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Matroid Bundles

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Abstract. Combinatorial vector bundles, or *matroid bundles*, are a combinatorial analog to real vector bundles. Combinatorial objects called *oriented matroids* play the role of real vector spaces. This combinatorial analogy is remarkably strong, and has led to combinatorial results in topology and bundle-theoretic proofs in combinatorics. This paper surveys recent results on matroid bundles, and describes a canonical functor from real vector bundles to matroid bundles.

1. Introduction

Matroid bundles are combinatorial objects that mimic real vector bundles. They were first defined in [MacPherson 1993] in connection with *combinatorial differential manifolds*, or *CD manifolds*. Matroid bundles generalize the notion of the "combinatorial tangent bundle" of a CD manifold. Since the appearance of McPherson's article, the theory has filled out considerably; in particular, matroid bundles have proved to provide a beautiful combinatorial formulation for characteristic classes.

We will recapitulate many of the ideas introduced by McPherson, both for the sake of a self-contained exposition and to describe them in terms more suited to our present context. However, we refer the reader to [MacPherson 1993] for background not given here. We recommend the same paper, as well as [Mnëv and Ziegler 1993] on the combinatorial Grassmannian, for related discussions.

We begin with a key intuitive point of the theory: the notion of an oriented matroid as a combinatorial analog to a vector space. From this we develop matroid bundles as a combinatorial bundle theory with oriented matroids as fibers. Section 2 will describe the category of matroid bundles and its relation to the category of real vector bundles. Section 3 gives examples of matroid bundles arising in both combinatorial and topological contexts, and Section 4 outlines some of the techniques that have been developed to study matroid bundles.

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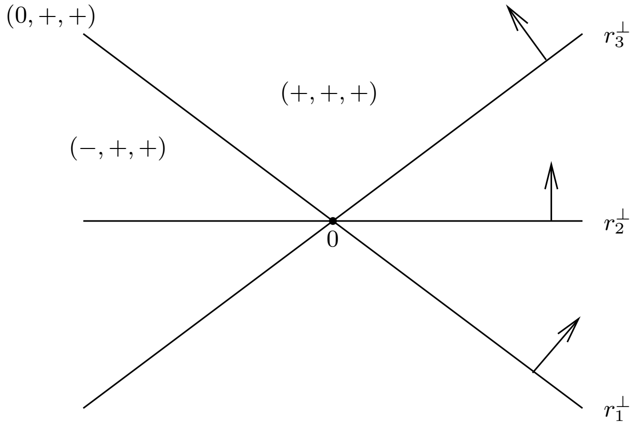

Figure 1. An arrangement of oriented hyperplanes in R^2 and some of the resulting sign vectors.

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1A. Oriented matroids. We give a brief introduction to oriented matroids, particularly to the idea of oriented matroids as "combinatorial vector spaces". See [Björner et al. 1993] for a more complete introduction to oriented matroids, and [MacPherson 1993, Appendix] for specific notions of importance here.

A rank-n oriented matroid can be considered as a combinatorial analog to an arrangement $\{r_i\}_{i\in E}$ of vectors in $\mathbb{R}^n$, or equivalently, to an arrangement $\{r_i^\perp\}_{i\in E}$ of oriented hyperplanes. (Here we allow the "degenerate hyperplane" $0^\perp = \mathbb{R}^n$.) The idea is as follows. An arrangement $\{r_i^\perp\}_{i\in E}$ of oriented hyperplanes in $\mathbb{R}^n$ partitions $\mathbb{R}^n$ into cones. Each cone C can be identified by a sign vector $v \in \{-,0,+\}^E$, where v_i indicates on which side of $r_i^\perp$ the cone C lies. (See Figure 1).

The set E together with the collection of sign vectors resulting from this arrangement is called a *realizable oriented matroid.* The sign vectors are called *covectors* of the oriented matroid. Every realizable oriented matroid has 0 as a covector. The hyperplanes describe a cell decomposition of the unit sphere in $\mathbb{R}^n$, with each cell labeled by a nonzero covector.

More generally, an *oriented matroid* M is a finite set E together with a collection $V^*(M)$ of signed sets in $\{-,0,+\}^E$, satisfying certain combinatorial axioms inspired by the case of realizable oriented matroids. (For a complete definition, see [Björner et al. 1993, Section 4.1].) In this more general context, we still have

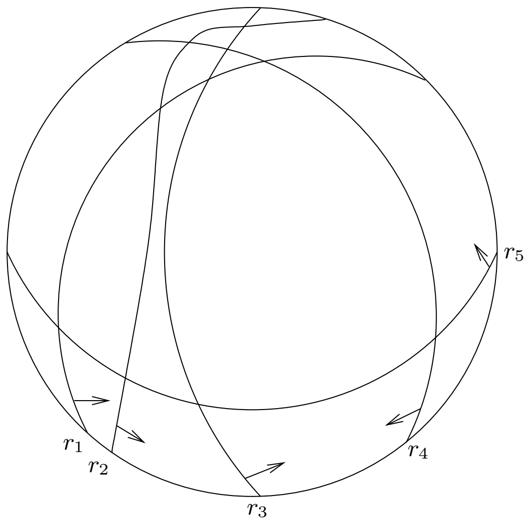

Figure 2. A rank-3 arrangement of five oriented pseudospheres.

a notion of the *rank* of an oriented matroid [MacPherson 1993, Section 5.3], and a beautiful theorem that gives this notion topological meaning.

The *Topological Representation Theorem* of Folkman and Lawrence [Björner et al. 1993, Section 1.4; Folkman and Lawrence 1978] says that the set of nonzero covectors of a rank-n oriented matroid describe a cell decomposition of S^{n-1}. More precisely: a *pseudosphere* in S^{n-1} is a subset S such that some homeomorphism of S^{n-1} takes S to an equator. Thus, a pseudosphere must partition S^{n-1} into two pseudohemispheres. An *oriented pseudosphere* is a pseudosphere together with a choice of positive pseudohemisphere. An *arrangement of oriented pseudospheres* is a set of oriented pseudospheres on S^{n-1} whose intersections behave topologically like intersections of equators. (For a precise definition, see [Björner et al. 1993, Definition 5.1.3].) For an example, see Figure 2.

An arrangement $\{S_i\}_{i\in E}$ of oriented psuedospheres in S^{n-1} determines a collection of signed sets in $\{-,0,+\}^E$ in the same way that an arrangement of oriented hyperplanes in $\mathbb{R}^n$ does. The Topological Representation Theorem states that any collection of signed sets arising in this way is the set of nonzero covectors of an oriented matroid, and that every oriented matroid arises in this way.

1B. Oriented matroids as "combinatorial vector spaces". A *strong map image* of an oriented matroid M is an oriented matroid N such that $V^*(N) \subseteq V^*(M)$. (Strong maps are called *strong quotients* in [Gelfand and MacPherson 1992]. See [Björner et al. 1993, Section 7.7] for more on strong maps.)

Consider a realizable rank-n oriented matroid M, realized as a set $R = \{r_1^\perp, r_2^\perp, \ldots, r_m^\perp\} \subset \mathbb{R}^n$. If V is a rank-k subspace in $\mathbb{R}^n$, consider the rank-k oriented matroid $\gamma_R(V)$ given by the intersections $\{V \cap r_i^\perp : i \in \{1,\ldots,m\}\}$. In terms of the vector picture of oriented matroids, $\gamma_R(V)$ is given by the orthogonal

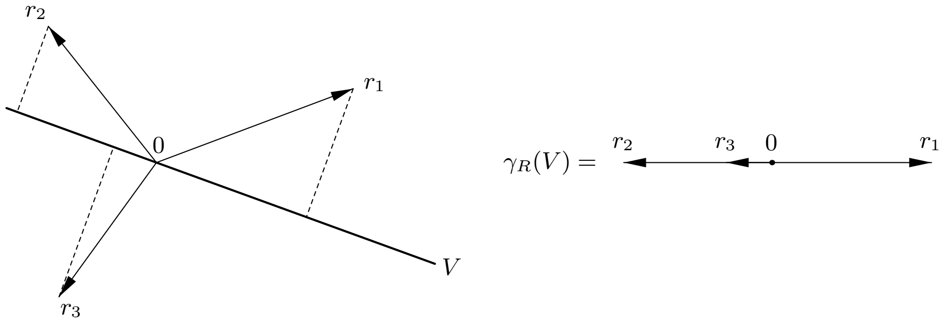

Figure 3. A strong map of realizable oriented matroids.

projections of the elements $\{r_1, \ldots, r_m\}$ onto V. The oriented matroid $\gamma_R(V)$ is a strong map image of M, and encodes considerable geometric data about V. For instance, the loops in $\gamma_R(V)$ are exactly those r_i such that $V \subseteq r_i^\perp$, and the cell decomposition of the unit sphere S_V in V given by the equators $S_V \cap r_i^\perp$ is canonically isomorphic to the cell complex of nonzero covectors of $\gamma_R(V)$. We will think of $\gamma_R(V)$ as a combinatorial model for V, and as a combinatorial "subspace" of M. Figure 3 shows a realization of a rank-2 oriented matroid, a 1-dimensional subspace V of $\mathbb{R}^2$, and the resulting oriented matroid $\gamma_R(V)$.

If M is not realizable, we will still use M as a combinatorial analog to $\mathbb{R}^n$, with the nonzero covectors in $V^*(M)$ playing the role of the unit sphere. Strong map images will be viewed as "pseudosubspaces".

1C. Matroid bundles. Consider a real rank-k vector bundle $\xi : E \to B$ over a compact base space. Choose a collection $\{e_1, \ldots, e_n\}$ of continuous sections of ξ such that at each point b in B, the vectors $\{e_1(b), \ldots, e_n(b)\}$ span the space $\xi^{-1}(b)$. The vectors $\{e_1(b), \ldots, e_n(b)\}$ determine a rank-k oriented matroid $M(b)$ with elements the integers $\{1, \ldots, n\}$. Note that any $b \in B$ has an open neighborhood U_b such that $M(b')$ *weak maps* to $M(b)$ for all $b' \in U_b$. (See [Björner et al. 1993, Section 7.7] for a definition of weak maps. Weak maps are called *specializations* in [MacPherson 1993] and *weak specializations* in [Gelfand and MacPherson 1992].)

Proposition 1.1. *Let $\xi : E \to B$ be a real vector bundle with B finite-dimensional and let $\mu : |T| \to B$ be a triangulation of B. Then there exists a simplicial subdivision T' of T and a spanning collection of sections of ξ such that for every simplex σ of T', the function M is constant on the relative interior of $\mu(|\sigma|)$.*

This is a corollary to the Combinatorialization Theorem in Section 2C.

Example. Figure 4 shows the Möbius strip as a line bundle over S^1, and a triangulation of S^1 with vertices a, b, c. The sections $\{\rho_1, \rho_2\}$ associate a single oriented matroid to the interior of each simplex.

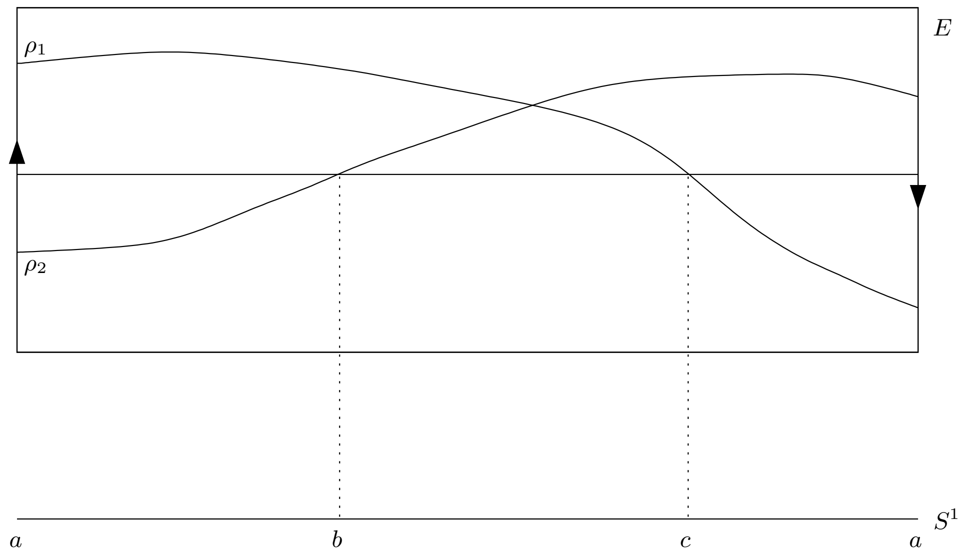

Figure 4. A spanning collection of sections for the Möbius strip.

Such a simplicial complex and the association of an oriented matroid to each cell give the motivating example of a matroid bundle:

Definition 1.2. A rank-k *matroid bundle* is a partially ordered set B (e.g., a simplicial complex with simplices ordered by inclusion) and a rank-k oriented matroid $\mathcal{M}(b)$ associated to each element b, so that $\mathcal{M}(b)$ weak maps to $M(b')$ whenever $b \geq b'$.

This is a simplification of the definition which appears in [MacPherson 1993]. Any matroid bundle in the sense of MacPherson gives a matroid bundle in the present sense. Conversely, given a matroid bundle $(B, \mathcal{M})$ in our current sense, consider the *order complex* ΔB of B, i.e., the simplicial complex of all chains in the partial order. The map associating to each simplex $b_1 \leq \cdots \leq b_m$ in ΔB the oriented matroid $\mathcal{M}(b_m)$ defines a matroid bundle in the sense of MacPherson.

A matroid bundle need not arise from a real vector bundle. For instance, a matroid bundle may include non-realizable oriented matroids as fibers. Section 3 will give examples of matroid bundles arising in combinatorics that do not correspond to any real vector bundles.

1D. What do we want from matroid bundles? The hope is that the category of matroid bundles is closely related to the category of real vector bundles, or perhaps to one of its weaker cousins, such as the category of piecewise-linear microbundles or the category of spherical quasifibrations. (These categories are described below.) Relating bundle theory to oriented matroids promises both combinatorial techniques for bundle theory and bundle-theoretic techniques for combinatorics.

We describe these categories very briefly here. Good sources for a more extended look at bundles are [Milnor and Stasheff 1974; Husemoller 1996]. The loose idea is as follows: a (topological) *bundle* is a map $\xi : E \to B$ of topological spaces such that for some open cover $\{U_i\}_{i \in I}$ of B, each restriction $\xi|_{\xi^{-1}(U_i)}$ "looks like" a projection $p : U_i \times F \to U_i$, for some space F. Different bundle theories arise from different notions of "looking like a projection". E is the *total space* of the bundle, and B is the *base space*. For any $b \in B$, the preimage $\xi^{-1}(b)$ is the *fiber* of ξ over b. A *morphism* from a bundle $\xi_1 : E_1 \to B_1$ to a bundle $\xi_2 : E_2 \to B_2$ is a commutative diagram

$$\begin{tikzcd} E_1 \arrow[r] \arrow[d, "\xi_1"'] & E_2 \arrow[d, "\xi_2"] \\ B_1 \arrow[r] & B_2 \end{tikzcd}$$

such that the map of total spaces preserves appropriate structure on fibers.

Three progressively weaker categories of bundles are of particular interest. The strongest is the category **Bun** of *real vector bundles*, in which $F \cong \mathbb{R}^k$ and for each U_i we must have a homeomorphism $h : U_i \times \mathbb{R}^k \to \xi^{-1}(U_i)$ such that

$$\begin{tikzcd} U_i \times \mathbb{R}^k \arrow[rr, "h"] \arrow[dr, "p"'] & & \xi^{-1}(U_i) \arrow[dl, "\xi"] \\ & U_i & \end{tikzcd}$$

commutes and h restricts to a linear isomorphism on each fiber. In the weaker category **PL** of *piecewise-linear microbundles*, F is still $\mathbb{R}^k$, but the maps h need only be piecewise-linear homeomorphisms with compatible 0 cross-sections. (See [Milnor 1961] for a precise definition.) A still weaker notion is that of a *quasifibration*, which must only "look like" a projection in that for each $x \in U_i$, $y \in \xi^{-1}(x)$, and $j \in \mathbb{N}$, the map of homotopy groups

$$p_* : \pi_j(p^{-1}(U_i), p^{-1}(x), y) \longrightarrow \pi_j(U_i, x)$$

is an isomorphism; see [Dold and Thom 1958, §§ 1.1, and 2.1]. From this condition it follows that each fiber has the same weak homotopy type. We will be interested in the category **Fib** of quasifibrations whose fibers are homotopy spheres. Any real vector bundle or PL microbundle has a canonical associated sphere bundle — essentially by taking a sphere around 0 in each fiber — which is a spherical quasifibration.

Associated to any good bundle theory is a *universal bundle* — that is, a bundle $\Xi : E_\infty \to B_\infty$ such that

1. for any bundle $\xi : E \to B$ there exists a morphism from ξ to Ξ, and
2. if $\xi_1 : E_1 \to B_1$ and $\xi_2 : E_2 \to B_2$ are bundles and $F : \xi_1 \to \xi_2$, $C_1 : \xi_1 \to \Xi$, and $C_2 : \xi_2 \to \Xi$ are morphisms, then there exists a *bundle homotopy* from

C_1 to $C_2 \circ F$, i.e., a morphism H from $\xi_1 \times \mathrm{id} : E_1 \times I \to B_1 \times I$ to Ξ such that $H|_{\xi_1 \times \{0\}} = C_1 \times *$ and $H|_{\xi_1 \times \{1\}} = (C_2 \circ F) \times *$.

In this situation B_∞ is called a *classifying space* for the category. For any bundle ξ and bundle map from ξ to the universal bundle, the map of base spaces is called a *classifying map*. It follows from the properties above that the universal bundle is unique up to bundle homotopy, and that for a fixed universal bundle and fixed real vector bundle, the classifying map is unique up to homotopy. In fact, every vector bundle over a base space B is characterized up to isomorphism by a homotopy class of maps from B to B_∞. Specifically, a bundle ξ over B with classifying map $c(\xi)$is isomorphic to the *pullback* of Ξ by $c(\xi)$, i.e., the bundle $\pi_1 : \{(b, v) : b \in B,\ v \in \Xi^{-1}(c(\xi)(b))\} \to B$. In this way isomorphism classes of bundles over a space B are in bijection with homotopy classes of maps $B \to B_\infty$. Thus if G_1 and G_2 are two categories of bundles with classifying spaces B^1_∞ and B^2_∞ then any map $B^1_\infty \to B^2_\infty$ gives a functor from isomorphism classes in G_1 to isomorphism classes in G_2.

For rank-k real vector bundles over paracompact base spaces, the classifying space (often called BO_k) is $G(k, \mathbb{R}^\infty)$, the space of all k-dimensional subspaces of $\mathbb{R}^\infty$. The universal bundle is the tautological bundle

$$\begin{array}{rcl} E_\infty = \{(V, x) : V \in G(k, \mathbb{R}^\infty),\ x \in V\} & \longrightarrow & G(k, \mathbb{R}^\infty), \\ (V, x) & \longrightarrow & V. \end{array}$$

(See [Milnor and Stasheff 1974, Chapter 5] for details.) The classifying spaces BPL_k for PL microbundles and BFib_k for spherical quasifibrations are harder to describe explicitly, and we won't attempt it here. (See [Milnor 1961, Chapter 5; Stasheff 1963] for constructions. We note in passing that BFib_k is isomorphic to the classifying space for rank-k spherical fibrations [Stasheff 1963] — see the related discussion in [Anderson and Davis $\geq$ 1999].) Since BO_k has a natural PL microbundle structure and BPL_k has an associated spherical quasifibration, there are canonical (up to homotopy) classifying maps $\mathrm{BO}_k \to \mathrm{BPL}_k \to \mathrm{BFib}_k$, giving canonical functors from real vector bundles to PL bundles to spherical quasifibrations.

How do matroid bundles fit into this picture? In Section 2A we will define morphisms of matroid bundles, leading to a category MB_k of matroid bundles. This category has a universal bundle, whose classifying space is called the *MacPhersonian* $\mathrm{MacP}(k, \infty)$. We can relate matroid bundles to other bundle theories by finding nice maps between $\mathrm{MacP}(k, \infty)$ and other classifying spaces.

Topologically, the category of matroid bundles is awkward in that the fibers are combinatorial objects — oriented matroids — which form no topological total space. In Section 4 we will discuss how the Topological Representation Theorem allows us to associate a spherical quasifibration (easily) and even a PL microbundle (gruelingly) to a matroid bundle, giving maps $\mathrm{MacP}(k, \infty) \to \mathrm{BFib}_k$ and $\mathrm{MacP}(k, \infty) \to \mathrm{BPL}_k$ and hence giving functors of bundle theories. Another key result is the Combinatorialization Theorem described in Section 2C, which implies a map $BO_k \to \mathrm{MacP}(k,\infty)$ and another functor.

Much of the progress on matroid bundles has been in the area of *characteristic classes*. A characteristic class for a bundle theory is a rule assigning to each bundle $\xi : E \to B$ an element $u(\xi)$ of $H^*(B)$ such that if

$$\begin{tikzcd} E_1 \arrow[r] \arrow[d, "\xi_1"'] & E_2 \arrow[d, "\xi_2"] \\ B_1 \arrow[r, "f"] & B_2 \end{tikzcd}$$

is a bundle map, then $u(\xi_1) = f^*(u(\xi_2))$. (See [Milnor and Stasheff 1974] for much more on characteristic classes.) From the definition of universal bundles, it follows that if B_∞ is the classifying space for a bundle theory, then the characteristic classes are in bijection with the elements of $H^*(B_\infty)$. (Note we have not specified coefficients for cohomology: different coefficients give different interesting characteristic classes.) Thus the maps $\mathrm{BO}_k \to \mathrm{MacP}(k,\infty)$, $\mathrm{MacP}(k,\infty) \to \mathrm{BPL}_k$, and $\mathrm{MacP}(k,\infty) \to \mathrm{BFib}_k$ give maps $H^*(\mathrm{BFib}_k) \to H^*(\mathrm{MacP}(k,\infty))$, $H^*(\mathrm{BPL}_k) \to H^*(\mathrm{MacP}(k,\infty))$, and $H^*(\mathrm{MacP}(k,\infty)) \to H^*(\mathrm{BO}_k)$ between the characteristic classes of the respective bundle theories. In various cases (e.g., with $\mathbb{Z}_2$ coefficients) these maps can be shown to be surjective. This gives new results on the topology of $\mathrm{MacP}(k,\infty)$ and connects matroid bundles to the many areas of topology that can be described in terms of characteristic classes.

2. Categories of Matroid Bundles and PL Vector Bundles

2A. The category of matroid bundles. Let B be the poset of cells in a PL cell complex $\mathcal{B}$. Any matroid bundle $(B,\mathcal{M})$ on B induces a canonical matroid bundle structure on the poset of cells of any PL subdivision of $\mathcal{B}$, by associating the oriented matroid $\mathcal{M}(\sigma)$ to each cell in the relative interior of a cell $\sigma \in B$. Two matroid bundles on PL cell complexes are defined to be *equivalent* if there exists a common PL subdivision of the cell complexes such that the resulting matroid bundles on this subdivision are identical.

For B an arbitrary poset, a matroid bundle $(B,\mathcal{M})$ induces a matroid bundle structure $(\Delta B, \mathcal{M}')$ on the cell complex $||\Delta B||$ by defining

$$\mathcal{M}'(\{b_1 < b_2 < \cdots < b_m\}) = \mathcal{M}(b_m).$$

We extend the above notion of equivalence by defining $(B,\mathcal{M})$ to be equivalent to $(\Delta B, \mathcal{M}')$.

Definition 2.1. If $(B_1,\mathcal{M}_1)$ and $(B_2,\mathcal{M}_2)$ are two matroid bundles, a *morphism* from $(B_1,\mathcal{M}_1)$ to $(B_2,\mathcal{M}_2)$ is a pair $(f,[C_f,\mathcal{M}_f])$, where f is a PL map from ΔB_1 to ΔB_2 and $[C_f,\mathcal{M}_f]$ is an equivalence class of matroid bundle structures on the mapping cylinder of f that restrict to structures equivalent to $(B_1,\mathcal{M}_1)$ and $(B_2,\mathcal{M}_2)$ at either end.

The *composition* of a morphism $(f, [C_f, \mathcal{M}_f])$ from $(B_1, \mathcal{M}_1)$ to $(B_2, \mathcal{M}_2)$ and a morphism $(g, [C_g, \mathcal{M}_g])$ from $(B_2, \mathcal{M}_2)$ to $(B_3, \mathcal{M}_3)$ is $(g \circ f, [C_{g\circ f}, \mathcal{M}_{g\circ f}])$, where $\mathcal{M}_{g\circ f}$ is determined by $\mathcal{M}_3$ on the simplices of B_3 and by $\mathcal{M}_f$ on the rest of the cells of $C_{g\circ f}$.

The set of rank-k matroid bundles and their morphisms form a category.

Definition 2.2. A morphism from $(B_1, \mathcal{M}_1)$ to $(B_2, \mathcal{M}_2)$ is an *isomorphism* if there exists a morphism from $(B_2, \mathcal{M}_2)$ to $(B_1, \mathcal{M}_1)$ such that the composition of these maps is the identity morphism.

We get a better relation to the category of rank-k real vector bundles by considering only isomorphism classes of matroid bundles:

Definition 2.3. MB_k will denote the category of isomorphism classes of rank-k matroid bundles and their morphisms.

The classifying space for matroid bundles. MB_k has a classifying space very similar in spirit (and, as we shall later see, in topology) to the classifying space $G(k, \mathbb{R}^\infty)$ for real rank-k vector bundles. Just as $G(k, \mathbb{R}^\infty)$ is the space of all rank-k subspaces of any $\mathbb{R}^n$, the classifying space for MB_k will be the set of all strong map images of any combinatorial model for any $\mathbb{R}^n$.

Definition 2.4. If M^n is a rank-n oriented matroid then define the *combinatorial Grassmannian* $\Gamma(k, M^n)$ to be the poset of all rank-k strong map images of M^n, with the partial order $M_1 \geq M_2$ if and only if M_1 weak maps to M_2.

In some papers the combinatorial Grassmannian is defined to be the order complex $\Delta\Gamma(k, M^n)$ of $\Gamma(k, M^n)$.

The combinatorial Grassmannian was first introduced in [MacPherson 1993] and was the subject of a previous survey article [Mnëv and Ziegler 1993], to which we refer the reader for further discussion.

A particularly useful case is when M^n is the coordinate oriented matroid:

Definition 2.5. Let M_n be the coordinate oriented matroid with elements $\{1, 2, \ldots, n\}$, i.e., the oriented matroid realized by the coordinate hyperplanes in $\mathbb{R}^n$. Then $\Gamma(k, M_n)$ is a *standard combinatorial Grassmannian*, or *MacPhersonian*, denoted $\mathrm{MacP}(k, n)$.

This case is especially important because of a nice alternate description:

Proposition 2.6 [Mnëv and Ziegler 1993]. $\mathrm{MacP}(k, n)$ *is the poset of all rank-k oriented matroids with elements* $\{1, 2, \ldots, n\}$, *ordered by weak maps.*

Note that if M_1 strong maps to M_2 then $\Gamma(k, M_2) \subseteq \Gamma(k, M_1)$ (and hence $\Delta\Gamma(k, M_2)$ is a subcomplex of $\Delta\Gamma(k, M_1)$). In particular:

- If $\{1, \ldots, n\}$ is the set of elements of M, $\Gamma(k, M)$ is a subposet of $\mathrm{MacP}(k, n)$.
- If M_2 is obtained from M_1 by deleting some elements, there is a natural embedding of $\Gamma(k, M_2)$ into $\Gamma(k, M_1)$. In particular, $\mathrm{MacP}(k, n) \hookrightarrow \mathrm{MacP}(k, n+1)$ for any k and n.

Thus the direct limit $\lim_{n\to\infty}\Gamma(k,M^n)$ in the category of posets and inclusions is $\bigcup_n \mathrm{MacP}(k,n)$, denoted $\mathrm{MacP}(k,\infty)$.

We can now rephrase the definition of matroid bundles:

Definition 2.7. A rank-k *matroid bundle* is a poset B and a poset map $\mathcal{M} : B \to \mathrm{MacP}(k,\infty)$.

Modifying definitions appropriately (to accomodate our combinatorial notion of bundles and bundle morphisms), we see:

Proposition 2.8. *The map* $\mathrm{id} : \mathrm{MacP}(k,\infty) \to \mathrm{MacP}(k,\infty)$ *is the universal bundle for* MB_k.

Proof. A matroid bundle $\mathcal{M} : B \to \mathrm{MacP}(k,\infty)$ determines a simplicial map from ΔB to $\Delta\,\mathrm{MacP}(k,\infty)$, and $\mathcal{M}$ induces a matroid bundle structure on the mapping cylinder, giving a classifying map. If $(f,[C_f,\mathcal{M}_f])$ is a matroid bundle morphism, then $(C_f,\mathcal{M}_f)$ determines a homotopy between the respective classifying maps. □

Thus $\mathrm{MacP}(k,\infty)$ is the classifying space for rank-k matroid bundles.

The cohomology of a poset P is defined to be the cohomology of its order complex ΔP. Thus we have:

Corollary 2.9. *The* characteristic classes *for* MB_k with coefficients in R *are the elements of the cohomology ring* $H^*(\Delta\,\mathrm{MacP}(k,\infty);R)$.

The finite combinatorial Grassmannians are of interest in their own right from several perspectives. The spaces $\Delta\Gamma(k,M^n)$ arise as the fibers of a combinatorial Grassmannian bundle in [MacPherson 1993], for instance, and $\Delta\Gamma(n-1,M^n)$ is closely related to the *extension space* $\mathcal{E}(M^n)$ discussed in Section 3.

2B. Relations between the real and combinatorial Grassmannians. We consider more closely the map

$$\gamma_R : G(k,\mathbb{R}^n) \to \Gamma(k,M^n)$$

introduced in Section 1B. The set of preimages of this map give a stratification of $G(k,\mathbb{R}^n)$ which is semialgebraic. This stratification has the property that if the closure of $\gamma_R^{-1}(M_1)$ intersects $\gamma_R^{-1}(M_2)$ then M_1 weak maps to M_2.

By the semi-algebraic triangulation theorem [Hironaka 1975], there exists a triangulation of $G(k,\mathbb{R}^n)$ refining this stratification, giving a simplicial map

$$\tilde{\gamma}_R : G(k,\mathbb{R}^n) \to \Delta\Gamma(k,M^n).$$

(This is described further in [MacPherson 1993] for $\mathrm{MacP}(k,n)$ and in [Anderson and Davis $\geq$ 1999] for more general M^n.) In the direct limit this gives a map $\tilde{\gamma} : G(k,\mathbb{R}^\infty) \to \Delta\,\mathrm{MacP}(k,\infty)$ of classifying spaces, and hence describes a map from the theory of real vector bundles to the theory of matroid bundles.

The hope is that the map $\tilde{\gamma}_R$ preserves a great deal of the topology of $G(k, \mathbb{R}^n)$. For instance, if the resulting map in cohomology were an isomorphism, then the process of making matroid bundles out of real vector bundles would preserve the theory of characteristic classes. There are numerous grounds for pessimism on this hope. These grounds are detailed in [Björner et al. 1993, Section 2.4]. To give two of the most glaring obstacles, this map is not surjective — it misses all the non-realizable oriented matroids — and the stratification of $G(k, \mathbb{R}^n)$ by preimages of γ_R can have strata with arbitrarily ugly topology [Mnëv 1988]. As an illustration of how bad the topology of combinatorial Grassmannians can be, we note examples by Mnëv and Richter-Gebert [1993] of non-realizable oriented matroids M^n with the property that $\Delta\Gamma(n-1, M^n)$ is disconnected.

This makes the array of positive results on γ_R rather surprising. For small values of k, for the first few homotopy groups, and for realizable oriented matroids we have results relating the real and combinatorial Grassmannians.

THEOREM 2.10 [Folkman and Lawrence 1978]. *$\Delta\Gamma(1, M^n)$ is homeomorphic to $G(1, \mathbb{R}^n)$. If M^n is realizable, $\gamma_R : G(1, \mathbb{R}^n) \to \Delta\Gamma(1, M^n)$ is a homeomorphism.*

THEOREM 2.11 [Babson 1993]. *$\Delta\Gamma(2, M^n)$ is homotopy equivalent to $G(2, \mathbb{R}^n)$.*

We will discuss the proof of Theorem 2.11 in Section 4B.

THEOREM 2.12 (compare [Mnëv and Ziegler 1993]). *Duality holds for the standard combinatorial Grassmannians: if $|E| = n$, then $\Delta\Gamma(k, E) \cong \Delta\Gamma(n-k, E)$.*

THEOREM 2.13 (compare [Mnëv and Ziegler 1993; Anderson 1998]).

1. *If $i = 0$ or $i = 1$ then $(\tilde{\gamma}_R)_* : \pi_i(G(k, \mathbb{R}^n)) \to \pi_i(\Delta\,\mathrm{MacP}(k, n))$ is an isomorphism. Further,*
$$(\tilde{\gamma}_R)_* : \pi_2(G(k, \mathbb{R}^n)) \to \pi_2(\Delta\,\mathrm{MacP}(k, n))$$
is a surjection.
2. *$\eta_* : \pi_i(\Delta\,\mathrm{MacP}(k, n)) \to \pi_i(\Delta\,\mathrm{MacP}(k, n+1))$ is an isomorphism if $n > k(i+2)$ and a surjection for $n > k(i+1)$.*

THEOREM 2.14 [Anderson and Davis $\geq$ 1999; Anderson et al. $\geq$ 1999].

1. *The maps $\gamma_R^* : H^*(\Delta\Gamma(k, M^n); \mathbb{Z}_2) \to H^*(G(k, \mathbb{R}^n); \mathbb{Z}_2)$ for realizable M^n and $\tilde{\gamma}^* : H^*(\Delta\,\mathrm{MacP}(k, \infty); \mathbb{Z}_2) \to H^*(G(k, \mathbb{R}^\infty); \mathbb{Z}_2)$ are split surjections.*
2. *The maps $\gamma_R^* : H^*(\Delta\Gamma(k, M^n); \mathbb{Q}) \to H^*(G(k, \mathbb{R}^n); \mathbb{Q})$ for realizable M^n and $\tilde{\gamma}^* : H^*(\Delta\,\mathrm{MacP}(k, \infty); \mathbb{Q}) \to H^*(G(k, \mathbb{R}^\infty); \mathbb{Q})$ are split surjections.*

Theorem 2.14 follows from the constructions of combinatorial sphere bundles associated to matroid bundles described in Section 4A. In terms of characteristic classes, this theorem implies that matroid bundles have well-defined Stiefel–Whitney and Pontrjagin classes.

Associated to a rank-k oriented real vector bundle $\xi : E \to B$ and its associated sphere bundle $E_0 \to B$ is a cohomology class $u \in H^k(E, E_0, \mathbb{Z})$ whose restriction to each fiber is the orientation class of that fiber. The *Thom isomorphism* of the vector bundle is the isomorphism

$$\begin{array}{rcl} \phi : H^i(B;\,\mathbb{Z}) & \longrightarrow & H^{i+k}(E, E_0;\,\mathbb{Z}) \\ x & \longrightarrow & \xi^*(x) \cup u \end{array}$$

[Milnor and Stasheff 1974, Chapter 9]. There is a unique class $e(\xi)$ in $H^k(B;\,\mathbb{Z})$ which is mapped to $u|_E$ under ξ^*. This is a characteristic class of ξ, called the *Euler class.*

The combinatorial sphere bundles associated to matroid bundles admit analogous constructions, giving a further result on characteristic classes.

THEOREM 2.15 [Anderson and Davis $\geq$ 1999]. *There is a well-defined Thom isomorphism and Euler class for matroid bundles.*

2C. The category of PL vector bundles. To compare real vector bundles and matroid bundles, we need to restrict to real vector bundles with triangulable base spaces. Specifically:

DEFINITION 2.16. Define VB_k to be the category whose objects are all rank-k real vector bundles over PL spaces and whose morphisms are the vector bundle maps preserving this PL structure.

This section will give a functor from VB_k to MB_k. Section 1C gave one way to associate a matroid bundle to a real vector bundle with a triangulated base space. Here we will use a related method that is less intuitive but more useful. Consider the map $\gamma_R : G(k, \mathbb{R}^n) \to \Gamma(k, M^n)$ for a realizable M and the direct limit map $\gamma : G(k, \mathbb{R}^\infty) \to \mathrm{MacP}(k, \infty)$. For any $V \in G(k, \mathbb{R}^\infty)$ there is a small neighborhood of V which is mapped by γ to $\mathrm{MacP}(k, \infty)_{\geq \gamma(V)}$.

If a real vector bundle $\xi : E \to B$ over a PL cell complex B has classifying map $c : B \to G(k, \mathbb{R}^\infty)$, the composition $\gamma \circ c$ associates an oriented matroid to each point in B. We call c *tame* if $\gamma \circ c$ is constant on the interior of each cell. Thus a tame classifying map defines a matroid bundle structure on the poset of cells of B.

COMBINATORIALIZATION THEOREM [Anderson and Davis $\geq$ 1999]. *Let* $\xi = (\pi : E \to \|B\|)$ *be a rank-k real vector bundle, where* B *is a finite-dimensional simplicial complex.*

1. ξ *has a classifying map which is tame with respect to some simplicial subdivision of* B.
2. *For* $i = 0, 1$, *let* $c_i : B \to G(k, V_i)$ *be a tame classifying maps for* ξ. *Then there is a tame classifying map* $h : B \times I \to G(k, V_0 \oplus V_1)$, *restricting to* c_i *on* $B \times \{i\}$.

A slightly more complicated form of this theorem holds for bundles over infinite-dimensional spaces: see [Anderson and Davis $\geq$ 1999] for details.

A classifying map to $G(k, \mathbb{R}^n)$ determines a spanning collection of n sections of the bundle, by projecting the unit coordinate vectors in $\mathbb{R}^n$ onto the images of the fibers. Thus Proposition 1.1 is a corollary to the above theorem.

THEOREM 2.17. 1. *Let $\xi : E \to B$ be an element of VB_k and let $\mu_0 : |T_0| \to B$ and $\mu_1 : |T_1| \to B$ be two triangulations of B. Let c_0, c_1 be two classifying maps for ξ such that $\gamma \circ c_i \circ \mu_i^{-1}$ is constant on the interior of each simplex. Then the matroid bundles arising from c_0 and c_1 are isomorphic.*

Thus every element ξ of VB_k gives rise to a unique element $F(\xi)$ of MB_k.

2. *If $\Xi : \xi_1 \to \xi_2$ is a morphism in VB_k, then there exists a simplicial decomposition C_Ξ on the mapping cylinder of the base spaces giving a morphism from $F(\xi_1)$ to $F(\xi_2)$ in MB_k (unique up to equivalence), denoted $F(\Xi)$.*

PROOF. 1. Since c_0 and c_1 are both classifying maps for ordinary vector bundles, we know there exists a homotopy $H : B \times I \to G(k, \mathbb{R}^\infty)$ from c_0 to c_1 in the category of ordinary vector bundles. $B \times I$ has a PL structure induced by B, and μ_0 and μ_1 give PL triangulations of $B \times \{0\}$ and $B \times \{1\}$, respectively. By [Hudson 1969, Corollary 1.6], there exists a triangulation $\mu : |T| \to B \times I$ which restricts to a subdivision of the given triangulations at either end. Note that the composition of $H|_{B\times\{0,1\}}$ with $\tilde{\gamma} : G(k, \mathbb{R}^\infty) \to \Delta\,\mathrm{MacP}(k, \infty)$ is simplicial with respect to T. By the Simplicial Approximation Theorem, there is a simplicial map homotopic to $\tilde{\gamma} \circ H$. Since the only simplicial approximation to a simplicial map is itself, this simplicial map must restrict to c_i on $B \times \{i\}$.

2. Because the map of base spaces is PL, as above we get a triangulation C_Ξ of the mapping cylinder which restricts to PL triangulations of the base spaces at either end. Any two such triangulations have a common simplicial subdivision. Again applying the simplicial approximation theorem, we get a matroid bundle structure $\mathcal{M}_\Xi$ on C_Ξ which restricts to the appropriate matroid bundle isomorphism classes at either end. Thus C_Ξ and $\mathcal{M}_\Xi$ define a matroid bundle morphism. □

The map F defined in the previous theorem is easily seen to be a covariant functor from VB_k to MB_k.

3. Examples of Matroid Bundles

Matroid bundles arise in various contexts besides that of real vector bundles. We give some examples here.

Combinatorial Grassmannians. For any oriented matroid M^n, the identity map on $\Gamma(k, M^n)$ defines a matroid bundle. As mentioned before, this bundle is of independent combinatorial interest, for instance in relation to the next example.

Extension spaces. Let M be an oriented matroid with elements E. A *nonzero extension* of M by x is an oriented matroid M' with elements $E \cup \{x\}$ such that $M' \backslash x = M$ and M is not a loop in M'. The nonzero extensions of M form a poset $\mathcal{E}(M)$, ordered by weak maps. Since nonzero extensions $M \cup x$ correspond exactly to orientations on corank-1 strong map images M/x of M, there is a canonical double cover $\mathcal{E}(M) \to \Gamma(\operatorname{rank} M - 1, M)$. Thus $\mathcal{E}(M)$ has a natural matroid bundle structure.

If M and all extensions of M are realizable, then the order complex $\Delta\mathcal{E}(M)$ is a $(\operatorname{rank} M - 1)$-sphere, and its matroid bundle structure arises from the standard tangent bundle structure on the sphere. For more general M, life is not nearly so simple. As mentioned before, Mnëv and Richter-Gebert [1993] have found examples for which $\Delta\mathcal{E}(M)$ is not even connected. The topology of general $\Delta\mathcal{E}(M)$ is a mystery of some importance — for instance, due to its connection to the Generalized Baues Conjecture [Mnëv and Ziegler 1993; Reiner 1999].

Combinatorial differential manifolds. The theory of matroid bundles arose out of the theory of *combinatorial differential manifolds*, or *CD manifolds*, the main subject of [MacPherson 1993]. A CD manifold consists of a simplicial complex equipped with an atlas of oriented matroid coordinate charts. Such an atlas yields a matroid bundle structure in a canonical way. All of this is described in detail in [MacPherson 1993]. We review the idea briefly here.

Let N be a differential manifold and $\eta : |T| \to N$ be a piecewise smooth triangulation of N (i.e., smooth on every closed simplex). Consider a point t in the interior of some simplex $|\sigma|$ of $|T|$. For each vertex s of $\overline{\operatorname{star}\sigma}$, the line segment from t to s in $|\sigma \cup \{s\}|$ gives a smooth path $p_s : [0,1] \to N$ from $\eta(t)$ to $\eta(s)$ in N. The tangent vectors $\{p'_s(0) : s \in \overline{\operatorname{star}\sigma}^0\} \subset T_{\eta(t)}(N)$, each defined up to a positive scalar, give an oriented matroid $M(t)$. This oriented matroid can be viewed as the combinatorial remnant of an embedding of $|\operatorname{star}\sigma|$ into $T_{\eta(t)}(N)$, and hence as a combinatorial coordinate chart.

A triangulation is *tame* if there exists a subdivision of T into regular cells so that $M \circ \eta$ is constant on each open cell. Thus associated to a tame triangulation there is a triple $(T, \hat{T}, \mathcal{M})$, with

- T a simplicial complex,
- $\hat{T}$ a regular subdivision of T,
- $\mathcal{M} : \hat{T} \to \operatorname{MacP}(n, |T^0|)$ a map associating to each cell of $\hat{T}$ an oriented matroid.

This triple constitutes a CD manifold.

More generally, a CD manifold is defined to be any triple $(T, \hat{T}, \mathcal{M})$ satisfying certain combinatorial axioms intended to make M look like an atlas of coordinate charts on T (see the complete definition in [MacPherson 1993]). A canonical construction of a matroid tangent bundle associated to any such atlas is given in [MacPherson 1993, Section 3.2].

CD manifolds are the basis for the combinatorial formula for Pontrjagin classes found in [Gelfand and MacPherson 1992]. This formula is discussed in more detail in Section 4C.

Not all CD manifolds arise from differential manifolds (for instance, there exist CD manifolds involving non-realizable oriented matroids). Indeed, nothing in the definition of CD manifolds promises immediately that the base space of a CD manifold is a topological manifold.

CONJECTURE 3.1. *If* $(X, \hat{X}, M)$ *is a CD manifold*, *then* X *is a PL manifold.*

In [Anderson 1999a] this conjecture was proved under the restriction that all oriented matroids involved are *Euclidean.* Euclidean oriented matroids [Björner et al. 1993, Section 10.5] are essentially oriented matroids with the intersection properties of a real vector arrangement. The class of Euclidean oriented matroids includes all realizable oriented matroids and all oriented matroids of rank less than 4.

4. Methods

We outline here some of the most important methods that have developed for studying matroid bundles.

4A. Sphere bundles. One disturbing aspect of the definition of matroid bundles is the absence of a topological total space. Our understanding of real vector bundles $\xi : E \to B$ follows largely from various constructions involving the total space E, or the sphere bundle $\{(b, v) : b \in B, v \in \xi^{-1}(b), v \neq 0\} \to B$. The Topological Representation Theorem leads to combinatorial analogs to these for matroid bundles. As mentioned in Section 1B, the intuition that makes an oriented matroid M a combinatorial model for a vector space also makes $V^*(M)\backslash 0$ a combinatorial model for the unit sphere in that vector space. We will discuss two ways to use these spheres to construct combinatorial sphere bundles over matroid bundles.

A spherical quasifibration associated to a matroid bundle.

DEFINITION 4.1. If $\mathcal{M} : B \to \mathrm{MacP}(k, \infty)$ is a matroid bundle, define

$$E(\mathcal{M}) = \{(b, X) : b \in B, X \in V^*(\mathcal{M}(b))\},$$
$$E_0(\mathcal{M}) = \{(b, X) : b \in B, X \in V^*(\mathcal{M}(b))\backslash 0\}.$$

Each of these sets is partially ordered by $(b_1, X_1) \geq (b_2, X_2)$ if and only if $b_1 \geq b_2$ and $X_1 \geq X_2$.

Note the following properties of $E(\mathcal{M})$ and $E_0(\mathcal{M})$:

- The projections $\pi' : E(\mathcal{M}) \to B$ and $\pi : E_0(\mathcal{M}) \to B$ onto the first component are poset maps. Thus they give simplicial maps $\Delta E(\mathcal{M}) \to \Delta B$ and $\Delta E_0(\mathcal{M}) \to \Delta B$.

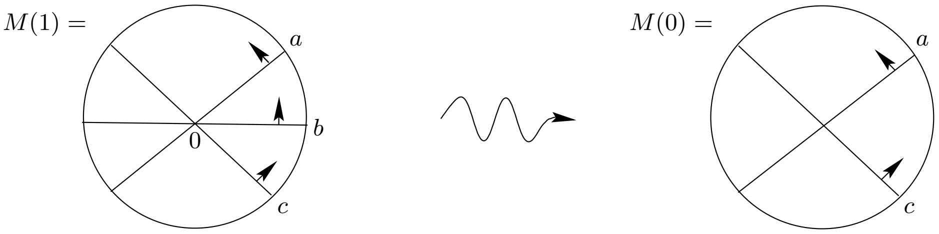

Figure 5. Fibers over elements of a poset $1 > 0$.

- For any element b of B, the fiber $\Delta\pi^{-1}(b)$ is the barycentric subdivision of the pseudosphere complex $V^*(M)\backslash 0$, hence is a sphere. The fiber $\Delta(\pi')^{-1}(b)$ is the cone on this sphere.
- If $\xi : \mathcal{E} \to \mathcal{B}$ is a vector bundle with a triangulation $\eta : ||B|| \to \mathcal{B}$ and a tame classifying map giving a matroid bundle $\mathcal{M} : B \to \mathrm{MacP}(k, \infty)$, then for any vertex b of ΔB, the unit sphere in $\xi^{-1}(\eta(b))$ is canonically isomorphic to the fiber over b in $\Delta E_0(\mathcal{M})$.
- The simplicial map $\Delta E_0(M) \to \Delta B$ is *not* necessarily a topological sphere bundle. For example, let B be the poset $1 > 0$ and $\mathcal{M} : B \to \mathrm{MacP}(2, 3)$ be as shown in Figure 5.

  The fiber over each vertex of ΔB is an S^1, but the fiber over the 1-simplex $\{1, 0\}$ is not an $S^1 \times I$, since it contains a 3-dimensional simplex $\{\{1, 0\}, a^-b^+c^+\} > \{\{1, 0\}, a^-b^0c^+\} > \{\{0\}, a^-b^0c^+\} > \{\{0\}, a^0b^0c^+\}\}$.

THEOREM 4.2 [Anderson and Davis $\geq$ 1999].

1. $\Delta E_0 \to \Delta B$ *is a spherical quasifibration.*
2. *A morphism of matroid bundles induces a morphism of the corresponding spherical quasifibrations.*
3. *Let $E_G \to G(k, \mathbb{R}^n)$ be the unit sphere bundle over $G(k, \mathbb{R}^n)$, and let M be a realizable rank-n oriented matroid. Then there is a map of spherical quasifibrations*

$$\begin{tikzcd}
E_G \arrow[r] \arrow[d] & \Delta E_0 \arrow[d] \\
G(k, \mathbb{R}^n) \arrow[r, "c"] & \Delta\Gamma(k, M)
\end{tikzcd}$$

The $\mathbb{Z}_2$ cohomology of $G(k, \mathbb{R}^n)$ is generated by the *Stiefel–Whitney classes* [Milnor and Stasheff 1974]. The classifying space BFib for spherical quasifibrations also has well-defined Stiefel–Whitney classes [Stasheff 1963]. Given a map of spherical quasifibrations, the induced map on the cohomology rings of the base spaces preserves Stiefel–Whitney classes. Thus as a corollary to Theorem 4.2 we have Theorem 2.14.1. See [Anderson and Davis $\geq$ 1999] for constructions of explicit combinatorial Stiefel–Whitney classes in $H^*(\Delta\Gamma(k, M); \mathbb{Z}_2)$.

In addition, a spherical quasifibration gives a Serre spectral sequence, whose collapsing gives the Thom isomorphism and Theorem 2.15.

A PL microbundle associated to a matroid bundle. A much more complicated construction associates to any matroid bundle a PL microbundle. For the full construction see [Anderson et al. $\geq$ 1999]. Here we will outline some key points of the construction.

THEOREM 4.3 [Anderson 1999b]. *Let $M_1 \rightsquigarrow M_2$ be a weak map of oriented matroids.*

1. *If $X \in V^*(M_1)$ then there is a unique maximal $g(X) \in V^*(M_2)$ such that $X \geq g(X)$.*
2. *The map $g : V^*(M_1) \to V^*(M_2)$ thus defined is a poset map.*
3. *If $Y \in V^*(M_2)$ then $g^{-1}(Y)$ is contractible.*
4. *If $\operatorname{rank}(M_1) = \operatorname{rank}(M_2)$ and $X \in V^*(M_1)\backslash 0$ then $g(X) \neq 0$.*

Thus a weak map $M_1 \rightsquigarrow M_2$ gives a simplicial map $g : V^*(M_1) \to V^*(M_2)$ of balls, and, if the oriented matroids have the same rank, a simplicial map $g : V^*(M_1)\backslash 0 \to V^*(M_2)\backslash 0$ of spheres.

THEOREM 4.4 [Anderson et al. $\geq$ 1999]. *If $M_1 \rightsquigarrow M_2 \rightsquigarrow M_3$ are oriented matroids and g_{ij} denotes the poset map $V^*(M_j)\backslash 0 \to V^*(M_i)\backslash 0$ as above, then $g_{32} \circ g_{21}(X) \leq g_{31}(X)$ for every $X \in V^*(M_1)\backslash 0$.*

If this last inequality were an equality, then g would give a functor from the category Γ_k of rank-k oriented matroids and weak maps to the category of PL spheres and PL maps, and the homotopy colimit of the image of this functor would give a PL sphere bundle over any matroid bundle. With the inequality we have, a much more delicate construction (detailed in [Anderson et al. $\geq$ 1999]) uses g to construct a PL sphere bundle over the second barycentric subdivision of the base space of any matroid bundle. Since PL microbundles have well-defined rational Pontrjagin classes and these classes generate $H^*(G(k, \mathbb{R}^\infty); \mathbb{Q})$, we have Theorem 2.14.2.

4B. Hairs. Babson [1993] generalized combinatorial Grassmannians to *combinatorial flag spaces*, and developed a new tool, *hairs*, to study rank-2 strong map images.

Let $G(v_1, \dots, v_m, \mathbb{R}^n)$ denote the topological space of all flags $V_1 \subset V_2 \subset \cdots \subset V_m \subseteq \mathbb{R}^n$ of subspaces in $\mathbb{R}^n$ with $\dim V_i = v_i$ for every i. As a combinatorial analog to these flag spaces, we define:

DEFINITION 4.5. If M^n is an oriented matroid and $\{v_1 < \cdots < v_m \leq n\}$ is a chain in $\mathbb{N}$, let

$$\Gamma(v_1, \dots, v_m, M^n) = \{(N_1, \dots, N_m) : N_i \in \Gamma(v_i, N_{i+1}) \text{ if } i < m \text{ and } N_m \in \Gamma(v_m, M)\}.$$

This is a poset, ordered by componentwise weak maps.

THEOREM 4.6 [Babson 1993]. *Let M^n be any oriented matroid of rank greater than* 1.

1. $\Delta\Gamma(1, 2, M^n)$ *is homotopy equivalent to* $G(1, 2, \mathbb{R}^n)$.
2. $\Delta\Gamma(2, M^n)$ *is homotopy equivalent to* $G(2, \mathbb{R}^n)$.

Let p_1 denote the projection $\Delta\Gamma(1, 2, M^n) \to \Delta\Gamma(1, M^n)$ and p_2 the projection $\Delta\Gamma(1, 2, M^n) \to \Delta\Gamma(2, M^n)$. The latter projection is easily seen to be a quasifibration. The crucial result in the proof of Theorem 4.6 is the following:

LEMMA 4.7. p_1 *is a quasifibration with fiber a homotopy* $\mathbb{RP}^{n-2}$.

The concept of *hairs* arises in the proof of this lemma. Note that an element of $\Gamma(1, M^n)$ is just a pair $\{X, -X\}$ of antipodal covectors, and an element of $\Gamma(1, 2, M^n)$ is such a pair together with an embedded circle in $\Delta V^*(M)$ containing X and $-X$. It is convenient to consider the space $\tilde{\Gamma}(1, 2, M^n)$ of such pairs together with an orientation on the circle. Then the projection $p : \tilde{\Gamma}(1, 2, M^n) \to \Gamma(1, 2, M^n)$ is a double cover, and one shows that $\tilde{p}_1 = p_1 \circ p$ is a quasifibration with fiber a homotopy sphere.

Since each such circle has antipodal symmetry, an element of $\tilde{\Gamma}(1, 2, M^n)$ can be represented by an oriented path of covectors from X to $-X$. Babson's proof factors $\tilde{p}_1$ through a series of intermediate spaces which, instead of recording a path from X to $-X$, record only a shorter path which starts at X and is contained in a rank-2 strong image of M^n. These paths are called *hairs*. At each stage of the factorization the ends of the hairs are cut off, until at the last stage they are simply a single covector (an element of $\Gamma(1, M^n)$). All the intermediate projections except the last are homotopy equivalences, while the last is easily seen to be a quasifibration.

To prove Theorem 4.6, fix a basis $\mathcal{B}$ for M^n, and let M_0 be the oriented matroid obtained from M^n by deleting all elements not in $\mathcal{B}$. Then $\Delta\Gamma(1, M_0)$, $\Delta\Gamma(2, M_0)$, and $\Delta\Gamma(1, 2, M_0)$ are easily seen to be homotopic to the analogous real flag spaces. The first part of Theorem 4.6 is proved by considering the diagram

$$\begin{tikzcd}
\Delta\Gamma(1,2,M^n) \arrow[r] \arrow[d, "d_{12}"'] & \Delta\Gamma(1,M^n) \arrow[d, "d_1"] \\
\Delta\Gamma(1,2,M_0) \arrow[r] & \Delta\Gamma(1,M_0)
\end{tikzcd}$$

where the vertical maps are obtained by deleting elements not in $\mathcal{B}$ from each oriented matroid. The map d_1 is easily seen to be a homotopy equivalence: both spaces are homeomorphic to $\mathbb{RP}^{n-1}$. Lemma 4.7 showed that the horizontal maps are quasifibrations with fiber $\mathbb{RP}^{n-2}$. One can also check that the induced maps on fibers are weak homotopy equivalences. A quasifibration has an associated long exact sequence of homotopy groups [Dold and Thom 1958], so we have a diagram with exact rows:

$$\begin{tikzcd}
\cdots \pi_i(p_1^{-1}(x), y) \arrow[r] \arrow[d] & \pi_i(\Delta\Gamma(1,2,M^n), y) \arrow[r] \arrow[d] & \pi_i(\Delta\Gamma(1,M^n), x) \arrow[r] \arrow[d] & \pi_{i-1}(p_1^{-1}(x), y) \cdots \arrow[d] \\
\cdots \pi_i(p_1^{-1}(x'), y') \arrow[r] & \pi_i(\Delta\Gamma(1,2,M_0), y') \arrow[r] & \pi_i(\Delta\Gamma(1,M_0), x') \arrow[r] & \pi_{i-1}(p_1^{-1}(x'), y') \cdots
\end{tikzcd}$$

where x and x' are base points in $\Delta\Gamma(1, M^n)$ and $\Delta\Gamma(1, M_0)$, respectively, and y and y' are base points in the fibers of p_1 over x and x', respectively. It follows that $d_{12*} : \pi_i(\Delta\Gamma(1, 2, M^n), y) \to \pi_i(\Delta\Gamma(1, 2, M_0), y')$ is an isomorphism.

Knowing this, one proves the second statement of Theorem 4.6 by considering the diagram

$$\begin{tikzcd}
\Delta\Gamma(2, M^n) \arrow[d, "d_2"'] & \Delta\Gamma(1,2,M^n) \arrow[l] \arrow[d, "d_{12}"] \\
\Delta\Gamma(2, M_0) & \Delta\Gamma(1,2,M_0). \arrow[l]
\end{tikzcd}$$

The horizontal maps are quasifibrations, and a diagram as before proves d_2 is a homotopy equivalence.

4C. Pontrjagin classes. The theorem of the previous section is the first step in an approach of Gelfand and MacPherson to finding a combinatorial formula for the rational Pontrjagin classes of a differential manifold ([Gelfand and MacPherson 1992]). As described in Section 3, a triangulation of a differential manifold yields a CD manifold, which in turn yields a matroid bundle. Associated to any matroid bundle $(B, \mathcal{M})$ there is a quasifibration $Y \to B$, where the fiber over a vertex b is $\Delta\Gamma(2, \mathcal{M}(b))$. In turn, there is a quasifibration $Z \to Y$ in which the fiber over a vertex (b, N) is $\Delta\Gamma(1, N)$. This is analogous to the association to a tangent bundle $TM \to M$ the Grassmannian 2-plane bundle $\mathcal{G}_2(TM) \to M$ and the circle bundle $\mathcal{G}_1(\mathcal{G}_2(TM)) \to \mathcal{G}_2(TM)$. If the matroid bundle arose from a differential manifold, we also get a *fixing cycle*, analogous to the orientation class of $\mathcal{G}_2(TM)$. These analogies allow one to reproduce in a combinatorial context a formula for Pontrjagin classes arising in Chern–Weil theory.

5. Areas for Further Research

1. We do not know the kernel of the map

$$\tilde{\gamma}^* : H^*(\Delta\Gamma(k, M^n), R) \to H^*(G(k, \mathbb{R}^n), R)$$

for any coefficients R. Any nontrivial elements would give exotic characteristic classes for matroid bundles which may have interesting combinatorial interpretations.

2. Is $\tilde{\gamma}^*$ surjective in integer coefficients? One motivation for this question is that integer characteristic classes are used to distinguish exotic differential structures

on spheres. Thus a positive answer to this question would suggest that CD manifolds can distinguish exotic differential structures.

3. Any smooth manifold has a tame triangulation, giving a CD manifold. Thus if Conjecture 3.1 is true in general, then CD manifolds lie somewhere between the differential and PL categories. The question is where. Does every PL manifold have a CD structure? Is every CD manifold smoothable?

4. The computational question of calculating the homotopy groups of some of the finite MacPhersonians becomes more enticing in light of the stability result, Theorem 2.2. Theorem 1.1 gives π_0 and π_1, but beyond this the question is open.

5. Under the right notion of "complex oriented matroid", one should get a useful theory of complex matroid bundles. There are two likely candidates for the fibers. Ziegler's notion of complex matroids [Ziegler 1993] has already proved to encode nontrivial aspects of complex structure, and gives a "combinatorial complex Grassmannian" by exactly the same construction as $\Gamma(k, M^n)$. Alternatively, one could define a complex oriented matroid to be a direct sum of two real oriented matroids. Neither avenue to complex matroid bundles has yet been explored.

6. To reiterate a question from [MacPherson 1993], what are the combinatorial analogs to such topological theories as transversality, cobordism, surgery, and so on?

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