilon\, T_A(r) - O(1), \tag{8.1}$$

where the counting function is replaced by the **truncated counting function**, which is defined for effective divisors $D$ by

$$N^{(1)}(D,r) := \sum_{w \in \mathbb{D}_r^\times} \min\{1, \operatorname{ord}_w f^*s\} \cdot \log \frac{r}{|w|}.$$

Of course, one can make the same definition in the context of number fields:

$$N^{(1)}(D,x) = \frac{1}{[k:\mathbb{Q}]} \sum_{v \notin S} \min\{1, \operatorname{ord}_v s(x)\} \cdot \log(R:\mathfrak{p}) \tag{8.2}$$

where as usual $\mathfrak{p}$ is the prime ideal in $R$ corresponding to the non-archimedean place $v$. One may then conjecture that (1.3) can be replaced by the stronger inequality

$$N^{(1)}(D,x) \geq h_{K+D}(x) - \varepsilon\, h_A(x) - C. \tag{8.3}$$

In Nevanlinna theory there is a stronger statement than (8.1):

$$m(D,r) + T_K(r) + N_{\text{Ram}}(r) \leq_{\text{exc}} \varepsilon\, T_A(r) + O(1), \tag{8.4}$$

Here $N_{\text{Ram}}$ is the **ramification term**: it counts the ramification of the holomorphic curve $f$; that is, it counts the zeroes of $f'$ (in local coordinates) in the same way that $N_f(\infty, r)$ counts the poles of a meromorphic function $f$. It is well known that (8.4) implies (8.1).

We claim that, in the notation of (1.3), if $x \in X(\overline{\mathbb{Q}})$, then the analogue of $N_{\text{Ram}}(r)$ should be $-d(x)$, where $d(x)$ is the **discriminant term**

$$d(x) := \frac{1}{[k:\mathbb{Q}]} \log |D_{K(x)}|.$$

Thus the arithmetic equivalent of (8.4) is the following:

Conjecture 8.5. *Let $X$ be a smooth projective curve over a number field $k$, let $D$ be an effective divisor on $X$ with no multiple points, let $K$ be a canonical divisor on $X$, let $A$ be an ample divisor on $X$, let $r$ be a positive integer, let $\varepsilon > 0$, and let $C$ be a constant. Then the inequality*

$$m(D,x) + h_K(x) \leq d(x) + \varepsilon\, h_A(x) + C \tag{8.5.1}$$

*holds for all but finitely many $x \in X(\overline{\mathbb{Q}})$ with $[K(x):k] \leq r$.*

Because of the sign change, it may seem unusual to suggest that $-d(x)$ is an analogue of $N_{\text{Ram}}(r)$. To support this assertion, however, we point out that (8.5.1) implies (8.3), corresponding to the fact that (8.4) implies (8.1):

PROPOSITION 8.6. *If Conjecture* 8.5 *holds*, *then* (8.3) *holds as well*.

This has not been proved elsewhere, so a proof appears in Appendix A.

In higher dimensions, there is likewise a modification of Conjecture 3.2:

CONJECTURE 8.7. *Let* $X$, $D$, $K$, $A$, *and* $\varepsilon$ *be as in Conjecture* 3.2, *and let* $r$ *be a positive integer*. *Then there exists a proper Zariski-closed subset* $Z \subseteq X$, *depending only on* $X$, $D$, $A$, *and* $\varepsilon$, *such that for all algebraic points* $x \in X(\bar{k})$ *with* $x \notin Z$ *and* $[K(x):k] \le r$,

$$m(D,x) + T_K(x) \le d(x) + \varepsilon\, h_A(x) + O(1).$$

To conclude this section, we mention the "abc conjecture" of Masser and Oesterlé. As was first observed by J. Noguchi [1996, (9.5)], it is the number-theoretic counterpart to Nevanlinna's Second Main Theorem with truncated counting functions, applied to the divisor $[0]+[1]+[\infty]$ on $\mathbb{P}^1$. In its simplest form it reads as follows.

CONJECTURE 8.8. *Let* $\varepsilon > 0$. *Then there exists a constant* $C$, *depending only on* $\varepsilon$, *such that for all relatively prime integers* $a,b,c \in \mathbb{Z}$ *with* $a+b+c=0$,

$$\max\{|a|,|b|,|c|\} \le C \prod_{p|abc} p^{1+\varepsilon}.$$

This conjecture, if proved, would have far-reaching consequences; for example, it would imply a weak form of Fermat's Last Theorem (now proved by Wiles).

In [Vojta 1987, pp. 71–72] it is shown that Conjecture 8.5 implies the abc conjecture.

It is also possible, via the variety $X \subseteq \mathbb{P}^2 \times \mathbb{P}^2$ defined by $ux^4+vy^4+wz^4=0$, to obtain from Conjecture 3.2 a weak form of the abc conjecture, for $\varepsilon > 26$. Here $X$ is a rational three-fold. Thus, versions of the Second Main Theorem, applied even to rational varieties, would give highly nontrivial consequences.

## Appendix A: Proof of Proposition 8.6

This appendix gives a proof of Proposition 8.6, because a proof has not appeared elsewhere. It will necessarily be more technical than the rest of the paper.

Recall that we are proving that the inequality

$$m(D,x) + h_K(x) \le d(x) + \varepsilon\, h_A(x) + O(1) \tag{A.1}$$

for algebraic points of bounded degree on a curve implies the inequality

$$N^{(1)}(D,x) \ge h_{K+D}(x) - \varepsilon\, h_A(x) - O(1). \tag{A.2}$$

for rational points on a curve.

This implication is proved by taking a cover $X'$ of $X$, highly ramified over $D$ but unramified elsewhere, and applying (A.1) to the pull-back of everything to $X'$. The counting functions end up being truncated because of the fact that the ramification of a number field is limited over any given place. The details of this construction are as follows.

First, we define a slightly different truncated counting function.

DEFINITION A.3. If $D$ is a divisor on a curve, then the **modified truncated counting function** is the function $N^\flat(D,x)$, defined for prime divisors $D$ by (8.2) and for arbitrary $D$ by linearity. (We will not define $N^\flat(D,x)$ on varieties of higher dimension, because the support of $D$ may be singular in that case.)

Next, we give an improved lemma of Chevalley-Weil type; compare [Vojta 1987, Thm. 5.1.6].

LEMMA A.4. *Let $\pi : X' \to X$ be a finite morphism of smooth projective curves over a global field $k$ of characteristic zero, let $R$ be the ramification divisor of $\pi$, and let $r$ be a positive integer. Then*

$$d(y) - d(\pi(y)) \le N^\flat(R,y) + O(1) \tag{A.4.1}$$

*for all $y \in X'(\bar{k})$ with $[K(y) : k] \le r$; here the constant in $O(1)$ depends on $\pi$, $r$, and the model used in defining $N^\flat$, but not on $y$.*

*Moreover, if for all $y \in X'$ the ramification index of $\pi$ at $y$ depends only on $\pi(y)$, so that $R = \pi^* B$ for some $\mathbb{Q}$-divisor $B$ on $X$, then*

$$d(y) - d(\pi(y)) \le N^\flat(B,\pi(y)) + O(1) \tag{A.4.2}$$

*for all $y$ as before.*

PROOF. To simplify the notation, we will assume that $k$ is a number field. Let $A$ be its ring of integers.

Let $\mathscr{X}'$ and $\mathscr{X}$ be regular models for $X'$ and $X$, over $\operatorname{Spec} A$, such that $\pi$ extends to a morphism $\mathscr{X}' \to \mathscr{X}$, also denoted $\pi$. Let $R$ also denote the ramification divisor of $\mathscr{X}'$ over $\mathscr{X}$. Let $S$ be the set of places of $k$ containing:

(i) all archimedean places;
(ii) all places of bad reduction of $\mathscr{X}'$ and $\mathscr{X}$;
(iii) all places where $\pi(\operatorname{Supp} R)$ is not étale over $\operatorname{Spec} A$;
(iv) all places where $\pi^{-1}(\pi(\operatorname{Supp} R))$ is not étale over $\operatorname{Spec} A$; and
(v) all places where $\pi$ fails to be a finite morphism.

This is a finite set. For places $v \in S$ the contribution to $d(y)$ is bounded; hence it suffices to show that the contribution to each side of (A.4.1) from places not in $S$ obeys the inequality. This will be done place by place, without any $O(1)$ term.

By making a base change, we may assume that $\pi(y)$ is rational over $k$.

Let $v$ be a place of $k$ not in $S$, and let $w$ be a place of $K(y)$ lying over $v$. Let $v$ also denote the point of $\operatorname{Spec} A$ corresponding to $v$; similarly let $C$ be the ring of integers of $K(y)$ and let $w$ also denote the point of $\operatorname{Spec} C$ corresponding to $w$. Let $\sigma$ be the section of the map $\mathscr{X} \to \operatorname{Spec} A$ corresponding to $\pi(y)$, and let $\tau : \operatorname{Spec} C \to \mathscr{X}'$ be the map corresponding to $y$. If $\tau(w)$ does not meet $R$, then the contribution at $w$ to the right-hand side of (A.4.1) is zero, but the contribution to the left-hand side is also zero, by [Vojta 1987, Lemma 5.1.8]. Thus we may assume that $\tau(w) \in \operatorname{Supp} R$.

Write $\xi = \sigma(v)$ and $\eta = \tau(w)$, so that $\xi = \pi(\eta)$. Let $n$ be the degree of $\pi$. After base change of $\pi : \mathscr{X}' \to \mathscr{X}$ to $\operatorname{Spec} \widehat{\mathscr{O}}_{\xi,\mathscr{X}}$, we have a finite morphism $\pi' : \operatorname{Spec} C' \to \operatorname{Spec} \widehat{\mathscr{O}}_{\xi,\mathscr{X}}$, where $C'$ is a semilocal ring. Let $\mathfrak{m}$ be the maximal ideal of $\widehat{\mathscr{O}}_{\xi,\mathscr{X}}$ and $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ the maximal ideals of $C'$. For sufficiently large $e$, we have

$$(\mathfrak{m}_1 \cdots \mathfrak{m}_r)^e \subseteq \mathfrak{m} \subseteq \mathfrak{m}_1 \cdots \mathfrak{m}_r;$$

therefore $C'$ is $\mathfrak{m}_1 \cdots \mathfrak{m}_r$-adically complete, and by [Matsumura 1986, Theorem 8.15] it follows that $C'$ is the product of the completions of its local rings at the maximal ideals. Thus

$$C' = \prod_{\alpha \in \pi^{-1}(\xi)} \widehat{\mathscr{O}}_{\alpha,\mathscr{X}'}.$$

Let $e$ be the degree of $\operatorname{Spec} \widehat{\mathscr{O}}_{\eta,\mathscr{X}'}$ over $\operatorname{Spec} \widehat{\mathscr{O}}_{\xi,\mathscr{X}}$. There is a unique branch of $\pi(\operatorname{Supp} R)$ passing through $\xi$ and a unique branch of $\pi^{-1}(\pi(\operatorname{Supp} R))$ passing through $\eta$. Therefore the multiplicity of that latter branch in $\pi^{-1}(\pi(\operatorname{Supp} R))$ (pulling back $\pi(\operatorname{Supp} R)$ as a divisor), must also equal $e$. If $e > 1$ then $R$ has a component with multiplicity $e-1$ passing through $\eta$; otherwise, $e-1 = 0$ and $R$ does not pass through $\eta$. Then the contribution at $w$ to the right-hand side of (A.4.1) is $(e-1)(\log q_w)/[K(y) : \mathbb{Q}]$, where $q_w$ is the number of elements of the residue field at $w$. But $\operatorname{Spec} \widehat{\mathscr{O}}_{\eta,\mathscr{X}'}$ has degree $e$ over $\operatorname{Spec} \widehat{\mathscr{O}}_{\xi,\mathscr{X}}$, so the local field $K(\eta)_w$ has degree at most $e$ over $k_v$. Thus the contribution at $\eta$ to the left-hand side is also at most $(e-1)(\log q_w)/[K(y) : \mathbb{Q}]$ (since wild ramification cannot occur). This is sufficient to imply (A.4.1).

Next we show (A.4.2). Again, we prove the inequality place by place, for places $v \notin S$, without the $O(1)$ term. We again assume that $\pi(y)$ is rational over $k$. Pick $v \in M_k \setminus S$. As before, we may assume that $\sigma(v)$ meets $\operatorname{Supp} B$. Then the contribution at $v$ to the right-hand side of (A.4.2) is

$$\frac{1}{[k : \mathbb{Q}]} \cdot \frac{e-1}{e} \log q_v.$$

For a place $w$ of $K(y)$ over $v$, let $e_{w/v}$ and $f_{w/v}$ denote the ramification index and residue field degree, respectively. Then the contribution at $v$ to the left-hand side of (A.4.2) is

$$\frac{1}{[K(y) : \mathbb{Q}]} \sum_{w \mid v} (e_{w/v} - 1) \log q_w.$$

Thus it will suffice to show that

$$\frac{1}{[K(y) : k]} \sum_{w \mid v} (e_{w/v} - 1) \log q_w \le \frac{e-1}{e} \log q_v.$$

But this follows from the easy facts

$$\sum_{w|v} e_{w/v} f_{w/v} = [K(y):k] \qquad \text{and} \qquad \log q_w = f_{v/w} \log q_v ,$$

and from the inequality $e_{v/w} \le e$ proved earlier. □

We may now continue with the proof of Proposition 8.6.

If $X$ has genus 0 and if $\deg D < 2$, the right-hand side of (A.2) is negative, and the result is trivial. Hence we may assume that $2g(X) - 2 + \deg D \ge 0$. Let $e$ be an integer, chosen large enough so that

$$h^0(X, n([e\varepsilon]A - D)) > 0 \tag{A.5}$$

for some $n > 0$, where $[e\varepsilon]$ denotes the greatest integer function. By [Vojta 1989a, Lemma 3.1], there is a cover $\pi : X' \to X$, where $X'$ is also nonsingular, which is unramified outside $\pi^{-1}(D)$ and ramified exactly to order $e$ at all points of $X'$ lying over $D$.

The $\mathbb{Q}$-divisor

$$D' := \frac{1}{e}\pi^* D$$

is an integral divisor on $X'$ with no multiple points. The ramification divisor $R$ of $\pi$ is given by

$$R = (e-1)D' = \frac{e-1}{e}\pi^* D \ . \tag{A.6}$$

Thus the canonical divisor on $X'$ satisfies the linear equivalence

$$K_{X'} \sim \pi^* K_X + \frac{e-1}{e}\pi^* D \ ,$$

so that

$$K_{X'} + D' \sim \pi^*(K_X + D) \ . \tag{A.7}$$

Lemma A.8. *In this situation, points $y \in X'(\bar{k})$ of bounded degree over $k$ satisfy*

$$N(D', y) + d(y) \le N^{(1)}(D, \pi(y)) + d(\pi(y)) + \varepsilon h_{\pi^* A}(y) + O(1) \ , \tag{A.8.1}$$

*where the constant in $O(1)$ depends on $X$, $X'$, $\pi$, the models used to define the counting functions, and the bound on the degree, but not on $y$.*

Proof. By (A.6), we may apply (A.4.2) to $X'$ with

$$B = \frac{e-1}{e} D$$

to obtain the inequality

$$d(y) \le d(\pi(y)) + N^\flat(B, \pi(y)) + O(1) \ . \tag{A.8.2}$$

By (A.5) we have $h^0(X', n([e\varepsilon]\pi^* A - \pi^* D)) > 0$ for some $n > 0$, so

$$\varepsilon\, h_{\pi^* A}(y) \ge h_{D'}(y) + O(1)$$

and therefore

$$\begin{aligned} N(D', y) &\le h_{D'}(y) + O(1) \\ &\le \varepsilon\, h_{\pi^* A}(y) + O(1)\,. \end{aligned}$$

By definition of $N^\flat$ we then have

$$\begin{aligned} N^\flat(B, \pi(y)) &= \frac{e-1}{e} N^\flat(D, \pi(y)) \\ &= \frac{e-1}{e} N^{(1)}(D, \pi(y)) + O(1) \\ &\le N^{(1)}(D, \pi(y)) + O(1) \\ &\le N^{(1)}(D, \pi(y)) - N(D', y) + \varepsilon h_{\pi^* A}(y) + O(1)\,. \end{aligned}$$

Combining this with (A.8.2) then gives (A.8.1). □

Since $\pi$ is a finite map and $A$ is ample, $\pi^* A$ is ample on $X'$. Thus, (A.1) applies to points $y$ on $X'$ lying over rational points on $X$, relative to the divisor $D'$, giving

$$m(D', y) + h_{K_{X'}}(y) \le d(y) + \varepsilon h_{\pi^* A}(y) + O(1)\,.$$

By the First Main Theorem, this is equivalent to

$$N(D', y) + d(y) \ge h_{K_{X'} + D'}(y) - \varepsilon\, h_{\pi^* A}(y) - O(1)\,. \tag{A.9}$$

By (A.8.1), (A.9), and (A.7), we then have

$$\begin{aligned} N^{(1)}(D, \pi(y)) + d(\pi(y)) &\ge N(D', y) + d(y) - \varepsilon\, h_{\pi^* A}(y) - O(1) \\ &\ge h_{K_{X'} + D'}(y) - 2\varepsilon\, h_{\pi^* A}(y) - O(1) \\ &\ge h_{K_X + D}(\pi(y)) - 2\varepsilon\, h_A(\pi(y)) - O(1)\,. \end{aligned}$$

Since $\pi(y)$ is rational, $d(\pi(y))$ is bounded; hence (A.2) follows after adjusting $\varepsilon$.

Thus, Proposition 8.6 is proved.

## References

[Babets 1984] V. A. Babets, "Theorems of Picard type for holomorphic mappings", *Sibirsk. Mat. Zh.* **25**:2 (1984), 35–41. In Russian; translated in *Siberian Math. J.* **25** (1984), 195–200.

[Baily and Borel 1966] J. Baily, W. L. and A. Borel, "Compactification of arithmetic quotients of bounded symmetric domains", *Ann. of Math.* (2) **84** (1966), 442–528.

[Bloch 1926] A. Bloch, "Sur les systèmes de fonctions uniformes satisfaisant à l'équation d'une variété algébrique dont l'irrégularité dépasse la dimension", *J. de Math.* **5** (1926), 19–66.

[Bombieri and van der Poorten 1988] E. Bombieri and A. J. van der Poorten, "Some quantitative results related to Roth's theorem", *J. Austral. Math. Soc. Ser. A* **45**:2 (1988), 233–248.

[Cartan 1933] H. Cartan, "Sur les zéros des combinaisons linéaires de $p$ fonctions holomorphes données", *Mathematica (Cluj)* **7** (1933), 5–29.

[Dethloff et al. 1995a] G. Dethloff, G. Schumacher, and P.-M. Wong, "Hyperbolicity of the complements of plane algebraic curves", *Amer. J. Math.* **117**:3 (1995), 573–599.

[Dethloff et al. 1995b] G. Dethloff, G. Schumacher, and P.-M. Wong, "On the hyperbolicity of the complements of curves in algebraic surfaces: the three-component case", *Duke Math. J.* **78**:1 (1995), 193–212.

[Dufresnoy 1944] J. Dufresnoy, "Théorie nouvelle des familles complexes normales. Applications à l'étude des fonctions algébroïdes", *Ann. Sci. École Norm. Sup.* (3) **61** (1944), 1–44.

[Erëmenko and Sodin 1991] A. È. Erëmenko and M. L. Sodin, "The value distribution of meromorphic functions and meromorphic curves from the point of view of potential theory", *Algebra i Analiz* **3**:1 (1991), 131–164. In Russian; translated in *St. Petersburg Math. J.* **3** (1992), 109–136.

[Faltings 1991] G. Faltings, "Diophantine approximation on abelian varieties", *Ann. of Math.* (2) **133**:3 (1991), 549–576.

[Faltings 1994] G. Faltings, "The general case of S. Lang's conjecture", pp. 175–182 in *Barsotti Symposium in Algebraic Geometry* (Abano Terme, 1991), edited by V. Christante and W. Messing, Perspectives in Math. **15**, Academic Press, San Diego, 1994.

[Faltings and Wüstholz 1994] G. Faltings and G. Wüstholz, "Diophantine approximations on projective spaces", *Invent. Math.* **116**:1-3 (1994), 109–138.

[Grauert 1989] H. Grauert, "Jetmetriken und hyperbolische Geometrie", *Math. Z.* **200**:2 (1989), 149–168.

[Green 1975] M. L. Green, "Some Picard theorems for holomorphic maps to algebraic varieties", *Amer. J. Math.* **97** (1975), 43–75.

[Green and Griffiths 1980] M. Green and P. Griffiths, "Two applications of algebraic geometry to entire holomorphic mappings", pp. 41–74 in *The Chern Symposium* (Berkeley, 1979), edited by W.-Y. Hsiang et al., Springer, New York, 1980.

[Griffiths and King 1973] P. Griffiths and J. King, "Nevanlinna theory and holomorphic mappings between algebraic varieties", *Acta Math.* **130** (1973), 145–220.

[Hinkkanen 1992] A. Hinkkanen, "A sharp form of Nevanlinna's second fundamental theorem", *Invent. Math.* **108**:3 (1992), 549–574.

[Kobayashi 1970] S. Kobayashi, *Hyperbolic manifolds and holomorphic mappings*, Pure and Applied Mathematics **2**, Marcel Dekker, New York, 1970.

[Lang 1970] S. Lang, *Algebraic number theory*, Addison-Wesley, Reading, MA, 1970. Reprinted by Springer, 1986.

[Lang 1971] S. Lang, "Transcendental numbers and diophantine approximations", *Bull. Amer. Math. Soc.* **77** (1971), 635–677.

[Lang 1983] S. Lang, *Fundamentals of Diophantine geometry*, Springer, New York, 1983.

[Lang 1991] S. Lang, *Number theory, III: Diophantine geometry*, Encyclopaedia of Mathematical Sciences **60**, Springer, Berlin, 1991.

[Lang and Cherry 1990] S. Lang and W. Cherry, *Topics in Nevanlinna theory*, Lecture Notes in Math. **1433**, Springer, Berlin, 1990.

[Masuda and Noguchi 1996] K. Masuda and J. Noguchi, "A construction of hyperbolic hypersurface of $\mathbb{P}^n(\mathbb{C})$", *Math. Ann.* **304**:2 (1996), 339–362.

[Matsumura 1986] H. Matsumura, *Commutative ring theory*, Cambridge Stud. Adv. Math. **8**, Cambridge University Press, Cambridge, 1986.

[Miles 1992] J. Miles, "A sharp form of the lemma on the logarithmic derivative", *J. London Math. Soc.* (2) **45**:2 (1992), 243–254.

[Nochka 1982] E. I. Nochka, "Defect relations for meromorphic curves", *Izv. Akad. Nauk Moldav. SSR Ser. Fiz.-Tekhn. Mat. Nauk* **1982**:1 (1982), 41–47, 79.

[Nochka 1983] E. I. Nochka, "On the theory of meromorphic curves", *Dokl. Akad. Nauk SSSR* **269**:3 (1983), 547–552. In Russian; translated in *Soviet Math. Dokl.* **27** (1983), 377–381.

[Noguchi 1981] J. Noguchi, "Lemma on logarithmic derivatives and holomorphic curves in algebraic varieties", *Nagoya Math. J.* **83** (1981), 213–233.

[Noguchi 1996] J. Noguchi, "On Nevanlinna's second main theorem", pp. 489–503 in *Geometric complex analysis* (Hayama, 1995), edited by J. Noguchi et al., World Sci. Publishing, Singapore, 1996.

[Noguchi 1998] J. Noguchi, "On holomorphic curves in semi-abelian varieties", *Math. Z.* **228**:4 (1998), 713–721.

[Osgood 1981] C. F. Osgood, "A number theoretic–differential equations approach to generalizing Nevanlinna theory", *Indian J. Math.* **23**:1-3 (1981), 1–15.

[Ru 1993] M. Ru, "Integral points and the hyperbolicity of the complement of hypersurfaces", *J. Reine Angew. Math.* **442** (1993), 163–176.

[Ru 1995] M. Ru, "Geometric and arithmetic aspects of $\mathbb{P}^n$ minus hyperplanes", *Amer. J. Math.* **117**:2 (1995), 307–321.

[Ru 1997] M. Ru, "On a general form of the second main theorem", *Trans. Amer. Math. Soc.* **349**:12 (1997), 5093–5105.

[Ru and Stoll 1991a] M. Ru and W. Stoll, "The second main theorem for moving targets", *J. Geom. Anal.* **1**:2 (1991), 99–138.

[Ru and Stoll 1991b] M. Ru and W. Stoll, "The Cartan conjecture for moving targets", pp. 477–508 in *Several complex variables and complex geometry* (Santa Cruz, CA, 1989), vol. 2, edited by E. Bedford et al., Proc. Sympos. Pure Math. **52**, Amer. Math. Soc., Providence, RI, 1991.

[Ru and Vojta 1997] M. Ru and P. Vojta, "Schmidt's subspace theorem with moving targets", *Invent. Math.* **127**:1 (1997), 51–65.

[Ru and Wong 1991] M. Ru and P.-M. Wong, "Integral points of $\mathbb{P}^n - \{2n+1$ hyperplanes in general position$\}$", *Invent. Math.* **106**:1 (1991), 195–216.

[Schmidt 1980] W. M. Schmidt, *Diophantine approximation*, Lecture Notes in Math. **785**, Springer, Berlin, 1980.

[Shiffman 1979] B. Shiffman, "On holomorphic curves and meromorphic maps in projective space", *Indiana Univ. Math. J.* **28**:4 (1979), 627–641.

[Shiffman 1996] B. Shiffman, "The second main theorem for log canonical $\mathbb{R}$-divisors", pp. 551–561 in *Geometric complex analysis* (Hayama, 1995), edited by J. Noguchi et al., World Sci. Publishing, Singapore, 1996.

[Siu 1995] Y.-T. Siu, "Hyperbolicity problems in function theory", pp. 409–513 in *Five decades as a mathematician and educator: on the 80th birthday of Professor Yung-Chow Wong*, edited by K.-Y. Chan and M.-C. Liu, World Sci. Publishing, Singapore, 1995.

[Siu and Yeung 1996a] Y.-T. Siu and S.-K. Yeung, "A generalized Bloch's theorem and the hyperbolicity of the complement of an ample divisor in an abelian variety", *Math. Ann.* **306**:4 (1996), 743–758.

[Siu and Yeung 1996b] Y.-T. Siu and S.-K. Yeung, "Hyperbolicity of the complement of a generic smooth curve of high degree in the complex projective plane", *Invent. Math.* **124**:1-3 (1996), 573–618.

[Soulé 1992] C. Soulé, *Lectures on Arakelov geometry*, Cambridge Stud. Adv. Math. **33**, Cambridge University Press, Cambridge, 1992. With the collaboration of D. Abramovich, J.-F. Burnol and J. Kramer.

[Steinmetz 1986] N. Steinmetz, "Eine Verallgemeinerung des zweiten Nevanlinnaschen Hauptsatzes", *J. Reine Angew. Math.* **368** (1986), 134–141.

[Tucker 1997] T. J. Tucker, "Moving targets on elliptic curves", preprint, University of Georgia, 1997.

[Vojta 1987] P. Vojta, *Diophantine approximations and value distribution theory*, Lecture Notes in Math. **1239**, Springer, Berlin, 1987.

[Vojta 1989a] P. Vojta, "Dyson's lemma for products of two curves of arbitrary genus", *Invent. Math.* **98**:1 (1989), 107–113.

[Vojta 1989b] P. Vojta, "A refinement of Schmidt's subspace theorem", *Amer. J. Math.* **111**:3 (1989), 489–518.

[Vojta 1992] P. Vojta, "A generalization of theorems of Faltings and Thue–Siegel–Roth–Wirsing", *J. Amer. Math. Soc.* **5**:4 (1992), 763–804.

[Vojta 1996a] P. Vojta, "Integral points on subvarieties of semiabelian varieties, I", *Invent. Math.* **126**:1 (1996), 133–181.

[Vojta 1996b] P. Vojta, "Roth's theorem with moving targets", *Internat. Math. Res. Notices* **1996**:3 (1996), 109–114.

[Vojta 1997] P. Vojta, "On Cartan's theorem and Cartan's conjecture", *Amer. J. Math.* **119**:1 (1997), 1–17.

[Vojta 1999] P. Vojta, "Integral points on subvarieties of semiabelian varieties, II", *Amer. J. Math.* **121**:2 (1999), 283–313.

[Ye 1995] Z. Ye, "On Nevanlinna's second main theorem in projective space", *Invent. Math.* **122**:3 (1995), 475–507.

[Zaĭdenberg 1988] M. G. Zaĭdenberg, "Stability of hyperbolic embeddedness and the construction of examples", *Mat. Sb.* (*N.S.*) **135(177)**:3 (1988), 361–372, 415. In Russian; translated in *Math. USSR Sbornik* **63** (1989), 351–361.

Paul Vojta
Department of Mathematics
University of California
970 Evans Hall #3840
Berkeley, CA, 94720-3840
United States
vojta@math.berkeley.edu