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Some Algebraic Properties of the Schechtman–Varchenko Bilinear Forms

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Abstract. We examine a bilinear form associated with a real arrangement of hyperplanes introduced in [Schechtman and Varchenko 1991]. Our main objective is to show that the linear algebraic properties of this bilinear form are related to the combinatorics and topology of the hyperplane arrangement. We will survey results and state a number of open problems which relate the determinant, cokernel structure and Smith normal form of the bilinear form to combinatorial and topological invariants of the arrangement including the characteristic polynomial, combinatorial structure of the intersection lattice and homology of the Milnor fibre.

1. The Varchenko B Matrices

Let $\mathcal{A} = \{H_1, \dots, H_l\}$ be an arrangement of hyperplanes in $\mathbb{R}^n$ and let $r(\mathcal{A}) = \{R_1, \dots, R_m\}$ denote the set of regions in the complement of the union of $\mathcal{A}$. Let $L(\mathcal{A})$ denote the collection of intersections of hyperplanes in $\mathcal{A}$. Included in $L(\mathcal{A})$ is $\mathbb{R}^n$, which we think of as the intersection of the empty set of hyperplanes. We order the elements of $L(\mathcal{A})$ by reverse inclusion thus making it into a poset. It is well known that this poset is a meet semilattice and is a geometric lattice if the arrangement is central. We will abbreviate $L(\mathcal{A})$ to L when the arrangement is clear.

For regions $S, T \in r(\mathcal{A})$, define $\mathcal{H}(S,T)$ to be the set of hyperplanes in $\mathcal{A}$ which separate S from T. Varchenko [1993] defines a matrix $B = B(\mathcal{A})$ with rows and columns indexed by the regions in $r(\mathcal{A})$ by saying that the S, T entry in B is $\prod_{H \in \mathcal{H}(S,T)} a_H$, where a_H is an indeterminate assigned to the hyperplane H. We will call $B = B(\mathcal{A})$ the *Varchenko matrix* of the arrangement $\mathcal{A}$.

Example 1.1. As a starting example, let $F = \{H_0, H_1, H_2\}$ be the arrangement in $\mathbb{R}^2$ where H_j is the line $y = (-1)^j x$ for $j = 0, 1$ and where H_2 is the line $y = 1$. Note that $r(F)$ consists of 7 regions. Let these regions be numbered $R_1, \dots, R_7$,

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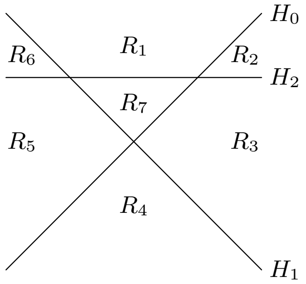
as follows:



Let a_j be the weight assigned to the hyperplane H_j. Then the matrix B is given by

$$B = \begin{pmatrix} 1 & a_2 & a_0a_2 & a_0a_1a_2 & a_0a_1 & a_1 & a_0 \\ a_2 & 1 & a_0 & a_0a_1 & a_0a_1a_2 & a_1a_2 & a_0a_2 \\ a_0a_2 & a_0 & 1 & a_1 & a_1a_2 & a_0a_1a_2 & a_2 \\ a_0a_1a_2 & a_0a_1 & a_1 & 1 & a_2 & a_0a_2 & a_1a_2 \\ a_0a_1 & a_0a_1a_2 & a_1a_2 & a_2 & 1 & a_0 & a_1 \\ a_1 & a_1a_2 & a_0a_1a_2 & a_0a_2 & a_0 & 1 & a_0a_1 \\ a_0 & a_0a_2 & a_2 & a_1a_2 & a_1 & a_0a_1 & 1 \end{pmatrix}.$$

EXAMPLE 1.2. An important example that we will return to several times is the arrangement $\mathcal{A}$ consisting of the $\binom{n}{2}$ hyperplanes $H_{i,j}$ in $\mathbb{R}^n$ given by

$$H_{i,j} = \{(x_1, \ldots, x_n) : \ x_i = x_j\}.$$

Note that $\mathcal{A}$ consists of the reflecting hyperplanes for the root system $\boldsymbol{A}_{n-1}$ and so we denote this arrangement by $\boldsymbol{A}_{n-1}$. Two points $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are in the same region of the complement if and only if the relative orders of their coordinates are the same. So, the permutations in S_n index the regions of the complement via the correspondence

$$\sigma \leftrightarrow \{(x_1, \ldots, x_n) : x_{\sigma 1} < x_{\sigma 2} < \cdots < x_{\sigma n}\}.$$

Let $a_{i,j} = a_{j,i}$ denote the weight assigned to the hyperplane $H_{i,j}$. For σ, τ in S_n, the σ, τ entry in B is the product of all $a_{i,j}$ such that i appears to the left of j in the one-line form of σ but to the right of j in the one-line form of τ. Another way of saying this is that $B_{\sigma,\tau}$ is the product of all $a_{i,j}$ such that $\{\sigma^{-1}(i), \sigma^{-1}(j)\}$ is an inversion of $\sigma\tau^{-1}$. In particular, if all parameters a_H are set equal to q, then the σ, τ entry of B is $q^{i(\sigma\tau^{-1})}$, where $i(\pi)$ denotes the number of inversions of π.

EXAMPLE 1.3. For each i, let O_i denote the hyperplane in $\mathbb{R}^n$ which consists of all vectors with a 0 in the i-th coordinate, and consider the arrangement $\{O_1, O_2, \ldots, O_n\}$. In this case, $r(\mathcal{A})$ has size 2^n and the individual regions can be indexed by sequences $S = (s_1, \ldots, s_n)$, where each s_i is either $+1$ or -1. The sequence S corresponds to the region R_S which contains all vectors $(x_1, \ldots, x_n)$ where $x_i < 0$ if $s_i = -1$ and $x_i > 0$ if $s_i = 1$. Given sequences S, T, the set of hyperplanes separating R_S and R_T is equal to the set of H_i such that the i-th coordinates of S and T differ. With $n = 3$ the matrix B appears below (relative to the ordering on regions which corresponds to the lexicographic order on the indexing sequences):

$$B = \begin{pmatrix} 1 & a_3 & a_2 & a_2a_3 & a_1 & a_1a_3 & a_1a_2 & a_1a_2a_3 \\ a_3 & 1 & a_2a_3 & a_2 & a_1a_3 & a_1 & a_1a_2a_3 & a_1a_3 \\ a_2 & a_2a_3 & 1 & a_3 & a_1a_2 & a_1a_2a_3 & a_1 & a_1a_3 \\ a_2a_3 & a_2 & a_3 & 1 & a_1a_2a_3 & a_1a_2 & a_1a_3 & a_1 \\ a_1 & a_1a_3 & a_1a_2 & a_1a_2a_3 & 1 & a_3 & a_2 & a_2a_3 \\ a_1a_3 & a_1 & a_1a_2a_3 & a_1a_2 & a_3 & 1 & a_2a_3 & a_2 \\ a_1a_2 & a_1a_2a_3 & a_1 & a_1a_3 & a_2 & a_2a_3 & 1 & a_3 \\ a_1a_2a_3 & a_1a_2 & a_1a_3 & a_1 & a_2a_3 & a_2 & a_3 & 1 \end{pmatrix}.$$

One of the first appearances of the B matrices was in the work of Schechtman and Varchenko [1991] on Drinfeld–Jimbo quantized Kac–Moody Lie algebras. To briefly describe this work, let H be a finite dimensional complex vector space equipped with a nondegenerate symmetric bilinear form $(\ , \ )$. We carry the form over to the dual space H^* in the usual way. Let $\alpha_1, \ldots, \alpha_r$ be linearly independent elements in H^* and let q be a nonzero complex number. Define U_qG to be the $\mathbb{C}$-algebra generated by the elements $1, e_i, f_i$ for $1 \le i \le r$ and H, subject to the relations

$$\begin{aligned} [h, e_i] &= \langle \alpha_i, h\rangle e_i, \\ [h, f_i] &= -\langle \alpha_i, h\rangle f_i, \\ [e_i, f_j] &= (q^{h_i/2} - q^{-h_i/2})\delta_{i,j}, \\ [H, H] &= 0. \end{aligned}$$

Here $\delta_{i,j}$ is the Kronecker delta and $q^{uh} = \exp(u(\ln q)h)$ so that we allow convergent power series in $h \in H$ as part of U_qG. It is possible to define, in addition, a comultiplication Δ which maps U_qG to $U_qG \otimes U_qG$ and an antipode ε on U_qG so that $(U_qG, \Delta, \varepsilon)$ is a Hopf algebra. We won't need the Hopf algebra structure here so we refer the reader to [Varchenko 1995, Chapter 4] for their definition.

Let U_qN_- be the subalgebra generated by $f_1, \ldots f_r$ and let L be the multilinear part of U_qN_-. In other words, L is spanned by words in $f_1, \ldots, f_r$ in which each f_i occurs exactly once. So L has dimension $r!$ with basis $f_\sigma := f_{\sigma 1} \ldots f_{\sigma r}$ for $\sigma \in S_r$.

Varchenko and Schechtman show that there is a natural contragredient form S on weight spaces of U_qN_-. Since L is one such weight space, one can ask for an explicit description of S relative to the basis $\{f_\sigma\}$ given above. An important result in [Schechtman and Varchenko 1991] is that

$$S(f_\sigma, f_\tau) = B(\sigma, \tau),$$

where B is the Varchenko matrix for the arrangement A_{r-1} (described in our Example 1.2 above) with parameters

$$a_{i,j} = (\alpha_i, \alpha_j).$$

Varchenko and Schechtman go on to show that the kernel of this contragredient form describes the Serre relations for the quantum Kac–Moody Lie algebra $U_q G$.

The main topic of this paper will be linear algebraic properties of the B matrices. We are going to report on a number of results and list a number of open questions that have to do with the nullspace of B and the Smith normal form of B. In addition, we will describe a new application of the Varchenko matrices to the problem of computing the Betti numbers of the Milnor fibre of the arrangement $\mathcal{A}$.

2. The Nullspace of the B Matrices

In this section, we will consider the nullspace of the Varchenko matrices. The reader may recall that the nullspaces are important in the work of Varchenko and Schechtman as they encode the analogues of the Serre relations for quantum Kac–Moody Lie algebras in the case that the arrangement is $\boldsymbol{A}_{n-1}$. In this section we will survey some known results and state some open problems relating to the nullspace of the Varchenko matrix.

A starting point for the study of the nullspace of B is the determinant of B. Note that $\det(B)$ is a polynomial in the a_H hence will vanish for certain choices of the a_H. The following result, due to Varchenko, gives an elegant factorization of $\det(B)$. As an immediate consequence of this theorem, one obtains a characterization of those values of the parameters for which B has a nontrivial nullspace.

Theorem 2.1 [Varchenko 1993, Theorem 1]. *The determinant of the bilinear form of the configuration $\mathcal{A}$ is*

$$\det(B) = \prod_{X \in L(\mathcal{A})^*} (1 - a_X^2)^{l(X)},$$

where $L(\mathcal{A})^*$ is the set of nonempty intersections in $L(\mathcal{A})$, where a_X is the product of the a_H over hyperplanes H containing X and where $l(X)$ is a nonnegative integer described explicitly in [Varchenko 1993].

In fact, Varchenko gives two methods to compute the exponents $l(X)$. The first, which precedes the statement of his theorem is more geometric, the second which comes later in the paper is more combinatorial. For completeness, we will briefly describe the second method. To compute $l(X)$, first choose a hyperplane $H \in \mathcal{A}$ which contains X. Then $l(X)$ is half the number of regions P which have the property that X is the minimal intersection containing $\bar{P} \cap H$.

EXAMPLE 2.2. Consider the B matrix for the arrangement given in Example 1.1 above. The poset of intersections consists of seven elements. Here are these seven intersections X_i, along with the exponents l_{X_i} computed according to the method of Varchenko described above.

$$\begin{aligned}
X_0 &= \mathbb{R}^2, & l_{X_0} &= 0,\\
X_1 &= H_0, & l_{X_1} &= 3,\\
X_2 &= H_1, & l_{X_2} &= 3,\\
X_3 &= H_2, & l_{X_3} &= 3,\\
X_4 &= H_1 \cap H_2, & l_{X_4} &= 0,\\
X_5 &= H_1 \cap H_0, & l_{X_5} &= 0,\\
X_6 &= H_2 \cap H_0, & l_{X_6} &= 0.
\end{aligned}$$

According to Varchenko's theorem, the determinant of B factors as

$$\det(B) = (1 - a_{X_1}^2)^3(1 - a_{X_2}^2)^3(1 - a_{X_3}^2)^3 = (1 - a_0^2)^3(1 - a_1^2)^3(1 - a_2^2)^3.$$

It is straightforward to check, using Maple or Mathematica for example, that this is a correct formula for the determinant.

Now consider this B matrix specialized so that all parameters are equal to q:

$$B = \begin{pmatrix}
1 & q & q^2 & q^3 & q^2 & q & q\\
q & 1 & q & q^2 & q^3 & q^2 & q^2\\
q^2 & q & 1 & q & q^2 & q^3 & q\\
q^3 & q^2 & q & 1 & q & q^2 & q^2\\
q^2 & q^3 & q^2 & q & 1 & q & q\\
q & q^2 & q^3 & q^2 & q & 1 & q^2\\
q & q^2 & q & q^2 & q & q^2 & 1
\end{pmatrix}.$$

The determinant of B is

$$\det(B) = (1 - q^2)^9.$$

Let G be the group (isomorphic to S_3) which permutes the three hyperplanes. Any permutation of the hyperplanes induces a permutation of the seven regions and so we get a representation of S_3 on the regions which commutes with the action of B. It is straightforward to check that the vector space spanned by regions, considered as a G-module, is isomorphic to one copy of the regular representation together with one copy of the trivial representation.

There are two values of q for which the determinant vanishes. For each value we can compute the representation of G on the nullspace of B. If $q = 1$, then B is the seven by seven matrix of all ones and the nullspace is easily seen to carry the regular representation. If $q = -1$ the situation is less clear. However, one can verify that the nullspace has dimension six and consists of two copies of the defining representation.

When you study the nullspace of B for values of the hyperplane weights where $\det(B) = 0$, the first question that arises is how to determine the dimension of the nullspace. Deformation theory arguments show that dimension of the nullspace is no larger than the sum of the exponents $l(X)$ taken over intersections X with $a_X^2 = 1$. To get more information in general is difficult. However, a case of special interest in which more general statements can be made is the case where all hyperplane weights are equal to the same number q. In this case, one can define the Smith normal form of B over $\mathbb{C}[q]$ which contains the information necessary to determine the dimension of the nullspace. The Smith normal form of B over $\mathbb{C}[q]$ is the topic of the next section in this paper. As we will see in the next section, it is possible for the dimension of the nullspace of $B(\zeta)$, for ζ a complex number, to be strictly smaller than the deformation theory bound stated above which in this case is equal to the multiplicity of $q - z$ as a divisor of $\det(B(q))$.

The question of dimension can be slightly broadened to ask for the "structure" of the nullspace (or more generally the cokernel) meaning a natural choice of basis for the nullspace (cokernel) as well.

PROBLEM 1. Let $\mathcal{A}$ be an arrangement and let $\{a_H\}$ be a choice of parameters for which the determinant of B vanishes. Find a natural basis for the nullspace (or for the cokernel) of B in terms of some kind of combinatorial and geometric information about the arrangement $\mathcal{A}$ and the parameters a_H.

One could ask what sort of information might go into a solution to Problem 1. Theorem 2.1 suggests that the necessary information may include facts about the combinatorial structure of $L(\mathcal{A})$ and in particular at intersections X in $L(\mathcal{A})$ for which $a_X^2 = 1$.

The previous example suggests another interesting question that one can ask about the nullspace of the Varchenko matrices. Suppose that a finite group G acts as a group of affine linear transformations and preserves the arrangement $\mathcal{A}$. Suppose also that the weight assigned to hyperplanes is G-invariant. Then the group G acts on the regions $r(A)$ and this action commutes with B. So one can ask for the G-module structure of the nullspace of B.

PROBLEM 2. Let $\mathcal{A}$ be an arrangement and let G be a finite group of affine-linear transformations which preserve the set $\mathcal{A}$. In addition assume that the weighting on hyperplanes is invariant under the action of G. What can you say about the G-module structure of the nullspace of B?

What sort of solution might you hope to find for Problem 2? As stated above in Problem 1, you would like to determine the nullspace of B in terms of some kind of combinatorial and geometric information related to the lattice $L(\mathcal{A})$ and the choice of parameters. You would hope for an answer to Problem 2 which extends this idea to include not just combinatorial information about $L(\mathcal{A})$ but also information about how G acts on $L(\mathcal{A})$.

One further problem naturally arises in the situation where there is a group G acting on the arrangement and where all hyperplane weights are equal to a single value q. Let V denote the vector space spanned by the set of regions $r(A)$. Recall that G acts on V and that this action commutes with the matrix B.

For each irreducible representation Ψ of G, let V_Ψ denote the Ψ-isotypic component of V. Because B commutes with the action of G, B preserves the subspace V_Ψ. Varchenko's theorem gives an elegant formula for the determinant of B as a linear map of V. A natural question is whether a correspondingly elegant formula holds for B as a linear transformation of V_Ψ.

PROBLEM 3. For Ψ an irreducible representation of G, let B_Ψ be the restriction of B to V_Ψ. Find a formula for $\det(B_\Psi)$.

A couple of comments about Problem 3. First, what kind of formula could you hope to achieve in answer to this question? Following the form of Varchenko's result, we would expect that the formula for $\det(B_\Psi)$ would depend on the intersection lattice $L(\mathcal{A})$ and some information about the action of G on $L(\mathcal{A})$. This information might include, for example, the action of G on the homology groups of the intervals of $L(\mathcal{A})$. In the best case, the formula would be a product over elements, or perhaps over orbits of elements, in the intersection lattice $L(\mathcal{A})$.

Second, it should be noted that there is a variant of Problem 3 which might have a more natural solution. Let Irr denote the set of irreducible representations of G. Let ρ be a virtual representation of G written as $\rho = \sum_{\Psi \in \mathrm{Irr}} c_\Psi \Psi$. Define

$$\det_\rho(B) = \prod_{\Psi \in \mathrm{Irr}} (\det(B_\Psi))^{c_\Psi}.$$

PROBLEM 3′. Compute $\det(B_\rho)$, for ρ in any set of representations that span the representation space.

Varchenko [1993] solves Problem 3 for the group of order 2 induced by negation in the central hyperplane arrangement case.

We conclude this section with a summary of what is known about these problems in a particularly interesting special case. Let $\mathcal{A}$ be the arrangement $\mathbf{A}_{n-1}$ given in Example 1.2, with all hyperplane weights set equal to the value q. As shown in that example, the regions in $r(\mathcal{A})$ are indexed by permutations in S_n. Moreover, for permutations σ, τ, the σ, τ entry of B is $q^{i(\sigma\tau^{-1})}$. It follows that B is the matrix for left multiplication by

$$\Gamma = \sum_{\beta \in S_n} q^{i(\beta)} \beta.$$

The group $G = S_n$ acts on the arrangement $\boldsymbol{A}_{n-1}$ and hence on the set of regions in $r(A)$. When the regions are indexed by permutations, this action of S_n corresponds to the right-regular representation.

As noted earlier, Problems 1 and 2 are particularly interesting in this case because the nullspace of B has an interpretation in terms of quantum Kac–Moody Lie algebras via results of Schechtman and Varchenko. Also in this case, there is a natural group of symmetries of the arrangement, namely the group S_n. So we will assume that the weighting on hyperplanes is invariant under S_n which in this case is equivalent to their all having the same value q.

Hanlon and Stanley [1998] have studied the S_n-module structure of the nullspace of B for this arrangement with weight q on all hyperplanes. We end this section by cataloguing a number of results and open problems that appear in their work.

The first step is to understand what Theorem 2.1 says in this case about the determinant of B. The intersection lattice of $\mathcal{A}$ is the partition lattice Π_n. It turns out that the exponent $l(X)$ is 0 unless the partition X has exactly one nontrivial block. If X has one nontrivial block of size α, then $l(X) = (\alpha-2)!\,(n-\alpha+1)!$. So in this case, Theorem 2.1 specializes to:

Corollary 2.3. *Let $\mathcal{A}$ be the arrangement $\boldsymbol{A}_{n-1}$ and let all hyperplanes have weight q. Then*

$$\det(B) = \prod_{\alpha=2}^{n} \left(1-q^{\alpha(\alpha-1)}\right)^{\binom{n}{\alpha}(\alpha-2)!(n-\alpha+1)!}. \tag{2–1}$$

This factorization of the determinant in this case was first proved by Zagier [1992], who came across the B matrix in this case in an entirely different context.

In view of Corollary 2.3, we need only consider values of q that are of the form $q = e^{2\pi i s/j(j-1)}$ for $j \in \{2, 3, \dots, n\}$. The first result that appears in [Hanlon and Stanley 1998] concerns the specialization of that form that is in some sense the most extreme. This result is stated in terms of the S_n module Lie_n which is the representation of S_n on the multilinear part of the free Lie algebra. There has been a great deal of study of this representation in part because it plays a role in many diverse mathematical situations.

Theorem 2.4 [Hanlon and Stanley 1998, Theorem 3.3]. *Let $q = e^{2\pi i/n(n-1)}$. Then*

$$\ker\left(\Gamma_n(q)\right) = \mathrm{ind}_{S_{n-1}}^{S_n}(\mathrm{Lie}_{n-1})/\mathrm{Lie}_n.$$

An interesting feature of Theorem 2.4 is the appearance of the module

$$W_n = \mathrm{ind}_{S_{n-1}}^{S_n}(\mathrm{Lie}_{n-1})/\mathrm{Lie}_n.$$

It is an old result that Lie_n is contained in $\mathrm{ind}_{S_{n-1}}^{S_n}(\mathrm{Lie}_{n-1})$. So this is a genuine (rather than virtual) module which is often called the *Whitehouse module*. This name refers to Sarah Whitehouse who came across the representation in quite a different context. The representation appears earlier in the work of Kontsevich [1993]. Whitehouse, together with Alan Robinson, investigated the homology of the space H_n of homeomorphically irreducible trees which have n labelled leaves. Here "labelled" is in the graph theory sense of the word, so the numbers $\{1, 2, \ldots, n\}$ are attached to the leaves. They say that one tree τ is a "face" of another tree ρ if τ is obtained from ρ by contracting an "internal edge", i.e., an edge which is not incident with a leaf. This notion of face gives us a complex whose boundary map commutes with the action of S_n induced by permutation of leaf labels. So one can ask for the S_n-module structure of the homology of this complex. The first result, due to Robinson and Whitehouse [1996], is that this homology vanishes except in degree $n-2$, and that in degree $n-2$ the dimension is $(n-2)!$. Whitehouse went on to show that the degree $n-2$ piece of the homology carries the S_n-module structure which we've called the Whitehouse module. Interestingly, this module has also come up in the work of Babson et al. [1999] on graphs that are not 2-connected. Recall that a graph G is "2-connected" if G is connected and remains so under the removal of any one vertex. Let N_n denote the collection of graphs with vertex set $\{1, 2, \ldots, n\}$ which are not 2-connected. If we identify a graph by its edge set, then the edge sets of the graphs in N_n form a simplicial complex which is clearly invariant under that action of S_n. An interesting result in [Babson et al. 1999] is that the simplicial homology of N_n is zero except in degree $n-2$ and in degree $n-2$ carries the S_n-module structure of the Whitehouse representation.

Hanlon and Stanley went on to state a conjecture which gives a description of the S_n-module structure of the kernel of B in a more general case than that covered in their Theorem 3.3. They provided a proof of this conjecture up to knowing a certain technical fact about the Smith normal form of the Varchenko matrices. This fact was subsequently proved by Denham, thereby giving a proof of the following result.

Theorem 2.5 [Denham 1999; Hanlon and Stanley 1998]. *Suppose that q is a root of exactly one of the factors on the right-hand side of* (2–1). *More precisely, suppose that $q = e^{2\pi i s/j(j-1)}$ for some $j \in \{2, 3, \ldots, n\}$ and some nonnegative integer s and that $q^{k(k-1)} \neq 1$ for $k \neq j$, $2 \leq k \leq n$. Then the S_n-module structure of* $\ker(\Gamma_n(q))$ *is*

$$\ker(B(q)) = \operatorname{ind}_{C_{j-1}}^{S_n}(q^j) / \operatorname{ind}_{C_j}^{S_n}(q^{j-1}),$$

where C_{j-1} is the subgroup of S_n generated by the $(j-1)$-cycle $z_{j-1} = (j-1, j-2, \ldots, 2, 1)$, and where q^j denotes the linear character of C_{j-1} whose value on z_{j-1} is q^j.

It is not immediately clear why the statement of Theorem 2.5 in the special case where $j = n$ and $s = 1$ agrees with the statement of Theorem 2.4. This follows from the well-known fact that $\mathrm{Lie}_n = \operatorname{ind}_{C_n}^{S_n}(q^{n-1})$ for q a primitive $n(n-1)^{st}$ root of unity. An interesting open problem is to extend these results to the case where $1 - q^{j(j-1)}$ vanishes for more than one j. The paper [Hanlon and Stanley 1998] contains tables which give the S_n-module structure of the kernel of B in these cases for small values of n.

We next turn to Problem $3'$ and see what is known here in the special case of $\mathcal{A} = \boldsymbol{A}_{n-1}$, with all hyperplane weights equal to q. Hanlon and Stanley [1998] have proved a result that gives a solution to Problem $3'$, but in order to state it we need some notation.

For each partition η, let P^η be the virtual character which has value 0 on conjugacy classes not indexed by η and which has value $\frac{n!}{c_\eta}$ on the conjugacy class indexed by η where c_η is the order of the centralizer in S_n of a permutation with cycle type η.

It is well-known that we can write P^η explicitly as

$$P^\eta = \sum_\lambda \chi^\lambda(\eta)\chi^\lambda,$$

where χ^λ is the irreducible character of S_n indexed by λ (see [James and Kerber 1981] or [Sagan 1991] for more on the irreducible representations of S_n). It is easy to see that the virtual characters P^η span the representation space of S_n.

The following result, which was originally conjectured by Stanley, give an elegant solution to Problem $3'$ in this case:

Theorem 2.6 [Hanlon and Stanley 1998], Theorem 3.7. *For each $\eta \vdash n$, let $D_\eta(q)$ denote $\det_{P^\eta}(B(q))$. Then*

(i) $D_\eta(q) = 1$ *unless η is of the form $l^d 1^{n-ld}$ for some l, d.*
(ii) $D_{(l^d 1^s)}(q) = \big(D_{(l^d 1)}(q)\big)^{s!}$ *for all l, d and all $s \geq 1$.*

Note that (a) and (b) reduce the determination of $D_\eta(q)$ to the cases where η is of the form l^d or $l^d 1$.

(iii) $D_{(l^d)}(q) = \prod_{m|l}\big(1 - q^{dm(n-1)}\big)^{\mu(m) l^d (d-1)!/m}$, where, in the exponent on the right-hand side, μ denotes the number-theoretic Möbius function.
(iv) $D_{(l^d,1)}(q) = D_{(l^d)}(q) D_{(l^d)}\left(q^{n/(n-2)}\right)^{-1}$.

3. The Smith Normal Form of $B(q)$

Let notation be as in the previous section, so that $\mathcal{A}$ is an arrangement of hyperplanes and B is the Varchenko matrix of the arrangement $\mathcal{A}$. In this section we will assume that all parameters a_H are equal to a parameter q. So B is the matrix with rows and columns indexed by regions whose R, S entry is $q^{n(R,S)}$ where $n(R, S)$ is the number of hyperplanes which separate R and S.

We can specialize Varchenko's Theorem (Theorem 2.1) by setting all $a_H = q$ to get a formula for the determinant of B:

$$\det(B) = \prod_{X \in L(\mathcal{A})^*} (1 - q^{2h(X)})^{l(X)},$$

where $h(X)$ is the number of hyperplanes in $\mathcal{A}$ which contain X. A consequence of this formula is that $\det(B)$ vanishes only at q being certain roots of unity.

Because the entries of B come from $\mathbb{C}[q]$ (which is a PID), we can define the Smith normal form of B over the ring $\mathbb{C}[q]$. Recall that the Smith normal form of a matrix M over a Euclidean domain R is a normal form for left and right multiplication by unimodular matrices. In other words, the Smith normal form of M, denoted $\mathrm{SNF}(M)$, is a particular choice from the set of matrices

$$\{UMV : U \text{ and } V \text{ are unimodular matrices over } R\}.$$

The matrix $\mathrm{SNF}(M)$ is a diagonal matrix with each entry along the diagonal dividing the next one. The diagonal entries are chosen up to multiplication by units in R. In our case, the units in $\mathbb{C}[q]$ are the nonzero complex numbers so we can assume that each diagonal entry in $\mathrm{SNF}(B)$ is a monic polynomial. Note also that if M is square, the unimodular conditions on U and V imply that $\det(\mathrm{SNF}(M)) = \det(M)$.

Let Θ denote the set of roots of $\det(B)$. By the divisibility condition on successive entries of $\mathrm{SNF}(B)$, the (i,i)-entry of $\mathrm{SNF}(B)$ is of the form

$$\prod_{z\in\Theta}(q-z)^{p_z^{(i)}},$$

where the exponents $p_z^{(i)}$ satisfy

$$p_z^{(1)} \le p_z^{(2)} \le \cdots \le p_z^{(N)}$$

for each $z \in \Theta$, where N denotes the number of regions in the complement of $\mathcal{A}$. Therefore, $\mathrm{SNF}(B)$ is completely determined by the sequences $\{p_z^{(i)}\}_{i=1}^N$ for each $z \in \Theta$. It is sometimes more convenient to work with an equivalent sequence of numbers $\sigma_z^{(i)}$, where $\sigma_z^{(i)}$ is equal to the number of j such that $p_z^{(j)}$ is equal to i. In terms of the Smith normal form, $\sigma_z^{(i)}$ is the number of diagonal entries in $\mathrm{SNF}(B)$ that are exactly divisible by $(q-z)^j$.

Example 3.1. Let $\mathcal{A}$ be the arrangement given in Example 2.2. The B matrix with all parameters set equal to q is given in that example, as is $\mathrm{SNF}(B)$. The sequences that determine $\mathrm{SNF}(B)$ as above are

$$\{p_1^{(i)}\} = \{p_{-1}^{(i)}\} = 0, 1, 1, 1, 2, 2, 2.$$

The σ sequences for this arrangement are

$$\{\sigma_1^{(i)}\} = \{\sigma_{-1}^{(i)}\} = 1, 3, 2.$$

This example demonstrates one of two facts about the $p_z^{(i)}$ that are straightforward to prove:

Proposition 3.2. *Let the sequences $p_z^{(i)}$ be those associated to the Smith normal form of the arrangement $\mathcal{A}$ in $\mathbb{R}^n$. Then*

(i) *If z and w are both primitive j-th roots of unity, then $p_z^{(i)} = p_w^{(i)}$ for every i.*
(ii) *For every root of unity z and every i, $p_z^{(i)} = p_{-z}^{(i)}$.*

The computation of the Smith normal form of the Varchenko matrices arises as our next problem of interest in this paper.

PROBLEM 4. (i) Determine the Smith normal form of B in terms of some information about the arrangement $\mathcal{A}$.
(ii) Is $\mathrm{SNF}(B)$ determined just by the intersection lattice of $\mathcal{A}$?

Aside from the intrinsic interest of computing the Smith normal form of the matrices B, this problem is directly relevant to the determination of the nullspace of B in the case that all parameters a_H are set equal to q. As noted in the last chapter, facts about the Smith normal form of B were needed by Denham [1999] to prove a conjecture of Hanlon and Stanley. In the next section, we will describe another application of the Smith normal form of B (due to Denham), this time to computing the homology of the Milnor fibre of the arrangement $\mathcal{A}$.

Note that the Smith normal form of B is determined by the numbers $p_z^{(i)}$ for every $z \in \Theta$ and every i. So an alternative formulation of Problem 4 is the following.

PROBLEM 4′. Determine the numbers $p_z^{(i)}$ for $i \geq 0$ and for each $z \in \Theta$. Equivalently, determine the numbers $\sigma_z^{(i)}$ for $i \geq 0$ and for each $z \in \Theta$.

One general result is known that gives an elegant partial solution to Problem 4′.

Theorem 3.3 [Denham and Hanlon 1997]. *For any arrangement $\mathcal{A}$ and any nonnegative integer i, $\sigma_1^{(i)} = \sigma_{-1}^{(i)}$ is the i-th Betti number of the arrangement. Equivalently,*

$$\sum_i \sigma_1^{(i)} \lambda^i = \sum_j \lambda^{p_1^{(j)}} = (-\lambda)^r \chi\Big(\frac{-1}{\lambda}\Big),$$

where $\chi(\lambda)$ is the characteristic polynomial of the lattice $L(\mathcal{A})$ and r is the rank of $L(\mathcal{A})$.

Example 3.4. To illustrate an instance of Problem 4, we consider the arrangements $\boldsymbol{A}_{n-1}$. For each n in the range $2 \leq n \leq 5$ there is a chart below which gives the numbers $\sigma_z^{(i)}$. To understand these charts, assume that n is fixed. By Corollary 3.1 we know that $z \in \Theta$ if and only if z is a primitive d-th root of unity for some d that divides $j(j-1)$ and some $j \leq n$. These numbers d index the rows of the charts below. In view of Proposition 3.2 (b) we do not include d of the form $2d'$ where d' is odd. So the number $\sigma_z^{(j)}$ that appears in the d, j entry of the chart below is the number of entries in $\mathrm{SNF}(B)$ that are exactly divisible by $(q - e^{2\pi i/d})^j$.

With these notational conventions, the Smith normal forms of the arrangements $\boldsymbol{A}_{n-1}$ for small values of n are as follows:

$$n=2\colon\ \begin{array}{cc} d & j=0 \quad 1 \\ 1 & \begin{pmatrix} 1 & 1 \end{pmatrix} \end{array} \qquad\qquad n=3\colon\ \begin{array}{cc} d & j=0 \quad 1 \quad 2 \\ \begin{array}{c} 1 \\ 3 \end{array} & \begin{pmatrix} 1 & 3 & 2 \\ 5 & 1 & 0 \end{pmatrix} \end{array}$$

$$n=4:\quad \begin{array}{c} d \\ 1 \\ 3 \\ 4 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccc} j=0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1 & 6 & 11 & 6 \\ 17 & 4 & 3 & 0 \\ 22 & 2 & 0 & 0 \\ 22 & 2 & 0 & 0 \end{pmatrix} \end{array} \qquad n=5:\quad \begin{array}{c} d \\ 1 \\ 3 \\ 4 \\ 5 \\ 12 \\ 20 \end{array} \begin{array}{c} \begin{array}{ccccc} j=0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 10 & 35 & 50 & 24 \\ 70 & 20 & 30 & 0 & 0 \\ 102 & 10 & 8 & 0 & 0 \\ 114 & 6 & 0 & 0 & 0 \\ 100 & 20 & 0 & 0 & 0 \\ 114 & 6 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Theorem 3.3 gives a description of the numbers that appear in the rows indexed by $d = 1$ in the tables above. More specifically, Theorem 3.3 equates the polynomial $Q(\lambda)$ obtained by using the numbers in the first row as coefficients to a polynomial obtained from the characteristic polynomial of the lattice of intersections of the arrangement $\boldsymbol{A}_{n-1}$. The lattice of intersections of the arrangement $\boldsymbol{A}_{n-1}$ is the partition lattice Π_n which is well-known to have characteristic polynomial $\chi(\lambda) = \prod_{i=1,n-1}(\lambda - i)$. So Theorem 3.3 states that the polynomial $Q(\lambda)$ is equal to

$$Q(\lambda) = (-\lambda)^{n-1} \prod_{i=1}^{n-1} \left(\frac{-1}{\lambda} - i \right) = \prod_{i=1}^{n-1} (1 + i\lambda).$$

A specific case of this factorization is $n = 5$ where Theorem 3.3 predicts that

$$1 + 10\lambda + 35\lambda^2 + 50\lambda^3 + 24\lambda^4 = (1+\lambda)(1+2\lambda)(1+3\lambda)(1+4\lambda).$$

It is easy to check that two sides of the above equality are in fact equal. Problem 4 asks for an extension of Theorem 3.3 to $d > 1$.

In order to refine Problem 4, we will take a different point of view on the Smith normal form. Let V be the module over $\boldsymbol{A} = \mathbb{C}[q]$ spanned by the regions in the complement of $\mathcal{A}$. Note that B determines a $\mathbb{C}$-linear transformation from V to V which commutes with the action of $\boldsymbol{A}$. Let $\zeta(q)$ be any polynomial in $\boldsymbol{A}$. Since B commutes with the action of $\boldsymbol{A}$ on V, multiplication by $\zeta(q)$ defines a map on C, the cokernel of B.

By definition, C is $V/\operatorname{im}(B)$. As a vector space over $\mathbb{C}$, C has dimension equal to the sum of the degrees of the entries in $\mathrm{SNF}(B)$. We will want to say a bit more about the structure of C. For each $z \in \Theta$, let $K_z^{(j)}$ denote the kernel in C of multiplication by $(q-z)^j$, and let $k_z^{(j)} = \dim_{\mathbb{C}}(K_z^{(j)})$. Then $k_z^{(j)}$ equals the number of i with $p_z^{(i)} \le j$. Note that

$$K_z^{(0)} \subseteq K_z^{(1)} \subseteq K_z^{(2)} \subseteq \cdots.$$

Let $L_z^{(j)} = K_z^{(j)}/K_z^{(j-1)}$. By the comments above, we see that $\dim_{\mathbb{C}}(L_z^{(j)})$ is equal to $\sigma_z^{(j)}$, the number of $p_z^{(i)}$ which are equal to j. It is not difficult to show that

$$C = \bigoplus_{z \in Z} \bigoplus_{j \ge 0} L_z^{(j)}.$$

Now suppose that G is a group of affine transformations defined over $\mathbb{C}$ that permutes the set of hyperplanes. Then G acts on the set of regions as well and this action commutes with the actions of both B and $\mathbb{C}[q]$. So G acts on the spaces $L_z^{(j)}$ for each $z \in \Theta$ and each j. Problem 1 asks one to determine the dimensions of the $L_z^{(j)}$, or equivalently to determine the character of the identity in G acting on each $L_z^{(j)}$. One can ask more generally about their G-module structures of the $L_z^{(j)}$.

PROBLEM 5. Determine the G-module structure of the spaces $L_z^{(j)}$ in terms of some information about the action of G on the arrangement $\mathcal{A}$.

A simpler, but still interesting, variant of Problem 5 is to determine the G-module structure of the entire cokernel of B.

EXAMPLE 3.5. To illustrate these concepts and problems we do an example. Let $\mathcal{A} = \{H_0, H_1, H_2\}$ be the two-dimensional arrangement where H_0 is the line $x = 0$, H_1 is the line $y = 1$ and H_2 is the line $y = -1$. Let G be the group of order four generated by h and v, where h is the reflection of $\mathbb{R}^2$ across the line $x = 0$ and v is reflection of $\mathbb{R}^2$ across the line $y = 0$.

Index the regions in the complement of the arrangement $R_1, \dots, R_6$ so that R_1, R_3, R_5 are those in the half-plane $x < 0$ arranged from top to bottom and R_2, R_4, R_6 are those in the half-plane $x > 0$ arranged from top to bottom. The B-matrix of this arrangement with respect to this ordering is

$$B = \begin{pmatrix} 1 & q & q & q^2 & q^2 & q^3 \\ q & 1 & q^2 & q & q^3 & q^2 \\ q & q^2 & 1 & q & q & q^2 \\ q^2 & q & q & 1 & q^2 & q \\ q^2 & q^3 & q & q^2 & 1 & q \\ q^3 & q^2 & q^2 & q & q & 1 \end{pmatrix}.$$

The Smith normal form of B is given by

$$\mathrm{SNF}(B) = \mathrm{diag}\big(1,\, 1-q^2,\, 1-q^2,\, 1-q^2,\, (1-q^2)^2,\, (1-q^2)^2\big),$$

so that the cokernel has dimension 14 (over $\mathbb{C}$.) A basis for the image of B is given by the six vectors

$$\begin{aligned}
\mu_1 &= (1,\, q,\, q,\, q^2,\, q^2,\, q^3),\\
\mu_2 &= (1-q^2)\cdot(0,\, 1,\, 1,\, 1+2q-q^2,\, 1+q,\, 1+2q+q^2-q^3),\\
\mu_3 &= (1-q^2)\cdot(0,\, -2,\, -1,\, -3q,\, -q,\, 1-4q^2),\\
\mu_4 &= (1-q^2)\cdot(0,\, 1,\, 0,\, q,\, 0,\, 1),\\
\mu_5 &= (1-q^2)^2\cdot(0,\, 0,\, 0,\, 1,\, 0,\, q-1),\\
\mu_6 &= (1-q^2)^2\cdot(0,\, 0,\, 0,\, 0,\, 0,\, 1).
\end{aligned}$$

So, the kernel of multiplication by $1 \pm q$ on $L^{(1)}_{\pm 1}$ has dimension 5 and has a basis given by

$$\begin{aligned}
\nu_2 &= (1 \mp q) \cdot (0, 1, 1, 1+2q-q^2, 1+q, 1+2q+q^2-q^3),\\
\nu_3 &= (1 \mp q) \cdot (0, 0, 1, 2+q-2q^2, 2+q, 3+4q-2q^2-2q^3),\\
\nu_4 &= (1 \mp q) \cdot (0, 1, 0, q, 0, 1),\\
\nu_5 &= (1-q^2)(1 \mp q) \cdot (0, 0, 0, 1, 0, q-1),\\
\nu_6 &= (1-q^2)(1 \mp q) \cdot (0, 0, 0, 0, 0, 1).
\end{aligned}$$

The group $G = \{e, h, v, hv\}$ acts on the ordered set of regions by

$$\begin{aligned}
e &= (R_1)(R_2)(R_3)(R_4)(R_5)(R_6),\\
h &= (R_1, R_2)(R_3, R_4)(R_5, R_6),\\
v &= (R_1, R_5)(R_2, R_6)(R_3)(R_4),\\
hv &= (R_1, R_6)(R_2, R_5)(R_3, R_4).
\end{aligned}$$

So, it is possible to explicitly calculate the action of any group element on the basis elements above. For example,

$$\begin{aligned}
v \cdot \nu_2 &= (1 \mp q) \cdot (1+q, 1+2q+q^2-q^3, 1, 1+2q-q^2, 0, 1)\\
&= (1 \mp q) \cdot (1+q) \cdot \mu_1 + (1+q-q^3) \cdot \nu_2 + (-2q-q^2+q^3) \cdot \nu_3\\
&\quad + (3q+q^2-q^3) \cdot \nu_4 + (-1-q-q^2) \cdot \nu_5 + (4-q)(1-q^2) \cdot \nu_6.
\end{aligned}$$

So the contribution to the trace of v that arises from the ν_2-diagonal term is $1+q-q^3 = 1$ since $(q-q^3)\nu_2 = 0$ in $K^{(1)}_z$.

A tedious calculation along the same lines as above shows that the characters of G acting on $L^{(j)}_z$ are

$$j=1\colon\ \begin{array}{c} z \\ +1 \\ -1 \end{array} \begin{array}{c} g=e \quad h \quad v \quad hv \\ \begin{pmatrix} 5 & -1 & 1 & -1 \\ 5 & 1 & 1 & 1 \end{pmatrix} \end{array} \qquad j=2\colon\ \begin{array}{c} z \\ +1 \\ -1 \end{array} \begin{array}{c} g=e \quad h \quad v \quad hv \\ \begin{pmatrix} 2 & -2 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{pmatrix} \end{array}$$

4. Local Systems and the Milnor Fibre

Earlier we cited a result of Schechtman and Varchenko which describes the Serre relations for a quantum Kac–Moody Lie algebra in terms of the Varchenko matrices B. In this section, we give a second application of the Varchenko matrix of an arrangement $\mathcal{A}$ — this time to an invariant of the singularity of a hyperplane arrangement at the origin.

Let $\mathcal{A}$ be an essential arrangement in $\mathbb{C}^n$ with defining polynomial $Q : \mathbb{C}^n \to \mathbb{C}$. Let $M = Q^{-1}(\mathbb{C}\backslash\{0\})$ denote its complement, and $N = Q^{-1}(0)$ the union of the hyperplanes. A local system on M is a representation of the fundamental group $\pi_1(M)$ in a complex vector space $\mathcal{W}$.

For local systems of rank one, various authors have studied the homology $H.(M, \mathcal{W})$, for the most part in connection with generalized hypergeometric functions: see [Kohno 1986; Gel'fand 1986], or the references provided in [Varchenko 1995]. When the local system is trivial, one recovers the ordinary homology $H.(M, \mathbb{C})$, which is isomorphic to the Whitney homology of the lattice $L(\mathcal{A})$.

When the defining polynomial Q is real, there is a strong deformation retract of M onto Salvetti's CW-complex $\mathfrak{X}$ [Salvetti 1987]. In this case, there are explicit chain complexes that calculate $H_\bullet(M, \mathcal{W})$ and $H_\bullet(\mathbb{C}^n, N, \mathcal{W})$. The B matrix appears as part of an important chain map between the two complexes.

Our interest in local systems over the arrangement's complement comes from a special case. It is well known that, if $\mathcal{A}$ is a central arrangement, the restriction of the defining polynomial to the complement, $Q : M \to \mathbb{C}^*$, gives M the structure of a fibre bundle over $\mathbb{C}^*$ [Milnor 1968]. The typical fibre $F = Q^{-1}(1)$ is a complex manifold of dimension $n-1$, known as the Milnor fibre of the polynomial Q (or of $\mathcal{A}$). F is homotopically equivalent to an infinite cyclic covering of M, so its singular homology is

$$H_\bullet(F, \mathbb{C}) = H_\bullet(M, \mathbb{C}\mathbb{Z}),$$

for a local system

$$\mathbb{C}\mathbb{Z} = \mathbb{C}[t, t^{-1}].$$

If the polynomial Q is real, then, we can exploit the complexes defined by (4–1) to study the homology of the Milnor fibre (Section 4C).

Milnor [1968] considered arbitrary polynomial maps $f : \mathbb{C}^n \to \mathbb{C}$ that vanish at 0. In particular, he showed that if the polynomial f has an isolated singularity at 0, then the Milnor fibre of f has the homotopy type of a bouquet of spheres. The Milnor fibre of defining polynomials of arrangements, then, continue to be of interest by forming a restricted family of examples in which the singularity is not isolated. Although the homotopy type of the Milnor fibre of a real arrangement is determined by the face lattice of an arrangement, an explicit description of the homology is only known in special cases; see [Cohen and Suciu 1998; Orlik and Randell 1993]. Moreover, it is not known if, in general, the intersection lattice of a complex hyperplane arrangement determines the Milnor fibre's homology.

In Section 4A, we introduce the Varchenko–Salvetti chain complexes. Section 4B indicates the role of the B matrices, and 4C specializes the construction to apply to the Milnor fibre.

4A. Salvetti's complex. In order to describe the general setup, recall that the fundamental group $\pi_1(M)$ is generated by loops $\{\alpha_H : H \in \mathcal{A}\}$ around each hyperplane [Randell 1982]. Let $\mathcal{W}$ be a (complex) local coefficient system defined by a representation $\rho : \pi_1(M) \to \mathrm{End}(\mathcal{W})$. For each $H \in \mathcal{A}$, let $b_H = \rho(\alpha_H)$. We require in what follows that the image of ρ be abelian. Subject to this constraint, one can choose any set of commuting endomorphisms $b_H \in \mathrm{End}(\mathcal{W})$, since then the representation ρ factors through $H_1(M,\mathbb{Z})$, and $H_1(M,\mathbb{Z})$ is freely generated by the cycles around each hyperplane.

$$\begin{tikzcd} \pi_1(M) \arrow[r, "\rho"] \arrow[d, "\mathrm{Ab}"'] & \mathrm{End}(\mathcal{W}) \\ H_1(M,\mathbb{Z}) \arrow[ur, dotted] & \end{tikzcd}$$

Salvetti's complex consists of cells indexed by pairs of faces of the real arrangement corresponding to $\mathcal{A}$. Let $\mathcal{L}$ denote the face lattice of the arrangement, ordered by reverse inclusion, so that $\mathcal{L}_k$ is the set of faces of codimension k. For a face P and a hyperplane H, let $P(H)=0$ if $P\subseteq H$, and $P(H)=\pm 1$ otherwise, depending on whether P lies on the positive or negative side of H. The face P is uniquely determined by these values. With this notation, we recall the definition of the vector product of faces with regions: for any $P\in\mathcal{L}$ and $R\in r(\mathcal{A})$, the region $PR\in r(\mathcal{A})$ is determined by

$$(PR)(H)=\begin{cases} R(H) & \text{if } P(H)=0,\\ P(H) & \text{otherwise,}\end{cases}$$

for each $H\in\mathcal{A}$.

EXAMPLE 4.1. We return to the arrangement of Example 1.1, defined by linear forms $f_0=x-y$, $f_1=x+y$, and $f_2=y-1$. Let $P=\{(x,y): x>1, y=1\}$. Then the triples $(F(H_0),F(H_1),F(H_2))$ for faces $F=P$, R_7, and PR_7 are

$$\begin{aligned} P &\longleftrightarrow (+1,+1,\ \ 0),\\ R_7 &\longleftrightarrow (-1,+1,-1),\\ PR_7 &\longleftrightarrow (+1,+1,-1),\end{aligned}$$

so $PR_7=R_1$.

The cells in dimension k of the Salvetti complexes $\mathfrak{X}$ and $\mathfrak{X}'$ are indexed by pairs consisting of a face in codimension k and a region containing the face. As vector spaces, let

$$C_k(\mathfrak{X})=C_k(\mathfrak{X}')=\mathbb{C}\left\{E(P,R): P\in\mathcal{L}_k,\ R\in\mathcal{L}_0,\ R\leq P\right\}\otimes_{\mathbb{C}}\mathcal{W}. \tag{4–1}$$

We use the symbol $\prec$ to denote the covering relation in the lattice $\mathcal{L}$. Coorient the faces of the real arrangement, and for faces $Q\prec P$, let $\varepsilon(P,Q)$ be $+1$ or -1 according to whether or not the coorientations P and Q agree. See [Varchenko 1995, (2.4.2)]. Given a pair $P\geq R$ and a face Q such that $Q\prec P$, define

$$B(R,Q;P)=\prod_{H\in\mathcal{A}:P(H)=0} b_H^{Q(H)R(H)/4}.$$

The boundary maps $\partial : C_k(\mathcal{X}) \to C_{k-1}(\mathcal{X})$ and $\partial' : C_k(\mathcal{X}') \to C_{k-1}(\mathcal{X}')$ are given by

$$\partial E(P,R) = \sum_{Q \prec P} \varepsilon(P,Q) B(R,Q;P) E(Q,QR) \tag{4–2}$$

and

$$\partial' E(P,R) = \sum_{R \leq Q \prec P} \varepsilon(P,Q) E(Q,R). \tag{4–3}$$

PROPOSITION 4.2 [Varchenko 1995]. *The chain complex with boundary map* (4–2) *computes the homology of the complement with local coefficient system* $\mathcal{W}$:

$$H_k(C_\bullet(\mathcal{X})) \cong H_k(M, \mathcal{W}).$$

The chain complex with boundary map (4–3) *computes the relative homology of the pair* $(\mathbb{C}^n, N)$ *with local coefficient system* $\mathcal{W}$:

$$H_k(C_\bullet(\mathcal{X}')) \cong H_k(\mathbb{C}^n, N, \mathcal{W}).$$

4B. Relation to the B matrix. For any face $P \in \mathcal{L}$, let $|P| \in L(\mathcal{A})$ denote the smallest subspace of $\mathbb{R}^n$ that contains P. For $0 \leq k \leq n$, define a map $S_k : C_k(\mathcal{X}) \to C_k(\mathcal{X})$ by

$$S_k(E(P,R)) = \sum_{S \leq P} B(R,S) E(P,S),$$

on the basis of $C_k(\mathcal{X})$ given in (4–1). Here,

$$B(R,S) = \prod_{H \in \mathcal{A} : P(H)=0} b_H^{R(H)S(H)/4},$$

where R and S are regions satisfying $R, S \leq P$. Note that the subarrangement $\mathcal{A}_{|P|}$ equals $\{H \in \mathcal{A} : P(H) = 0\}$, and its regions can be identified with those regions $R \leq P$. With this in mind, one finds that

$$S_k = \bigoplus_{P \in \mathcal{L}_k} \Bigl(\prod_{H \in \mathcal{A}_{|P|}} b_H^{-1/4} \Bigr) B(\mathcal{A}_{|P|}),$$

where the weights of the hyperplanes of $\mathcal{A}_{|P|}$ are taken to be $a_H = b_H^{1/2}$.

One can verify that $S_{k-1}\partial_k = \partial'_k S_k$; that is, $S_\bullet : C_\bullet(\mathcal{X}) \to C_\bullet(\mathcal{X}')$ is a chain map [Varchenko 1995].

4C. Application to the Milnor fibre. In this section, $\mathcal{A}$ is a real, central arrangement. The (complex) homology of the Milnor fibre of $\mathcal{A}$ is isomorphic to $H_\bullet(M, \mathbb{C}[t,t^{-1}])$, where $\pi_1(M)$ acts on $\mathbb{C}[t,t^{-1}]$ by $b_H = t$, for all $H \in \mathcal{A}$. Since the determinants of the B matrices are nonzero over $\mathbb{C}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$, the chain map $S_\bullet$ is an injection, and the short exact sequence

$$0 \to C_\bullet(\mathcal{X}) \xrightarrow{S} C_\bullet(\mathcal{X}') \to \operatorname{coker} S_\bullet \to 0 \tag{4–4}$$

gives rise to a long exact sequence in homology. Since the homology of $C_\bullet(\mathcal{X}')$ is zero for central arrangements [Varchenko 1995], one obtains an isomorphism

$$H_{k-1}(F, \mathbb{C}) \cong H_{k-1}(C_\bullet(\mathcal{X})) \cong H_k(\operatorname{coker} S_\bullet)$$

for $1 \leq k \leq n$. The first isomorphism is useful for calculation, since we have an explicit description for the chain complex on the right-hand side, so calculating homology reduces to linear algebra suitable, in small enough cases, for a computer algebra program. The second isomorphism reduces the problem further to a chain complex that is finite-dimensional over $\mathbb{C}$; however, as we see in Section 3, the cokernel of $S_\bullet$ is still poorly understood.

We can write

$$\operatorname{coker} S_k = \bigoplus_{P \in \mathcal{L}_k} \operatorname{coker} B_{|P|}$$

as vector spaces, although the boundary map does not preserve this decomposition. Since multiplication by the determinant of a matrix map annihilates its cokernel, and $H_{k-1}(F, \mathbb{C})$ is isomorphic to a subquotient of $\operatorname{coker} S_k$, the determinant of S_k annihilates $H_{k-1}(F, \mathbb{C})$. However, a more precise statement can be made.

Proposition 4.3. *For a real, central arrangement of l hyperplanes and k for which $1 \leq k \leq n$, $H_{k-1}(F, \mathbb{C})$ consists of a direct sum of modules of the form*

$$\mathbb{C}[t, t^{-1}]/(\Phi_d(t)),$$

where $d \mid \gcd(h(X), l)$ for some $X \in L_{\leq k}$ that satisfies $l(X) \neq 0$. (Recall that $h(X)$ is the number of hyperplanes containing X, while $l(X)$ is introduced in Theorem 2.1.)

Proof. A cyclic (monodromy) group $\mathbb{Z}/l\mathbb{Z}$ acts on $H_\bullet(F, \mathbb{C})$ via multiplication by t. A complete discussion can be found in [Dimca 1992]. That is, t^l acts as the identity, and $H_\bullet(F, \mathbb{C})$ contains no nilpotent elements. The proposition is proven by comparing these constraints on $H_{k-1}(F, \mathbb{C})$ with the expression for the determinant of S_k. □

The proposition shows, for example, that multiplication by $1-t$ kills $H_{k-1}(F, \mathbb{C})$ for $0 \leq k < k_0$, where k_0 is the smallest codimension of a flat X that is not, in some sense, a "general position" intersection of hyperplanes ($p(X) \neq 0$), and for which $h(X)$ divides the number of hyperplanes in the arrangement. This is of interest because the direct summand of $H_{k-1}(F, \mathbb{C})$ annihilated by $1-t$ is known [Orlik and Terao 1992; Cohen and Suciu 1995], while the higher torsion is more elusive.

We conclude this section with a remark and some numerical data. First, one can use the complex $C_\bullet(\mathcal{X})$ to calculate group homology when our arrangement is a real, $K(\pi, 1)$ arrangement. For example, the fundamental group of the arrangement $\boldsymbol{A}_{n-1}$ is the pure braid group P_n, and in this case $C_\bullet(\mathcal{X})$ coincides with a construction from D. B. Fuks. In this case, $H_k(F,\mathbb{C}) = H_k(P'_n,\mathbb{C})$, where P'_n is the derived subgroup of the pure braid group. As in Section 2, one can attempt to make use the S_n-module structure. Using Fuks' complex, E. V. Frenkel [1988] has found an expression for the S_n-invariant part of $H_k(P'_n,\mathbb{C})$. In a parallel vein, Cohen and Suciu [1998] construct a complex that is useful for group homology calculations for supersolvable, rather than real, arrangements.

The reader is invited to find a pattern in our computations of the homology of the Milnor fibre for the arrangement $\boldsymbol{A}_{n-1}$, in parallel with our Example 3.4. The table below shows the characteristic polynomial of the monodromy operator on $H_p(F,\mathbb{C})$; in particular, the Betti numbers of F are just the degrees of the polynomials. These polynomials were also calculated for $n \leq 5$ in [Cohen and Suciu 1998].

p	$n=2$	3	4	5
0	$1-t$	$1-t$	$1-t$	$1-t$
1		$(1-t)(1-t^3)$	$(1-t)^4(1-t^3)$	$(1-t)^9$
2			$(1-t)^3(1-t^3)(1-t^6)^2$	$(1-t)^{24}(1-t^2)^2$
3				$(1-t)^{16}(1-t^2)^2(1-t^{10})^6$

p	$n=$ 6	7
0	$1-t$	$1-t$
1	$(1-t)^{14}$	$(1-t)^{20}$
2	$(1-t)^{70}(1-t^3)$	$(1-t)^{154}(1-t^3)$
3	$(1-t)^{134}(1-t^3)^{14}(1-t^5)^6$	$(1-t)^{560}(1-t^3)^{20}$
4	$(1-t)^{77}(1-t^3)^{13}(1-t^5)^6(1-t^{15})^{24}$	$(1-t)^{923}(1-t^3)^{121}$
5		$(1-t)^{498}(1-t^3)^{102}(1-t^{21})^{120}$

5. Factorizations of B

In this section, we will give two combinatorial recipes for writing the B matrix of an n-dimensional arrangement as a product. The first, given in Section 5B, shows that

$$B = M^t \cdot \operatorname{diag}(A_0, A_1, \ldots, A_n) \cdot M,$$

where M is invertible as a matrix of polynomials in the weights $\{a_0, \ldots, a_{l-1}\}$, and the dimension of A_k is the $n-k$th Betti number of the arrangement's complex complement.

The second factorization is only defined for central arrangements. It depends on an order of the hyperplanes, $\mathcal{A} = \{H_0, H_1, \ldots, H_{l-1}\}$. The factorization has the form

$$B = B[a_{l-1} \leftarrow 0]M_{l-1},$$

where $B[a_{l-1} \leftarrow 0]$ denotes the matrix obtained from B by specializing the weight of H_{l-1} to zero. Proceeding inductively,

$$B = M_0 M_1 \cdots M_{l-1},$$

where M_k is a matrix whose entries are polynomials in the weights $\{a_0, a_1, \ldots, a_k\}$.

Both factorizations are applications of the Möbius function of the same type of poset, studied first by Edelman [1984]. We begin with its description.

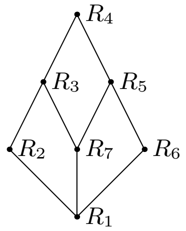
5A. Posets on the set of regions. As before, let $r(\mathcal{A})$ denote the set of regions of a real arrangement $\mathcal{A}$, and let $\mathcal{H}(R,S)$ denote the set of hyperplanes in $\mathcal{A}$ that separate regions $R, S \in r(\mathcal{A})$. For a fixed region $R \in r(\mathcal{A})$, Edelman's poset $P(\mathcal{A}, R)$ has underlying set $r(\mathcal{A})$, ordered by $S <_R T$ if and only if $\mathcal{H}(R,S) \subset \mathcal{H}(R,T)$. It is not hard to verify that R is the unique bottom element of $P(\mathcal{A}, R)$, and that the poset is graded by $|\mathcal{H}(R,S)|$, for $S \in P(\mathcal{A}, R)$. See [Edelman 1984] for details.

Let Z be any flat in $\mathbb{R}^n$ that intersects $\mathcal{A}$ transversely. By this, we mean an affine subspace Z for which, for $X \in L(\mathcal{A})$, if $X \cap Z \neq \varnothing$ then $\operatorname{codim}(X \cap Z) = \operatorname{codim} X + \operatorname{codim} Z$. Partition the regions of $\mathcal{A}$ by setting

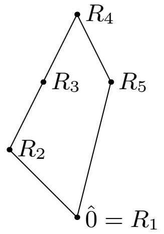
$$r_1 = \{S \in r(\mathcal{A}) : S \cap Z \neq \varnothing\},$$

and $r_0 = r(\mathcal{A}) \backslash r_1$. For any region $R \in r(\mathcal{A})$, define a new poset $P(\mathcal{A}, Z, R)$ on $r_1 \cup \{\hat{0}\}$ using the ordering $<_R$, with an added bottom element $\hat{0}$. We shall be most interested in such posets for $R \in r_0$, in which case one simply has the subposet of $P(\mathcal{A}, R)$ restricted to $r_1 \cup \{R\}$, under the identification $\hat{0} = R$.

EXAMPLE 5.1. Let Z be the line $x = -2$ in the arrangement of Example 1.1, and let $R = R_1$. Then $r_1 = \{R_2, R_3, R_4, R_5\}$, and the posets $P(\mathcal{A}, R)$ and $P(\mathcal{A}, Z, R)$ are as follows:



$P(\mathcal{A}, R)$



$P(\mathcal{A}, Z, R)$

For $a, b \in \{0, 1\}$, let B_{ab} be the submatrix of the B matrix whose rows and columns are indexed by r_a and r_b, respectively. The factorization results of this section depend on the following important observation.

LEMMA 5.2. *Under the hypotheses above, $B_{10} = B_{11} U$, where U is a $|r_1| \times |r_0|$ matrix with entries*

$$U(i,j) = -\mu_j(\hat{0}, i) B(i,j),$$

for $i \in r_1$ and $j \in r_0$. μ_j is the Möbius function of the poset $P(\mathcal{A}, Z, j)$.

Proof. The claim is equivalent to the equation

$$\sum_{j\in r_1} -\mu_k(j)B(i,j)B(j,k) = B(i,k). \tag{5–1}$$

Observe that, for any $j \in r_1$,

$$B(i,j)B(j,k) = B(i,k)\prod_H a_H^2,$$

where the product is taken over hyperplanes $H \in \mathcal{H}(i,j)\cap\mathcal{H}(j,k)$. With this in mind, let

$$r_{ik}(S) = \{j \in r_1 : \mathcal{H}(i,j)\cap\mathcal{H}(j,k) = S\},$$

for sets $S \subseteq \mathcal{A}$. It is enough to show that, for every $S \subseteq \mathcal{A}$,

$$\sum_{j\in r_{ik}(S)} \mu_k(j) = \begin{cases} -1 & \text{if } S = \varnothing, \\ 0 & \text{otherwise.} \end{cases} \tag{5–2}$$

Let $\mathcal{A}' = \mathcal{A}\backslash\mathcal{H}(i,k)$, and consider the order-preserving surjection of posets $\pi : P(\mathcal{A}, Z, k) \to P(\mathcal{A}', Z, \pi(k))$ induced by the inclusion of regions of $\mathcal{A}$ into regions of $\mathcal{A}'$. For any region $j \in r_1$, the hyperplanes that separate j from k in $\mathcal{A}'$ equal $\mathcal{H}(i,j)\cap\mathcal{H}(j,k)$. It follows that, for $j, j' \in r_1$, $\pi(j) = \pi(j')$ if and only if $j, j' \in r_{ik}(S)$ for some $S \subseteq \mathcal{A}$.

For any $S \subseteq \mathcal{A}$, if $r_{ik}(S)$ is nonempty, let $x = \pi(r_{ik}(S))$. Let μ' denote the Möbius function of $P(\mathcal{A}', Z, \pi(k))$. A routine argument shows that the fibres $\pi^{-1}(x)$ each contain upper bounds, for all $x \in P(\mathcal{A}', Z, \pi(k))$. It follows that π induces a closure operation on $P(\mathcal{A}, Z, k)$ by letting $\bar{\jmath}$ be the maximum element satisfying $\pi(\bar{\jmath}) = \pi(j)$. Using a well-known property of closure operations (see [Stanley 1986]),

$$\mu'_{\pi(k)}(x) = \sum_{j\in\pi^{-1}(x)} \mu_k(j) = \sum_{j\in r_{ik}(S)} \mu_k(j).$$

Since no hyperplanes of $\mathcal{A}'$ separate regions k and i, $\pi(i)$ is the only element of $P(\mathcal{A}', Z, k)$ that covers $\hat{0}$. It follows that the Möbius function satisfies

$$\mu'_{\pi(k)}(x) = \begin{cases} -1 & \text{if } x = \pi(i), \\ 0 & \text{for any other region.} \end{cases}$$

$x = \pi(i)$ exactly when $S = \varnothing$, so equation (5–2) is proven. □

The next proposition gives a more explicit description of the matrix U in a special case. Their proofs will appear in [Denham ≥ 1999]. Let $\mathcal{A}$ be a central arrangement and let f be a linear functional for which $H = \ker f \in \mathcal{A}$. Let $Z = \ker f + 1$, a hyperplane parallel to H. The induced arrangement $\mathcal{A}^Z$ is known as the *decone* of $\mathcal{A}$ with respect to H [Orlik and Terao 1992]. Inclusion identifies the regions of $\mathcal{A}^Z$ with regions of $\mathcal{A}$; moreover, if $R \in r(\mathcal{A})$, either R or $-R$ intersects Z.

For $R \in r(\mathcal{A})\backslash r(\mathcal{A}^Z)$ and $S \in r(\mathcal{A}^Z)$, let

$$W_R(S) = \bar{S} \cap \{H \in \mathcal{A}^Z : H \notin \mathcal{H}(-R, S)\},$$

where $\bar{S}$ denotes the topological closure of S in Z. $W_R(S)$ should be thought of as the topological space consisting of the walls around chamber S that do not separate S from $-R$, in the deconed arrangement.

Lemma 5.3. *Let $\mathcal{A}$ be a central arrangement, and take Z as defined above, $R \in r(\mathcal{A})\backslash r(\mathcal{A}^Z)$ and $S \in r(\mathcal{A}^Z)$. Let μ_R be the Möbius function of the poset $P(\mathcal{A}, Z, R)$. Then*

$$\mu_R(\hat{0}, S) = \chi(W_R(S)),$$

the reduced Euler characteristic of $W_R(S)$.

Definition 5.4. For any hyperplane $H \in \mathcal{A}$, define a map of sets $f_H : r(\mathcal{A}) \to L(\mathcal{A}^H)$ by letting $f_H(R) = \left|\overline{R} \cap H\right|$, the intersection of all hyperplanes containing $\overline{R} \cap H$.

It is shown in [Denham $\geq$ 1999] that the Möbius function of $P(\mathcal{A}, Z, R)$ has a simple description in these terms.

Proposition 5.5. *For $R \in r(\mathcal{A})\backslash r(\mathcal{A}^Z)$ and $S \in r(\mathcal{A}^Z)$, $\mu_R(\hat{0}, S) = 0$ unless $f_H(R) \geq f_H(S)$. Moreover, if $f_H(R) \geq f_H(S)$, let d be the codimension of $f_H(R)$ in H. If S is bounded,*

$$\mu_R(\hat{0}, S) = \begin{cases} (-1)^{d-1} & \text{if } -R = S; \\ 0 & \text{otherwise.} \end{cases}$$

If S is unbounded,

$$\mu_R(\hat{0}, S) = \begin{cases} 0 & \text{if } -R = S; \\ (-1)^{d} & \text{otherwise.} \end{cases}$$

5B. Induction on dimension. Our first decomposition is not unique: it depends on an arbitrary choice of a flag of flats in general position with respect to the arrangement $\mathcal{A}$. By this, we mean affine subspaces Z_d for which

$$Z_0 \supset Z_1 \supset \cdots \supset Z_n,$$

$\operatorname{codim} Z_d = d$, and for $X \in L(\mathcal{A})$, $X \cap Z_d = \varnothing$ if $\operatorname{codim} X > n - d$; otherwise, $\operatorname{codim}(X \cap Z) = \operatorname{codim} X + d$, for $1 \leq d \leq n$.

Let

$$r_d = \{S \in r(\mathcal{A}) : S \cap Z_d \neq \varnothing,\ S \cap Z_{d+1} = \varnothing\}$$

for $0 \leq d < n$, and let r_n consist of the single region containing the point Z_n. Let $B_{a,b}$ be the submatrix of B with rows and columns indexed by r_a and r_b, respectively. For $0 \leq d < n$, B_{dd} can be identified with the B matrix of the arrangement $\mathcal{A}^{Z_d}$, and Lemma 5.2 applies. It states that $B_{d+1,d} = B_{d+1,d+1} U_d$, where U_d is the matrix with entries

$$U_d(i, j) = -\mu_j(\hat{0}, i) B(i, j)$$

for $i \in r_{d+1}$ and $j \in r_d$, and μ_j is the Möbius function of $P(\mathcal{A}^{Z_d}, Z_{d+1}, j)$.

If one orders the basis $r(\mathcal{A})$ so that r_0 precedes r_1, one finds that

$$B = M_0^t \cdot \operatorname{diag}(B_{00} - U_0^t B_{11} U_0, B_{11}) \cdot M_0,$$

where

$$M_0 = \begin{pmatrix} I_{|r_0|} & 0 \\ U_0 & I_{|r_1|} \end{pmatrix}.$$

More generally:

Theorem 5.6. *Order the regions of a real arrangement $\mathcal{A}$ so that $R < S$ if $R \in r_a$ and $S \in r_b$ with $a < b$, in the notation above. Subject to this ordering, $B = M^t \cdot \operatorname{diag}(A_0, A_1, \dots, A_n) \cdot M$, where*

$$A_d = B_{dd} - U_d^t B_{d+1,d+1} U_d$$

is an $r_d \times r_d$ matrix for $0 \le d < n$, $A_n = (1)$, and

$$M = \overrightarrow{\prod_{0 \le d < n}} \begin{pmatrix} I_{|r_d|} & 0 \\ U_d & I_{|r_{d+1}|} \end{pmatrix}. \qquad \square$$

By the assumption that Z_1 is in general position, the intersection semilattice of $\mathcal{A}^{Z_1}$ is a truncation of that of $\mathcal{A}$: $L(\mathcal{A}^{Z_1}) = L_{\le n-1}(\mathcal{A})$. By induction, $L(\mathcal{A}^{Z_d}) = L_{\le n-d}(\mathcal{A})$.

Let μ be the Möbius function of $L(\mathcal{A})$, and let

$$b_k = \sum_{X \in L_k(\mathcal{A})} (-1)^k \mu(\hat{0}, X).$$

It is well known [Orlik and Terao 1992] that $|r(\mathcal{A})| = \sum_{k=0}^n b_k$. Since the inclusion map from $\mathcal{A}^{Z_d}$ to $\mathcal{A}$ identifies the restriction of μ with the Möbius function of $L(\mathcal{A}^{Z_d})$, $\left|r(\mathcal{A}^{Z_d})\right| = \sum_{k=0}^{n-d} b_k$; consequently, $|r_d| = b_{n-d}$.

It can be shown that the matrix A_k, is equivalent to the bilinear form B^k defined in [Varchenko 1993, Section 20], under a suitable change of basis. This leads to another proof of Theorem 3.3.

5C. Induction on the number of hyperplanes. The second factorization requires that $\mathcal{A}$ be a real, central arrangement. In order to give a description, let $H = \ker f$ be a hyperplane in $\mathcal{A}$, and let $Z = \ker f + 1$ be a hyperplane parallel to H.

Proposition 5.7. *B factors as $B = B[a_H \leftarrow 0] M_H$, where the matrix*

$$M_H(R,S) = \begin{cases} 1 & \text{if } R = S, \\ -\mu_S(\hat{0}, R) B(R,S) & \text{if } H \text{ separates } R \text{ from } S, \\ 0 & \text{otherwise}, \end{cases}$$

and μ_S is the Möbius function of the poset $P(\mathcal{A}, \pm Z, S)$, where the sign is chosen so that $S \cap \pm Z = \varnothing$.

Proof. For convenience, we reorder the basis of the space of regions so that $r(\mathcal{A}^Z)$ appears first in order. Let $2m = |\mathcal{A}|$, and let $\sigma : r(\mathcal{A}) \to [2m]$ be a bijection for which $\sigma(R) \leq m$ and $\sigma(-R) = \sigma(R) + m$ for $R \in r(\mathcal{A}^Z)$. Let Q be the corresponding permutation matrix, $Q(R, i) = \delta_{\sigma(R),i}$, for $R \in r(\mathcal{A})$ and $i \in [2m]$. In the notation of Lemma 5.2,

$$Q^{-1}BQ = \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix}.$$

Since $B(-R, -S) = B(R, S)$, we have $B_{00} = B_{11}$ and $B_{01} = B_{10}$. Using Lemma 5.2, it follows that

$$Q^{-1}BQ = \begin{pmatrix} B_{00} & 0 \\ 0 & B_{11} \end{pmatrix} \begin{pmatrix} I_m & U \\ U & I_m \end{pmatrix} = Q^{-1}B[a_H \leftarrow 0]M_H Q.$$

□

The proposition relates two B matrices, one with the hyperplanes weighted arbitrarily, and the other with one hyperplane given weight zero. One can apply the proposition to each hyperplane in succession to obtain the following.

Theorem 5.8. *Let $\mathcal{A} = \{H_0, H_1, \ldots, H_{l-1}\}$ be a real, central arrangement of hyperplanes. For $0 \leq d \leq l-1$, let Z_d be a parallel translate of H_d. Then $B(\mathcal{A}) = M_0 M_1 \cdots M_{l-1}$, where M_d is a matrix over $\mathbb{Z}[a_0, \ldots, a_d]$. Explicitly,*

(i) $M_d(S, S) = 1$,

(ii) $M_d(R, S) = -\mu_S(\hat{0}, R)B(R, S)$ *if* $d = \max\{k : H_k \in \mathcal{H}(R, S)\}$,

(iii) $M_d(R, S) = 0$ *otherwise.*

Here again, μ_S is the Möbius function of $P(\mathcal{A}, \pm Z_d, S)$, with the sign chosen so that $S \cap Z_d = \varnothing$.

Example 5.9. Consider the arrangement $\boldsymbol{A}_2$ from Example 1.2. Order the hyperplanes H_{12}, H_{23}, H_{13}, and order the regions $123, 213, 132, 231, 312, 321$. Then $B = M_0 M_1 M_2$, where

$$M_1 = \begin{pmatrix} 1 & 0 & a_{23} & & & \\ 0 & 1 & 0 & & & \\ a_{23} & a_{12}a_{23} & 1 & & & \\ & & & 1 & a_{12}a_{23} & a_{23} \\ & & & 0 & 1 & 0 \\ & & & a_{23} & 0 & 1 \end{pmatrix}$$

and

$$M_2 = \begin{pmatrix} 1 & & & 0 & 0 & -a_{12}a_{23}a_{13} \\ & 1 & & a_{13} & 0 & a_{23}a_{13} \\ & & 1 & 0 & a_{13} & a_{12}a_{13} \\ a_{12}a_{13} & a_{13} & 0 & 1 & & \\ a_{23}a_{13} & 0 & a_{13} & & 1 & \\ -a_{12}a_{23}a_{13} & 0 & 0 & & & 1 \end{pmatrix},$$

and the off-diagonal entries of M_0 are all zero, except for a_{12} in entries $(123, 213)$, $(213, 123)$, $(312, 321)$, and $(321, 312)$.

We conclude with an example of the sort of information this theorem gives us. In the notation of Proposition 5.7,

$$Q^{-1}M_dQ = \begin{pmatrix} I & U_d \\ U_d & I \end{pmatrix},$$

so that $\det M_d = \det(I - U_d^2)$. One can use Proposition 5.5 to show that $I - U_d^2$ is upper-triangular. Then keeping track of the diagonal entries shows that

$$\det M_d = \det(I - U_d^2) = \prod_X (1 - a_X^2)^{l(X)},$$

where the product is taken over all $X \in L(\mathcal{A})$ for which $d = \max\{k : H_k \le X\}$. This gives another proof of Varchenko's formula, Theorem 2.1.

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