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Combinatorial Representation Theory

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Abstract. We survey the field of combinatorial representation theory, describe the main results and main questions and give an update of its current status. Answers to the main questions are given in Part I for the fundamental structures, S_n and $\mathrm{GL}(n,\mathbb{C})$, and later for certain generalizations, when known. Background material and more specialized results are given in a series of appendices. We give a personal view of the field while remaining aware that there is much important and beautiful work that we have been unable to mention.

Contents

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Appendices

Introduction

In January 1997, during the special year in combinatorics at MSRI, at a dessert party at Hélène's house, Gil Kalai, in his usual fashion, began asking very pointed questions about exactly what all the combinatorial representation theorists were investigating. After several unsuccessful attempts at giving answers that Gil would find satisfactory, it was decided that some talks should be given in order to explain to other combinatorialists what the specialty is about and what its main questions are.

In the end, Arun gave two talks at MSRI in which he tried to clear up the situation. After the talks several people suggested that it would be helpful if someone would write a survey article containing what had been covered in the two talks and including further interesting details. After some arm twisting it was agreed that Arun and Hélène would write such a paper on combinatorial representation theory. What follows is our attempt to define the field of combinatorial representation theory, describe the main results and main questions and give an update of its current status.

Of course this is wholly impossible. Everybody in the field has their own point of view and their own preferences of questions and answers. Furthermore, there is much too much material in the field to possibly collect it all in a single article (even conceptually). We therefore feel that we must stress the obvious: in this article we give a personal viewpoint on the field while remaining aware that there is much important and beautiful work that we have not been able to mention.

On the other hand, we have tried very hard to give a focused approach and to make something that will be useful to both specialists and non specialists, for understanding what we do, for learning the concepts of the field, and for tracking down history and references. We have chosen to write in an informal style in the hope that this way we can better convey the conceptual aspects of the field. Readers should keep this in mind and refer to the notes and references and the appendices when there are questions about the precision in definitions and statements of results. We have included a table of contents at the beginning of the paper which should help with navigation. The survey articles [Hanlon 1984; Howe 1995; Stanley 1983] are excellent complements to this one. Having made these points, and put in a lot of work, we leave it to you the reader, with the earnest hope that you find it useful.

We would like to thank the many people in residence at the special year 1996–97 in combinatorics at MSRI, for their interest, their suggestions, and for continually encouraging us to explain and write about the things that we enjoy doing. We both are extremely indebted to our graduate advisors, A. Garsia and H. Wenzl, who (already many years ago) introduced us to and taught us this wonderful field.

PART I

1. What is Combinatorial Representation Theory?

What do we mean by "combinatorial representation theory"? First and foremost, combinatorial representation theory is representation theory. The adjective "combinatorial" will refer to the way in which we answer representation theoretic questions; we will discuss this more fully later. For the moment we begin with:

What is Representation Theory?

If representation theory is a black box, or a machine, then the input is an algebra A. The output of the machine is information about the modules for A.

An **algebra** is a vector space A over $\mathbb{C}$ with a multiplication.

An important example: define the **group algebra** of a group G to be

$$A = \mathbb{C}G = \mathbb{C}\text{-span } \{g \in G\},$$

so that the elements of G form a basis of A. The multiplication in the group algebra is inherited from the multiplication in the group.

We want to study the algebra A via its actions on vector spaces.

An **A-module** is a finite-dimensional vector space M over $\mathbb{C}$ with an A action.

See Appendix A1 for a complete definition. We shall use the words module and **representation** interchangeably. Representation theorists are always trying to break up modules into pieces.

An A-module M is **indecomposable** if $M \not\cong M_1 \oplus M_2$ where M_1 and M_2 are nonzero A-modules.

An A-module M is **irreducible** or **simple** if it has no submodules.

In reference to modules the words "irreducible" and "simple" are used completely interchangeably.

The algebra A is **semisimple** if

$$\textbf{indecomposable} = \textbf{irreducible}$$

for A-modules.

The nonsemisimple case, where indecomposable is not the same as irreducible, is called modular representation theory. We will not consider this case much in these notes. However, before we banish it completely we describe the flavor of modular representation theory.

A **composition series** for M is a sequence

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_k = 0$$

of submodules of M such that each M_i/M_{i+1} is simple.

The Jordan–Hölder theorem says that two different composition series of M will always produce the same multiset $\{M_i/M_{i+1}\}$ of simple modules. Modular representation theorists are always trying to determine this multiset of *composition factors of M*.

Remarks

(1) We shall not make life difficult in this article but one should note that it is common to work over general fields rather than just using the field $\mathbb{C}$.

(2) If one is bold one can relax things in the definition of module and let M be infinite-dimensional.

(3) Of course the definition of irreducible modules is not correct since 0 and M are always submodules of M. So we are ignoring these two submodules in this definition. But conceptually the definition is the right one, we want a simple module to be something that has no submodules.

(4) The definition of semisimple above is not technically correct; see Appendix A1 for the proper definition. However, the power of semisimplicity is exactly that it makes all indecomposable modules irreducible. So "indecomposable = irreducible" is really the right way to think of semisimplicity.
(5) A good reference for the basics of representation theory is [Curtis and Reiner 1962]. The book [Bourbaki 1958] contains a completely general and comprehensive treatment of the theory of semisimple algebras. Appendix A1 to this article also contains a brief (and technically correct) introduction with more specific references.

Main Questions in Representation Theory

I. What are the irreducible A-modules?

What do we mean by this question? We would like to be able to give some kind of answer to the following more specific questions.

(a) How do we index/count them?

(b) What are their dimensions?

The dimension of a module is just its dimension as a vector space.

(c) What are their characters?

The character of a module M is the function $\chi_M : A \to \mathbb{C}$, where $\chi_M(a)$ is the trace of the linear transformation determined by the action of a on M. More precisely,

$$\chi_M(a) = \sum_{b_i \in B} ab_i\big|_{b_i},$$

where the sum is over a basis B of the module M and $ab_i\big|_{b_i}$ denotes the coefficient of b_i in ab_i when we expand in terms of the basis B.

C. How do we construct the irreducible modules?

S. Special/Interesting representations M

(a) How does M decompose into irreducibles?

If we are in the semisimple case then M will always be a direct sum of irreducible modules. If we group the irreducibles of the same type together we can write

$$M \cong \bigoplus_{\lambda} (V^\lambda)^{\oplus c_\lambda},$$

where the modules V^λ are the irreducible A-modules and c_λ is the number of times an irreducible of type V^λ appears as a summand in M. It is common to abuse notation and write

$$M = \sum_\lambda c_\lambda V^\lambda.$$

(b) What is the character of M?

Special modules often have particularly nice formulas describing their characters. It is important to note that having a nice character formula for M does not necessarily mean that it is easy to see how M decomposes into irreducibles. Thus this question really is different from the previous one.

(c) How do we find interesting representations?

Sometimes special representations turn up by themselves and other times one has to work hard to construct the right representation with the right properties. Often very interesting representations come from other fields.

(d) Are they useful?

A representation may be particularly interesting just because of its structure while other times it is a special representation that helps to prove some particularly elusive theorem. Sometimes these representations lead to a completely new understanding of previously known facts. A famous example (which unfortunately we won't have space to discuss; see [Humphreys 1972]) is the Verma module, which was discovered in the mid 1960s and completely changed representation theory.

M. The modular case

In the modular case we have the following important question in addition to those above.

(a) What are the indecomposable representations?

(b) What are the structures of their composition series?

For each indecomposable module M there is a multiset of irreducibles $\{M_i/M_{i+1}\}$ determined by a composition series of M. One would like to determine this multiset.

Even better (especially for combinatorialists), the submodules of M form a lattice under inclusion of submodules and one would like to understand this lattice. This lattice is always a modular lattice and we may imagine that each edge of the Hasse diagram is labeled by the simple module N_1/N_2 where N_1 and N_2 are the modules on the ends of the edge. With this point of view the various composition series of M are the maximal chains in this lattice of modules. The Jordan–Hölder theorem says that every maximal chain in the lattice of submodules of M has the same multiset of labels on its edges. What modular representation theorists try to do is determine the set of labels on a maximal chain.

Remark. The abuse of notation that allows us to write $M = \sum_\lambda c_\lambda V^\lambda$ has been given a formal setting called the *Grothendieck ring*. In other words, the formal object that allows us to write such identities has been defined carefully. See [Serre 1971] for precise definitions of the Grothendieck ring.

Answers Should Be of the Form...

Now we come to the adjective "Combinatorial." It refers to the way in which we give the answers to the main questions of representation theory.

I. What are the irreducible A-modules?

(a) How do we index/count them?

We want to answer with a **bijection**

$$\textbf{nice combinatorial objects } \lambda \quad \overset{1-1}{\longleftrightarrow} \quad \textbf{irreducible representations } V^\lambda.$$

(b) What are their dimensions?

We should answer with a **formula** of the form

$$\dim(V^\lambda) = \#\ \textbf{of nice combinatorial objects.}$$

(c) What are their characters?

We want a **character formula** of the type

$$\chi(a) = \sum_T \mathrm{wt}^a(T),$$

where the sum runs over all T in a set of nice combinatorial objects and wt^a is a weight on these objects which depends on the element $a \in A$ where we are evaluating the character.

C. How do we construct the irreducible modules?

We want to give constructions that have a **very explicit and very combinatorial flavor.** What we mean by this will be clearer from the examples; see **C(i)**, **C(ii)** in Section 2 (page 31).

S. Special/Interesting representations M

(a) How does M decompose into irreducibles?

If M is an interesting representation we want to determine **the positive integers** c_λ in the decomposition

$$M \cong \bigoplus_\lambda (V^\lambda)^{\oplus c_\lambda}$$

in the form

$$c_\lambda = \#\ \textbf{of nice combinatorial objects.}$$

In the formula for the decomposition of M the sum is over all λ which are objects indexing the irreducible representations of A.

(b) What is the character of M?

As in case **I(c)**, we want a **character formula** of the type

$$\chi_M(a) = \sum_T \mathrm{wt}^a(T),$$

where the sum runs over all T in some set of nice combinatorial objects and wt^a is a weight on these objects which depends on the element $a \in A$ where we are evaluating the character.

(c) How do we find interesting representations?

It is particularly pleasing when interesting representations arise in **other parts of combinatorics!** One such example is a representation on the homology of the partition lattice, which also, miraculously, appears as a representation on the free Lie algebra. We won't have space to discuss this here, see the original references [Hanlon 1981; Joyal 1986; Klyachko 1974; Stanley 1982], the article [Garsia 1990] for some further basics, and [Barcelo 1990] for a study of how it can be that this representation appears in two completely different places. Another fascinating example can be found in Sundaram's work [1994]. In this paper she study the homology representations of the symmetric group on Cohen–Macaulay subposets of the partition lattice. The coefficients discussed in **(S2)** of Section 2 (page 32) also appear in her work.

(d) Are they useful?

Are they useful for **solving combinatorial problems?** How about for **making new ones?** Sometimes a representation is exactly what is most helpful for solving a combinatorial problem. One example of this is in the recent solution of the last few plane partition conjectures. See [Stanley 1971; Macdonald 1995, Chap. I, § 5, Ex. 13–18] for the statement of the problem and [Kuperberg 1994c; 1994b; 1996a; Stembridge 1994a; 1994b; 1995] for the solutions. These solutions were motivated by the method of Proctor [1984].

The main point of all this is that a combinatorialist thinks in a special way (nice objects, bijections, weighted objects, etc.) and this method of thinking should be an integral part of the form of the solution to the problem.

2. Answers for S_n, the Symmetric Group

Most people in the field of combinatorial representation theory agree that the field begins with the fundamental results for the symmetric group S_n. We give the answers to the main questions for the case of S_n. The precise definitions of all the objects used below can be found in Appendix A2. As always, by a representation of the symmetric group we mean a representation of its group algebra $A = \mathbb{C}S_n$.

I. What are the irreducible S_n-modules?

(a) How do we index/count them?

There is a **bijection**

$$\textbf{partitions } \lambda \textbf{ of } n \quad \overset{1-1}{\longleftrightarrow} \quad \textbf{irreducible representations } S^\lambda.$$

(b) What are their dimensions?

The **dimension** of the irreducible representation S^λ is

$$\dim(S^\lambda) = \#\textbf{ of standard tableaux of shape } \lambda = \frac{n!}{\prod_{x\in\lambda} h_x},$$

where h_x is the hook length at the box x in λ (see Appendix A2).

(c) What are their characters?

Let $\chi^\lambda(\mu)$ be the character of the irreducible representation S^λ evaluated at a permutation of cycle type $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$. Then the **character** $\chi^\lambda(\mu)$ is given by

$$\chi^\lambda(\mu) = \sum_T \mathrm{wt}^\mu(T),$$

where the sum is over all standard tableaux T of shape λ and

$$\mathrm{wt}^\mu(T) = \prod_{i=1}^n f(i,T),$$

with

$$f(i,T) = \begin{cases} -1 & \text{if } i \notin B(\mu) \text{ and } i+1 \text{ is SW of } i, \\ 0 & \text{if } i,\, i{+}1 \notin B(\mu),\ i{+}1 \text{ is NE of } i, \text{ and } i{+}2 \text{ is SW of } i{+}1, \\ 1, & \text{otherwise,} \end{cases}$$

and $B(\mu) = \{\mu_1 + \mu_2 + \cdots + \mu_k \mid 1 \le k \le l\}$. In the formula for $f(i,T)$, SW means strictly south and weakly west and NE means strictly north and weakly east.

C. How do we construct the irreducible modules?

There are several interesting constructions of the irreducible S^λ.

(i) Via **Young symmetrizers.** Let T be a tableau. Set

$$\begin{aligned} R(T) &= \text{ permutations that fix the rows of } T\text{, as sets;} \\ C(T) &= \text{ permutations that fix the columns of } T\text{, as sets;} \\ P(T) &= \sum_{w\in R(T)} w, \qquad \text{and} \qquad N(T) = \sum_{w\in C(T)} \varepsilon(w) w, \end{aligned}$$

where $\varepsilon(w)$ is the sign of the permutation w. Then

$$S^\lambda \cong \mathbb{C}S_n P(T) N(T),$$

where the action of the symmetric group is by left multiplication.

(ii) Young's seminormal construction. Let

$$S^\lambda = \mathbb{C}\text{-span-}\{v_T \mid T \text{ is a standard tableau of shape } \lambda\},$$

so that the vectors v_T are a basis of S^λ. The **action** of S_n on S^λ is given by

$$s_i v_T = (s_i)_{TT} v_T + (1 + (s_i)_{TT}) v_{s_i T},$$

where s_i is the transposition $(i, i{+}1)$ and

$$(s_i)_{TT} = \frac{1}{c(T(i+1)) - c(T(i))},$$

with the following notation:

$T(i)$ denotes the box containing i in T;

$c(b)$ is the **content** of the box b, that is, $j-i$ if b has position (i,j) in λ;

$s_i T$ is the same as T except that the entries i and $i{+}1$ are switched;

$v_{s_i T} = 0$ if $s_i T$ is not a standard tableau.

There are other important constructions of the irreducible representations S^λ. We do not have room to discuss these constructions here, see Remarks (10)–(12) on page 34 and Appendix A3. The main ones are:

(iii) Young's orthonormal construction,

(iv) the Kazhdan–Lusztig construction, and

(v) the Springer construction.

S. Particularly interesting representations

(S1) Let $k + l = n$. The module $S^\lambda\big\downarrow^{S_n}_{S_k \times S_l}$ is the same as S^λ except that we only look at the action of the subgroup $S_k \times S_l$. Then

$$S^\lambda\big\downarrow^{S_n}_{S_k \times S_l} = \bigoplus_{\mu \vdash k, \nu \vdash l} (S^\mu \otimes S^\nu)^{\oplus c^\lambda_{\mu\nu}} = \sum_{\mu,\nu} c^\lambda_{\mu\nu} (S^\mu \otimes S^\nu),$$

where $\mu \vdash k$ means that μ is a partition of k, as usual, and the positive integers $c^\lambda_{\mu\nu}$ are the **Littlewood–Richardson coefficients**: $c^\lambda_{\mu\nu}$ is the number of column strict fillings of λ/μ of content ν such that the word of the filling is a lattice permutation. See Appendix A2.

(S2) Let $\mu = (\mu_1, \ldots, \mu_l)$ be a partition of n. Let $S_\mu = S_{\mu_1} \times \cdots \times S_{\mu_l}$. The module $\mathbf{1}\big\uparrow^{S_n}_{S_\mu}$ is the vector space

$$\mathbf{1}\big\uparrow^{S_n}_{S_\mu} = \mathbb{C}(S_n/S_\mu) = \mathbb{C}\text{-span}\{wS_\mu \mid w \in S_n\}$$

where the action of S_n on the cosets is by left multiplication. Then

$$\mathbf{1}\big\uparrow^{S_n}_{S_\mu} = \sum_\lambda K_{\lambda\mu} S^\lambda,$$

where

$K_{\lambda\mu} = \#$ **of column strict tableaux of shape λ and weight μ.**

This representation also occurs in the following context:

$$\mathbf{1}\big\uparrow_{S_\mu}^{S_n} \cong H^*(\mathcal{B}_u),$$

where u is a unipotent element of $\mathrm{GL}(n,\mathbb{C})$ with Jordan decomposition μ and $\mathcal{B}_u$ is the variety of Borel sugroups in $\mathrm{GL}(n,\mathbb{C})$ containing u. This representation is related to the Springer construction mentioned as item **C(v)** on the preceding page. See Appendix A3 for further details.

(S3) If $\mu, \nu \vdash n$, the tensor product S_n-module $S^\mu \otimes S^\nu$ is defined by $w(m\otimes n) = wm \otimes wn$, for all $w \in S_n$, $m \in S^\mu$ and $n \in S^\nu$. There are **positive integers** $\gamma_{\mu\nu\lambda}$ such that

$$S^\mu \otimes S^\nu = \sum_{\lambda\vdash n} \gamma_{\mu\nu\lambda} S^\lambda.$$

Except for a few special cases the $\gamma_{\mu\nu\lambda}$ are still unknown. See [Remmel 1992] for a combinatorial description of the cases for which the coefficients $\gamma_{\mu\nu\lambda}$ are known.

Remarks

(1) The bijection in **(Ia)**, page 31, between irreducible representations and partitions, is due to Frobenius [1900]. Frobenius is the founder of representation theory and the symmetric group was one of the first examples that he worked out.

(2) The formula in **(Ib)**, page 31, for the dimension of S^λ as the number of standard tableaux is immediate from the work of Frobenius, but it really came into the fore with the work of Young [1901; 1902; 1928; 1930a; 1930b; 1931; 1934a; 1934b]. The "hook formula" for $\dim(S^\lambda)$ is due to Frame, Robinson, and Thrall [Frame et al. 1954].

(3) The formula for the characters of the symmetric group given in **(Ic)**, page 31, is due to Fomin and Greene [1998]. For them, this formula arose by application of their theory of noncommutative symmetric functions. Roichman [1997] discovered this formula independently in the more general case of the Iwahori–Hecke algebra. The formula for the Iwahori–Hecke algebra is exactly the same as the one for the S_n case except that the 1 appearing in case 3 of the definition of $f(i,T)$ should be changed to a q.

(4) There is a different and more classical formula for the characters than the one given in **(Ic)**, called the Murnaghan–Nakayama rule [Murnaghan 1937; Nakayama 1941]. We describe it in Theorem A2.2 (page 57). Once the formula in **(Ic)** is given it is not hard to show combinatorially that it is equivalent to the Murnaghan–Nakayama rule, but if one does not know the formula it is nontrivial to guess it from the Murnaghan–Nakayama rule.

(5) We do not know if anyone has compared the algorithmic complexity of the formula given in **(Ic)** with that of the Murnaghan–Nakayama rule. One would expect that they have the same complexity: the formula above is a sum over more objects than the sum in the Murnaghan–Nakayama rule but these objects are easier to create and many of them have zero weight.

(6) One of the beautiful things about the formula for the character of S^λ given in **(Ic)** is that it is a sum over the same set that we have used to describe the dimension of S^λ.

(7) The construction of S^λ by Young symmetrizers is due to Young [1901; 1902]. It is used so often and has so many applications that it is considered classical. A review and generalization of this construction to skew shapes appears in [Garsia and Wachs 1989].

(8) The seminormal form construction of S^λ is also due to Young [1931; 1934b], although it was discovered some thirty years after the Young symmetrizer construction.

(9) Young's orthonormal construction differs from the seminormal construction only by multiplication of the basis vectors by certain constants. A comprehensive treatment of all three constructions of Young is given in [Rutherford 1948].

(10) The Kazhdan–Lusztig construction uses the Iwahori–Hecke algebra in a crucial way. It is combinatorial but relies crucially on certain polynomials which seem to be impossible to compute in practice except for very small n; see [Brenti 1994] for further information. This construction has important connections to geometry and other parts of representation theory. The paper [Garsia and McLarnan 1988] and the book [Humphreys 1990] give elementary treatments of the Kazhdan–Lusztig construction.

(11) Springer's construction is a geometric construction. In this construction the irreducible module S^λ is realized as the top cohomology group of a certain variety; see [Springer 1978; Chriss and Ginzburg 1997], and Appendix A3.

(12) There are many ways of constructing new representations from old ones. Among the common techniques are *restriction*, *induction*, and *tensoring*. The special representations **(S1)**, **(S2)**, and **(S3)** given on pages 32–33 are particularly nice examples of these constructions. One should note that tensoring of representations works for group algebras (and Hopf algebras) but not for general algebras.

3. Answers for $\mathrm{GL}(n, \mathbb{C})$, the General Linear Group

The results for the general linear group are just as beautiful and just as fundamental as those for the symmetric group. The results are surprisingly similar and yet different in many crucial ways. We shall see that the results for $\mathrm{GL}(n, \mathbb{C})$ have been generalized to a very wide class of groups whereas the results for S_n have only been generalized successfully to groups that look very similar to symmetric groups. The representation theory of $\mathrm{GL}(n,\mathbb{C})$ was put on a very firm footing from the fundamental work of Schur [1901; 1927].

I. What are the irreducible $\mathrm{GL}(n,\mathbb{C})$-modules?

(a) How do we index/count them?

There is a **bijection**

$$\textbf{partitions } \lambda \textbf{ with at most } n \textbf{ rows} \quad \overset{1-1}{\longleftrightarrow} \quad \textbf{irreducible polynomial representations } V^\lambda.$$

See Appendix A4 for a definition and discussion of what it means to be a *polynomial* representation.

(b) What are their dimensions?

The **dimension** of the irreducible representation V^λ is

$$\begin{aligned}\dim(V^\lambda) &= \#\ \textbf{of column strict tableaux of shape } \lambda \\ &\qquad\qquad \textbf{filled with entries from } \{1,2,\ldots,n\} \\ &= \prod_{x\in\lambda} \frac{n+c(x)}{h_x},\end{aligned}$$

where $c(x)$ is the content of the box x and h_x is the hook length at the box x.

(c) What are their characters?

Let $\chi^\lambda(g)$ be the character of the irreducible representation V^λ evaluated at an element $g \in \mathrm{GL}(n,\mathbb{C})$. The **character** $\chi^\lambda(g)$ is given by

$$\begin{aligned}\chi^\lambda(g) &= \sum_T x^T \\ &= \frac{\sum_{w\in S_n} \varepsilon(w) w x^{\lambda+\delta}}{\sum_{w\in S_n} \varepsilon(w) w x^{\delta}} = \frac{\det(x_i^{\lambda_j+n-j})}{\det(x_i^{n-j})},\end{aligned}$$

where the sum is over all column strict tableaux T of shape λ filled with entries from $\{1,2,\ldots,n\}$ and

$$x^T = x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n},$$

where μ_i is the number of i's in T and $x_1, x_2, \ldots, x_n$ are the eigenvalues of the matrix g. (Let us not worry at the moment about the quotient of sums on the second line of the display above. It is routine to rewrite it as the second expression on that line, which is one of the standard expressions for the Schur function; see [Macdonald 1995, Chap. I, § 3].)

C. How do we construct the irreducible modules?

There are several interesting constructions of the irreducible V^λ.

(C1) Via **Young symmetrizers.** Recall that the irreducible S^λ of the symmetric group S_k was constructed via Young symmetrizers in the form

$$S^\lambda \cong \mathbb{C}S_n P(T)N(T).$$

We can construct the irreducible $\mathrm{GL}(n,\mathbb{C})$-module in a similar form. If λ is a **partition** of k then

$$V^\lambda \cong V^{\otimes k} P(T)N(T).$$

This important construction is detailed in Appendix A5.

(C2) Gelfand–Tsetlin bases. This construction of the irreducible $\mathrm{GL}(n,\mathbb{C})$ representations V^λ is analogous to Young's seminormal construction of the irreducible representations S^λ of the symmetric group. Let

$$V^\lambda = \text{span-}\left\{ v_T \;\middle|\; \begin{array}{c} T \text{ is a column strict tableau of shape } \lambda \\ \text{filled with elements of } \{1,2,\ldots,n\} \end{array} \right\},$$

so that the vectors v_T are a basis of V^λ. Define an action of symbols $E_{k-1,k}$, for $2 \le k \le n$, on the basis vectors v_T by

$$E_{k-1,k} v_T = \sum_{T^-} a_{T^-T}(k) v_{T^-},$$

where the sum is over all column strict tableaux T^- which are obtained from T by changing a k to a $k-1$ and the coefficients $a_{T^-T}(k)$ are given by

$$a_{T^-T}(k) = -\frac{\prod_{i=1}^{k}(T_{ik} - T_{j,k-1} + j - k)}{\prod_{\substack{i=1 \\ i\neq j}}^{k-1}(T_{i,k-1} - T_{j,k-1} + j - k)},$$

where j is the row number of the entry where T^- and T differ and T_{ik} is the position of the rightmost entry $\le k$ in row i of T. Similarly, define an action of symbols $E_{k,k-1}$, for $2 \le k \le n$, on the basis vectors v_T by

$$E_{k,k-1} v_T = \sum_{T^+} b_{T^+T}(k) v_{T^+},$$

where the sum is over all column strict tableaux T^+ which are obtained from T by changing a $k-1$ to a k and the coefficients $b_{T^+T}(k)$ are given by

$$b_{T^+T}(k) = \frac{\prod_{i=1}^{k-2}(T_{i,k-2} - T_{j,k-1} + j - k)}{\prod_{\substack{i=1 \\ i\neq j}}^{k-1}(T_{i,k-1} - T_{j,k-1} + j - k)},$$

where j is the row number of the entry where T^+ and T differ and T_{ik} is the position of the rightmost entry $\le k$ in row i.

Since

$$g_i(z) = \begin{pmatrix} 1 & 0 & & \cdots & & 0 \\ 0 & \ddots & & & & \\ & & 1 & & & \vdots \\ \vdots & & & z & & \\ & & & & 1 & \\ & & & & \ddots & 0 \\ 0 & & \cdots & & 0 & 1 \end{pmatrix}, \quad \text{for} \quad z \in \mathbb{C}^*,$$

$$g_{i-1,i}(z) = \begin{pmatrix} 1 & 0 & \cdots & & & 0 \\ 0 & \ddots & & & & \\ & & 1 & z & & \vdots \\ \vdots & & 0 & 1 & & \\ & & & & \ddots & 0 \\ 0 & & \cdots & & 0 & 1 \end{pmatrix}, \quad \text{for} \quad z \in \mathbb{C},$$

$$g_{i,i-1}(z) = \begin{pmatrix} 1 & 0 & \cdots & & & 0 \\ 0 & \ddots & & & & \\ & & 1 & 0 & & \vdots \\ \vdots & & z & 1 & & \\ & & & & \ddots & 0 \\ 0 & & \cdots & & 0 & 1 \end{pmatrix}, \quad \text{for} \quad z \in \mathbb{C},$$

generate $\mathrm{GL}(n, \mathbb{C})$, the action of these matrices on the basis vectors v_T will determine the action of all of $\mathrm{GL}(n, \mathbb{C})$ on the space V^λ. The action of these generators is given by

$$\begin{aligned} g_i(z) v_T &= z^{(\#\text{ of } i\text{'s in } T)} v_T, \\ g_{i-1,i}(z) v_T &= e^{zE_{i-1,i}} v_T = (1 + zE_{i-1,i} + \tfrac{1}{2!} z^2 E_{i-1,i}^2 + \cdots) v_T, \\ g_{i,i-1}(z) v_T &= e^{zE_{i,i-1}} v_T = (1 + zE_{i,i-1} + \tfrac{1}{2!} z^2 E_{i,i-1}^2 + \cdots) v_T. \end{aligned}$$

(C3) The Borel–Weil–Bott construction. Let λ be a partition of n. Then λ defines a character (one-dimensional representation) of the group T_n of diagonal matrices in $G = \mathrm{GL}(n, \mathbb{C})$. This character can be extended to the group $B = B_n$ of upper triangular matrices in $G = \mathrm{GL}(n, \mathbb{C})$ by letting it act trivially on U_n the group of upper unitriangular matrices in $G = \mathrm{GL}(n, \mathbb{C})$. Then the fiber product

$$\mathcal{L}_\lambda = G \times_B \lambda$$

is a line bundle on G/B. Finally,

$$V^\lambda \cong H^0(G/B, \mathcal{L}_\lambda),$$

where $H^0(G/B, \mathcal{L}_\lambda)$ is the **space of global sections** of the line bundle $\mathcal{L}_\lambda$. More details on the construction of the character λ and the line bundle $\mathcal{L}_\lambda$ are given in Appendix A6.

S. Special/Interesting representations

(S1) Let

$$\mathrm{GL}(k) \times \mathrm{GL}(l) = \begin{pmatrix} \begin{pmatrix} \mathrm{GL}(k,\mathbb{C}) \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} \mathrm{GL}(l,\mathbb{C}) \end{pmatrix} \end{pmatrix} \subseteq \mathrm{GL}(n),$$

where $k + l = n$. Then

$$V^\lambda \big\downarrow^{\mathrm{GL}(n)}_{\mathrm{GL}(k)\times\mathrm{GL}(l)} = \sum_{\mu,\nu} c^\lambda_{\mu\nu} (V^\mu \otimes V^\nu),$$

where the $c^\lambda_{\mu\nu}$ are the **Littlewood–Richardson coefficients** that appeared in the decomposition of $S^\lambda \big\downarrow^{S_n}_{S_k \times S_l}$ in terms of $S^\mu \otimes S^\nu$ (page 32). We may write this expansion in the form

$$V^\lambda \big\downarrow^{\mathrm{GL}(n)}_{\mathrm{GL}(k)\times\mathrm{GL}(l)} = \sum_{F \text{ fillings}} V^{\mu(F)} \otimes V^{\nu(F)}.$$

We could do this precisely if we wanted. We won't do it now, but the point is that it may be nice to write this expansion as a sum over combinatorial objects. This will be the form in which this will be generalized later.

(S2) Let V^μ and V^ν be irreducible polynomial representations of $\mathrm{GL}(n)$. Then

$$V^\mu \otimes V^\nu = \sum_\lambda c^\lambda_{\mu\nu} V^\lambda,$$

where $\mathrm{GL}(n)$ acts on $V^\mu \otimes V^\nu$ by $g(m \otimes n) = gm \otimes gn$, for $g \in \mathrm{GL}(n,\mathbb{C})$, $m \in V^\mu$ and $n \in V^\nu$. Amazingly, the coefficients $c^\lambda_{\mu\nu}$ are the **Littlewood–Richardson coefficients** again, the same that appeared in the **(S1)** case immediately above and in the **(S1)** case for the symmetric group (page 32).

Remarks

(1) There is a strong similarity between the results for the symmetric group and the results for $\mathrm{GL}(n,\mathbb{C})$. One might wonder whether there is any connection between these two pictures.

There are **two distinct** ways of making concrete connections between the representation theories of $\mathrm{GL}(n,\mathbb{C})$ and the symmetric group. In fact these two are so different that **different symmetric groups** are involved.

(a) *If λ is a partition of n* then the "zero weight space", or $(1, 1, \ldots, 1)$ weight space, of the irreducible $\mathrm{GL}(n,\mathbb{C})$-module V^λ is isomorphic to the irreducible module S^λ for the group S_n, where the action of S_n is determined by the fact that S_n is the subgroup of permutation matrices in $\mathrm{GL}(n, \mathbb{C})$. This relationship is reflected in the combinatorics: the standard tableaux of shape λ are exactly the column strict tableaux of shape λ which are of weight $\nu = (1, 1, \ldots, 1)$.

(b) Schur–Weyl duality (see Appendix A5) says that the action of the symmetric group S_k on $V^{\otimes k}$ by permutation of the tensor factors generates the full centralizer of the $\mathrm{GL}(n, \mathbb{C})$-action on $V^{\otimes k}$ where V is the standard n-dimensional representation of $\mathrm{GL}(n, \mathbb{C})$. By double centralizer theory, this duality induces a correspondence between the irreducible representations of $\mathrm{GL}(n, \mathbb{C})$ which appear in $V^{\otimes k}$ and the irreducible representations of S_k which appear in $V^{\otimes k}$. These representations are indexed by *partitions* λ *of* k.

(2) Note that the word *character* has two different and commonly used meanings and the use of the word character in **(C3)** (page 37) is different from that in Section 1. In **(C3)** above the word character means *one-dimensional representation*. This terminology is used particularly (but not exclusively) in reference to representations of abelian groups, like the group T_n in **(C3)**. In general one has to infer from the context which meaning is intended.

(3) The indexing and the formula for the characters of the irreducible representations is due to Schur [1901].

(4) The formula for the dimensions of the irreducibles as the number of column strict tableaux follows from the work of Kostka [1882] and Schur [1901]. The "hook-content" formula appears in [Macdonald 1995, Chap. I, § 3, Ex. 4], where the book [Littlewood 1940] is quoted.

(5) The construction of the irreducibles by Young symmetrizers appeared in 1939 in the influential book [Weyl 1946]. It was generalized to the symplectic and orthogonal groups by H. Weyl in the same book. Further important information about this construction in the symplectic and orthogonal cases is found in [Berele 1986a] and [King and Welsh 1993]. It is not known how to generalize this construction to arbitrary complex semisimple Lie groups.

(6) The Gelfand–Tsetlin basis construction originates in [Gel'fand and Tsetlin 1950a]. A similar construction was given for the orthogonal group at the same time [Gel'fand and Tsetlin 1950b] and was generalized to the symplectic group by Zhelobenko [1987; 1970]. This construction does not generalize well to other complex semisimple groups since it depends crucially on a tower $G \supseteq G_1 \supseteq \cdots \supseteq G_k \supseteq \{1\}$ of "nice" Lie groups such that all the combinatorics is controllable.

(7) The Borel–Weil–Bott construction is not a combinatorial construction of the irreducible module V^λ. It is very important because it is a construction that generalizes well to all other compact connected real Lie groups.

(8) The facts about the special representations which we have given above are found in [Littlewood 1940].

4. Answers for Finite-Dimensional Complex Semisimple Lie Algebras $\mathfrak{g}$

Although the foundations for generalizing the $\mathrm{GL}(n,\mathbb{C})$ results to all complex semisimple Lie groups and Lie algebras were laid in the fundamental work of Weyl [1925; 1926], it was only recently that a complete generalization of the tableaux results for $\mathrm{GL}(n,\mathbb{C})$ was obtained [Littelmann 1995]. The results which we state below are generalizations of those given for $\mathrm{GL}(n,\mathbb{C})$ in the last section; partitions get replaced by points in a lattice called P^+, and column strict tableaux get replaced by paths. See Appendix A7 for some basics on complex semisimple Lie algebras.

I. What are the irreducible $\mathfrak{g}$-modules?

(a) How do we index/count them?

There is a **bijection**

$$\lambda \in P^+ \quad \overset{1-1}{\longleftrightarrow} \quad \text{irreducible representations } V^\lambda,$$

where P^+ is the cone of dominant integral weights for $\mathfrak{g}$ (see Appendix A8).

(b) What are their dimensions?

The **dimension** of the irreducible representation V^λ is

$$\dim(V^\lambda) = \#\text{ of paths in } \mathcal{P}\pi_\lambda \; = \prod_{\alpha>0} \frac{\langle \lambda+\rho,\, \alpha\rangle}{\langle \rho, \alpha\rangle},$$

where

$\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$ is the half sum of the positive roots,

π_λ is the straight line path from 0 to λ, and

$\mathcal{P}\pi_\lambda = \{f_{i_1}\cdots f_{i_k}\pi_\lambda \mid 1 \le i_1,\dots,i_k \le n\}$, where $f_1,\dots,f_n$ are the path operators introduced in [Littelmann 1995].

We shall not define the operators f_i; we will just say that they act on paths and are partial permutations in the sense that if f_i acts on a path π then the result is either 0 or another path. See Appendix A8 (page 65) for a few more details.

(c) What are their characters?

The **character** of the irreducible module V^λ is

$$\mathrm{char}(V^\lambda) = \sum_{\eta\in\mathcal{P}\pi_\lambda} e^{\eta(1)} = \frac{\sum_{w\in W}\varepsilon(w)e^{w(\lambda+\rho)}}{\sum_{w\in W}\varepsilon(w)e^{w\rho}},$$

where $\eta(1)$ is the **endpoint** of the path η. These expressions live in the group algebra of the weight lattice P, $\mathbb{C}[P] = \mathrm{span}\{e^\mu \mid \mu \in P\}$, where e^μ is a formal variable indexed by μ and the multiplication is given by $e^\mu e^\nu = e^{\mu+\nu}$, for $\mu,\nu\in P$. See Appendix A7 for more details.

S. Special/Interesting representations

(S1) Let $\mathfrak{l} \subseteq \mathfrak{g}$ be a Levi subalgebra of $\mathfrak{g}$ (this is a Lie algebra corresponding to a subgraph of the Dynkin diagram associated with $\mathfrak{g}$). The subalgebra $\mathfrak{l}$ corresponds to a subset J of the set $\{\alpha_1, \ldots, \alpha_n\}$ of simple roots. The **restriction rule** from $\mathfrak{g}$ to $\mathfrak{l}$ is

$$V^\lambda \big\downarrow^{\mathfrak{g}}_{\mathfrak{l}} = \sum_\eta V^{\eta(1)},$$

where

the sum is over all paths $\eta \in \mathcal{P}\pi_\lambda$ such that $\eta \in \bar{C}_{\mathfrak{l}}$, and

$\eta \in \bar{C}_{\mathfrak{l}}$ means that $\langle \eta(t), \alpha_i \rangle \geq 0$, for all $t \in [0,1]$ and all $\alpha_i \in J$.

(S2) The **tensor product** of two irreducible modules is given by

$$V^\mu \otimes V^\nu = \sum_\eta V^{\mu+\eta(1)},$$

where the sum is over all paths $\eta \in \mathcal{P}\pi_\nu$ such that $\pi_\mu * \eta \in \bar{C}$,

π_μ and π_ν are straight line paths from 0 to μ and 0 to ν, respectively,

$\mathcal{P}\pi_\nu$ is as in (Ib) on page 40,

$\pi_\mu * \eta$ is the path obtained by attaching η to the end of π_μ, and

$(\pi_\mu * \eta) \in \bar{C}$ means that $\langle (\pi_\mu * \eta)(t), \alpha_i \rangle \geq 0$, for all $t \in [0,1]$ and all simple roots α_i.

Remarks

(1) The indexing of irreducible representations given in **(Ia)**, page 40, is due to Cartan and Killing, the founders of the theory, from around the turn of the century. Introductory treatments of this result can be found in [Fulton and Harris 1991; Humphreys 1972].

(2) The first equality in **(Ib)**, page 40, is due to Littelmann [1994], but his later article [1995] has some improvements and can be read independently, so we recommend the later article. This formula for the dimension of the irreducible representation, the number of paths in a certain set, is exactly analogous to the formula in the $\mathrm{GL}(n, \mathbb{C})$ case, the number of tableaux which satisfy a certain condition. The second equality is the Weyl dimension formula, originally proved in [Weyl 1925; 1926]. It can be proved easily from the Weyl character formula given in **(Ic)** on page 40; see [Humphreys 1972; Stembridge 1994a, Lemma 2.5]. This product formula is an analogue of the "hook-content" formula given in the $\mathrm{GL}(n, \mathbb{C})$ case.

(3) A priori, it might be possible that the set $\mathcal{P}\pi_\lambda$ is an infinite set, at least the way that we have defined it. In fact, this set is always finite and there is a description of the paths that are contained in it. The paths in this set are called *Lakshmibai–Seshadri paths*; see [Littelmann 1995]. The explicit

description of these paths is a generalization of the types of indexings that were used in the "standard monomial theory" of Lakshmibai and Seshadri [1991].

(4) The first equality in **(Ic)** is due to Littelmann [1995]. This formula, a weighted sum over paths, is an analogue of the formula for the irreducible character of $\mathrm{GL}(n,\mathbb{C})$ as a weighted sum of column strict tableaux. The second equality in **(Ic)** is the celebrated Weyl character formula, originally proved in [Weyl 1925; 1926]. Modern treatments can be found in [Bröcker and tom Dieck 1985; Humphreys 1972; Varadarajan 1984].

(5) The general restriction formula **(S1)** on the preceding page is due to Littelmann [1995]. This is an analogue of the rule given in **(S1)** of the $\mathrm{GL}(n,\mathbb{C})$ results, page 38. In this case the formula is as a sum over paths which satisfy certain conditions whereas in the $\mathrm{GL}(n,\mathbb{C})$ case the formula is a sum over column strict fillings which satisfy a certain condition.

(6) The general tensor product formula in **(S2)** on the preceding page is due to Littelmann [1995]. This formula is an analogue of the formula given in **(S2)** of the $\mathrm{GL}(n,\mathbb{C})$ results, page 38.

(7) The results of Littelmann given above are some of the most exciting results of combinatorial representation theory in recent years. They were very much inspired by some very explicit conjectures of Lakshmibai that arose out of the "standard monomial theory" developed by Lakshmibai and Seshadri [1991]. Although Littelmann's theory is actually much more general than we have stated above, the special set of paths $\mathcal{P}\pi_\lambda$ used in **(Ib)** and **(Ic)** is a modified description of the same set which appeared in Lakshmibai's conjecture. Another important influence on Littelmann in his work was Kashiwara's work [1990] on crystal bases.

PART II

5. Generalizing the S_n Results

Having the above results for the symmetric group in hand we would like to try to generalize as many of the S_n results as we can to other similar groups and algebras. Work along this line began almost immediately after the discovery of the S_n results and it continues today. In the current state of results this has been largely

1. **successful** for the complex reflection groups $G(r,p,n)$ and their "Hecke algebras",
2. **successful** for tensor power centralizer algebras and their q-analogues, and
3. **unsuccessful** for general Weyl groups and finite Coxeter groups.

Some partial results giving answers to the main questions for the complex reflection groups $G(r,p,n)$, their "Hecke algebras", and some tensor power centralizer algebras can be found in Appendices B1–B8. Here we limit ourselves to a brief description of these objects and some notes and references.

5.1. The Reflection Groups $G(r, p, n)$

A **finite Coxeter group** is a finite group generated by reflections in $\mathbb{R}^n$. In other words, take a bunch of linear transformations of $\mathbb{R}^n$ which are reflections (in the sense of reflections and rotations in the orthogonal group) and see what group they generate. If the group is finite then it is a finite Coxeter group. Actually, this definition of finite Coxeter group is not the usual one (which is given in Appendix B1), but since we have the following theorem we are not too far astray.

THEOREM 5.1. *A group is a finite group generated by reflections if and only if it is a finite Coxeter group.*

The finite Coxeter groups have been classified completely and there is one group of each of the following "types"

$$A_n,\ B_n,\ D_n,\ E_6,\ E_7,\ E_8,\ F_4,\ H_3,\ H_4,\ \text{or}\ I_2(m).$$

The finite *crystallographic* reflection groups are called *Weyl groups* because of their connection with Lie theory. These are the finite Coxeter groups of types

$$A_n,\ B_n,\ D_n,\ E_6,\ E_7,\ E_8,\ F_4,\ \text{and}\ G_2 = I_2(6).$$

A **complex reflection group** is a group generated by complex reflections, that is, invertible linear transformations of $\mathbb{C}^n$ which have finite order and which have exactly one eigenvalue that is not 1. Every finite Coxeter group is also a finite complex reflection group. The finite complex reflection groups have been classified by Shephard and Todd [1954] and each such group is one of the groups

(a) $G(r, p, n)$, where r, p, n are positive integers such that p divides r, or
(b) one of 34 "exceptional" finite complex reflection groups.

The groups $G(r, p, n)$ are very similar to the symmetric group S_n in many ways and this is probably why generalizing the S_n theory has been so successful for these groups. The symmetric groups and the finite Coxeter groups of types B_n, and D_n are all special cases of the groups $G(r, p, n)$.

I. What are the irreducible modules?

The indexing, dimension formulas and character formulas for the representations of the groups $G(r, p, n)$ are originally due to

– Young [1901] for finite Coxeter groups of types B_n and D_n;
– Specht [1932] for the group $G(r, 1, n)$.

We do not know who first did the general $G(r, p, n)$ case but it is easy to generalize Young and Specht's results to this case. See [Ariki 1995; Halverson and Ram 1998] for recent accounts.

Essentially what one does to determine the indexing, dimensions and the characters of the irreducible modules is to use Clifford theory to reduce the $G(r,1,n)$ case to the case of the symmetric group S_n. Then one can use Clifford theory again to reduce the $G(r,p,n)$ case to the $G(r,1,n)$ case. The original reference for Clifford theory is [Clifford 1937]; the book by Curtis and Reiner [1981; 1987] has a modern treatment. The articles [Stembridge 1989b; Halverson and Ram 1998] explain how the reduction from $G(r,p,n)$ to $G(r,1,n)$ is done. The dimension and character theory for the case $G(r,1,n)$ has an excellent modern treatment in [Macdonald 1995, Chap. I, App. B].

C. How do we construct the irreducible modules?

The construction of the irreducible representations by Young symmetrizers was extended to the finite Coxeter groups of types B_n and D_n by Young himself [1930b]. The authors don't know when the general case was first treated in the literature, but it is not difficult to extend Young's results to the general case $G(r,p,n)$. The $G(r,p,n)$ case does appear periodically in the literature, see [Allen 1997], for example.

Young's seminormal construction was generalized to the "Hecke algebras" of $G(r,p,n)$ in [Ariki and Koike 1994; Ariki 1995]. One can easily set $q=1$ in the constructions of Ariki and Koike and obtain the appropriate analogues for the groups $G(r,p,n)$. We do not know if the analogue of Young's seminormal construction for the groups $G(r,p,n)$ appeared in the literature previous to the work of Ariki and Koike on the "Hecke algebra" case.

S. Special/Interesting representations

We do not know if the analogues of the S_n results, **(S1)**–**(S3)** of Section 2 (page 32), have explicitly appeared in the literature. It is easy to use symmetric functions and the character formulas of Specht (see [Macdonald 1995, Chap. I, App. B]) to derive formulas for the $G(r,1,n)$ case in terms of the symmetric group results. Then one proceeds as described above to compute the necessary formulas for $G(r,p,n)$ in terms of the $G(r,1,n)$ results. See [Stembridge 1989b] for how this is done.

5.2. The "Hecke algebras" of reflection groups

In group theory, a Hecke algebra is a specific kind of double coset algebra associated with a finite group G and a subgroup B of G; see Appendix B3 for the definition. In connection with finite Coxeter groups and the groups $G(r,p,n)$ certain algebras can be defined that are analogous in some sense to Hecke algebras.

The **Iwahori–Hecke algebra** of a finite Coxeter group W is a certain algebra that is a q-analogue, or q-deformation, of the group algebra W. (See Appendix B3 for a proper definition.) It has a basis T_w, for $w \in W$ (so it is the same dimension as the group algebra of W), but multiplication depends on a particular number $q \in \mathbb{C}$, which can be chosen arbitrarily. These algebras are true Hecke algebras only when W is a finite Weyl group.

The **"Hecke algebras" of the groups** $G(r,p,n)$ are q-analogues of the group algebras of the groups $G(r,p,n)$; see Appendix B4 for a discusson of how they are defined. They were defined only recently, as follows:

- Ariki and Koike [1994] for the case $G(r,1,n)$;
- Broué and Malle [1993] and Ariki [1995] for the general case.

It is important to note that these algebras are not true Hecke algebras. In group theory a Hecke algebra is very specific kind of double coset algebra and the "Hecke algebras" of the groups $G(r,p,n)$ do not fit this mold.

Appendix B4 gives some partial answers to the main questions for these algebras; here we give references to the literature for these answers.

I. What are the irreducibles?

Results of [Ariki and Koike 1994; Ariki 1995] say that the "Hecke algebras" of $G(r,p,n)$ are q-deformations of the group algebras of the groups $G(r,p,n)$. Thus, it follows from the Tits deformation theorem [Carter 1985, Chap. 10, 11.2; Curtis and Reiner 1987, § 68.17] that the indexings and dimension formulas for the irreducible representations of these algebras must be the same as the indexings and dimension formulas for the groups $G(r,p,n)$. Finding analogues of the character formulas requires a bit more work and a Murnaghan–Nakayama type rule for the "Hecke algebras" of $G(r,p,n)$ was given by Halverson and Ram [1998]. As far as we know, the formula for the irreducible characters of S_n as a weighted sum of standard tableaux which we gave in the symmetric group section has not yet been generalized to the case of $G(r,p,n)$ and its "Hecke algebras".

C. How do we construct the irreducible modules?

Analogues of Young's seminormal representations have been given by

- Hoefsmit [1974] and Wenzl [1988a], independently, for Iwahori–Hecke algebras of type A_{n-1};
- Hoefsmit [1974], for Iwahori–Hecke algebras of types B_n and D_n;
- Ariki and Koike [1994] for the "Hecke algebras" of $G(r,1,n)$;
- Ariki [1995] for the general "Hecke algebras" of $G(r,p,n)$.

There seems to be more than one appropriate choice for the analogue of Young symmetrizers for Hecke algebras. The definitions in the literature are due to

- Gyoja [1986], for the Iwahori–Hecke algebras of type A_{n-1};
- Dipper and James [1986] and Murphy [1992; 1995] for the Iwahori–Hecke algebras of type A_{n-1};
- King and Wybourne [1992] and Duchamp et al. [1995] for the Iwahori–Hecke algebras of type A_{n-1};

– Dipper, James, and Murphy [1992; 1995] for the Iwahori–Hecke algebras of type B_n;
– Pallikaros [1994], for the Iwahori–Hecke algebras of type D_n;
– Mathas [1997] and Murphy [$\geq$ 1999], for the "Hecke algebras" of $G(r,p,n)$.

The paper [Graham and Lehrer 1996] also contains important ideas in this direction.

S. Special/Interesting representations

It follows from the Tits deformation theorem (or rather, an extension of it) that the results for the "Hecke algebras" of $G(r,p,n)$ must be the same as for the case of the groups $G(r,p,n)$.

5.3. Tensor Power Centralizer Algebras

A **tensor power centralizer algebra** is an algebra that is isomorphic to $\mathrm{End}_G(V^{\otimes k})$ for some group (or Hopf algebra) G and some representation V of G. In this definition

$$\mathrm{End}_G(V^{\otimes k}) = \{T \in \mathrm{End}(V^{\otimes k}) \mid Tgv = gTv, \text{ for all } g \in G \text{ and all } v \in V^{\otimes k}\}.$$

Some examples of tensor power centralizer algebras have been particularly important:

(a) The **group algebras, $\mathbb{C}S_k$, of the symmetric groups S_k.**
(b) **Iwahori–Hecke algebras, $H_k(q)$, of type** A_{k-1} were introduced by Iwahori [1964] in connection with $\mathrm{GL}(n, \mathbb{F}_q)$. Jimbo [1986] realized that they arise as tensor power centralizer algebras for quantum groups.
(c) **Temperley–Lieb algebras**, $\mathrm{TL}_k(x)$, were introduced independently by several people. Some of the discoverers were Rumer, Teller, and Weyl [Rumer et al. 1932], Penrose [1969; 1971], Temperley and Lieb [1971], Kaufmann [1987] and Jones [1983]. The work of V. Jones was crucial in making them so important in combinatorial representation theory today.
(d) **Brauer algebras**, $B_k(x)$, were defined in [Brauer 1937]. Brauer also proved that they are tensor power centralizers.
(e) **Birman–Murakami–Wenzl algebras**, $\mathrm{BMW}_k(r,q)$, are due to Birman and Wenzl [1989] and Murakami [1987]. It was realized early on [Reshetikhin 1987; Wenzl 1990] that they arise as tensor power centralizers but there was no proof in the literature for some time. See the references in [Chari and Pressley 1994, § 10.2].
(f) **Spider algebras** were written down combinatorially and studied by G. Kuperberg [1994a; 1996b].
(g) **Rook monoid algebras** are the "group algebras" of some very natural monoids, and L. Solomon (unpublished work) realized these as tensor power centralizers. Their combinatorics has also been studied in [Garsia and Remmel 1986].

(h) **Solomon–Iwahori algebras** were introduced in [Solomon 1990]. The fact that they are tensor power centralizer algebras is an unpublished result of Solomon [1995].

(i) **Rational Brauer algebras** were introduced in a nice combinatorial form in [Benkart et al. 1994] and in other forms in [Koike 1989; Procesi 1976] and other older invariant theory works [Weyl 1946]. All of these works were related to tensor power centralizers and/or fundamental theorems of invariant theory.

(j) The **q-rational Brauer algebras** were introduced by Kosuda and Murakami [1992; 1993] and subsequently studied in [Leduc 1994; Halverson 1996; 1995].

(k) **partition algebras** were introduced by V. Jones [1994] and have been studied subsequently by P. Martin [1996].

The following paragraphs give references on the results associated with objects (b)–(e); see also Appendices B5 to B8 for the results themselves and missing definitions. We will not discuss objects (f)–(k) further; see the references given in the individual entries above.

I. What are the irreducibles?

Indexing of the representations of tensor power centralizer algebras follows from double centralizer theory [Weyl 1946] and a good understanding of the indexings and tensor product rules for the group or algebra that it is centralizing: $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{O}(n,\mathbb{C})$, $U_q\mathfrak{sl}(n)$, etc. The references for resulting indexings and dimension formulas for the irreducible representations are as follows:

Temperley–Lieb algebras. These results are classical and can be found in [Goodman et al. 1989].

Brauer algebras. These results were known to Brauer [1937] and Weyl [1946]. An important combinatorial point of view was given by Berele [1986b; 1986a] and further developed by Sundaram [1990c; 1990b; 1990a].

Iwahori–Hecke algebras of type A_{n-1}. These results follow from the Tits deformation theorem and the corresponding results for the symmetric group.

Birman–Murakami–Wenzl algebras. These results follow from the Tits deformation theorem and the corresponding results for the Brauer algebra.

The indexings and dimension formulas for the Temperley–Lieb and Brauer algebras also follow easily by using the techniques of the Jones basic construction [Wenzl 1988b; Halverson and Ram 1995].

The references for the irreducible characters of the various tensor power centralizer algebras are as follows:

Temperley–Lieb algebras. Character formulas can be derived easily by using Jones Basic Construction techniques [Halverson and Ram 1995].

Iwahori–Hecke algebras of type A_{n-1}. The analogue of the formula for the irreducible characters of S_n as a weighted sum of standard tableaux was found by

Roichman [1997]. Murnaghan–Nakayama type formulas were found by several authors [King and Wybourne 1992; Van der Jeugt 1991; Vershik and Kerov 1988; Ueno and Shibukawa 1992; Ram 1991].

Brauer algebras and Birman–Murakami–Wenzl algebras. Murnaghan–Nakayama type formulas were derived in [Ram 1995] and [Halverson and Ram 1995], respectively.

Brauer algebra and Birman–Murakami–Wenzl algebra analogues of the formula for the irreducible characters of the symmetric groups as a weighted sum of standard tableaux have not appeared in the literature.

C. How do we construct the irreducibles?

Temperley–Lieb algebras. An application of the Jones Basic Construction [Wenzl 1988b; Halverson and Ram 1995] gives a construction of the irreducible representations of the Temperley–Lieb algebras. This construction is classical and has been rediscovered by many people. In this case the construction is an analogue of the Young symmetrizer construction. The analogue of the seminormal construction appears in [Goodman et al. 1989].

Iwahori–Hecke algebras of type A_{n-1}. The analogue of Young's seminormal construction for this case is due, independently, to Hoefsmit [1974] and Wenzl [1988a]. Various analogues of Young symmetrizers have been given by Gyoja [1986], Dipper and James [1986], Murphy [1992; 1995], King and Wybourne [1992] and Duchamp et al. [1995].

Brauer algebras. Analogues of Young's seminormal representations have been given, independently, by Nazarov [1996] and Leduc and Ram [1997]. An analogue of the Young symmetrizer construction can be obtained by applying the Jones Basic Construction to the classical Young symmetrizer construction and this is the one that has been used by many authors [Benkart et al. 1990; Hanlon and Wales 1989; Kerov 1992; Graham and Lehrer 1996]. The actual element of the algebra that is the analogue of the Young symmetrizer involves a central idempotent for which there is no known explicit formula and this is the reason that most authors work with a quotient formulation of the appropriate module.

Birman–Murakami–Wenzl algebras. Analogues of Young's seminormal representations have been given by Murakami [1990] and Leduc and Ram [1997], using different methods: Murakami uses the physical theory of Boltzmann weights, whereas Leduc and Ram use the theory of ribbon Hopf algebras and quantum groups. Exactly in the same way as for the Brauer algebra, an analogue of the Young symmetrizer construction can be obtained by applying the Jones Basic Construction to the Young symmetrizer constructions for the Iwahori–Hecke algebra of type A_{n-1}. As in the Brauer algebra case one should work with a quotient formulation of the module to avoid using a central idempotent for which there is no known explicit formula.

5.4. Reflection Groups of Exceptional Type

Generalizing the S_n theory to finite Coxeter groups of exceptional type, finite complex reflection groups of exceptional type and the corresponding Iwahori–Hecke algebras, has been largely unsuccessful. This is not to say that there haven't been some very nice partial results, only that at the moment nobody has any understanding of how to make a good combinatorial theory to encompass all the classical and exceptional types at once. Two amazing partial results along these lines are the **Springer construction** and the **Kazhdan–Lusztig construction**.

The Springer construction is a construction of the irreducible representations of the crystallographic reflection groups on cohomology of unipotent varieties [Springer 1978]. It is a geometric construction and not a combinatorial construction. See Appendix A3 for more information in the symmetric group case. It is possible that this construction may be combinatorialized in the future, but to date no one has done this.

The Kazhdan–Lusztig construction [1979] is a construction of certain representations called *cell representations* and it works for all finite Coxeter groups. The cell representations are almost irreducible but unfortunately not irreducible in general, and nobody understands how to break them up into irreducibles, except in a case by case fashion. The other problem with these representations is that they depend crucially on certain polynomials, the Kazhdan–Lusztig polynomials, which seem to be exceedingly difficult to compute or understand well except in very small cases; see [Brenti 1994] for more information. See [Carter 1985] for a summary and tables of the known facts about representation of finite Coxeter groups of exceptional type.

Remarks

(1) As mentioned in Section 5.1.2, a Hecke algebra is a specific double coset algebra that depends on a group G and a subgroup B. Iwahori [1964] studied these algebras in the case that G is a finite Chevalley group and B is a Borel subgroup of G, and defined what are now called *Iwahori–Hecke algebras*. These are q-analogues of the group algebras of finite Weyl groups. The work of Iwahori yields a presentation for these algebras which can easily be extended to define Iwahori–Hecke algebras for all Coxeter groups but, except for the original Weyl group case, these have never been realized as true Hecke algebras, i.e., double coset algebras corresponding to an appropriate G and B. The "Hecke algebras" corresponding to the groups $G(r,p,n)$ are q-analogues of the group algebras of $G(r,p,n)$. Although these algebras are not true Hecke algebras either, Broué and Malle [1993] have shown that many of these algebras arise in connection with nondefining characteristic representations of finite Chevalley groups and Deligne–Lusztig varieties.

(2) There is much current research on generalizing symmetric group results to affine Coxeter groups and affine Hecke algebras. The case of affine Coxeter groups was done by Kato [1983] using ideas from Clifford theory. The case of affine Hecke algebras has been intensely studied [Lusztig 1987a; 1987b; 1987b; 1989a; 1988; 1989b; 1995b; 1995a, Kazhdan and Lusztig 1987; Ginsburg 1987; Chriss and Ginzburg 1997] Most of this work is very geometric and relies on methods involving intersection cohomology and K-theory. Hopefully some of this work will be made combinatorial in the near future.
(3) Wouldn't it be great if we had a nice combinatorial representation theory for finite simple groups!

6. Generalizations of $\mathrm{GL}(n,\mathbb{C})$ Results

There have been successful generalizations of the $\mathrm{GL}(n,\mathbb{C})$ results **(Ia)–(Ic)** and **(S1)–(S2)** of Section 3 (pages 35–38) to the following classes of groups and algebras.

(1) **Connected complex semisimple Lie groups**. Examples: $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{SO}(n,\mathbb{C})$, $\mathrm{Sp}(2n,\mathbb{C})$, $\mathrm{PGL}(n,\mathbb{C})$, $\mathrm{PSO}(2n,\mathbb{C})$, $\mathrm{PSp}(2n,\mathbb{C})$.
(2) **Compact connected real Lie groups.** Examples: $\mathrm{SU}(n,\mathbb{C})$, $\mathrm{SO}(n,\mathbb{R})$, $\mathrm{Sp}(n)$, where $\mathrm{Sp}(n) = Sp(2n,\mathbb{C}) \cap U(2n,\mathbb{C})$.
(3) **Finite-dimensional complex semisimple Lie algebras.** Examples: $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{so}(n,\mathbb{C})$, $\mathfrak{sp}(2n,\mathbb{C})$. See Appendix A7 for the complete list of the finite-dimensional complex semisimple Lie algebras.
(4) **Quantum groups corresponding to complex semisimple Lie algebras.**

The method of generalizing the $\mathrm{GL}(n,\mathbb{C})$ results to these objects is to reduce them all to case (3) and then solve case (3). The results for case (3) are given in Section 4. The reduction of cases (1) and (2) to case (3) are outlined in [Serre 1987], and given in more detail in [Varadarajan 1984] and [Bröcker and tom Dieck 1985]. The reduction of (4) to (3) is given in [Chari and Pressley 1994] and in [Jantzen 1996].

Partial Results for Further Generalizations

Some partial results along the lines of the results **(Ia)–(Ic)** and **(S1)–(S2)** of Section 3 have been obtained for the following groups and algebras.

(1) **Kac–Moody Lie algebras and groups**
(2) **Yangians**
(3) **Simple Lie superalgebras**

Other groups and algebras, for which the combinatorial representation theory is not understood very well, are

(4) **Finite Chevalley groups**
(5) **$\mathfrak{p}$-adic Chevalley groups**
(6) **Real reductive Lie groups**
(7) **The Virasoro algebra**

There are many many possible ways that we could extend this list but probably these four cases are the most fundamental cases where the combinatorial representation theory has not been formulated. There has been intense work on all of these cases, but hardly any by combinatorialists. Thus many beautiful results are known, but very few of them have been stated or interpreted through a combinatorialist's eyes. They are gold veins yet to be mined!

Remarks

(1) An introductory reference to Kac–Moody Lie algebras is [Kac 1990]. This book contains a good description of the basic representation theory of these algebras. We don't know of a good introductory reference for the Kac–Moody groups case. We would suggest beginning with [Kostant and Kumar 1986] and following the references given there.
(2) The basic introductory reference for Yangians and their basic representation theory is [Chari and Pressley 1994, Chapter 12]. See also the references given there.
(3) The best introductory reference for Lie superalgebras is [Scheunert 1979]. For an update on the combinatorial representation theory of these cases see [Sergeev 1984; Berele and Regev 1987; Benkart et al. 1998; Serganova 1996].
(4) Finding a general combinatorial representation theory for finite Chevalley groups has been elusive for many years. After the fundamental work of J. A. Green [1955], which established a combinatorial representation theory for $\mathrm{GL}(n, \mathbb{F}_q)$, there has been a concerted effort to extend these results to other finite Chevalley groups. G. Lusztig [1978; 1977; 1984; 1974] has made important contributions to this field; in particular, the results of Deligne and Lusztig [1976] are fundamental. However, this is a geometric approach rather than a combinatorial one and there is much work to be done for combinatorialists, even in interpreting the known results from a combinatorial viewpoint. A good introductory treatment of this theory is the book by Digne and Michel [1991]. The original work of Green is treated in [Macdonald 1995, Chap. IV].
(5) The representation theory of $\mathfrak{p}$-adic Lie groups has been studied intensely by representation theorists but essentially not at all by combinatorialists. It is clear that there is a beautiful (although possibly very difficult) combinatorial representation theory lurking here. The best introductory reference to this work is the paper of R. Howe [1994] on $\mathfrak{p}$-adic $\mathrm{GL}(n)$. Recent results of G.

Lusztig [1995a] are a very important step in providing a general combinatorial representation theory for $\mathfrak{p}$-adic groups.

(6) The best places to read about the representation theory of real reductive groups are the books [Knapp and Vogan 1995; Vogan 1987; 1981; Wallach 1988; 1992].

(7) The Virasoro algebra is a Lie algebra that seems to turn up in every back alley of representation theory. One can only surmise that it must have a beautiful combinatorial representation theory that is waiting to be clarified. A good place to read about the Virasoro algebra is in [Feĭgin and Fuchs 1990].

APPENDICES

A1. Basic Representation Theory

An *algebra* A is a vector space over $\mathbb{C}$ with a multiplication that is associative, distributive, has an identity and satisfies the equation

$$(ca_1)a_2 = a_1(ca_2) = c(a_1a_2), \quad \text{for all } a_1, a_2 \in A \text{ and } c \in \mathbb{C}.$$

An A-*module* is a vector space M over $\mathbb{C}$ with an A-action $A \times M \to M$, $(a, m) \mapsto am$, that satisfies

$$\begin{aligned} 1m &= m, \\ a_1(a_2m) &= (a_1a_2)m, \\ (a_1 + a_2)m &= a_1m + a_2m, \\ a(c_1m_1 + c_2m_2) &= c_1(am_1) + c_2(am_2). \end{aligned}$$

for all $a, a_1, a_2 \in A$, $m, m_1, m_2 \in M$ and $c_1, c_2 \in \mathbb{C}$. We shall use the words module and *representation* interchangeably.

A module M is *indecomposable* if there do not exist non zero A-modules M_1 and M_2 such that

$$M \cong M_1 \oplus M_2.$$

A module M is *irreducible* or *simple* if the only submodules of M are the zero module 0 and M itself. A module M is *semisimple* if it is the direct sum of simple submodules.

An algebra is *simple* if the only ideals of A are the zero ideal 0 and A itself. The *radical* $\mathrm{rad}(A)$ of an algebra A is the intersection of all the maximal left ideals of A. An algebra A is *semisimple* if all its modules are semisimple. An algebra A is *Artinian* if every decreasing sequence of left ideals of A stabilizes, that is for every chain

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$$

of left ideals of A there is an integer m such that $A_i = A_m$ for all $i \geq m$.

The following statements follow directly from the definitions.

Let A be an algebra.

(a) *Every irreducible A-module is indecomposable.*

(b) *The algebra A is semisimple if and only if every indecomposable A-module is irreducible.*

The proofs of the following statements are more involved and can be found in [Bourbaki 1958, Chap. VIII, § 6 n°4 and § 5 n°3].

THEOREM A1.1. (a) *If A is an Artinian algebra, the radical of A is the largest nilpotent ideal of A.*
(b) *An algebra A is semisimple if and only if A is Artinian and* $\mathrm{rad}(A) = 0$.
(c) *Every semisimple algebra is a direct sum of simple algebras.*

The case when A is not necessarily semisimple is often called *modular representation theory.* Let M be an A-module. A *composition series* of M is a chain

$$M = M_k \supseteq M_{k-1} \supseteq \cdots \supseteq M_1 \supseteq M_0 = 0,$$

such that, for each $1 \leq i \leq k$, the modules M_i/M_{i-1} are irreducible. The irreducible modules M_i/M_{i-1} are the *factors* of the composition series. The following theorem is proved in [Curtis and Reiner 1962, (13.7)].

THEOREM A1.2 (JORDAN–HÖLDER). *If there exists a composition series for M then any two composition series must have the same multiset of factors* (*up to module isomorphism*).

An important combinatorial point of view is as follows: The analogue of the subgroup lattice of a group can be studied for any A-module M. More precisely, the *submodule lattice* $L(M)$ of M is the lattice defined by the submodules of M with the order relations given by inclusions of submodules. The composition series are maximal chains in this lattice.

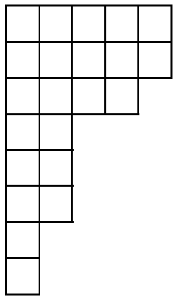
References. All these results can be found in [Bourbaki 1958, Chap. VIII; Curtis and Reiner 1962].

A2. Partitions and Tableaux

Partitions. A *partition* is a sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ of integers such that

$$\lambda_1 \geq \cdots \geq \lambda_n \geq 0.$$

We write $\lambda \vdash N$ if $N = \lambda_1 + \cdots + \lambda_n$. It is conventional to identify a partition with its Ferrers diagram, which has λ_i boxes in the i-th row. For example, the partition $\lambda = (55422211)$ has the Ferrers diagram shown at the top of the next page.



$\lambda = (55422211)$

We number the rows and columns of the Ferrers diagram as is conventionally done for matrices. If x is the box in λ in position (i, j), the *content* of x is $c(x) = j - i$ and the *hook length* of x is $h_x = \lambda_i - j + \lambda'_j - i + 1$, where λ'_j is the length of the j-th column of λ.

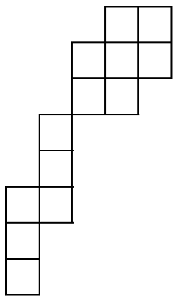
0	1	2	3	4
−1	0	1	2	3
−2	−1	0	1	
−3	−2			
−4	−3			
−5	−4			
−6				
−7				

Contents

12	9	5	4	2
11	8	4	3	1
9	6	2	1	
6	3			
5	2			
4	1			
2				
1				

Hook lengths

If μ and λ are partitions such that the Ferrers diagram of μ is contained in that of λ, we write $\mu \subseteq \lambda$ and we denote the difference of the Ferrers diagrams by λ/μ. We refer to λ/μ as a *shape* or, more specifically, a *skew shape.*



$\lambda/\mu = (55422211)/(32211)$

Tableaux. Suppose that λ has k boxes. A *standard tableau* of shape λ is a filling of the Ferrers diagram of λ with $1, 2, \ldots, k$ such that the rows and columns are increasing from left to right and from top to bottom respectively.

1	2	5	9	13
3	6	10	14	16
4	8	15	17	
7	12			
11	20			
18	21			
19				
22				

Let λ/μ be a shape. A *column strict tableau* of shape λ/μ filled with $1, 2, \ldots n$ is a filling of the Ferrers diagram of λ/μ with elements of the set $\{1, 2, \ldots, n\}$ such that the rows are weakly increasing from left to right and the columns are strictly increasing from top to bottom. The *weight* of a column strict tableau T is the sequence of positive integers $\nu = (\nu_1, \ldots, \nu_n)$, where ν_i is the number of i's in T.

1	1	1	2	3
2	2	4	4	5
3	3	5	5	
6	7			
7	8			
8	9			
10				
11				

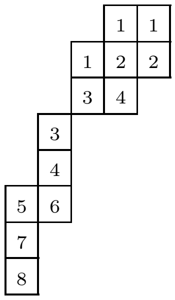
Shape $\lambda = (55422211)$

Weight $\nu = (33323122111)$

The *word* of a column strict tableau T is the sequence

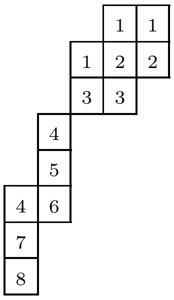
$$w = w_1 w_2 \cdots w_p$$

obtained by reading the entries of T from right to left in successive rows, starting with the top row. A word $w = w_1 \cdots w_p$ is a *lattice permutation* if for each $1 \le r \le p$ and each $1 \le i \le n-1$ the number of occurrences of the symbol i in $w_1 \cdots w_r$ is not less than the number of occurrences of $i+1$ in $w_1 \cdots w_r$.



$w = 1122143346578$

Not a lattice permutation



$w = 1122133456478$

Lattice permutation

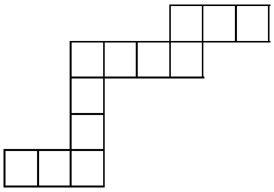
A *border strip* is a skew shape λ/μ that

(a) is connected (two boxes are connected if they share an edge), and
(b) does not contain a 2×2 block of boxes.

The weight of a border strip λ/μ is given by

$$\mathrm{wt}(\lambda/\mu) = (-1)^{r(\lambda/\mu)-1},$$

where $r(\lambda/\mu)$ is the number of rows in λ/μ.



$\lambda/\mu = (86333)/(5222)$

$\mathrm{wt}(\lambda/\mu) = (-1)^{5-1}$

Let λ and $\mu = (\mu_1, \ldots, \mu_l)$ be partitions of n. A μ-*border strip tableau* of shape λ is a sequence of partitions

$$T = (\varnothing = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(l-1)} \subseteq \lambda^{(l)} = \lambda)$$

such that, for each $1 \le i \le l$,

(a) $\lambda^{(i)}/\lambda^{(i-1)}$ is a border strip, and
(b) $|\lambda^{(i)}/\lambda^{(i-1)}| = \mu_i$.

The weight of a μ-border strip tableau T of shape λ is

$$\mathrm{wt}(T) = \prod_{i=1}^{l-1} \mathrm{wt}(\lambda^{(i)}/\lambda^{(i-1)}). \qquad \text{(A2–1)}$$

Theorem A2.1 (Murnaghan–Nakayama rule). *Let λ and μ be partitions of n and let $\chi^\lambda(\mu)$ denote the irreducible character of the symmetric group S_n indexed by λ evaluated at a permutation of cycle type μ. Then*

$$\chi^\lambda(\mu) = \sum_T \mathrm{wt}(T),$$

where the sum is over all μ-border strip tableaux T of shape λ and $\mathrm{wt}(T)$ *is as given in* (A2–1).

References. All these facts can be found in [Macdonald 1995, Chap. I]. The proof of theorem A2.2 is given in [Macdonald 1995, Chap. I, § 7, Ex. 5].

A3. The Flag Variety, Unipotent Varieties, and Springer Theory for $\mathrm{GL}(n, \mathbb{C})$

Borel subgroups, Cartan subgroups, and unipotent elements. Define

$$B_n = \left\{ \begin{pmatrix} * & * & \cdots & * \\ 0 & * & & \vdots \\ \vdots & & \ddots & * \\ 0 & \cdots & 0 & * \end{pmatrix} \right\}, \quad T_n = \left\{ \begin{pmatrix} * & 0 & \cdots & 0 \\ 0 & * & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & * \end{pmatrix} \right\}, \quad U_n = \left\{ \begin{pmatrix} 1 & * & \cdots & * \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & * \\ 0 & \cdots & 0 & 1 \end{pmatrix} \right\}$$

as the subgroups of $\mathrm{GL}(n, \mathbb{C})$ consisting of upper triangular, diagonal, and upper unitriangular matrices, respectively.

A *Borel subgroup* of $\mathrm{GL}(n, \mathbb{C})$ is a subgroup that is conjugate to B_n.

A *Cartan subgroup* of $\mathrm{GL}(n, \mathbb{C})$ is a subgroup that is conjugate to T_n.

A matrix $u \in \mathrm{GL}(n, \mathbb{C})$ is *unipotent* if it is conjugate to an upper unitriangular matrix.

The flag variety. There are one-to-one correspondences among the following sets:

(1) $\mathcal{B} = \{\text{Borel subgroups of } \mathrm{GL}(n, \mathbb{C})\}$,
(2) G/B, where $G = \mathrm{GL}(n, \mathbb{C})$ and $B = B_n$,
(3) $\{\text{flags } 0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = \mathbb{C}^n \text{ such that } \dim(V_i) = i\}$.

Each of these sets naturally has the structure of a complex algebraic variety, that is called the *flag variety.*

The unipotent varieties. Given a unipotent element $u \in \mathrm{GL}(n, \mathbb{C})$ with Jordan blocks given by the partition $\mu = (\mu_1, \ldots, \mu_l)$ of n, define an algebraic variety

$$\mathcal{B}_\mu = \mathcal{B}_u = \{\text{Borel subgroups of } \mathrm{GL}(n, \mathbb{C}) \text{ that contain } u\}.$$

By conjugation, the structure of the subvariety $\mathcal{B}_u$ of the flag variety depends only on the partition μ. Thus $\mathcal{B}_\mu$ is well defined, as an algebraic variety.

Springer theory. It is a deep theorem of Springer [1978] (which holds in the generality of semisimple algebraic groups and their corresponding Weyl groups) that there is an action of the symmetric group S_n on the cohomology $H^*(\mathcal{B}_u)$ of the variety $\mathcal{B}_u$. This action can be interpreted nicely as follows. The imbedding

$$\mathcal{B}_u \subseteq \mathcal{B} \quad \text{induces a surjective map} \quad H^*(\mathcal{B}) \longrightarrow H^*(\mathcal{B}_u).$$

It is a famous theorem of Borel that there is a ring isomorphism

$$H^*(\mathcal{B}) \cong \mathbb{C}[x_1, \dots, x_n]/I^+, \tag{A3–1}$$

where I^+ is the ideal generated by symmetric functions without constant term. It follows that $H^*(\mathcal{B}_u)$ is also a quotient of $\mathbb{C}[x_1, \dots, x_n]$. From the work of Kraft [1981], DeConcini and Procesi [1981] and Tanisaki [1982], one has that the ideal $\mathcal{T}_u$ which it is necessary to quotient by in order to obtain an isomorphism

$$H^*(\mathcal{B}_u) \cong \mathbb{C}[x_1, \dots, x_n]/\mathcal{T}_u,$$

can be described explicitly.

The symmetric group S_n acts on the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ by permuting the variables. It turns out that the ideal $\mathcal{T}_u$ remains invariant under this action, thus yielding a well defined action of S_n on $\mathbb{C}[x_1, \dots, x_n]/\mathcal{T}_u$. This action coincides with the Springer action on $H^*(\mathcal{B}_u)$. Hotta and Springer [1977] have established that, if u is a unipotent element of shape μ then, for every permutation $w \in S_n$,

$$\sum_i q^i \varepsilon(w) \operatorname{trace}(w^{-1}, H^{2i}(\mathcal{B}_u)) = \sum_{\lambda \vdash n} \tilde{K}_{\lambda\mu}(q) \chi^\lambda(w),$$

where

$\varepsilon(w)$ is the sign of the permutation w,
$\operatorname{trace}(w^{-1}, H^{2i}(\mathcal{B}_u))$ is the trace of the action of w^{-1} on $H^{2i}(\mathcal{B}_u)$,
$\chi^\lambda(w)$ is the irreducible character of the symmetric group evaluated at w, and
$\tilde{K}_{\lambda\mu}(q)$ is a variant of the Kostka–Foulkes polynomial; see [Macdonald 1995, Chap. III, § 7, Ex. 8, and § 6].

It follows from this discussion and some basic facts about the polynomials $\tilde{K}_{\lambda\mu}(q)$ that the top degree cohomology group in $H^*(\mathcal{B}_\mu)$ is a realization of the irreducible representation of S_n indexed by μ,

$$S^\mu \cong H^{\text{top}}(\mathcal{B}_\mu).$$

This construction of the irreducible modules of S_n is the *Springer construction.*

References. See [Macdonald 1995, Chap. II, § 3, Ex. 1] for a description of the variety $\mathcal{B}_u$ and its structure. The theorem of Borel stated in (A3–1) is given in [Borel 1953; Bernšteĭn et al. 1973]. The references quoted in the text above will provide a good introduction to the Springer theory. The beautiful combinatorics of Springer theory has been studied by Barcelo [1993], Garsia and Procesi [1992], Lascoux [1991], Lusztig and Spaltenstein [1985], Shoji [1979], Spaltenstein [1976], Weyman [1989], and others.

A4. Polynomial and Rational Representations of $\mathrm{GL}(n, \mathbb{C})$

If V is a $\mathrm{GL}(n,\mathbb{C})$-module of dimension d then, by choosing a basis of V, we can define a map

$$\begin{array}{rcl} \rho_V : \mathrm{GL}(n,\mathbb{C}) & \longrightarrow & \mathrm{GL}(d,\mathbb{C}) \\ g & \longmapsto & \rho(g), \end{array}$$

where $\rho(g)$ is the transformation of V that is induced by the action of g on V. Let

g_{ij} denote the (i,j) entry of the matrix g, and
$\rho(g)_{kl}$ denote the (k,l) entry of the matrix $\rho(g)$.

The map ρ depends on the choice of the basis of V, but the following definitions do not.

The module V is a *polynomial representation* if there are polynomials $p_{kl}(x_{ij})$, for $1 \le k,l \le d$, such that

$$\rho(g)_{kl} = p_{kl}(g_{ij}), \quad \text{for all } 1 \le k,l \le d.$$

In other words $\rho(g)_{jk}$ is the same as the polynomial p_{kl} evaluated at the entries g_{ij} of the matrix g.

The module V is a *rational representation* if there are rational functions (quotients of two polynomials) $p_{kl}(x_{ij})/q_{kl}(x_{ij})$, for $1 \le k,l \le d$, such that

$$\rho(g)_{kl} = p_{kl}(g_{ij})/q_{kl}(g_{ij}), \quad \text{for all } 1 \le k,l \le n.$$

Clearly, every polynomial representation is a rational one.

The theory of rational representations of $\mathrm{GL}(n,\mathbb{C})$ can be reduced to the theory of polynomial representations of $\mathrm{GL}(n,\mathbb{C})$. This is accomplished as follows. The determinant $\det : \mathrm{GL}(n,\mathbb{C}) \to \mathbb{C}$ defines a 1-dimensional (polynomial) representation of $\mathrm{GL}(n,\mathbb{C})$. Any integral power

$$\begin{array}{rrcl} \det^k : & \mathrm{GL}(n,\mathbb{C}) & \longrightarrow & \mathbb{C} \\ & g & \longmapsto & \det(g)^k \end{array}$$

of the determinant also determines a 1-dimensional representation of $\mathrm{GL}(n,\mathbb{C})$. **All irreducible rational representations $\mathrm{GL}(n,\mathbb{C})$ can be constructed in the form**

$$\det{}^k \otimes V^\lambda,$$

for some $k \in \mathbb{Z}$ and some irreducible polynomial representation V^λ of $\mathrm{GL}(n,\mathbb{C})$.

There exist representations of $\mathrm{GL}(n,\mathbb{C})$ that are not rational representations, for example

$$g \mapsto \begin{pmatrix} 1 & \ln|\det(g)| \\ 0 & 1 \end{pmatrix}.$$

There is no known classification of representations of $\mathrm{GL}(n,\mathbb{C})$ that are not rational.

References. See [Stembridge 1989a; 1987] for a study of the combinatorics of the rational representations of $\mathrm{GL}(n,\mathbb{C})$.

A5. Schur–Weyl Duality and Young Symmetrizers

Let V be the usual n-dimensional representation of $\mathrm{GL}(n,\mathbb{C})$ on column vectors of length n, that is

$$V = \mathrm{span}\{b_1,\dots,b_n\} \quad \text{where} \quad b_i = (0,\dots,0,1,0,\dots,0)^t,$$

and the 1 in b_i appears in the i-th entry. Then

$$V^{\otimes k} = \mathrm{span}\{b_{i_1}\otimes\cdots\otimes b_{i_k} \mid 1 \le i_1,\dots,i_k \le n\}$$

is the span of the words of length k in the letters b_i (except that the letters are separated by tensor symbols). The general linear group $\mathrm{GL}(n,\mathbb{C})$ and the symmetric group S_k act on $V^{\otimes k}$ by

$$g(v_1\otimes\cdots\otimes v_k) = gv_1\otimes\cdots\otimes gv_k, \quad \text{and} \quad (v_1\otimes\cdots\otimes v_k)\sigma = v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(k)},$$

where $g \in \mathrm{GL}(n,\mathbb{C})$, $\sigma \in S_k$, and $v_1,\dots,v_k \in V$. We have chosen to make the S_k-action a right action here; one could equally well choose the action of S_k to be a left action, but then the formula would be

$$\sigma(v_1\otimes\cdots\otimes v_k) = v_{\sigma^{-1}(1)}\otimes\cdots\otimes v_{\sigma^{-1}(k)}.$$

The following theorem is the amazing relationship between the group S_k and the group $\mathrm{GL}(n,\mathbb{C})$ discovered by Schur [1901] and exploited with such success by Weyl [1946].

Theorem A5.1 (Schur–Weyl duality).

(a) *The action of* S_k *on* $V^{\otimes k}$ *generates* $\mathrm{End}_{Gl(n,\mathbb{C})}(V^{\otimes k})$.
(b) *The action of* $\mathrm{GL}(n,\mathbb{C})$ *on* $V^{\otimes k}$ *generates* $\mathrm{End}_{S_k}(V^{\otimes k})$.

This theorem has an corollary that provides an intimate correspondence between the representation theory of S_k and *some* of the representations of $\mathrm{GL}(n,\mathbb{C})$ (the ones indexed by partitions of k).

Corollary A5.2. *As* $\mathrm{GL}(n,\mathbb{C}) \times S_k$ *bimodules*

$$V^{\otimes k} \cong \bigoplus_{\lambda \vdash k} V^\lambda \otimes S^\lambda,$$

where V^λ *is the irreducible* $\mathrm{GL}(n,\mathbb{C})$*-module and* S^λ *is the irreducible* S_k*-module indexed by* λ*.*

If λ is a partition of k, the irreducible $\mathrm{GL}(n,\mathbb{C})$-representation V^λ is given by

$$V^\lambda \cong V^{\otimes k} P(T) N(T),$$

where T is a tableau of shape λ and $P(T)$ and $N(T)$ are as defined in Section 2, Question **C** (page 31).

A6. The Borel–Weil–Bott Construction

Let $G = \mathrm{GL}(n,\mathbb{C})$ and let $B = B_n$ be the subgroup of upper triangular matrices in $\mathrm{GL}(n,\mathbb{C})$. A *line bundle on* G/B is a pair $(\mathcal{L}, p)$ where $\mathcal{L}$ is an algebraic variety and p is a map (morphism of algebraic varieties)

$$p : \mathcal{L} \longrightarrow G/B,$$

such that the fibers of p are lines and such that $\mathcal{L}$ is a locally trivial family of lines. In this definition, *fibers* means the sets $p^{-1}(x)$ for $x \in G/B$ and *lines* means one-dimensional vector spaces. For the definition of *locally trivial family of lines* see [Shafarevich 1994, Chap. VI, § 1.2]. By abuse of language, a line bundle $(\mathcal{L}, p)$ is simply denoted by $\mathcal{L}$. Conceptually, a line bundle on G/B means that we are putting a one-dimensional vector space over each point in G/B.

A *global section* of the line bundle $\mathcal{L}$ is a map (morphism of algebraic varieties)

$$s : G/B \to \mathcal{L}$$

such that $p \circ s$ is the identity map on G/B. In other words a global section is any possible "right inverse map" to the line bundle.

Each partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ determines a character (one-dimensional representation) of the group T_n of diagonal matrices in $\mathrm{GL}(n,\mathbb{C})$ via

$$\lambda\left(\begin{pmatrix} t_1 & 0 & \cdots & 0 \\ 0 & t_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & t_n \end{pmatrix}\right) = t_1^{\lambda_1} t_2^{\lambda_2} \cdots t_n^{\lambda_n}.$$

Extend this character to be a character of $B = B_n$ by letting λ ignore the strictly upper triangular part of the matrix, that is $\lambda(u) = 1$, for all $u \in U_n$. Let $\mathcal{L}_\lambda$ be the fiber product $G \times_B \lambda$, that is, the set of equivalence classes of pairs (g, c), with $g \in G$ and $c \in \mathbb{C}^*$, under the equivalence relation

$$(gb, c) \sim (g, \lambda(b^{-1})c), \quad \text{for all } b \in B.$$

Then $\mathcal{L}_\lambda = G \times_B \lambda$ with the map

$$\begin{array}{rccc} p: & G \times_B \lambda & \longrightarrow & G/B \\ & (g, c) & \longmapsto & gB \end{array}$$

is a line bundle on G/B.

The Borel–Weil–Bott theorem says that the irreducible representation V^λ of $GL_n(\mathbb{C})$ is

$$V^\lambda \cong H^0(G/B, \mathcal{L}_\lambda),$$

where $H^0(G/B, \mathcal{L}_\lambda)$ is the space of global sections of the line bundle $\mathcal{L}_\lambda$.

References. See [Fulton and Harris 1991] and G. Segal's article in [Carter et al. 1995] for further information and references on this very important construction.

A7. Complex Semisimple Lie Algebras

A *finite-dimensional complex semisimple Lie algebra* is a finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ such that $\mathrm{rad}(\mathfrak{g}) = 0$. The following theorem classifies all finite-dimensional complex semisimple Lie algebras.

THEOREM A7.1. (a) *Every finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$ is a direct sum of complex simple Lie algebras.*

(b) *There is one complex simple Lie algebra corresponding to each of the following types*

$$A_{n-1},\ B_n,\ C_n,\ D_n,\ E_6,\ E_7,\ E_8,\ F_4,\ G_2.$$

The complex simple Lie algebras of types A_n, B_n, C_n and D_n are the ones of *classical type* and they are:

$$\begin{aligned} \text{Type } A_{n-1}: &\qquad \mathfrak{sl}(n, \mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \mathrm{Tr}(A) = 0\}, \\ \text{Type } B_n: &\qquad \mathfrak{so}(2n+1, \mathbb{C}) = \{A \in M_{2n+1}(\mathbb{C}) \mid A + A^t = 0\}, \\ \text{Type } C_n: &\qquad \mathfrak{sp}(2n, \mathbb{C}) = \{A \in M_{2n}(\mathbb{C}) \mid AJ + JA^t = 0\}, \\ \text{Type } D_n: &\qquad \mathfrak{so}(2n) = \{A \in M_{2n}(\mathbb{C}) \mid A + A^t = 0\}, \end{aligned}$$

where J is the matrix of a skew-symmetric form on a $2n$-dimensional space.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. A *Cartan subalgebra* of $\mathfrak{g}$ is a maximal abelian subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. If V is a finite-dimensional $\mathfrak{g}$-module and $\mu : \mathfrak{h} \to \mathbb{C}$ is any linear function, define

$$V_\mu = \{v \in V \mid hv = \mu(h)v \quad \text{for all } h \in \mathfrak{h}\}.$$

The space V_μ is the *μ-weight space* of V. It is a nontrivial theorem [Serre 1987] that

$$V = \bigoplus_{\mu \in P} V_\mu,$$

where P is a $\mathbb{Z}$-lattice in $\mathfrak{h}^*$ which can be identified with the $\mathbb{Z}$-lattice P defined in Appendix A8. The vector space $\mathfrak{h}^*$ is the space of linear functions from $\mathfrak{h}$ to $\mathbb{C}$.

Let $\mathbb{C}[P]$ be the group algebra of P. It can be given explicitly as

$$\mathbb{C}[P] = \mathbb{C}\text{-span}\{e^\mu \mid \mu \in P\}, \quad \text{with multiplication } e^\mu e^\nu = e^{\mu+\nu}, \text{ for } \mu, \nu \in P,$$

where the e^μ are formal variables indexed by the elements of P. The *character* of a $\mathfrak{g}$-module is

$$\mathrm{char}(V) = \sum_{\mu \in P} \dim(V_\mu) e^\mu.$$

References. Theorem (A7.1) is due to the founders of the theory, Cartan and Killing, from the late 1800's. The beautiful text of Serre [1987] gives a review of the definitions and theory of complex semisimple Lie algebras. See [Humphreys 1972] for further details.

A8. Roots, Weights and Paths

To each of the "types", A_n, B_n, etc., there is an associated hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^n$ (Figure 1). The space $\mathbb{R}^n$ has the usual Euclidean inner product $\langle \cdot, \cdot \rangle$. For each hyperplane in the arrangement $\mathcal{A}$ we choose two vectors orthogonal to the hyperplane and pointing in opposite directions. This set of chosen vectors is called the *root system* R associated to $\mathcal{A}$ (Figure 2, left). There is a convention for choosing the lengths of these vectors but we shall not worry about that here.

Choose a chamber (connected component) C of $\mathbb{R}^n \setminus \bigcup_{H \in \mathcal{A}} H$ (Figure 2, right). For each root $\alpha \in R$ we say that α is *positive* if it points toward the same side of the hyperplane as C is and *negative* if points toward the opposite side. It is standard notation to write

$$\begin{cases} \alpha > 0, & \text{if } \alpha \text{ is a positive root,} \\ \alpha < 0, & \text{if } \alpha \text{ is a negative root.} \end{cases}$$

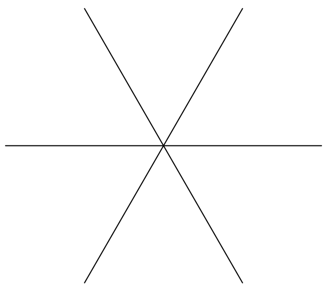


Figure 1. Hyperplane arrangement for A_2.

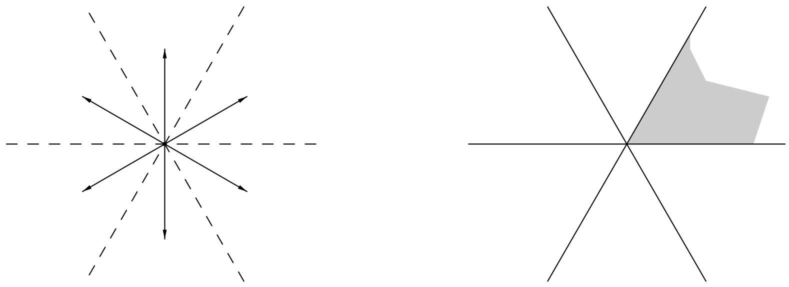


Figure 2. Left: Root system for A_2. Right: A chamber for A_2.

The positive roots associated to hyperplanes that form the walls of C are the *simple roots* $\{\alpha_1, \ldots, \alpha_n\}$. The *fundamental weights* are the vectors $\{\omega_1, \ldots, \omega_n\}$ in $\mathbb{R}^n$ such that

$$\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}, \quad \text{where} \quad \alpha_j^\vee = \frac{2\alpha_j}{\langle \alpha_j, \alpha_j \rangle}.$$

Then

$$P = \sum_{i=1}^{r} \mathbb{Z}\omega_i \quad \text{and} \quad P^+ = \sum_{i=1}^{n} \mathbb{N}\,\omega_i, \quad \text{where} \quad \mathbb{N} = \mathbb{Z}_{\geq 0},$$

are the lattice of *integral weights* and the cone of *dominant integral weights*, respectively (Figure 3). There is a one-to-one correspondence between the irreducible representations of $\mathfrak{g}$ and the elements of the cone P^+ in the lattice P.

If λ is a point in P^+, the *straight line path* from 0 to λ is the map

$$\begin{array}{rccc} \pi_\lambda : & [0,1] & \longrightarrow & \mathbb{R}^n \\ & t & \longmapsto & t\lambda. \end{array}$$

The set $\mathcal{P}\pi_\lambda$ is given by

$$\mathcal{P}\pi_\lambda = \{ f_{i_1} \cdots f_{i_k} \pi_\lambda \mid 1 \leq i_1, \ldots, i_k \leq n \}$$

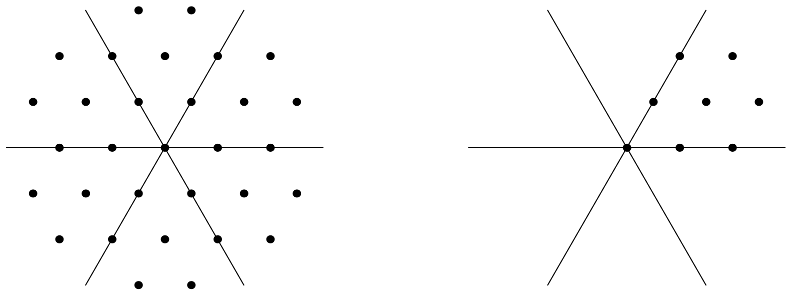


Figure 3. Left: The lattice of integral weights. Right: The cone of dominant integral weights.

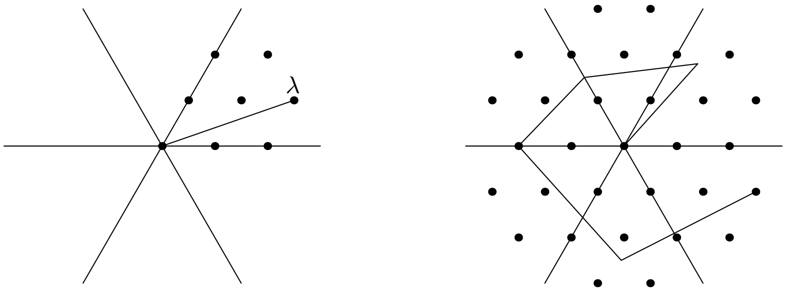


Figure 4. Left: Straight line path from 0 to λ. Right: A path in $\mathcal{P}\pi_\lambda$.

where $f_1, \ldots, f_n$ are the path operators introduced in [Littelmann 1995]. These paths are always piecewise linear and end at a point in P. An example is shown in Figure 4, right.

References. The basics of root systems can be found in [Humphreys 1972]. The reference for the path model is [Littelmann 1995].

B1. Coxeter Groups, Groups Generated by Reflections, and Weyl Groups

A *Coxeter group* is a group W presented by generators $S = \{s_1, \ldots, s_n\}$ and relations

$$\begin{aligned} s_i^2 &= 1, && \text{for } 1 \le i \le n, \\ (s_i s_j)^{m_{ij}} &= 1, && \text{for } 1 \le i \ne j \le n, \end{aligned}$$

where each m_{ij} is either ∞ or a positive integer greater than 1.

A *reflection* is a linear transformation of $\mathbb{R}^n$ that is a reflection in some hyperplane.

A *finite group generated by reflections* is a finite subgroup of $\mathrm{GL}(n, \mathbb{R})$ generated by reflections.

Theorem B1.1. *The finite Coxeter groups are exactly the finite groups generated by reflections.*

A finite Coxeter group is *irreducible* if it cannot be written as a direct product of finite Coxeter groups.

Theorem B1.2 (Classification of finite Coxeter groups).

(a) *Every finite Coxeter group can be written as a direct product of irreducible finite Coxeter groups.*
(b) *There is one irreducible finite Coxeter group corresponding to each of the following "types"*

$$A_{n-1},\ B_n,\ D_n,\ E_6,\ E_7,\ E_8,\ F_4,\ H_3,\ H_4,\ I_2(m).$$

The irreducible finite Coxeter groups of *classical type* are those of types A_{n-1}, B_n, and D_n and the others are the irreducible finite Coxeter groups of *exceptional type*.

(a) The group of type A_{n-1} is the symmetric group S_n.
(b) The group of type B_n is the hyperoctahedral group $(\mathbb{Z}/2\mathbb{Z}) \wr S_n$, the wreath product of the group of order 2 and the symmetric group S_n. It has order $2^n n!$.
(c) The group of type D_n is a subgroup of index 2 in the Coxeter group of type B_n.
(d) The group of type $I_2(m)$ is a dihedral group of order $2m$.

A finite group W generated by reflections in $\mathbb{R}^n$ is *crystallographic* if there is a lattice in $\mathbb{R}^n$ stable under the action of W. The crystallographic finite Coxeter groups are also called *Weyl groups*. The irreducible Weyl groups are the irreducible finite Coxeter groups of types

$$A_{n-1},\ \ B_n,\ \ D_n,\ \ E_6,\ \ E_7,\ \ E_8,\ \ F_4,\ \ G_2 = I_2(6).$$

References. The most comprehensive reference for finite groups generated by reflections is [Bourbaki 1968]. See also the [Humphreys 1990].

B2. Complex Reflection Groups

A *complex reflection* is an invertible linear transformation of $\mathbb{C}^n$ of finite order which has exactly one eigenvalue that is not 1. A *complex reflection group* is a group generated by complex reflections in $\mathbb{C}^n$. The finite complex reflection groups have been classified by Shepard and Todd [1954]. Each finite complex reflection group is either

(a) $G(r,p,n)$ for some positive integers r,p,n such that p divides r, or
(b) one of 34 other "exceptional" finite complex reflection groups.

Let r,p,d and n be positive integers such that $pd = r$. The complex reflection group $G(r,p,n)$ is the set of $n \times n$ matrices such that

(a) The entries are either 0 or r-th roots of unity,
(b) There is exactly one nonzero entry in each row and each column,
(c) The d-th power of the product for the nonzero entries is 1.

The group $G(r,p,n)$ is a normal subgroup of $G(r,1,n)$ of index p and

$$|G(r,p,n)| = dr^{n-1}n!.$$

In addition:

(a) $G(1,1,n) \cong S_n$ the symmetric group or *Weyl group of type A_{n-1}*,
(b) $G(2,1,n)$ is the *hyperoctahedral group* or *Weyl group of type B_n*,

(c) $G(r,1,n) \cong (\mathbb{Z}/r\mathbb{Z}) \wr S_n$, the wreath product of the cyclic group of order r with S_n,
(d) $G(2,2,n)$ is the *Weyl group of type* D_n.

Partial Results for $G(r,1,n)$

Here are the answers to the main questions **(Ia)**–**(Ic)** (page 27) for the groups $G(r,1,n) \cong (\mathbb{Z}/r\mathbb{Z}) \wr S_n$. For the general $G(r,p,n)$ case see [Halverson and Ram 1998].

I. What are the irreducible $G(r,1,n)$-modules?

(a) How do we index/count them?

There is a **bijection**

$$\textbf{\textit{r}-tuples } \lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \textbf{ of partitions such that } \textstyle\sum_{i=1}^{r} |\lambda^{(i)}| = n$$
$$\overset{1-1}{\longleftrightarrow} \quad \textbf{irreducible representations } C^\lambda.$$

(b) What are their dimensions?

The **dimension** of the irreducible representation C^λ is

$$\dim(C^\lambda) = \#\textbf{ of standard tableaux of shape } \lambda = n! \prod_{i=1}^{r} \prod_{x \in \lambda^{(i)}} \frac{1}{h_x},$$

where h_x is the hook length at the box x. A standard tableau of shape $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ is any filling of the boxes of the $\lambda^{(i)}$ with the numbers $1, 2, \dots, n$ such that the rows and the columns of each $\lambda^{(i)}$ are increasing.

(c) What are their characters?

A Murnaghan–Nakayama type rule for the characters of the groups $G(r,1,n)$ was originally given by Specht [1932]. See also [Osima 1954; Halverson and Ram 1998].

References. The original paper of Shepard and Todd [1954] remains a basic reference. Further information about these groups can be found in [Halverson and Ram 1998]. The articles [Orlik and Solomon 1980; Lehrer 1995; Stembridge 1989b; Malle 1995] contain other recent work on the combinatorics of these groups.

B3. Hecke Algebras and "Hecke Algebras" of Coxeter Groups

Let G be a finite group and let B be a subgroup of G. The *Hecke algebra* of the pair (G,B) is the subalgebra

$$\mathcal{H}(G,B) = \left\{ \sum_{g \in G} a_g g \;\middle|\; a_g \in \mathbb{C}, \text{ and } a_g = a_h \text{ if } BgB = BhB. \right\}$$

of the group algebra of G. The elements

$$T_w = \frac{1}{|B|} \sum_{g \in BwB} g,$$

as w runs over a set of representatives of the double cosets $B \backslash G/B$, form a basis of $\mathcal{H}(G, B)$.

Let G be a finite Chevalley group over the field $\mathbb{F}_q$ with q elements and fix a Borel subgroup B of G. The pair (G, B) determines a pair (W, S) where W is the Weyl group of G and S is a set of simple reflections in W (with respect to B). The *Iwahori–Hecke algebra* corresponding to G is the Hecke algebra $\mathcal{H}(G, B)$. In this case the basis elements T_w are indexed by the elements w of the Weyl group W corresponding to the pair (G, B) and the multiplication is given by

$$T_s T_w = \begin{cases} T_{sw}, & \text{if } l(sw) > l(w), \\ (q-1)T_w + qT_{sw}, & \text{if } l(sw) < l(w), \end{cases}$$

if s is a simple reflection in W. In this formula $l(w)$ is the *length* of w, i.e., the minimum number of factors needed to write w as a product of simple reflections.

A particular example of the Iwahori–Hecke algebra occurs when $G = \mathrm{GL}(n, \mathbb{F}_q)$ and B is the subgroup of upper triangular matrices. Then the Weyl group W, is the symmetric group S_n, and the simple reflections in the set S are the transpositions $s_i = (i, i{+}1)$, for $1 \le i \le n-1$. In this case the algebra $\mathcal{H}(G, B)$ is the *Iwahori–Hecke algebra of type* A_{n-1} and (as we will see in Theorems B5.1 and B5.2) can be presented by generators $T_1, \ldots, T_{n-1}$ and relations

$$\begin{aligned} T_i T_j &= T_j T_i, && \text{for } |i-j| > 1, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, && \text{for } 1 \le i \le n-2, \\ T_i^2 &= (q-1)T_i + q, && \text{for } 2 \le i \le n. \end{aligned}$$

See Appendix B5 for more facts about the Iwahori–Hecke algebras of type A. In particular, these Iwahori–Hecke algebras also appear as tensor power centralizer algebras; see Theorem B5.3. This is rather miraculous: the Iwahori–Hecke algebras of type A are the only Iwahori–Hecke algebras which arise naturally as tensor power centralizers.

In view of the multiplication rules for the Iwahori–Hecke algebras of Weyl groups it is easy to define a "Hecke algebra" for all Coxeter groups (W, S), just by defining it to be the algebra with basis T_w, for $w \in W$, and multiplication

$$T_s T_w = \begin{cases} T_{sw}, & \text{if } l(sw) > l(w), \\ (q-1)T_w + qT_{sw}, & \text{if } l(sw) < l(w), \end{cases}$$

if $s \in S$. These algebras are not true Hecke algebras except when W is a Weyl group.

References. For references on Hecke algebras see [Curtis and Reiner 1981, § 11]. For references on Iwahori–Hecke algebras see [Bourbaki 1968, Chap. IV, § 2, Ex. 23–25; Curtis and Reiner 1987, § 67; Humphreys 1990, Chap. 7]. The article [Curtis 1988] is also very informative.

B4. "Hecke algebras" of the Groups $G(r,p,n)$

Let q and $u_0, u_1, \ldots, u_{r-1}$ be indeterminates. Let $H_{r,1,n}$ be the algebra over the field $\mathbb{C}(u_0, u_1, \ldots, u_{r-1}, q)$ given by generators $T_1, T_2, \ldots, T_n$ and relations

(1) $T_iT_j = T_jT_i$, for $|i-j| > 1$,

(2) $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$, for $2 \le i \le n-1$,

(3) $T_1T_2T_1T_2 = T_2T_1T_2T_1$,

(4) $(T_1 - u_0)(T_1 - u_1)\cdots(T_1 - u_{r-1}) = 0$,

(5) $(T_i - q)(T_i + q^{-1}) = 0$, for $2 \le i \le n$.

Upon setting $q = 1$ and $u_{i-1} = \xi^{i-1}$, where ξ is a primitive r-th root of unity, one obtains the group algebra $\mathbb{C}G(r,1,n)$. In the special case where $r = 1$ and $u_0 = 1$, we have $T_1 = 1$, and $H_{1,1,n}$ is isomorphic to an Iwahori–Hecke algebra of type A_{n-1}. The case $H_{2,1,n}$ when $r = 2$, $u_0 = p$, and $u_1 = p^{-1}$, is isomorphic to an Iwahori–Hecke algebra of type B_n.

Now suppose that p and d are positive integers such that $pd = r$. Let $x_0^{1/p}, \ldots, x_{d-1}^{1/p}$ be indeterminates, let $\varepsilon = e^{2\pi i/p}$ be a primitive p-th root of unity and specialize the variables $u_0, \ldots, u_{r-1}$ according to the relation

$$u_{ld+kp+1} = \varepsilon^l x_k^{1/p},$$

where the subscripts on the u_i are taken mod r. The *"Hecke algebra" $H_{r,p,n}$ corresponding to the group $G(r,p,n)$* is the subalgebra of $H_{r,1,n}$ generated by the elements

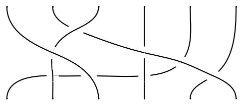
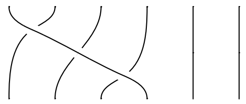
$$a_0 = T_1^p, \quad a_1 = T_1^{-1}T_2T_1, \quad \text{and} \quad a_i = T_i, \quad \text{for } 2 \le i \le n.$$

Upon specializing $x_k^{1/p} = \xi^{kp}$, where ξ is a primitive r-th root of unity, $H_{r,p,n}$ becomes the group algebra $\mathbb{C}G(r,p,n)$. Thus $H_{r,p,n}$ is a "q-analogue" of the group algebra of the group $G(r,p,n)$.

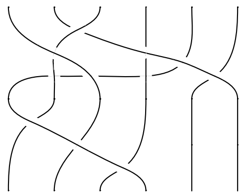
References. The algebras $H_{r,1,n}$ were first constructed by Ariki and Koike [1994]. They were classified as cyclotomic Hecke algebras of type B_n by Broué and Malle [1993] and the representation theory of $H_{r,p,n}$ was studied by Ariki [1995]. See [Halverson and Ram 1998] for information about the characters of these algebras.

B5. The Iwahori–Hecke Algebras $H_k(q)$ of Type A

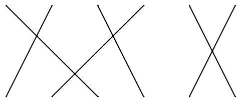
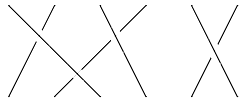
A *k-braid* is viewed as two rows of k vertices, one above the other, and k strands that connect top vertices to bottom vertices in such a way that each vertex is incident to precisely one strand. Strands cross over and under one another in three-space as they pass from one vertex to the next.

$t_1 =$  , $t_2 =$  .

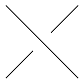
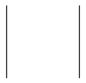
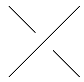
We multiply k-braids t_1 and t_2 using the concatenation product given by identifying the vertices in the top row of t_2 with the corresponding vertices in the bottom row of t_1 to obtain the product t_1t_2.

$t_1t_2 =$ 

Given a permutation $w \in S_k$ we will make a k-braid T_w by tracing the edges in order from left to right across the top row. Any time an edge that we are tracing crosses an already traced edge we raise the pen briefly so that the edge being traced goes under the edge that is already there. Applying this process to all of the permutations in S_k produces a set of $k!$ braids.

$w =$  $T_w =$ 

Fix $q \in \mathbb{C}$. The *Iwahori–Hecke algebra* $H_k(q)$ *of type* A_{k-1} is the span of the $k!$ braids produced by tracing permutations in S_k with multiplication determined by the braid multiplication and the identity

 $= (q-1)$  $+\ q$  .

This identity can be applied in any local portion of the braid.

THEOREM B5.1. *The algebra $H_k(q)$ is the associative algebra over $\mathbb{C}$ presented by generators $T_1, \ldots, T_{k-1}$ and relations*

$$\begin{aligned} &T_iT_j = T_jT_i, && \textit{for } |i-j|>1,\\ &T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, && \textit{for } 1 \le i \le n-2,\\ &T_i^2 = (q-1)T_i + q, && \textit{for } 2 \le i \le n. \end{aligned}$$

The Iwahori–Hecke algebra of type A is a q-analogue of the group algebra of the symmetric group. If we allow ourselves to be imprecise (about the limit) we can write

$$\lim_{q\to 1} H_k(q) = \mathbb{C}S_k.$$

Let q be a power of a prime and let $G = \mathrm{GL}(n, \mathbb{F}_q)$ where $\mathbb{F}_q$ is the finite field with q elements. Let B be the subgroup of upper triangular matrices in G and let $\mathbf{1}_B^G$ be the trivial representation of B induced to G, i.e., the G-module given by

$$\mathbf{1}_B^G = \mathbb{C}\text{-span}\{gB \mid g \in G\},$$

where G acts on the cosets by left multiplication. Using the description of $H_n(q)$ as a double coset algebra (Appendix B3), one gets an action of $H_n(q)$ on $\mathbf{1}_B^G$, by right multiplication. This action commutes with the G action.

Theorem B5.2. (a) *The action of $H_n(q)$ on $\mathbf{1}_B^G$ generates* $\mathrm{End}_G(\mathbf{1}_B^G)$.
(b) *The action of G on $\mathbf{1}_B^G$ generates* $\mathrm{End}_{H_n(q)}(\mathbf{1}_B^G)$.

This theorem gives a "duality" between $\mathrm{GL}(n, \mathbb{F}_q)$ and $H_n(q)$ that is similar to a Schur–Weyl duality, but differs in a crucial way: the representation $\mathbf{1}_B^G$ is not a tensor power representation, and thus this is not yet realizing $H_n(q)$ as a tensor power centralizer.

The following result gives a true analogue of the Schur–Weyl duality for the Iwahori–Hecke algebra of type A: it realizes $H_k(q)$ as a tensor power centralizer. Assume that $q \in \mathbb{C}$ is not 0 and is not a root of unity. Let $U_q\mathfrak{sl}_n$ be the Drinfel'd–Jimbo quantum group of type A_{n-1} and let V be the n-dimensional irreducible representation of $U_q\mathfrak{sl}_n$ with highest weight ω_1. There is an action of $H_k(q^2)$ on $V^{\otimes k}$ which commutes with the $U_q\mathfrak{sl}_n$ action; see [Chari and Pressley 1994].

Theorem B5.3. (a) *The action of $H_k(q^2)$ on $V^{\otimes k}$ generates* $\mathrm{End}_{U_q\mathfrak{sl}_n}(V^{\otimes k})$.
(b) *The action of $U_q\mathfrak{sl}_n$ on $V^{\otimes k}$ generates* $\mathrm{End}_{H_k(q^2)}(V^{\otimes k})$.

Theorem B5.4. *The Iwahori–Hecke algebra of type A_{k-1}, $H_k(q)$, is semisimple if and only if $q \neq 0$ and q is not a j-th root of unity for any $2 \leq j \leq n$.*

Partial Results for $H_k(q)$

The following results give answers to the main questions **(Ia)**–**(Ic)** for the Iwahori–Hecke algebras of type A hold when q is such that $H_k(q)$ is semisimple.

I. What are the irreducible $H_k(q)$-modules?

(a) How do we index/count them?

There is a **bijection**

$$\textbf{partitions } \lambda \textbf{ of } n \quad \overset{1-1}{\longleftrightarrow} \quad \textbf{irreducible representations } H^\lambda.$$

(b) What are their dimensions?

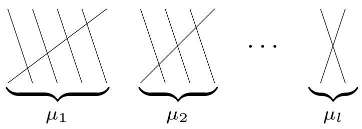
The **dimension** of the irreducible representation H^λ is given by

$$\dim(H^\lambda) = \#\ \textbf{of standard tableaux of shape}\ \lambda = \frac{n!}{\prod_{x\in\lambda} h_x},$$

where h_x is the hook length at the box x in λ.

(c) What are their characters?

For each partition $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$ of k, let $\chi^\lambda(\mu)$ be the **character** of the irreducible representation H^λ evaluated at the element T_{γ_μ} where γ_μ is the permutation

$\gamma_\mu =$  .

Then **the character** $\chi^\lambda(\mu)$ is given by

$$\chi^\lambda(\mu) = \sum_T \mathrm{wt}^\mu(T),$$

where the sum is over all standard tableaux T of shape λ and

$$\mathrm{wt}^\mu(T) = \prod_{i=1}^n f(i,T),$$

where

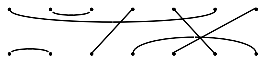
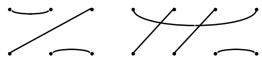
$$f(i,T) = \begin{cases} -1, & \text{if } i \notin B(\mu) \text{ and } i+1 \text{ is SW of } i, \\ 0, & \text{if } i, i+1 \notin B(\mu),\ i+1 \text{ is NE of } i, \text{ and } i+2 \text{ is SW of } i+1, \\ q, & \text{otherwise,} \end{cases}$$

and $B(\mu) = \{\mu_1 + \mu_2 + \cdots + \mu_k \mid 1 \le k \le l\}$. In the formula for $f(i,T)$, SW means strictly south and weakly west and NE means strictly north and weakly east.

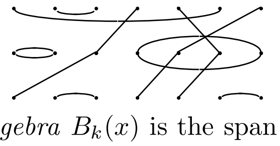
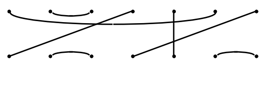
References. The book [Chari and Pressley 1994] contains a treatment of the Schur–Weyl duality type theorem given above. See also the references there. Several basic results on the Iwahori–Hecke algebra are given in [Goodman et al. 1989]. The theorem giving the explicit values of q such that $H_k(q)$ is semisimple is due to Gyoja and Uno [1989]. The character formula given above is due to Roichman [1997]. See [Ram 1998] for an elementary proof.

B6. The Brauer Algebras $B_k(x)$

Fix $x \in \mathbb{C}$. A *Brauer diagram on k dots* is a graph on two rows of k-vertices, one above the other, and k edges such that each vertex is incident to precisely one edge. The product of two k-diagrams d_1 and d_2 is obtained by placing d_1 above d_2 and identifying the vertices in the bottom row of d_1 with the corresponding vertices in the top row of d_2. The resulting graph contains k paths and some number c of closed loops. If d is the k-diagram with the edges that are the paths in this graph but with the closed loops removed, then the product $d_1 d_2$ is given by $d_1 d_2 = x^c d$. For example, if

$d_1 =$  and $d_2 =$  ,

then

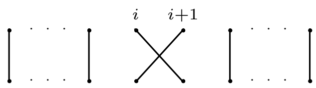
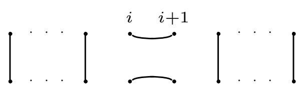
$d_1 d_2 =$  $= x^2$  .

The *Brauer algebra* $B_k(x)$ is the span of the k-diagrams with multiplication given by the linear extension of the diagram multiplication. The dimension of the Brauer algebra is

$$\dim(B_k(x)) = (2k)!! = (2k-1)(2k-3)\cdots 3\cdot 1,$$

since the number of k-diagrams is $(2k)!!$.

The diagrams in $B_k(x)$ that have all their edges connecting top vertices to bottom vertices form a symmetric group S_k. The elements

$s_i =$  and $e_i =$  ,

for $1 \le i \le k-1$, generate the Brauer algebra $B_k(x)$.

THEOREM B6.1. *The Brauer algebra $B_k(x)$ has a presentation as an algebra by generators $s_1, s_2, \ldots, s_{k-1}$, $e_1, e_2, \ldots, e_{k-1}$ and relations*

$$s_i^2 = 1, \quad e_i^2 = xe_i, \quad e_i s_i = s_i e_i = e_i, \quad \textit{for } 1 \le i \le k-1,$$
$$s_i s_j = s_j s_i, \quad s_i e_j = e_j s_i, \quad e_i e_j = e_j e_i, \quad \textit{if } |i-j| > 1,$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad e_i e_{i+1} e_i = e_i, \quad e_{i+1} e_i e_{i+1} = e_{i+1}, \quad \textit{for } 1 \le i \le k-2,$$
$$s_i e_{i+1} e_i = s_{i+1} e_i, \quad e_{i+1} e_i s_{i+1} = e_{i+1} s_i, \quad \textit{for } 1 \le i \le k-2.$$

There are two different Brauer algebra analogues of the Schur Weyl duality theorem, Theorem A5.1. In the first one the orthogonal group $\mathrm{O}(n,\mathbb{C})$ plays the same role that $\mathrm{GL}(n,\mathbb{C})$ played in the S_k-case, and in the second, the symplectic group $\mathrm{Sp}(2n,\mathbb{C})$ takes the $\mathrm{GL}(n,\mathbb{C})$ role.

Let $\mathrm{O}(n,\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid AA^t = I\}$ be the orthogonal group and let V be the usual n-dimensional representation of the group $\mathrm{O}(n,\mathbb{C})$. There is an action of the Brauer algebra $B_k(n)$ on $V^{\otimes k}$ which commutes with the action of $\mathrm{O}(n,\mathbb{C})$ on $V^{\otimes k}$.

THEOREM B6.2. (a) *The action of $B_k(n)$ on $V^{\otimes k}$ generates* $\mathrm{End}_{\mathrm{O}(n)}(V^{\otimes k})$.
(b) *The action of* $\mathrm{O}(n,\mathbb{C})$ *on* $V^{\otimes k}$ *generates* $\mathrm{End}_{B_k(n)}(V^{\otimes k})$.

Let $\mathrm{Sp}(2n,\mathbb{C})$ be the symplectic group and let V be the usual $2n$-dimensional representation of the group $\mathrm{Sp}(2n,\mathbb{C})$. There is an action of the Brauer algebra $B_k(-2n)$ on $V^{\otimes k}$ which commutes with the action of $\mathrm{Sp}(2n,\mathbb{C})$ on $V^{\otimes k}$.

THEOREM B6.3. (a) *The action of* $B_k(-2n)$ *on* $V^{\otimes k}$ *generates* $\mathrm{End}_{\mathrm{Sp}(2n,\mathbb{C})}(V^{\otimes k})$.
(a) *The action of* $\mathrm{Sp}(2n,\mathbb{C})$ *on* $V^{\otimes k}$ *generates* $\mathrm{End}_{B_k(-2n)}(V^{\otimes k})$.

THEOREM B6.4. *The Brauer algebra* $B_k(x)$ *is semisimple if*

$$x \notin \{-2k+3, -2k+2, \dots, k-2\}.$$

Partial Results for $B_k(x)$

The following results giving answers to the main questions **(Ia)**–**(Ic)** for the Brauer algebras hold when x is such that $B_k(x)$ is semisimple.

I. What are the irreducible $B_k(x)$-modules?

(a) How do we index/count them?

There is a **bijection**

$$\textbf{Partitions of } k-2h\textbf{, for } 0 \le h \le \lfloor k/2 \rfloor \;\overset{1-1}{\longleftrightarrow}\; \textbf{Irreducible representations } B^\lambda\textbf{.}$$

(b) What are their dimensions?

The **dimension** of the irreducible representation B^λ is

$$\begin{aligned}\dim(B^\lambda) &= \#\textbf{ of up-down tableaux of shape } \lambda \textbf{ and length } k\\ &= \binom{k}{2h}(2h-1)!!\,\frac{(k-2h)!}{\prod_{x\in\lambda} h_x},\end{aligned}$$

where h_x is the hook length at the box x in λ. An up-down tableau of shape λ and length k is a sequence $(\varnothing=\lambda^{(0)}, \lambda^{(1)}, \dots \lambda^{(k)}=\lambda)$ of partitions, such that each partition in the sequence differs from the previous one by either adding or removing a box.

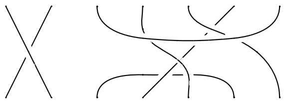
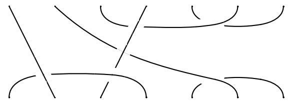
(c) What are their characters?

A Murnaghan–Nakayama type rule for the characters of the Brauer algebras was given in [Ram 1995].

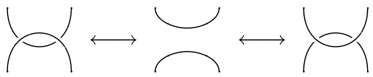
References. The Brauer algebra was defined originally by R. Brauer [1937]; it is also treated in [Weyl 1946]. The Schur–Weyl duality type theorems are also proved in Brauer's original paper [1937]. See also [Ram 1995] for a detailed description of these Brauer algebra actions. The theorem giving values of x for which the Brauer algebra is semisimple is due to Wenzl [1988b].

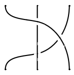
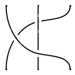
B7. The Birman–Murakami–Wenzl Algebras $\mathrm{BMW}_k(r,q)$

A *k-tangle* is viewed as two rows of k vertices, one above the other, and k strands that connect vertices in such a way that each vertex is incident to precisely one strand. Strands cross over and under one another other in three-space as they pass from one vertex to the next. For example, here are two 7-tangles:

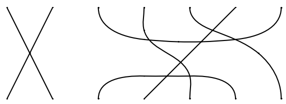
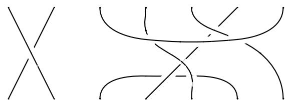
$t_1 =$  , $t_2 =$  .

We multiply k-tangles t_1 and t_2 using the concatenation product given by identifying the vertices in the top row of t_2 with the corresponding vertices in the bottom row of t_1 to obtain the product tangle $t_1 t_2$. Then we allow the following "moves":

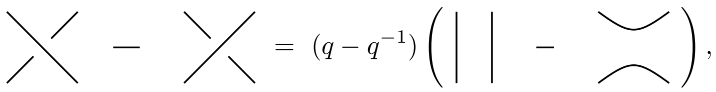
Reidemeister move II: 

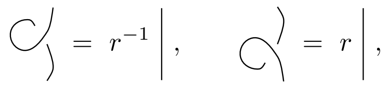
Reidemeister move III:  $\longleftrightarrow$ 

Given a Brauer diagram d we will make a tangle T_d tracing the edges in order from left to right across the top row and then from left to right across the bottom row. Any time an edge that we are tracing crosses and edge that has been already traced we raise the pen briefly so that the edge being traced goes under the edge that is already there. Applying this process to all of the Brauer diagrams on k dots produces a set of $(2k)!!$ tangles.

$d =$  $T_d =$ 

Fix numbers $r, q \in \mathbb{C}$. The *Birman–Murakami–Wenzl algebra* $\mathrm{BMW}_k(r,q)$ is the span of the $(2k)!!$ tangles produced by tracing the Brauer diagrams with multiplication determined by the tangle multiplication, the Reidemeister moves and the following tangle identities.





 $= x, \quad \text{where} \quad x = \dfrac{r - r^{-1}}{q - q^{-1}} + 1.$

The Reidemeister moves and the tangle identities can be applied in any appropriate local portion of the tangle.

THEOREM B7.1. *Fix numbers $r, q \in \mathbb{C}$. The Birman–Murakami–Wenzl algebra* $\mathrm{BMW}_k(r,q)$ *is the algebra generated over* $\mathbb{C}$ *by* $1, g_1, g_2, \ldots, g_{k-1}$, *which are assumed to be invertible, subject to the relations*

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1},$$
$$g_i g_j = g_j g_i \quad \text{if } |i-j| \geq 2,$$
$$(g_i - r^{-1})(g_i + q^{-1})(g_i - q) = 0,$$
$$E_i g_{i-1}^{\pm 1} E_i \;=\; r^{\pm 1} E_i \quad \text{and} \quad E_i g_{i+1}^{\pm 1} E_i \;=\; r^{\pm 1} E_i,$$

where E_i is defined by the equation

$$(q - q^{-1})(1 - E_i) = g_i - g_i^{-1}.$$

The BMW-algebra is a q-analogue of the Brauer algebra in the same sense that the Iwahori–Hecke algebra of type A is a q-analogue of the group algebra of the symmetric group. If we allow ourselves to be imprecise (about the limit) we can write

$$\lim_{q \to 1} \mathrm{BMW}_k(q^{n+1}, q) = B_k(n).$$

It would be interesting to sharpen the following theorem to make it an if and only if statement.

THEOREM B7.2 [Wenzl 1990]. *The Birman–Murakami–Wenzl algebra is semisimple if q is not a root of unity and $r \neq q^{n+1}$ for any $n \in \mathbb{Z}$.*

Partial Results for $\mathrm{BMW}_k(r,q)$

The following results hold when r and q are such that $\mathrm{BMW}_k(r,q)$ is semisimple.

I. What are the irreducible $\mathrm{BMW}_k(r,q)$-modules?

(a) How do we index/count them?

There is a **bijection**

partitions of $k - 2h$, for $0 \leq h \leq \lfloor k/2 \rfloor$ $\overset{1-1}{\longleftrightarrow}$ **Irreducible representations W^λ.**

(b) What are their dimensions?

The **dimension** of the irreducible representation W^λ is

$$\begin{aligned}\dim(W^\lambda) &= \#\ \textbf{of up-down tableaux of shape } \lambda \textbf{ and length } k \\ &= \binom{k}{2h}(2h-1)!!\frac{(k-2h)!}{\prod_{x\in\lambda} h_x},\end{aligned}$$

where h_x is the hook length at the box x in λ, and up-down tableaux are as in the case **(Ib)** of the Brauer algebra (page 74).

(c) What are their characters?

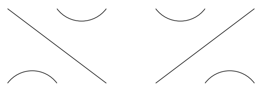
A Murnaghan–Nakayama rule for the irreducible characters of the BMW-algebras was given in [Halverson and Ram 1995].

References. The Birman–Murakami–Wenzl algebra was defined independently by Birman and Wenzl [1989] and by Murakami [1987]. See [Chari and Pressley 1994] for references to the analogue of Schur–Weyl duality for the BMW-algebras. The articles [Halverson and Ram 1995; Leduc and Ram 1997; Murakami 1990; Reshetikhin 1987; Wenzl 1990] contain further important information about the BMW-algebras.

Kaufmann [1990] first formalized the tangle description of the BMW-algebra.

B8. The Temperley–Lieb Algebras $\mathrm{TL}_k(x)$

A TL_k-*diagram* is a Brauer diagram on k dots that can be drawn with no crossings of edges.



The *Temperley–Lieb algebra* $\mathrm{TL}_k(x)$ is the subalgebra of the Brauer algebra $B_k(x)$ that is the span of the TL_k-diagrams.

THEOREM B8.1. *The Temperley–Lieb algebra* $\mathrm{TL}_k(x)$ *is the algebra over* $\mathbb{C}$ *given by generators* $E_1, E_2, \dots, E_{k-1}$ *and relations*

$$E_iE_j = E_jE_i \quad \text{if } |i-j| > 1,$$
$$E_iE_{i\pm 1}E_i = E_i,$$
$$E_i^2 = xE_i.$$

THEOREM B8.2. *Let* $q \in \mathbb{C}^*$ *be such that* $q + q^{-1} + 2 = 1/x^2$ *and let* $H_k(q)$ *be the Iwahori–Hecke algebra of type* A_{k-1}. *Then the map*

$$\begin{array}{ccc} H_k(q) & \longrightarrow & \mathrm{TL}_k(x) \\ T_i & \longmapsto & \dfrac{q+1}{x}E_i - 1 \end{array}$$

is a surjective homomorphism and the kernel of this homomorphism is the ideal generated by the elements

$$T_iT_{i+1}T_i + T_iT_{i+1} + T_{i+1}T_i + T_i + T_{i+1} + 1, \quad \text{for } 1 \le i \le n-2.$$

The Schur–Weyl duality theorem for S_n has the following analogue for the Temperley–Lieb algebras. Let $U_q\mathfrak{sl}_2$ be the Drinfeld–Jimbo quantum group corresponding to the Lie algebra $\mathfrak{sl}_2$ and let V be the 2-dimensional representation of

$U_q\mathfrak{sl}_2$. There is an action, see [CP], of the Temperley–Lieb algebra $\mathrm{TL}_k(q+q^{-1})$ on $V^{\otimes k}$ which commutes with the action of $U_q\mathfrak{sl}_2$ on $V^{\otimes k}$.

THEOREM B8.3. (a) *The action of* $\mathrm{TL}_k(q+q^{-1})$ *on* $V^{\otimes k}$ *generates* $\mathrm{End}_{U_q\mathfrak{sl}_2}(V^{\otimes k})$.
(a) *The action of* $U_q\mathfrak{sl}_2$ *on* $V^{\otimes k}$ *generates* $\mathrm{End}_{\mathrm{TL}_k(q+q^{-1})}(V^{\otimes k})$.

THEOREM B8.4. *The Temperley–Lieb algebra is semisimple if and only if* $1/x^2 \neq 4\cos^2(\pi/l)$, *for any* $2 \leq l \leq k$.

Partial Results for $\mathrm{TL}_k(x)$

The following results giving answers to the main questions **(Ia)–(Ic)** for the Temperley–Lieb algebras hold when x is such that $\mathrm{TL}_k(x)$ is semisimple.

I. What are the irreducible $\mathrm{TL}_k(x)$-modules?

(a) How do we index/count them?

There is a **bijection**

$$\textbf{partitions of } k \textbf{ with } \leq 2 \textbf{ rows} \quad \overset{1-1}{\longleftrightarrow} \quad \textbf{irreducible representations } T^{\lambda}.$$

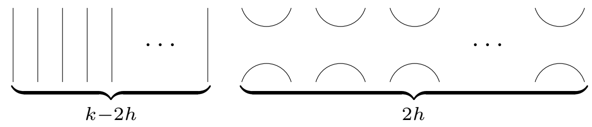
(b) What are their dimensions?

The **dimension** of the irreducible representation $T^{(k-l,l)}$ is

$$\begin{aligned}\dim(T^{(k-l,l)}) &= \#\ \textbf{of standard tableaux of shape } (k-l,l) \\ &= \binom{k}{l} - \binom{k}{l-1}.\end{aligned}$$

(c) **What are their characters?**

The **character** of the irreducible representation $T^{(k-l,l)}$ evaluated at the element

$d_{2h} =$ 

is

$$\chi^{(k-l,l)}(d_{2h}) = \begin{cases} \binom{k-2h}{l-h} - \binom{k-2h}{l-h-1}, & \text{if } l \geq h, \\ 0, & \text{if } l < h. \end{cases}$$

There is an algorithm for writing the character $\chi^{(k-l,l)}(a)$ of a general element $a \in \mathrm{TL}_k(x)$ as a linear combination of the characters $\chi^{(k-l,l)}(d_{2h})$.

References. The book [Goodman et al. 1989] contains a comprehensive treatment of the basic results on the Temperley–Lieb algebra. The Schur–Weyl duality theorem is treated in [Chari and Pressley 1994]; see also the references there. The character formula given above is derived in [Halverson and Ram 1995].

B9. Complex Semisimple Lie Groups

We shall not define Lie groups and Lie algebras. We recall simply that a complex Lie group is a differential $\mathbb{C}$-manifold, that a real Lie group is a differential $\mathbb{R}$-manifold and that every Lie group has an associated Lie algebra; see [Carter et al. 1995].

If G is a complex Lie group, the word *representation* usually refers to a *holomorphic representation*, i.e., the homomorphism

$$\rho : G \to \mathrm{GL}(V)$$

determined by the module V should be a morphism of (complex) analytic manifolds. Strictly speaking there are representations which are not holomorphic but there is a good theory only for holomorphic representations, so one usually abuses language and assumes that representation means holomorphic representation. The terms holomorphic representation and *complex analytic representation* are used interchangeably. Similarly, if G is a real Lie group then representation usually means *real analytic representation*. See [Varadarajan 1984, p. 102] for further details. Every holomorphic representation of $\mathrm{GL}(n, \mathbb{C})$ is also a rational representation; see [Fulton and Harris 1991].

A *complex semisimple Lie group* is a connected complex Lie group G such that its Lie algebra $\mathfrak{g}$ is a complex semisimple Lie algebra.

References

[Allen 1997] E. E. Allen, "New bases for the decompositions of the regular representations of wreath products of S_n", preprint, Wake Forest University, 1997.

[Ariki 1995] S. Ariki, "Representation theory of a Hecke algebra of $G(r, p, n)$", *J. Algebra* **177**:1 (1995), 164–185.

[Ariki and Koike 1994] S. Ariki and K. Koike, "A Hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$ and construction of its irreducible representations", *Adv. Math.* **106**:2 (1994), 216–243.

[Barcelo 1990] H. Barcelo, "On the action of the symmetric group on the free Lie algebra and the partition lattice", *J. Combin. Theory Ser. A* **55**:1 (1990), 93–129.

[Barcelo 1993] H. Barcelo, "Young straightening in a quotient S_n-module", *J. Algebraic Combin.* **2**:1 (1993), 5–23.

[Benkart et al. 1990] G. M. Benkart, D. J. Britten, and F. W. Lemire, *Stability in modules for classical Lie algebras — a constructive approach*, Mem. Amer. Math. Soc. **430**, Amer. Math. Soc., 1990.

[Benkart et al. 1994] G. Benkart, M. Chakrabarti, T. Halverson, R. Leduc, C. Lee, and J. Stroomer, "Tensor product representations of general linear groups and their connections with Brauer algebras", *J. Algebra* **166**:3 (1994), 529–567.

[Benkart et al. 1998] G. Benkart, C. L. Shader, and A. Ram, "Tensor product representations for orthosymplectic Lie superalgebras", *J. Pure and Applied Algebra* **130** (1998), 1–48.

[Berele 1986a] A. Berele, "Construction of Sp-modules by tableaux", *Linear and Multilinear Algebra* **19**:4 (1986), 299–307.

[Berele 1986b] A. Berele, "A Schensted-type correspondence for the symplectic group", *J. Combin. Theory Ser. A* **43**:2 (1986), 320–328.

[Berele and Regev 1987] A. Berele and A. Regev, "Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras", *Adv. in Math.* **64**:2 (1987), 118–175.

[Bernšteĭn et al. 1973] I. N. Bernšteĭn, I. M. Gel'fand, and S. I. Gel'fand, "Schubert cells and the cohomology of the spaces G/P", *Uspehi Mat. Nauk* **28**:3 (1973), 3–26. In Russian; translation in *Russian Math. Surveys* **28**:3 (1973), 1–26.

[Birman and Wenzl 1989] J. S. Birman and H. Wenzl, "Braids, link polynomials and a new algebra", *Trans. Amer. Math. Soc.* **313**:1 (1989), 249–273.

[Borel 1953] A. Borel, "Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts", *Ann. of Math.* (2) **57** (1953), 115–207.

[Bourbaki 1958] N. Bourbaki, *Algèbre, Chap. VIII: Modules et anneaux semi-simples*, Actualités Sci. Ind. **1261**, Hermann, Paris, 1958.

[Bourbaki 1968] N. Bourbaki, *Groupes et algèbres de Lie, Chap. IV: Groupes de Coxeter et systèmes de Tits; Chap. V: Groupes engendrés par des réflexions; Chap. VI: systèmes de racines*, Actualités Sci. Ind. **1337**, Hermann, Paris, 1968.

[Brauer 1937] R. Brauer, "On algebras which are connected with the semisimple continous groups", *Ann. Math.* **38** (1937), 854–872.

[Brenti 1994] F. Brenti, "A combinatorial formula for Kazhdan-Lusztig polynomials", *Invent. Math.* **118**:2 (1994), 371–394.

[Bröcker and tom Dieck 1985] T. Bröcker and T. tom Dieck, *Representations of compact Lie groups*, Graduate Texts in Mathematics **98**, Springer, New York, 1985.

[Broué and Malle 1993] M. Broué and G. Malle, "Zyklotomische Heckealgebren", pp. 119–189 in *Représentations unipotentes génériques et blocs des groupes réductifs finis*, Astérisque **212**, Soc. math. France, Paris, 1993.

[Carter 1985] R. W. Carter, *Finite groups of Lie type: Conjugacy classes and complex characters*, John Wiley, New York, 1985.

[Carter et al. 1995] R. Carter, G. Segal, and I. Macdonald, *Lectures on Lie groups and Lie algebras*, London Math. Soc. Student Texts **32**, Cambridge University Press, Cambridge, 1995. With a foreword by Martin Taylor.

[Chari and Pressley 1994] V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge University Press, Cambridge, 1994.

[Chriss and Ginzburg 1997] N. Chriss and V. Ginzburg, *Representation theory and complex geometry*, Birkhäuser, Boston, 1997.

[Clifford 1937] A. H. Clifford, "Representations induced in an invariant subgroup", *Ann. of Math.* **38** (1937), 533–550.

[Curtis 1988] C. W. Curtis, "Representations of Hecke algebras", pp. 13–60 in *Orbites unipotentes et représentations, I*, Astérisque **168**, Soc. math. France, Paris, 1988.

[Curtis and Reiner 1962] C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Pure and Applied Mathematics **11**, Wiley/Interscience, New York, 1962. Reprinted by Wiley, 1998.

[Curtis and Reiner 1981] C. W. Curtis and I. Reiner, *Methods of representation theory*, vol. I, Pure and Applied Mathematics, John Wiley, New York, 1981.

[Curtis and Reiner 1987] C. W. Curtis and I. Reiner, *Methods of representation theory*, vol. II, Pure and Applied Mathematics, John Wiley, New York, 1987.

[De Concini and Procesi 1981] C. De Concini and C. Procesi, "Symmetric functions, conjugacy classes and the flag variety", *Invent. Math.* **64**:2 (1981), 203–219.

[Deligne and Lusztig 1976] P. Deligne and G. Lusztig, "Representations of reductive groups over finite fields", *Ann. of Math.* (2) **103**:1 (1976), 103–161.

[Digne and Michel 1991] F. Digne and J. Michel, *Representations of finite groups of Lie type*, London Math. Soc. Student Texts **21**, Cambridge University Press, Cambridge, 1991.

[Dipper and James 1986] R. Dipper and G. James, "Representations of Hecke algebras of general linear groups", *Proc. London Math. Soc.* (3) **52**:1 (1986), 20–52.

[Dipper and James 1992] R. Dipper and G. James, "Representations of Hecke algebras of type B_n", *J. Algebra* **146**:2 (1992), 454–481.

[Dipper et al. 1995] R. Dipper, G. James, and E. Murphy, "Hecke algebras of type B_n at roots of unity", *Proc. London Math. Soc.* (3) **70**:3 (1995), 505–528.

[Duchamp et al. 1995] G. Duchamp, D. Krob, A. Lascoux, B. Leclerc, T. Scharf, and J.-Y. Thibon, "Euler–Poincaré characteristic and polynomial representations of Iwahori–Hecke algebras", *Publ. Res. Inst. Math. Sci.* **31**:2 (1995), 179–201.

[Feĭgin and Fuchs 1990] B. L. Feĭgin and D. B. Fuchs, "Representations of the Virasoro algebra", pp. 465–554 in *Representation of Lie groups and related topics*, edited by A. M. Vershik and D. P. Zhelobenko, Adv. Stud. Contemp. Math. **7**, Gordon and Breach, New York, 1990.

[Fomin and Greene 1998] S. Fomin and C. Greene, "Noncommutative Schur functions and their applications", *Discrete Math.* **193**:1-3 (1998), 179–200.

[Frame et al. 1954] J. S. Frame, G. d. B. Robinson, and R. M. Thrall, "The hook graphs of the symmetric groups", *Canadian J. Math.* **6** (1954), 316–324.

[Frobenius 1900] F. G. Frobenius, "Über die Charaktere der symmetrischen Gruppe", *Sitzungsber. König. Preuss. Akad. Wiss. Berlin* (1900), 516–534. Reprinted as pp. 148–166 in his *Gesammelte Abhandlungen*, vol. 3, edited by J.-P. Serre, Berlin, Springer, 1968.

[Fulton and Harris 1991] W. Fulton and J. Harris, *Representation theory*, Graduate Texts in Mathematics **129**, Springer, New York, 1991.

[Garsia 1990] A. M. Garsia, "Combinatorics of the free Lie algebra and the symmetric group", pp. 309–382 in *Analysis, et cetera*, edited by P. H. Rabinowitz and E. Zehnder, Academic Press, Boston, MA, 1990.

[Garsia and McLarnan 1988] A. M. Garsia and T. J. McLarnan, "Relations between Young's natural and the Kazhdan–Lusztig representations of S_n", *Adv. in Math.* **69**:1 (1988), 32–92.

[Garsia and Procesi 1992] A. M. Garsia and C. Procesi, "On certain graded S_n-modules and the q-Kostka polynomials", *Adv. Math.* **94**:1 (1992), 82–138.

[Garsia and Remmel 1986] A. M. Garsia and J. B. Remmel, "q-counting rook configurations and a formula of Frobenius", *J. Combin. Theory Ser. A* **41**:2 (1986), 246–275.

[Garsia and Wachs 1989] A. M. Garsia and M. L. Wachs, "Combinatorial aspects of skew representations of the symmetric group", *J. Combin. Theory Ser. A* **50**:1 (1989), 47–81.

[Gel'fand and Tsetlin 1950a] I. M. Gel'fand and M. L. Tsetlin, "Finite-dimensional representations of the group of unimodular matrices", *Doklady Akad. Nauk SSSR* (*N.S.*) **71** (1950), 825–828. In Russian.

[Gel'fand and Tsetlin 1950b] I. M. Gel'fand and M. L. Tsetlin, "Finite-dimensional representations of groups of orthogonal matrices", *Doklady Akad. Nauk SSSR* (*N.S.*) **71** (1950), 1017–1020. In Russian.

[Ginsburg 1987] V. Ginsburg, " "Lagrangian" construction for representations of Hecke algebras", *Adv. in Math.* **63**:1 (1987), 100–112.

[Goodman et al. 1989] F. M. Goodman, P. de la Harpe, and V. F. R. Jones, *Coxeter graphs and towers of algebras*, Mathematical Sciences Research Institute Publications **14**, Springer, New York, 1989.

[Graham and Lehrer 1996] J. J. Graham and G. I. Lehrer, "Cellular algebras", *Invent. Math.* **123**:1 (1996), 1–34.

[Green 1955] J. A. Green, "The characters of the finite general linear groups", *Trans. Amer. Math. Soc.* **80** (1955), 402–447.

[Gyoja 1986] A. Gyoja, "A q-analogue of Young symmetrizer", *Osaka J. Math.* **23**:4 (1986), 841–852.

[Gyoja and Uno 1989] A. Gyoja and K. Uno, "On the semisimplicity of Hecke algebras", *J. Math. Soc. Japan* **41**:1 (1989), 75–79.

[Halverson 1995] T. Halverson, "A q-rational Murnaghan-Nakayama rule", *J. Combin. Theory Ser. A* **71**:1 (1995), 1–18.

[Halverson 1996] T. Halverson, "Characters of the centralizer algebras of mixed tensor representations of $\mathrm{GL}(r,\mathbb{C})$ and the quantum group $\mathcal{U}_q(\mathfrak{gl}(r,\mathbb{C}))$", *Pacific J. Math.* **174**:2 (1996), 359–410.

[Halverson and Ram 1995] T. Halverson and A. Ram, "Characters of algebras containing a Jones basic construction: the Temperley–Lieb, Okasa, Brauer, and Birman–Wenzl algebras", *Adv. Math.* **116**:2 (1995), 263–321.

[Halverson and Ram 1998] T. Halverson and A. Ram, "Murnaghan–Nakayama rules for characters of Iwahori–Hecke algebras of the complex reflection groups $G(r,p,n)$", *Canad. J. Math.* **50**:1 (1998), 167–192.

[Hanlon 1981] P. Hanlon, "The fixed point partition lattices", *Pacific J. Math.* **96**:2 (1981), 319–341.

[Hanlon 1984] P. Hanlon, "An introduction to the complex representations of the symmetric group and general linear Lie algebra", pp. 1–18 in *Combinatorics and algebra* (Boulder, CO, 1983), edited by C. Greene, Contemp. Math. **34**, Amer. Math. Soc., Providence, 1984.

[Hanlon and Wales 1989] P. Hanlon and D. Wales, "Eigenvalues connected with Brauer's centralizer algebras", *J. Algebra* **121**:2 (1989), 446–476.

[Hoefsmit 1974] P. N. Hoefsmit, *Representations of Hecke algebras of finite groups with* BN-pairs of classical type, Ph.D. thesis, University of British Columbia, 1974.

[Hotta and Springer 1977] R. Hotta and T. A. Springer, "A specialization theorem for certain Weyl group representations and an application to the Green polynomials of unitary groups", *Invent. Math.* **41**:2 (1977), 113–127.

[Howe 1994] R. Howe, "Hecke algebras and p-adic GL_n", pp. 65–100 in *Representation theory and analysis on homogeneous spaces* (New Brunswick, NJ, 1993), edited by S. Gindikin et al., Contemp. Math. **177**, Amer. Math. Soc., Providence, RI, 1994.

[Howe 1995] R. Howe, "Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond", pp. 1–182 in *The Schur lectures* (Tel Aviv, 1992), edited by I. Piatetski-Shapiro and S. Gelbart, Israel Math. Conf. Proc. **8**, Bar-Ilan Univ., Ramat Gan, 1995.

[Humphreys 1972] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics **9**, Springer, New York, 1972.

[Humphreys 1990] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics **29**, Cambridge University Press, Cambridge, 1990.

[Iwahori 1964] N. Iwahori, "On the structure of a Hecke ring of a Chevalley group over a finite field", *J. Fac. Sci. Univ. Tokyo Sect. I* **10** (1964), 215–236.

[Jantzen 1996] J. C. Jantzen, *Lectures on quantum groups*, vol. 6, Graduate Studies in Math., Amer. Math. Soc., Providence, RI, 1996.

[Jimbo 1986] M. Jimbo, "A q-analogue of $U(\mathfrak{gl}(N+1))$, Hecke algebra, and the Yang-Baxter equation", *Lett. Math. Phys.* **11**:3 (1986), 247–252.

[Jones 1983] V. F. R. Jones, "Index for subfactors", *Invent. Math.* **72**:1 (1983), 1–25.

[Jones 1994] V. F. R. Jones, "The Potts model and the symmetric group", pp. 259–267 in *Subfactors: proceedings of the Taniguchi Symposium on Operator Algebras* (Kyuzeso, 1993), edited by H. Araki et al., World Sci. Publishing, River Edge, NJ, 1994.

[Joyal 1986] A. Joyal, "Foncteurs analytiques et espèces de structures", pp. 126–159 in *Combinatoire énumérative* (Montreal, 1985), edited by G. L. et P. Leroux, Lecture Notes in Math. **1234**, Springer, Berlin, 1986.

[Kac 1990] V. G. Kac, *Infinite-dimensional Lie algebras*, Third ed., Cambridge University Press, Cambridge, 1990.

[Kashiwara 1990] M. Kashiwara, "Crystalizing the q-analogue of universal enveloping algebras", *Comm. Math. Phys.* **133**:2 (1990), 249–260.

[Kato 1983] S.-i. Kato, "A realization of irreducible representations of affine Weyl groups", *Nederl. Akad. Wetensch. Indag. Math.* **45**:2 (1983), 193–201.

[Kauffman 1987] L. H. Kauffman, "State models and the Jones polynomial", *Topology* **26**:3 (1987), 395–407.

[Kauffman 1990] L. H. Kauffman, "An invariant of regular isotopy", *Trans. Amer. Math. Soc.* **318**:2 (1990), 417–471.

[Kazhdan and Lusztig 1979] D. Kazhdan and G. Lusztig, "Representations of Coxeter groups and Hecke algebras", *Invent. Math.* **53**:2 (1979), 165–184.

[Kazhdan and Lusztig 1987] D. Kazhdan and G. Lusztig, "Proof of the Deligne–Langlands conjecture for Hecke algebras", *Invent. Math.* **87**:1 (1987), 153–215.

[Kerov 1992] S. V. Kerov, "Characters of Hecke and Birman–Wenzl algebras", pp. 335–340 in *Quantum groups* (Leningrad, 1990), edited by P. P. Kulish, Lecture Notes in Math. **1510**, Springer, Berlin, 1992.

[King and Welsh 1993] R. C. King and T. A. Welsh, "Construction of orthogonal group modules using tableaux", *Linear and Multilinear Algebra* **33**:3-4 (1993), 251–283.

[King and Wybourne 1992] R. C. King and B. G. Wybourne, "Representations and traces of the Hecke algebras $H_n(q)$ of type A_{n-1}", *J. Math. Phys.* **33**:1 (1992), 4–14.

[Klyachko 1974] A. A. Klyachko, "Елементы Ли в тенсорои алгебре (Lie elements in the tensor algebra)", *Sib. Mat. Zh.* **15** (1974), 1296–1304. Translated in *Siberian Math. J.*, **15** (1974), 1296–1304.

[Knapp and Vogan 1995] A. W. Knapp and J. Vogan, David A., *Cohomological induction and unitary representations*, Princeton Mathematical Series **45**, Princeton University Press, Princeton, NJ, 1995.

[Koike 1989] K. Koike, "On the decomposition of tensor products of the representations of the classical groups: by means of the universal characters", *Adv. Math.* **74**:1 (1989), 57–86.

[Kostant and Kumar 1986] B. Kostant and S. Kumar, "The nil Hecke ring and cohomology of G/P for a Kac–Moody group G", *Adv. in Math.* **62**:3 (1986), 187–237.

[Kostka 1882] C. Kostka, "Über den Zusammenhang zwischen einigen Formen von symmetrischen Funktionen", *J. reine ungew. math.* **93** (1882), 89–123.

[Kosuda and Murakami 1992] M. Kosuda and J. Murakami, "The centralizer algebras of mixed tensor representations of $U_q(\mathfrak{gl}_n)$ and the HOMFLY polynomial of links", *Proc. Japan Acad. Ser. A Math. Sci.* **68**:6 (1992), 148–151.

[Kosuda and Murakami 1993] M. Kosuda and J. Murakami, "Centralizer algebras of the mixed tensor representations of quantum group $U_q(\mathfrak{gl}(n,\mathbb{C}))$", *Osaka J. Math.* **30**:3 (1993), 475–507.

[Kraft 1981] H. Kraft, "Conjugacy classes and Weyl group representations", pp. 191–205 in *Young tableaux and Schur functions in algebra and geometry* (Toruń, Poland, 1980), Astérisque **87–88**, Soc. Math. France, Paris, 1981.

[Kuperberg 1994a] G. Kuperberg, "The quantum G_2 link invariant", *Internat. J. Math.* **5**:1 (1994), 61–85.

[Kuperberg 1994b] G. Kuperberg, "Self-complementary plane partitions by Proctor's minuscule method", *European J. Combin.* **15**:6 (1994), 545–553.

[Kuperberg 1994c] G. Kuperberg, "Symmetries of plane partitions and the permanent-determinant method", *J. Combin. Theory Ser. A* **68**:1 (1994), 115–151.

[Kuperberg 1996a] G. Kuperberg, "Four symmetry classes of plane partitions under one roof", *J. Combin. Theory Ser. A* **75**:2 (1996), 295–315.

[Kuperberg 1996b] G. Kuperberg, "Spiders for rank 2 Lie algebras", *Comm. Math. Phys.* **180**:1 (1996), 109–151.

[Lakshmibai and Seshadri 1991] V. Lakshmibai and C. S. Seshadri, "Standard monomial theory", pp. 279–322 in *Proceedings of the Hyderabad Conference on Algebraic Groups* (Hyderabad, 1989), edited by S. Ramanan, Manoj Prakashan, Madras, 1991.

[Lascoux 1991] A. Lascoux, "Cyclic permutations on words, tableaux and harmonic polynomials", pp. 323–347 in *Proceedings of the Hyderabad Conference on Algebraic Groups* (Hyderabad, 1989), edited by S. Ramanan, Manoj Prakashan, Madras, 1991.

[Leduc 1994] R. Leduc, *A two-parameter version of the centralizer algebra of the mixed tensor representations of the general linear group and quantum general linear group*, Ph.D. thesis, University of Wisconsin–Madison, 1994.

[Leduc and Ram 1997] R. Leduc and A. Ram, "A ribbon Hopf algebra approach to the irreducible representations of centralizer algebras: the Brauer, Birman–Wenzl, and type A Iwahori–Hecke algebras", *Adv. Math.* **125**:1 (1997), 1–94.

[Lehrer 1995] G. I. Lehrer, "Poincaré polynomials for unitary reflection groups", *Invent. Math.* **120**:3 (1995), 411–425.

[Littelmann 1994] P. Littelmann, "A Littlewood-Richardson rule for symmetrizable Kac–Moody algebras", *Invent. Math.* **116**:1-3 (1994), 329–346.

[Littelmann 1995] P. Littelmann, "Paths and root operators in representation theory", *Ann. of Math.* (2) **142**:3 (1995), 499–525.

[Littlewood 1940] D. E. Littlewood, *The theory of group characters and matrix representations of groups*, Oxford University Press, New York, 1940.

[Lusztig 1974] G. Lusztig, *The discrete series of* GL_n *over a finite field*, Annals of Mathematics Studies **81**, Princeton University Press, Princeton, NJ, 1974.

[Lusztig 1977] G. Lusztig, "Classification des représentations irréductibles des groupes classiques finis", *C. R. Acad. Sci. Paris Sér. A-B* **284**:9 (1977), A473–A475.

[Lusztig 1978] G. Lusztig, *Representations of finite Chevalley groups*, CBMS Regional Conference Series in Mathematics **39**, Amer. Math. Soc., Providence, 1978.

[Lusztig 1984] G. Lusztig, *Characters of reductive groups over a finite field*, Annals of Mathematics Studies **107**, Princeton University Press, Princeton, NJ, 1984.

[Lusztig 1987a] G. Lusztig, "Cells in affine Weyl groups, II", *J. Algebra* **109**:2 (1987), 536–548.

[Lusztig 1987b] G. Lusztig, "Cells in affine Weyl groups, III", *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **34**:2 (1987), 223–243.

[Lusztig 1988] G. Lusztig, "Cuspidal local systems and graded Hecke algebra", *Inst. Hautes Études Sci. Publ. Math.* **67** (1988), 145–202.

[Lusztig 1989a] G. Lusztig, "Affine Hecke algebras and their graded version", *J. Amer. Math. Soc.* **2**:3 (1989), 599–635.

[Lusztig 1989b] G. Lusztig, "Representations of affine Hecke algebras", pp. 73–84 in *Orbites unipotentes et représentations, II*, Astérisque **171-172**, Soc. math. France, Paris, 1989.

[Lusztig 1995a] G. Lusztig, "Classification of unipotent representations of simple p-adic groups", *Internat. Math. Res. Notices* **11** (1995), 517–589.

[Lusztig 1995b] G. Lusztig, "Cuspidal local systems and graded Hecke algebras, II", pp. 217–275 in *Representations of groups* (Banff, AB, 1994), edited by B. N. Allison and G. H. Cliff, CMS Conf. Proc. **16**, Amer. Math. Soc., Providence, RI, 1995.

[Lusztig and Spaltenstein 1985] G. Lusztig and N. Spaltenstein, "On the generalized Springer correspondence for classical groups", pp. 289–316 in *Algebraic groups and related topics* (Kyoto/Nagoya, 1983), edited by R. Hotta, Adv. Stud. Pure Math. **6**, North-Holland, Amsterdam, 1985.

[Macdonald 1995] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford Univ. Press, 1995.

[Malle 1995] G. Malle, "Unipotente Grade imprimitiver komplexer Spiegelungsgruppen", *J. Algebra* **177**:3 (1995), 768–826.

[Martin 1996] P. Martin, "The structure of the partition algebras", *J. Algebra* **183**:2 (1996), 319–358.

[Mathas 1997] A. Mathas, "Canonical bases and the decomposition matrices of Ariki–Koike algebras", preprint, 1997.

[Murakami 1987] J. Murakami, "The Kauffman polynomial of links and representation theory", *Osaka J. Math.* **24**:4 (1987), 745–758.

[Murakami 1990] J. Murakami, "The representations of the q-analogue of Brauer's centralizer algebras and the Kauffman polynomial of links", *Publ. Res. Inst. Math. Sci.* **26**:6 (1990), 935–945.

[Murnaghan 1937] F. D. Murnaghan, "The characters of the symmetric group", *Amer. J. Math.* **59** (1937), 739–753.

[Murphy 1992] G. E. Murphy, "On the representation theory of the symmetric groups and associated Hecke algebras", *J. Algebra* **152**:2 (1992), 492–513.

[Murphy 1995] G. E. Murphy, "The representations of Hecke algebras of type A_n", *J. Algebra* **173**:1 (1995), 97–121.

[Murphy $\geq$ 1999] G. E. Murphy, "Representations of Hecke algebras of $C_r \wr S_n$ at roots of unity". In preparation.

[Nakayama 1941] T. Nakayama, "On some modular properties of irreducible representations of symmetric groups, I and II", *Jap. J. Math.* **17** (1941), 165–184 and 411–423.

[Nazarov 1996] M. Nazarov, "Young's orthogonal form for Brauer's centralizer algebra", *J. Algebra* **182**:3 (1996), 664–693.

[Orlik and Solomon 1980] P. Orlik and L. Solomon, "Unitary reflection groups and cohomology", *Invent. Math.* **59**:1 (1980), 77–94.

[Osima 1954] M. Osima, "On the representations of the generalized symmetric group", *Math. J. Okayama Univ.* **4** (1954), 39–56.

[Pallikaros 1994] C. Pallikaros, "Representations of Hecke algebras of type D_n", *J. Algebra* **169**:1 (1994), 20–48.

[Penrose 1969] R. Penrose, "Angular momentum: an approach to combinatorial space-time", in *Quantum theory and beyond*, edited by T. A. Bastin, Cambridge Univ. Press, 1969.

[Penrose 1971] R. Penrose, "Applications of negative dimensional tensors", pp. 221–244 in *Combinatorial mathematics and its applications* (Oxford, 1969), edited by D. J. A. Welsh, Academic Press, London, 1971.

[Procesi 1976] C. Procesi, "The invariant theory of $n \times n$ matrices", *Advances in Math.* **19**:3 (1976), 306–381.

[Proctor 1984] R. A. Proctor, "Bruhat lattices, plane partition generating functions, and minuscule representations", *European J. Combin.* **5**:4 (1984), 331–350.

[Ram 1991] A. Ram, "A Frobenius formula for the characters of the Hecke algebras", *Invent. Math.* **106**:3 (1991), 461–488.

[Ram 1995] A. Ram, "Characters of Brauer's centralizer algebras", *Pacific J. Math.* **169**:1 (1995), 173–200.

[Ram 1998] A. Ram, "An elementary proof of Roichman's rule for irreducible characters of Iwahori–Hecke algebras of type A", pp. 335–342 in *Mathematical essays in honor* *of Gian-Carlo Rota* (Cambridge, MA, 1996), edited by B. E. Sagan and R. P. Stanley, Progress in Mathematics **161**, Birkhäuser, Boston, 1998.

[Remmel 1992] J. B. Remmel, "Formulas for the expansion of the Kronecker products $S_{(m,n)} \otimes S_{(1^{p-r},r)}$ and $S_{(1^k 2^l)} \otimes S_{(1^{p-r},r)}$", *Discrete Math.* **99**:1-3 (1992), 265–287.

[Reshetikhin 1987] N. Reshetikhin, "Quantized universal enveloping algebras, the Yang–Baxter equation and invariants of links I", Preprint E-4-87, LOMI, Leningrad, 1987.

[Roichman 1997] Y. Roichman, "A recursive rule for Kazhdan–Lusztig characters", *Adv. Math.* **129**:1 (1997), 25–45.

[Rumer et al. 1932] G. Rumer, E. Teller, and H. Weyl, "Eine für die Valenztheorie geeignete Basis der binären Vektor-invarianten", *Nachr. Ges. Wiss. Göttingen Math.-Phys. Kl.* (1932), 499–504. Reprinted as pp. 380–385 in Weyl's *Gesammelte Abhandlungen*, v. 3, edited by K. Chandrasekharan, Springer, Berlin, 1968.

[Rutherford 1948] D. E. Rutherford, *Substitutional Analysis*, Edinburgh U. Press, 1948.

[Scheunert 1979] M. Scheunert, *The theory of Lie superalgebras: an introduction*, Lecture Notes in Math. **716**, Springer, Berlin, 1979.

[Schur 1901] I. Schur, *Über eine Klasse von Matrizen, die sich einer gegeben Matrix zuordnen lassen*, Dissertation, 1901. Reprinted as pp. 1–72 in his *Gesammelte Abhandlungen*, vol. 1, edited by A. Brauer and H. Rohrbach, Springer, Berlin, 1973.

[Schur 1927] I. Schur, "Über die rationalen Darstellungen der allgemeinen linearen Gruppe", *Sitzungsber. Preuss. Akad. Wiss. Berlin Math.-Phys. Kl.* (1927), 58–75. Reprinted as pp. 68–85 in his *Gesammelte Abhandlungen*, vol. 3, edited by A. Brauer and H. Rohrbach, Springer, Berlin, 1973.

[Serganova 1996] V. Serganova, "Kazhdan–Lusztig polynomials and character formula for the Lie superalgebra $\mathfrak{gl}(m|n)$", *Selecta Math.* (*N.S.*) **2**:4 (1996), 607–651.

[Sergeev 1984] A. N. Sergeev, "Representations of the Lie superalgebras $\mathfrak{gl}(n, m)$ and $Q(n)$ in a space of tensors", *Funktsional. Anal. i Prilozhen.* **18**:1 (1984), 80–81. In Russian; translation in *Funct. Anal. App.* **18**:1 (1984), 70–72.

[Serre 1971] J.-P. Serre, *Représentations linéaires des groupes finis*, 2nd ed., Hermann, Paris, 1971. Translated as *Linear representations of finite groups*, Graduate Texts in Math. **42**, Springer, New York, 1977.

[Serre 1987] J.-P. Serre, *Complex semisimple Lie algebras*, Springer, New York, 1987.

[Shafarevich 1994] I. R. Shafarevich, *Basic algebraic geometry,* 2 : *Schemes and complex manifolds*, Second ed., Springer, Berlin, 1994.

[Shephard and Todd 1954] G. C. Shephard and J. A. Todd, "Finite unitary reflection groups", *Canadian J. Math.* **6** (1954), 274–304.

[Shoji 1979] T. Shoji, "On the Springer representations of the Weyl groups of classical algebraic groups", *Comm. Algebra* **7**:16 (1979), 1713–1745. Correction in **7**:18 (1979), 2027–2033.

[Solomon 1990] L. Solomon, "The Bruhat decomposition, Tits system and Iwahori ring for the monoid of matrices over a finite field", *Geom. Dedicata* **36**:1 (1990), 15–49.

[Solomon 1995] L. Solomon, "Abstract No. 900-16-169", *Abstracts American Math. Soc.* **16**:2 (Spring 1995).

[Spaltenstein 1976] N. Spaltenstein, "On the fixed point set of a unipotent transformation on the flag manifold", *Proc. Kon. Nederl. Akad. Wetensch.* **79** (1976), 452–456.

[Specht 1932] W. Specht, "Eine Verallgemeinerung der symmetrischen Gruppe", *Schriften Math. Seminar* (*Berlin*) **1** (1932), 1–32.

[Springer 1978] T. A. Springer, "A construction of representations of Weyl groups", *Invent. Math.* **44**:3 (1978), 279–293.

[Stanley 1971] R. Stanley, "Theory and application of plane partitions", *Stud. Appl. Math.* **50** (1971), 167–188 and 259–279.

[Stanley 1982] R. P. Stanley, "Some aspects of groups acting on finite posets", *J. Combin. Theory Ser. A* **32**:2 (1982), 132–161.

[Stanley 1983] R. P. Stanley, "$\mathrm{GL}(n,\mathbb{C})$ for combinatorialists", pp. 187–199 in *Surveys in combinatorics* (Southampton, 1983), edited by E. K. Lloyd, London Math. Soc. Lecture Note Ser. **82**, Cambridge Univ. Press, Cambridge, 1983.

[Stembridge 1987] J. R. Stembridge, "Rational tableaux and the tensor algebra of $\mathfrak{gl}_n$", *J. Combin. Theory Ser. A* **46**:1 (1987), 79–120.

[Stembridge 1989a] J. R. Stembridge, "A combinatorial theory for rational actions of GL_n", pp. 163–176 in *Invariant theory* (Denton, TX, 1986), edited by R. Fossum et al., Contemp. Math. **88**, Amer. Math. Soc., Providence, RI, 1989.

[Stembridge 1989b] J. R. Stembridge, "On the eigenvalues of representations of reflection groups and wreath products", *Pacific J. Math.* **140**:2 (1989), 353–396.

[Stembridge 1994a] J. R. Stembridge, "On minuscule representations, plane partitions and involutions in complex Lie groups", *Duke Math. J.* **73**:2 (1994), 469–490.

[Stembridge 1994b] J. R. Stembridge, "Some hidden relations involving the ten symmetry classes of plane partitions", *J. Combin. Theory Ser. A* **68**:2 (1994), 372–409.

[Stembridge 1995] J. R. Stembridge, "The enumeration of totally symmetric plane partitions", *Adv. Math.* **111**:2 (1995), 227–243.

[Sundaram 1990a] S. Sundaram, "The Cauchy identity for $\mathrm{Sp}(2n)$", *J. Combin. Theory Ser. A* **53**:2 (1990), 209–238.

[Sundaram 1990b] S. Sundaram, "Orthogonal tableaux and an insertion algorithm for $\mathrm{SO}(2n+1)$", *J. Combin. Theory Ser. A* **53**:2 (1990), 239–256.

[Sundaram 1990c] S. Sundaram, "Tableaux in the representation theory of the classical Lie groups", pp. 191–225 in *Invariant theory and tableaux* (Minneapolis, 1988), IMA Vol. Math. Appl. **19**, Springer, New York, 1990.

[Sundaram 1994] S. Sundaram, "The homology representations of the symmetric group on Cohen-Macaulay subposets of the partition lattice", *Adv. Math.* **104**:2 (1994), 225–296.

[Tanisaki 1982] T. Tanisaki, "Defining ideals of the closures of the conjugacy classes and representations of the Weyl groups", *Tôhoku Math. J.* (2) **34**:4 (1982), 575–585.

[Temperley and Lieb 1971] H. N. V. Temperley and E. H. Lieb, "Relations between the 'percolation' and 'colouring' problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the 'percolation' problem", *Proc. Roy. Soc. London Ser. A* **322** (1971), 251–280.

[Ueno and Shibukawa 1992] K. Ueno and Y. Shibukawa, "Character table of Hecke algebra of type A_{N-1} and representations of the quantum group $U_q(\mathfrak{gl}_{n+1})$", pp. 977–984 in *Infinite analysis* (Kyoto, 1991), vol. 2, edited by A. Tsuchiya et al., Adv. Ser. Math. Phys. **16**, World Sci. Publishing, River Edge, NJ, 1992.

[Van der Jeugt 1991] J. Van der Jeugt, "An algorithm for characters of Hecke algebras $H_n(q)$ of type A_{n-1}", *J. Phys. A* **24**:15 (1991), 3719–3725.

[Varadarajan 1984] V. S. Varadarajan, *Lie groups, Lie algebras, and their representations*, Graduate Texts in Math. **102**, Springer, New York, 1984. Reprint of the 1974 edition.

[Vershik and Kerov 1988] A. M. Vershik and S. V. Kerov, "Characters and realizations of representations of the infinite-dimensional Hecke algebra, and knot invariants", *Dokl. Akad. Nauk SSSR* **301**:4 (1988), 777–780. In Russian; translated in *Soviet Math. Dokl.* **38** (1989), 134–137.

[Vogan 1981] J. Vogan, David A., *Representations of real reductive Lie groups*, Progress in Mathematics **15**, Birkhäuser, Boston, 1981.

[Vogan 1987] J. Vogan, David A., *Unitary representations of reductive Lie groups*, Annals of Mathematics Studies **118**, Princeton University Press, Princeton, NJ, 1987.

[Wallach 1988] N. R. Wallach, *Real reductive groups*, vol. 1, Academic Press, Boston, MA, 1988.

[Wallach 1992] N. R. Wallach, *Real reductive groups*, vol. 2, Academic Press, Boston, MA, 1992.

[Wenzl 1988a] H. Wenzl, "Hecke algebras of type A_n and subfactors", *Invent. Math.* **92**:2 (1988), 349–383.

[Wenzl 1988b] H. Wenzl, "On the structure of Brauer's centralizer algebras", *Ann. of Math.* (2) **128**:1 (1988), 173–193.

[Wenzl 1990] H. Wenzl, "Quantum groups and subfactors of type B, C, and D", *Comm. Math. Phys.* **133**:2 (1990), 383–432.

[Weyl 1925] H. Weyl, "Theorie der Darstellung kontinuerlicher halb-einfacher Gruppen durch lineare Transformationen I", *Mathematische Zeitschrift* **23** (1925), 271–309.

[Weyl 1926] H. Weyl, "Theorie der Darstellung kontinuerlicher halb-einfacher Gruppen durch lineare Transformationen II, III", *Mathematische Zeitschrift* **24** (1926), 328–376, 377–395.

[Weyl 1946] H. Weyl, *The classical groups: their invariants and representations*, 2nd ed., Princeton University Press, Princeton, NJ, 1946.

[Weyman 1989] J. Weyman, "The equations of conjugacy classes of nilpotent matrices", *Invent. Math.* **98**:2 (1989), 229–245.

[Young 1901] A. Young, "On quantitative substitutional analysis, I", *Proc. London Math. Soc.* **33** (1901), 97–146. Reprinted as pp. 42–91 in [Young 1977].

[Young 1902] A. Young, "On quantitative substitutional analysis, II", *Proc. London Math. Soc.* **34** (1902), 361–397. Reprinted as pp. 92–128 in [Young 1977].

[Young 1928] A. Young, "On quantitative substitutional analysis, III", *Proc. London Math. Soc.* (2) **28** (1928), 255–292. Reprinted as pp. 352–389 in [Young 1977].

[Young 1930a] A. Young, "On quantitative substitutional analysis, IV", *Proc. London Math. Soc.* (2) **31** (1930), 353–372. Reprinted as pp. 396–415 in [Young 1977].

[Young 1930b] A. Young, "On quantitative substitutional analysis, V", *Proc. London Math. Soc.* (2) **31** (1930), 273–288. Reprinted as pp. 416–431 in [Young 1977].

[Young 1931] A. Young, "On quantitative substitutional analysis, VI", *Proc. London Math. Soc.* (2) **34** (1931), 196–230. Reprinted as pp. 432–466 in [Young 1977].

[Young 1934a] A. Young, "On quantitative substitutional analysis, VII", *Proc. London Math. Soc.* (2) **36** (1934), 304–368. Reprinted as pp. 494–558 in [Young 1977].

[Young 1934b] A. Young, "On quantitative substitutional analysis, VIII", *Proc. London Math. Soc.* (2) **37** (1934), 441–495. Reprinted as pp. 559–613 in [Young 1977].

[Young 1977] A. Young, *The Collected Papers of Alfred Young,* 1873–1940, Mathematical Exposition **21**, U. of Toronto Press, 1977.

[Zhelobenko 1970] D. P. Zhelobenko, Компактные группы Ли и их представления, Nauka, Moscow, 1970. Translated as *Compact Lie groups and their representations*, Translations of Mathematical Monographs **40**, Amer. Math. Soc., Providence, 1973.

[Zhelobenko 1987] D. P. Zhelobenko, "An analogue of the Gel'fand–Tsetlin basis for symplectic Lie algebras", *Uspekhi Mat. Nauk* **42**:6 (= 258) (1987), 193–194. In Russian; translation in *Russ. Math. Surveys* **42**:6 (1987), 247–248.

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