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The Generalized Baues Problem

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Abstract. We survey the generalized Baues problem of Billera and Sturmfels. The problem is one of discrete geometry and topology, and asks about the topology of the set of subdivisions of a certain kind of a convex polytope. Along with a discussion of most of the known results, we survey the motivation for the problem and its relation to triangulations, zonotopal tilings, monotone paths in linear programming, oriented matroid Grassmannians, singularities, and homotopy theory. Included are several open questions and problems.

1. Introduction

The generalized Baues problem, or GBP for short, is a question arising in the work of Billera and Sturmfels [1992, p. 545] on *fiber polytopes*; see also [Billera et al. 1994, § 3]. The question asks whether certain partially ordered sets whose elements are subdivisions of polytopes, endowed with a certain topology [Björner 1995], have the homotopy type of spheres. Cases are known [Rambau and Ziegler 1996] where this fails to be true, but the general question of when it is true or false remains an exciting subject of current research.

The goal of this survey is to review the motivation for fiber polytopes and the GBP, and discuss recent progress on the GBP and the open questions remaining. Some recommended summary sources on this subject are the introductory chapters in the doctoral theses [Rambau 1996; Richter-Gebert 1992], Lecture 9 in [Ziegler 1995], and the paper [Sturmfels 1991]. The articles [Billera et al. 1990; 1993], though not discussed in the text, are nonetheless also relevant to the GBP.

Before diving into the general setting of fiber polytopes and the GBP, it is worthwhile to ponder three motivating classes of examples.

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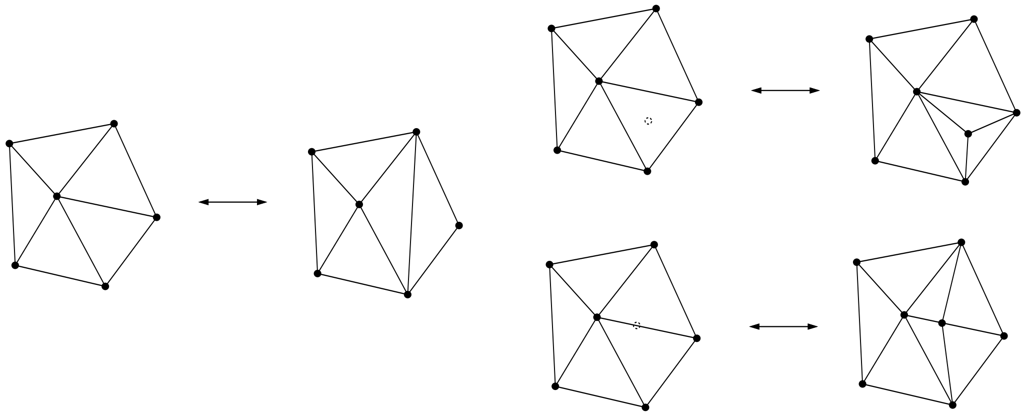

Figure 1. Typical bistellar operations (also known as perestroikas or modifications) in $\mathbb{R}^2$. Left: diagonal flip. Right: vertex insertion/removal.

Triangulations. Let $\mathcal{A}$ denote a finite set of points in $\mathbb{R}^d$. A *triangulation* of $\mathcal{A}$ is, roughly speaking, a polyhedral subdivision of the convex hull of $\mathcal{A}$ into simplices, each having the property that their vertices lie in $\mathcal{A}$. Note that not every point of $\mathcal{A}$ need appear as a vertex of one of the simplices in the triangulation. The set of all triangulations of $\mathcal{A}$ is in general a difficult object to compute, but one that arises in many applications; see [de Loera 1995b]. One approach to the study and computation of triangulations is to consider an extra structure on them, namely the connections between them by certain moves called *bistellar operations* (or *perestroikas* or *modifications*). For triangulations of $\mathcal{A}$ in $\mathbb{R}^2$, typical bistellar operations are shown in Figure 1, where points of $\mathcal{A}$ that are not being used as a vertex in the triangulation are shown dotted. Figure 2, adapted from [de Loera 1995b], depicts the set of all triangulations of a particular configuration of six points in $\mathbb{R}^2$, and the bistellar moves which connect them. We remark that the precise coordinates of the points of $\mathcal{A}$ are important in determining which triangulations and bistellar operations are possible, since we are talking about triangulations using straight geometric simplices. This is different from the point of view in the theory of triangulated *planar maps* (see [Goulden and Jackson 1983, § 2.9], for example) and also different from the bistellar equivalences of triangulations of PL-manifolds as considered by Nabutovsky [1996] or Pachner [1991].

The most well-studied example of triangulations occurs when $\mathcal{A}$ is the vertex set of a convex n-gon in $\mathbb{R}^2$. It is well-known that the number of triangulations is the *Catalan number* $\frac{1}{n-1}\binom{2n-4}{n-2}$ (see, for example, [Stanton and White 1986, § 3.1; Stanley 1999, Exercise 6.19] for this and for bijections between triangulations and other standard objects counted by the Catalan number). This is essentially the only nontrivial example of an infinite family of point configurations whose number of triangulations is known (but see Conjecture 6.5). The only possible bistellar operations in this case are the *diagonal flips* from Figure 1, and it is easy to see that any two triangulations can be connected by a sequence

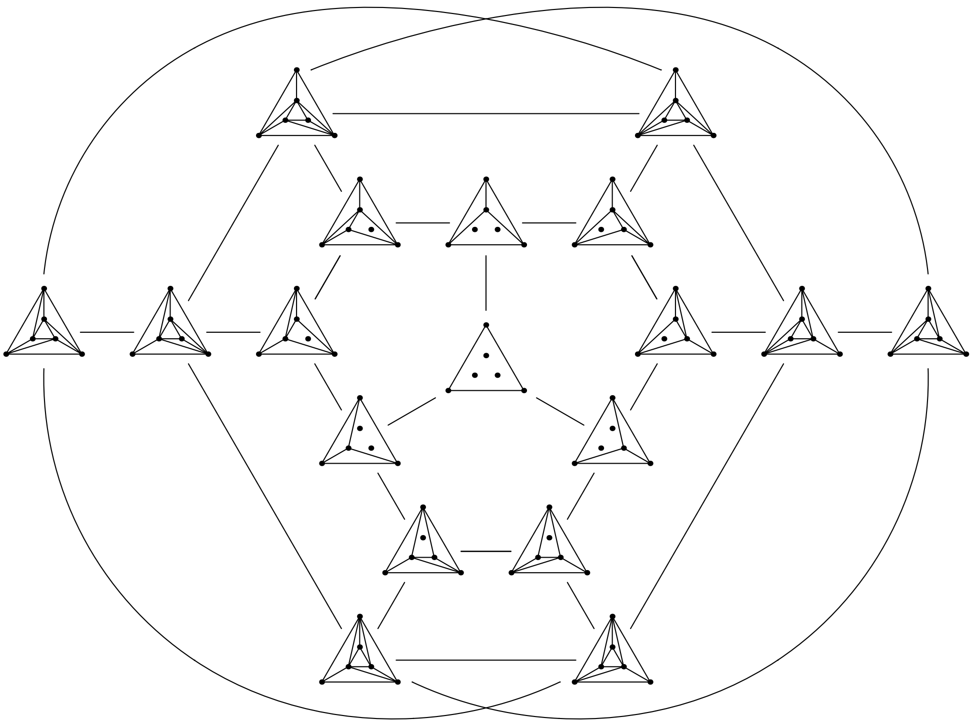

Figure 2. All triangulations and bistellar operations for a set $\mathcal{A}$ of 6 points in $\mathbb{R}^2$. The points of $\mathcal{A}$ form the vertices of two homothetic and concentric equilateral triangles. Adapted from [de Loera 1995b].

of such flips. There is a well-known bijection between triangulations of an n-gon and nonassociative bracketings of a product $a_1 a_2 \cdots a_{n-1}$, and under this identification bistellar operations correspond to "rebracketings". From this point of view, the graph of triangulations of an n-gon and diagonal flips was perhaps first studied in the 1950's by Tamari [1951] and later in collaboration with others [Tamari 1962; Friedman and Tamari 1967; Huang and Tamari 1972; Huguet and Tamari 1978]. These authors distinguished a direction on each rebracketing and defined a poset on the triangulations having these directed edges as its cover relations. They were able to show that this *Tamari poset* is a lattice [Friedman and Tamari 1967; Huang and Tamari 1972]. Its Hasse diagram is depicted in Figure 3 for $n = 6$, for a choice of a particular convex 6-gon whose vertices lie on a semicircle.

These authors seem also to have been aware (without proof) that this graph appears to be the 1-skeleton of a cellular $(n-4)$-sphere, and proved results about how its "facial" structure interacts with the Tamari lattice structure. Meanwhile, similar issues of associativity appeared in the early 1960's in Stasheff's work [1963] on homotopy associativity. Stasheff vindicated this apparent sphericity

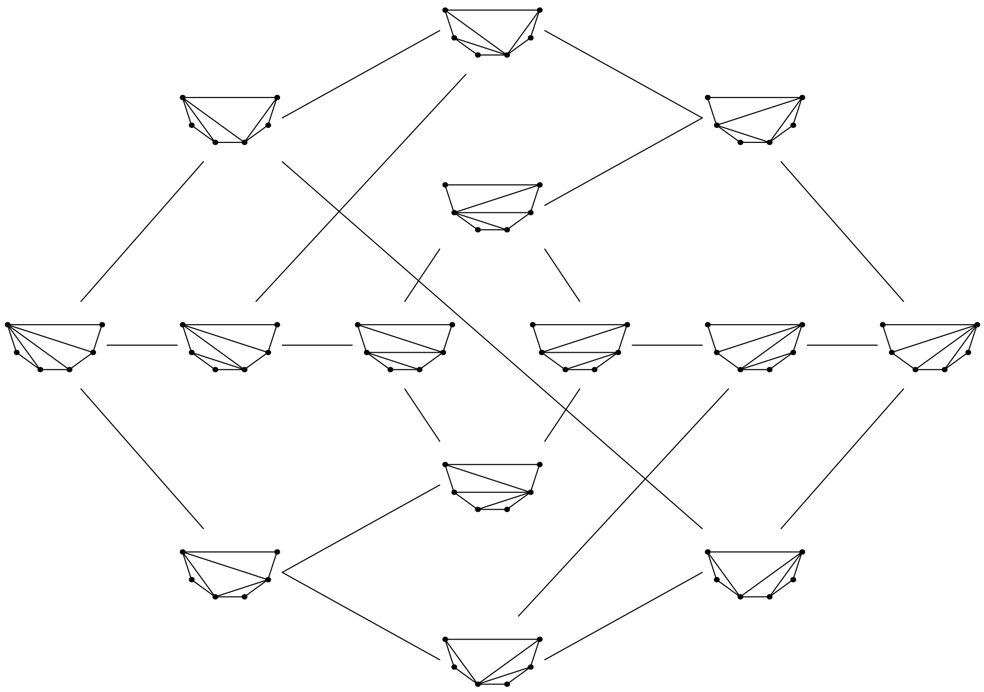

Figure 3. Triangulations of a convex 6-gon: the associahedron.

by showing (essentially) that the set of all *polygonal subdivisions* of an n-gon indexes the cells in a *regular cell complex* [Björner 1995, (12.3)] homeomorphic to the $(n-4)$-sphere. Note that in this way of thinking, a diagonal flip bistellar operation corresponds to a polygonal subdivision whose maximal cells are all triangles except for one quadrangle (containing the flipping diagonal), and less refined subdivisions of the n-gon correspond to higher dimensional cells in the sphere. In an unpublished work (see [Kapranov 1993, p. 120]), Milnor produced a set of vertex coordinates for the vertices of this $(n-4)$-sphere which embed it as the boundary complex of an $(n-3)$-dimensional polytope. Unfortunately, the existence of this polytopal embedding seems to have been unknown in the combinatorial geometry community, and was rediscovered in the mid 1980's after Perles posed the problem of whether this complex was polytopal; see [Lee 1989]. Independently, Haiman [1984] and Lee [1989] constructed this polytope, which Haiman dubbed the *associahedron*. In [Kapranov and Voevodsky 1991; Gel'fand et al. 1994] it is sometimes called the *Stasheff polytope*. Kapranov and Saito [1997] document its occurrence in other surprising geometric contexts.

The associahedron also makes its appearance in computer science, where triangulations of an n-gon show up in the equivalent guise of *binary trees*, and bistellar operations correspond to an operation on binary trees called *rotation*. Here Sleator, Tarjan and Thurston [Sleator et al. 1988] were able to determine the diameter of the 1-skeleton of the associahedron (it is at most $2n - 10$ for $n \geq 13$ and is exactly $2n - 10$ for infinitely many values of n). Pallo [1987; 1988; 1990; 1993] studied computational aspects of this 1-skeleton and in particular computed the *Möbius function* [Stanley 1997, § 3.7] of the Tamari lattice.

In one of Lee's constructions of the associahedron, he employs the method of *Gale diagrams* [Lee 1989, § 4]. Around the same time, Gelfand, Kapranov and Zelevinsky were using these methods for studying triangulations as part of their theory of $\mathcal{A}$-*discriminants*, $\mathcal{A}$-*resultants*, and $\mathcal{A}$-*determinants* (see [Gel'fand et al. 1994] and the references therein). Briefly, the principal $\mathcal{A}$-*determinant* is a polynomial $E_{\mathcal{A}}$ in a variable set $\{c_a\}_{a \in \mathcal{A}}$ indexed by $\mathcal{A}$, which vanishes whenever the sparse d-variate polynomial in $x_1, \ldots, x_d$

$$f := \sum_{a \in \mathcal{A}} c_a \boldsymbol{x}^a$$

has a root $(x_1, \ldots, x_n)$ in common with all of the derived polynomials

$$x_1 \frac{\partial f}{\partial x_1}, \;\ldots,\; x_d \frac{\partial f}{\partial x_d}.$$

Their work showed that the *Newton polytope* of $E_{\mathcal{A}}$, that is, the convex hull in $\mathbb{R}^{\mathcal{A}}$ of the set of exponent vectors of the monomials having nonzero coefficients in $E_{\mathcal{A}}$, is an $(n - d - 1)$-dimensional polytope whose vertices correspond to a subset of the triangulations of $\mathcal{A}$ called the *regular* (and later called *coherent*) triangulations. A triangulation T of $\mathcal{A}$ is *coherent* if there exists a choice of *heights* α_a in $\mathbb{R}$ for each $a \in \mathcal{A}$ which induces T in the following fashion: after "lifting" the points a in $\mathbb{R}^d$ to the points $(a, \alpha_a) \in \mathbb{R}^{d+1}$ and taking the convex hull to form a polytope P_α, the "lower" facets of P_α (i.e., those facets whose normal vector has negative $(d+1)$-coordinate) project to the maximal simplices of T under the projection $\mathbb{R}^{d+1} \to \mathbb{R}^d$. Figure 4, borrowed from [Rambau 1996],

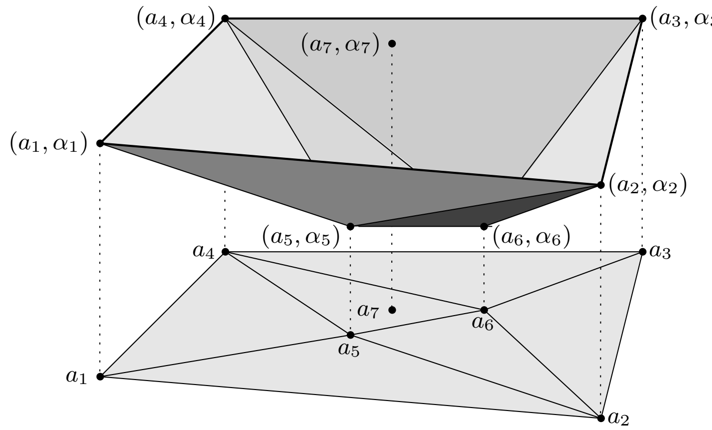

Figure 4. A coherent triangulation induced by a choice of heights. From [Rambau 1996].

Figure 5. The two incoherent triangulations lurking among those in Figure 2 (on the far left and far right of that figure).

illustrates a coherent triangulation of a set $\mathcal{A}$ in $\mathbb{R}^2$ along with a choice of heights α which induces it.

After seeing the definition, it is perhaps not obvious that one can have *incoherent* triangulations! However, the standard examples, discussed extensively in [Connelly and Henderson 1980; Schönhardt 1928], already occur as two of the triangulations appearing in Figure 2. We have isolated these two triangulations and depicted them separately in Figure 5. It is a nontrivial exercise to check the impossibility of assigning six heights to these points in such a way as to induce either of these triangulations. In general, checking whether a triangulation is coherent involves checking whether there exists a solution to a certain system of linear inequalities in the heights α_a, where the coefficients in the inequalities depend upon the coordinates of the points in $\mathcal{A}$; see [Hastings 1998, Chapter 2; de Loera 1995b, § 1.3].

Gelfand, Kapranov and Zelevinsky called the Newton polytope of $E_{\mathcal{A}}$ the *secondary polytope* $\Sigma(\mathcal{A})$. Knowing that the vertices of $\Sigma(\mathcal{A})$ correspond to the coherent triangulations of $\mathcal{A}$, it is perhaps not surprising that the higher dimensional faces of $\Sigma(\mathcal{A})$ correspond to *coherent subdivisions*, that is, subdivisions into polytopes which are not necessarily simplices, but induced in a similar fashion by a choice of heights α_a for a in $\mathcal{A}$.

Theorem 1.1 [Gel'fand et al. 1994, Chapter 7, Theorem 2.4]. *The faces of the secondary polytope* $\Sigma(\mathcal{A})$ *are indexed by the coherent subdivisions of* $\mathcal{A}$, *and reverse inclusion of faces of* $\Sigma(\mathcal{A})$ *corresponds to refinement of subdivisions.*

In particular, they showed that every bistellar operation between coherent triangulations corresponds to a coherent subdivision and hence forms an edge in the secondary polytope $\Sigma(\mathcal{A})$. This has a strong consequence: it implies that the subgraph of coherent triangulations and bistellar operations is connected (and even $(n-d-1)$-vertex-connected in the graph-theoretic sense by Balinski's Theorem [Ziegler 1995, 3.5]). Polytopality of $\Sigma(\mathcal{A})$ also has nice implications for computing the particular coherent triangulation induced by a choice of heights α_a, such as the *Delaunay triangulation* of $\mathcal{A}$ arising in computational geometry applications; see [Edelsbrunner and Shah 1992].

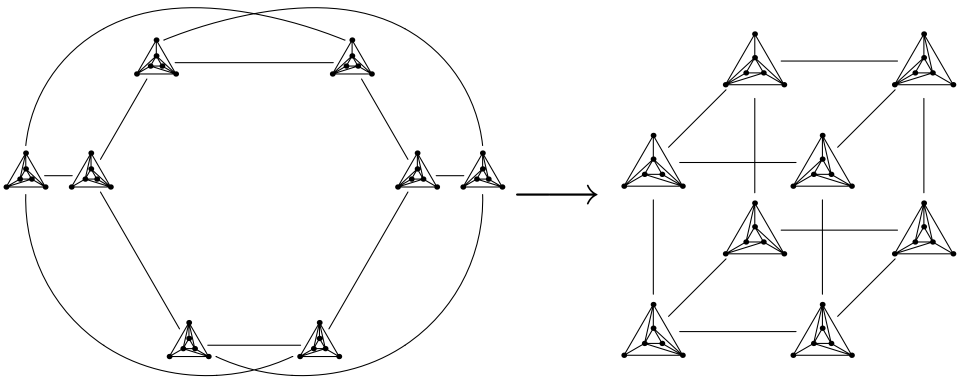

Figure 6. The incoherent triangulations and their neighbors: two equivalent pictures.

We remark that in the case where $\mathcal{A}$ is the set of vertices of a convex n-gon, every subdivision is coherent, and hence the secondary polytope $\Sigma(\mathcal{A})$ is the associahedron encountered earlier.

The fact that the subgraph of coherent triangulations and bistellar operations is highly connected and forms the 1-skeleton of a cellular (even polytopal) sphere raises the following basic question:

Question 1.2. *Is the graph of all triangulations of $\mathcal{A}$ and their bistellar operations connected?*

A glance at Figure 2 illustrates that even in small cases where there are incoherent triangulations, the graph still appears to be connected. We can provide some motivation for the Generalized Baues Problem by performing the following mental exercise while staring at Figure 2. First picture the planar subgraph of coherent triangulations, by ignoring the two vertices corresponding to the incoherent triangulations in Figure 2 (call them T_1 and T_2). When one imagines this planar subgraph as a two-dimensional spherical cell complex, that is the boundary of the three-dimensional secondary polytope $\Sigma(\mathcal{A})$, the union of the neighbors of T_1 and T_2 form the vertices of a hexagonal cell, corresponding to the unbounded region in the planar embedding; see Figure 6. Now "inflate" this hexagonal cell on the 2-sphere into a cubical 3-dimensional cell with the extra two vertices corresponding to T_1, T_2. This gives a 3-dimensional cell complex which is still homotopy equivalent (but not homeomorphic) to a 2-sphere.

Roughly speaking, the Baues question in this context asks whether this behavior is general: Do the *incoherent* triangulations and subdivisions of $\mathcal{A}$ attach themselves to the spherical boundary of $\Sigma(\mathcal{A})$ in such a way as to not change its homotopy type?

Zonotopal tilings. Consider Figure 7, similar to [Billera and Sturmfels 1992, Figure 1], depicting the tilings of a centrally symmetric octagon having unit side

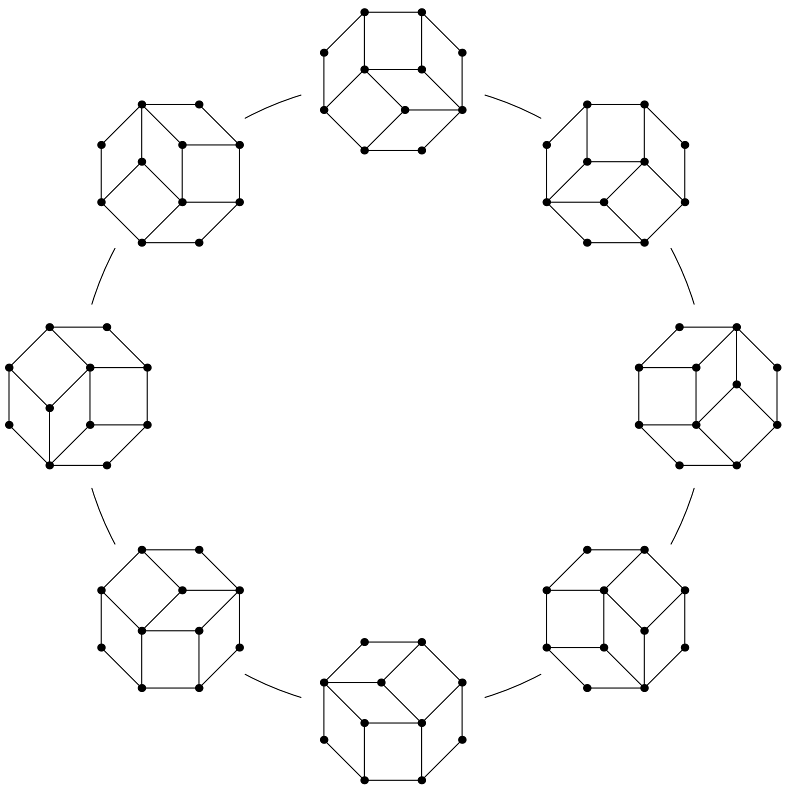

Figure 7. The rhombic tilings of an octagon.

lengths by unit rhombi. As in the case of triangulations of a point set, we have drawn in edges between the tilings corresponding to certain natural operations connecting them, illustrated in Figure 8. Similarly, the graph whose vertices are the tilings of a 10-gon and whose edges are these operations is depicted in Figure 9, which may not look very planar, but is in fact the 1-skeleton of a 3-dimensional polyotope.

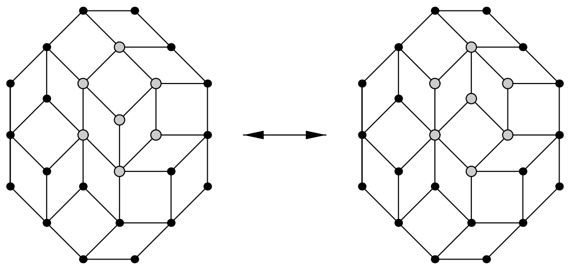

Figure 8. A typical cube flip, also known as a mutation, triangle switch, 1-move, braid relation, Yang–Baxter relation, elementary flip, or localized phason.

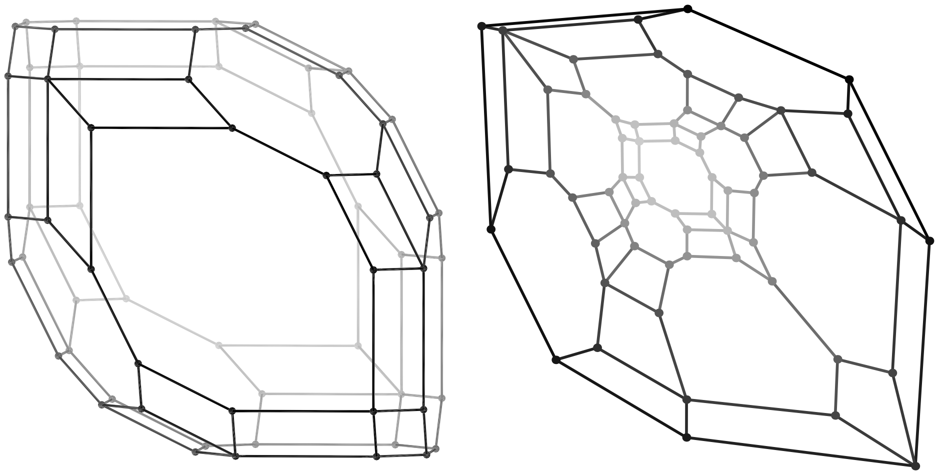

Figure 9. The graph of tilings of a decagon, seen in $\mathbb{R}^3$ from infinity (left) and from a nearby point. Compare [Ziegler 1993, Figure 3].

These operations have been given various names in the literature, depending upon the context in which the tilings arise. In the crystalline physics literature [Destainville et al. 1997; Mosseri and Bailly 1993], where the set of tilings is a model for the possible states of a crystalline solid, these moves are called *elementary flips* or *localized phasons*. Rather than considering tilings of a $2n$-gon, an equivalent (and useful) viewpoint comes from consideration of *arrangements of pseudolines* (see [Björner et al. 1993, Chapter 6] for definition, background and references). An arrangement of n affine pseudolines in the plane labelled $1, 2, \ldots, n$ counterclockwise gives rise to a rhombic tiling of a centrally symmetric $2n$-gon which is "dual" to the line arrangement in the sense of planar maps; see Figure 10.

In the pseudoline picture, the move depicted in Figure 8 corresponds to moving one pseudoline locally across the nearby crossing point of two other pseudolines;

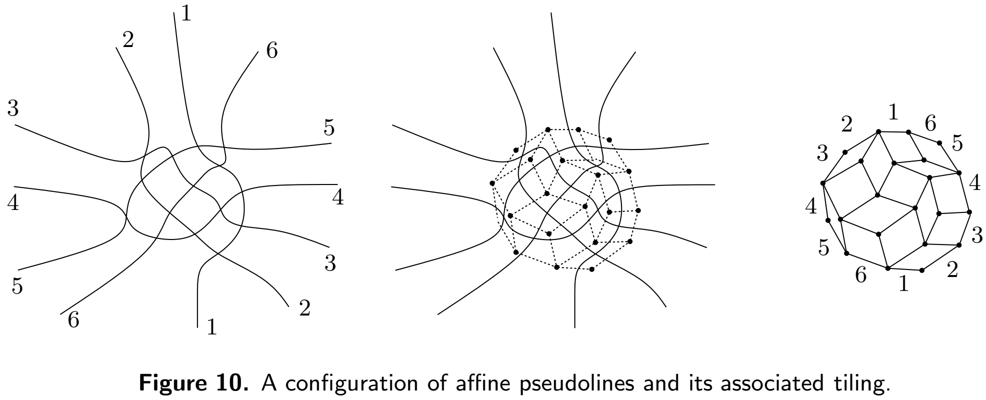

Figure 10. A configuration of affine pseudolines and its associated tiling.

such moves are often called *mutations* or *triangle-switches* or *1-moves*. When one thinks of such a pseudoline arrangement as a degenerate *braid diagram* recording a *reduced decomposition* of a permutation (see [Björner et al. 1993, § 6.4]), such moves are sometimes called *braid relations* or *Yang–Baxter relations*.

Rather than restricting our attention to tilings of centrally symmetric polygons, we can more generally consider the set of *zonotopal subdivisions* of a *zonotope*. A *zonotope* Z in $\mathbb{R}^d$ is the Minkowski sum of a set V of line segments in $\mathbb{R}^d$, and a zonotopal subdivision of Z is, roughly speaking, a subdivision of Z into smaller zonotopes, each a translate of a zonotope generated by a subset of V, and which intersect pairwise along common faces (possibly empty). The subdivision is *cubical* if it is as refined as possible, that is each smaller zonotope in the subdivision is a translate of a cube generated by a linearly independent subset of V. In the case where Z is a centrally symmetric $2n$-gon, V is a set of n line segments whose slopes match the slopes of the polygon edges. Cubical tilings in this case coincide with the rhombic tilings depicted earlier, and the "cube flip" moves which formed the edges in the graphs of Figures 7 and 9 correspond to zonotopal subdivisions of Z in which all of the smaller zonotopes are cubes except for one which is hexagonal.

Note that the graph of tilings in Figure 7 is circular, and the graph of tilings in Figure 9 appears to be planar and possibly even polytopal. This reflects the fact that for centrally symmetric octagons and decagons, all zonotopal subdivisions are *coherent* in a sense which will be described below. A special case of Billera and Sturmfels' fiber polytope construction [Billera and Sturmfels 1992, § 5] states that the subset of coherent zonotopal subdivisions of a d-dimensional zonotope having n generators index the faces an $(n-d)$-dimensional polytope (which happens to be itself a zonotope). Thus the graphs in Figures 7 and 9 are the 1-skeleta of these *fiber zonotopes*.

Coherence of a zonotopal subdivision is defined similarly to coherence of a triangulation. A zonotopal subdivision T of a zonotope Z in $\mathbb{R}^d$ having generating line segments V is *coherent* if there exists a choice of segments $\hat{V}$ in $\mathbb{R}^{d+1}$ which project down to V under the forgetful projection $\mathbb{R}^{d+1} \to \mathbb{R}^d$ and induce T in the following fashion: the "upper facets" of the zonotope $\hat{Z}$ generated by $\hat{V}$ project to the maximal cells of T under the map $\mathbb{R}^{d+1} \to \mathbb{R}^d$. An example is shown in Figure 11. Again, it is not obvious that incoherent zonotopal subdivisions can exist, but it can be shown for example, that the tiling of a 12-gon depicted in Figure 10 is incoherent for certain choices of the slopes of edges in the 12-gon, using essentially the same arguments as in [Björner et al. 1993, Example 1.11.2]. As with coherence of triangulations, checking coherence of a particular tiling is a problem of existence of a solution to a system of linear inequalities, and the system of inequalities in this case strongly depends upon the slopes of the segments V (although not upon the length of these segments). Again as in the case of triangulations, the fact that the graph of coherent tilings and cube flips is the 1-skeleton of a polytope has strong consequences for its connectivity.

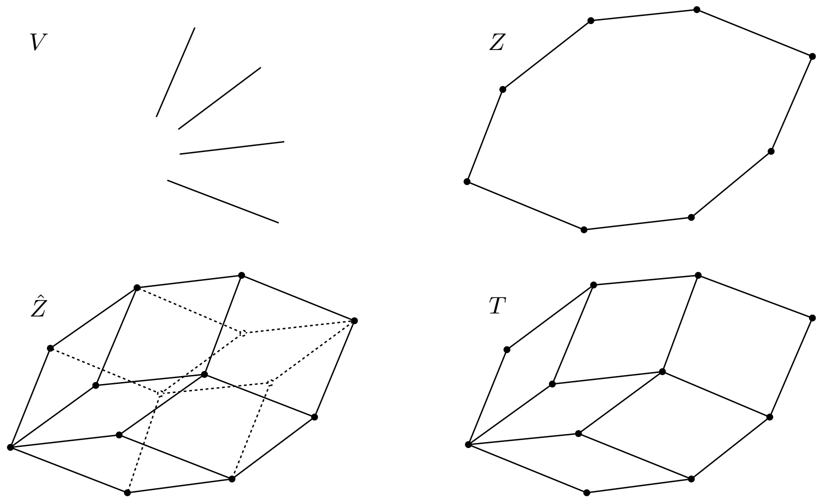

Figure 11. A coherent tiling T, induced by lifting into $\mathbb{R}^3$ the generating segments V of the 2-dimensional zonotope Z, then projecting the upper facets of the resulting 3-dimensional zonotope $\hat{Z}$ back into the plane.

This raises the analogous question to Question 1.2:

QUESTION 1.3. *Is the graph of all cubical tilings of a zonotope and their cube flips connected*?

One can also view coherence of two-dimensional tilings in terms of pseudolines and straight lines. A coherent tiling is one whose pseudoline arrangement is isomorphic to a (straight) line arrangement in which each line has slope perpendicular to the slope of the edge in the polygon to which it corresponds (that is, to the edge of the polygon labelled with the same number in Figure 10). Some of this viewpoint is explained in the instructions for the delightful puzzle Hexa-Grid [MRI n.d.], which supplies foam rubber versions of the rhombic tiles occurring in Figure 10, and asks the consumer to assemble them into a tiling of a zonotopal 12-gon!

In studying tilings and zonotopal subdivisions of higher dimensional zonotopes, the oriented matroid point of view has become indispensable; see [Björner et al. 1993, § 2.2; Ziegler 1996]. The Bohne–Dress Theorem [Bohne 1992; Richter-Gebert and Ziegler 1994] states that zonotopal subdivisions of Z biject with the *single-element liftings* of the *realized* oriented matroid $\mathcal{M}$ associated with the generating segments V, or using oriented matroid duality, to the *single-element extensions* of the dual oriented matroid $\mathcal{M}^*$ (see [Björner et al. 1993, § 7.1]). From this point of view, the subset of coherent zonotopal subdivisions of Z corresponds to the *coherent liftings* of V [Billera and Sturmfels 1992, § 5]. If one views realized oriented matroids and their liftings in terms of sphere and pseudosphere arrangements, then the notion of a coherent lifting was explored in the work of Bayer and Brandt [1997] on *discriminantal arrangements*, generalizing earlier work of Manin and Schechtman [1989]. The discriminantal arrangement associated to Z in [Bayer and Brandt 1997] is nothing more than the hyperplane arrangement which is the polar dual of the fiber zonotope associated to Z in [Billera and Sturmfels 1992].

Monotone paths. Let P be a polytope in $\mathbb{R}^d$, and f a linear functional in $(\mathbb{R}^d)^*$ that achieves its minimum and maximum values uniquely on P, say at two vertices $v_{\min}, v_{\max}$. A path from $v_{\min}$ to $v_{\max}$ in the 1-skeleton of the boundary of P will be called *f-monotone* if every step in the path is along an edge which strictly increases the value of f (as in the paths produced by the *simplex algorithm* for linear programming- see [Ziegler 1995, Lecture 3.2]). We wish to consider the structure of the set of all f-monotone paths. Note that these paths are exactly the subject of Ziegler's *strict monotone Hirsch conjecture* [Ziegler 1995, Conjecture 3.9].

Just as in the case of triangulations of $\mathcal{A}$ or tilings of a zonotope Z, there is a natural set of moves which connect f-monotone paths: if two paths agree in most of their steps and differ only by following opposite paths around some 2-dimensional face of P, we say that the two paths differ by a *polygon move*. Figure 12 illustrates the graph of f-monotone paths and polygon moves where P is the 3-cube $[0,1]^3 \subset \mathbb{R}^3$ and $f(x_1, x_2, x_3) = x_1 + x_2 + x_3$.

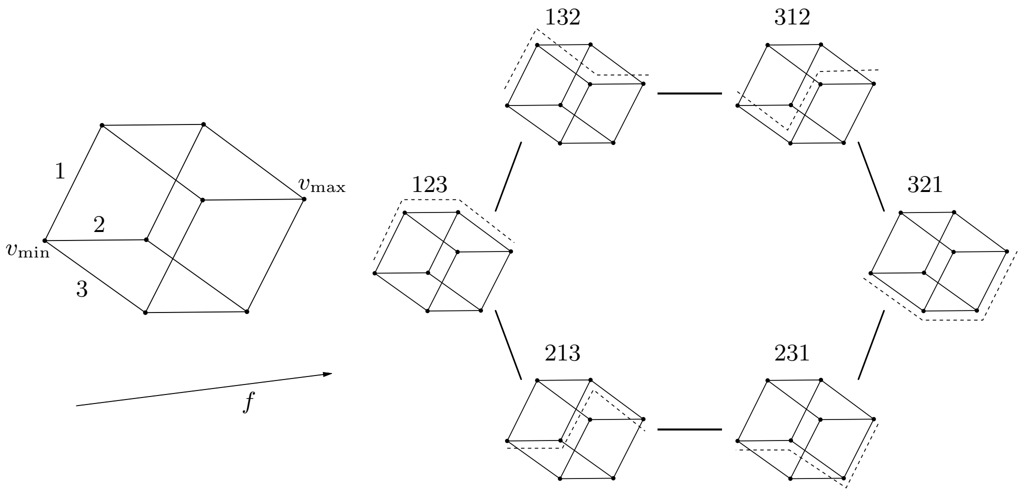

Figure 12. Monotone paths in the 3-cube. Edges pointing in the original three coordinate directions have been labelled 1, 2, 3, and then monotone paths are labelled by the sequence of directions of the steps.

In general if P is the n-cube $[0,1]^n \subset \mathbb{R}^n$ and $f(\boldsymbol{x}) = \sum_{i=1}^n x_i$, the f-monotone paths biject with permutations of $\{1, 2, \ldots, n\}$: one obtains a permutation by recording which coordinate axis is parallel to each step of the path in sequence, as in Figure 12. The polygon moves across square faces of the n-cube then correspond to *adjacent transpositions* of the permutations, and the whole graph

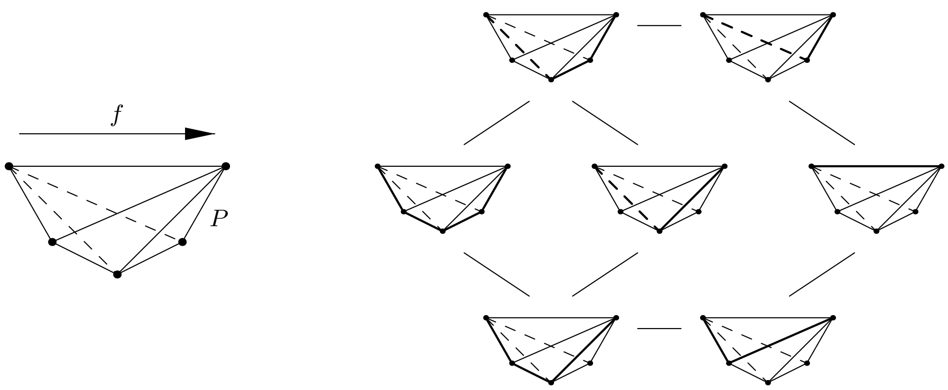

Figure 13. Monotone paths (thick lines) in a cyclic 3-polytope with five vertices.

is isomorphic to the 1-skeleton of the well-known *permutohedron* [Ziegler 1995, Example 0.10]; see also [Milgram 1966].

What happens for other polytopes P and functionals f? Figure 13 shows the graph of f-monotone paths in a (cyclic) 3-polytope with five vertices in $\mathbb{R}^3$ which is the convex hull of the points $\{(t, t^2, t^3) : t = -2, -1, 0, 1, 2\}$, and $f(x_1, x_2, x_3) = x_1$.

Although this graph is connected, it is perhaps disappointing that it is not circular as in the case of Figure 12. Once again, geometry comes to the rescue in singling out a well-behaved subset of f-monotone paths. Say that an f-monotone path γ on P is *coherent* if there exists some linear functional $g \in (\mathbb{R}^d)^*$ which induces γ in the following way: each point of γ (not necessarily a vertex) is the g-maximal point among all those points of P with the same f value, or in other words, γ is the union over all points x in $f(P) \subset \mathbb{R}$ of the g-maximal points in the fibers $f^{-1}(x)$. With this definition, the monotone path in the middle of the graph in Figure 13 is incoherent. To see this, assume there is some functional g inducing this monotone path, and identify g as the dot product with some fixed vector. Then this vector must point roughly toward the front (the visible side) of the polytope P in order to induce the right portion of the path, but also point toward the back (the invisible side) in order to induce the left portion of the path; contradiction. The remaining six paths in Figure 13 are easily seen to be coherent (by imagining appropriate functionals g) and the subgraph on the corresponding six vertices is indeed circular.

In general, it follows as a special case of Billera and Sturmfels' fiber polytope construction [Billera and Sturmfels 1992, § 7] that the graph of coherent f-monotone paths in a polytope P is the 1-skeleton of a polytope called the *monotone path polytope*. Higher dimensional faces of the monotone path polytope correspond to objects called *coherent cellular strings* on P with respect to f. A *cellular string* on P with respect to f is a sequence $(F_1, \ldots, F_r)$ of boundary faces of P with the following properties:

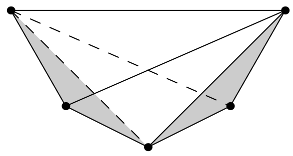

Figure 14. An incoherent cellular string.

- $v_{\min} \in F_1$ and $v_{\max} \in F_r$.
- f is not constant on any face F_i.
- For each i, the f-maximizing face of F_i is the f-minimizing face of F_{i+1}.

A cellular string $(F_1, \dots, F_r)$ is *coherent* if there exists some functional $g \in (\mathbb{R}^d)^*$ such that the union $\bigcup_{i=1}^r F_i$ equals the union over all points x in $f(P)$ of the g-maximal points in the fibers $f^{-1}(x)$.

As an exercise to get a feeling for how the cellular strings fit into the graph of f-monotone paths, and to further motivate the Baues problem, we invite the reader to try the following labelling exercise. Label each edge in the graph of Figure 13 by a cellular string containing mostly 1-faces along with exactly one triangular 2-face corresponding to the polygon move for that edge. Having done this, there is only one other possible cellular string, consisting of two triangles and pictured in Figure 14. This cellular string should label a square 2-cell attached to the four leftmost vertices and edges in Figure 13 (see also the middle picture in Figure 15). Notice that the resulting 2-dimensional cell complex is homotopy equivalent to the circular subgraph indexed by coherent cellular strings. This raises the following question.

Question 1.4. *For a polytope P and functional f, is the graph of f-monotone paths and their polygon moves connected? Is it part of a complex homotopy equivalent to a $(d-2)$-sphere?*

This question includes the original question asked by Baues [1980] as a special case. Specifically, Baues asked if the poset of cellular strings on the *permutohedron* with respect to a generic linear functional f has the homotopy type of a sphere (after endowing the poset with a certain topology: see Section 2). His question is a natural extension of ideas of Adams [1956] and Milgram [1966] involving edge paths in polytopes as models for loop spaces and iterated loop spaces: see [Rambau 1996, § 1.2] for a nice sketch of the ideas involved.

As we will see in Section 4, this original Baues question was answered positively by Billera, Kapranov and Sturmfels [Billera et al. 1994], who resolved it not only for cellular strings on the permutohedron, but on arbitrary polytopes. Another proof, for the case of arbitrary zonotopes, was given by Björner [1992].

2. Fiber Polytopes and the Baues Problem

The theory of fiber polytopes [Billera and Sturmfels 1992] provides a common framework in which to discuss triangulations, tilings, and monotone paths, and also a common notion of coherence for these objects. The fiber polytope $\Sigma(P \xrightarrow{\pi} Q)$, is a polytope naturally associated to any linear projection of polytopes $\pi : P \to Q$. Let P be a d'-dimensional polytope in $\mathbb{R}^{d'}$, Q a d-dimensional polytope in $\mathbb{R}^d$ and $\pi : \mathbb{R}^{d'} \to \mathbb{R}^d$ a linear map with $\pi(P) = Q$. A *polytopal subdivision* of Q is a polytopal complex which subdivides Q. A polytopal subdivision of Q is π-*induced* if

(i) it is of the form $\{\pi(F) : F \in \mathcal{F}\}$ for some specified collection $\mathcal{F}$ of faces of P having all $\pi(F)$ distinct, and
(ii) $\pi(F) \subseteq \pi(F')$ implies $F = F' \cap \pi^{-1}(\pi(F))$, and in particular $F \subseteq F'$.

It is possible that different collections $\mathcal{F}$ of faces of P project to the same subdivision $\{\pi(F) : F \in \mathcal{F}\}$ of Q, so we distinguish these subdivisions by labelling them with the family $\mathcal{F}$. We partially order the π-induced subdivisions of Q by $\mathcal{F}_1 \leq \mathcal{F}_2$ if and only if $\bigcup \mathcal{F}_1 \subseteq \bigcup \mathcal{F}_2$. The resulting partially ordered set is denoted by $\omega(P \xrightarrow{\pi} Q)$ and called the *Baues poset*. The minimal elements in this poset are the *tight* subdivisions, that is those for which F and $\pi(F)$ have the same dimension for all F in $\mathcal{F}$.

We next explain how π-induced subdivisions of Q generalize triangulations, tilings, and monotone paths. This is perhaps easiest to see for monotone paths and cellular strings. Given a polytope P and functional f, let Q be the 1-dimensional polytope $f(P)$ in $\mathbb{R}^1$. Then a cellular string $(F_1, \ldots, F_r)$ on P with respect to f gives rise to a family $\mathcal{F}$ satisfying the definition for a π-induced subdivision of Q as follows: $\mathcal{F}$ consist of the F_i's along with their f-minimizing and f-maximizing faces. Tight π-induced subdivisions of Q correspond to monotone paths on P.

For triangulations and tilings, there is a concealed projection of polytopes lurking in the background. Given a point set $\mathcal{A}$ in $\mathbb{R}^d$ with cardinality n, let Q denote its convex hull. There is a natural surjection $\pi : \Delta^{n-1} \to Q$ from a simplex Δ^{n-1} having n vertices, which sends each vertex of the simplex to one of the points of $\mathcal{A}$. One can then check that the π-induced subdivisions of Q as defined above correspond to the following notion of a subdivision of $\mathcal{A}$, which replaces the naive definition given in Section 1. A subdivision of $\mathcal{A}$ is a collection of pairs $\{(Q_\alpha, \mathcal{A}_\alpha)\}$, where

- $\mathcal{A}_\alpha$ are subsets of $\mathcal{A}$,
- each Q_α is the convex hull of $\mathcal{A}_\alpha$ and is d-dimensional,
- the union of the Q_α covers Q, and
- for any α, β, the intersection $Q_\alpha \cap Q_\beta$ is a face F (possibly empty) of each, and $\mathcal{A}_\alpha \cap F = \mathcal{A}_\beta \cap F$.

Tight π-induced subdivisions of Q correspond to triangulations of $\mathcal{A}$. Furthermore, the Baues poset corresponds to the natural *refinement* ordering on subdivisions of $\mathcal{A}$: $\{(Q_\alpha, \mathcal{A}_\alpha)\} \leq \{(Q'_\beta, \mathcal{A}'_\beta)\}$ if and only if for every α there exists some β with $\mathcal{A}_\alpha \subseteq \mathcal{A}'_\beta$ (and hence also $Q_\alpha \subseteq Q'_\beta$).

Let Z be a zonotope in $\mathbb{R}^d$ generated by n line segments V. Without loss of generality we may assume that these segments all have one endpoint at the origin, and we can think of them as vectors pointing in a certain direction rather than segments. There is then a natural surjection $\pi : I^n \to Z$ of the n-cube I^n in $\mathbb{R}^n$ onto Z which sends the standard basis vectors in $\mathbb{R}^n$ onto the vectors V. Then the π-induced subdivisions of Q as defined above correspond to a notion of zonotopal subdivision of Z explained carefully in [Richter-Gebert and Ziegler 1994, Definitions 1.3 and 1.4], and which replaces the naive definition we gave in Section 1. Tight π-induced subdivisions then correspond to cubical tilings of Z, and the Baues poset corresponds to a natural *refinement* ordering on zonotopal subdivisions. As was remarked in Section 1, the Bohne–Dress Theorem shows that the zonotopal subdivisions, or equivalently π-induced subdivisions of Z, are the same as single-element liftings of the realized oriented matroid $\mathcal{M}$ corresponding to the vectors V. It is furthermore true that the Baues poset corresponds to the usual *weak map* ordering [Björner et al. 1993, § 7.2] on single-element liftings of $\mathcal{M}$, or equivalently on the single-element extensions of the dual $\mathcal{M}^*$.

Returning to our general set-up of a projection $\pi : P \to Q$, we wish to define when a π-induced subdivision is π-*coherent*, generalizing the notion of coherence for triangulations, tilings, and monotone paths. There is more than one way to say this, and we start with one of the descriptions from [Billera and Sturmfels 1992]. Choose a linear functional $g \in (\mathbb{R}^{d'})^*$. For each point q in Q, the fiber $\pi^{-1}(q)$ is a convex polytope which has a unique face $\overline{F_q}$ on which the value of g is minimized. This face lies in the relative interior of a unique face F_q of P and the collection of faces $\mathcal{F} = \{F_q\}_{q \in Q}$ projects under π to a subdivision of Q. Subdivisions of Q which arise from a functional g in this fashion are called π-*coherent*. Note that this definition of π-coherence clearly generalizes our earlier notion of coherence for cellular strings on P.

It is possible to rephrase the definition of π-coherent subdivisions given in [Billera and Sturmfels 1992] as follows (see also [Ziegler 1995, § 9.1]). Having chosen the functional $g \in (\mathbb{R}^{d'})^*$ as above, form the graph of the linear map $\hat{\pi} : P \to \mathbb{R}^{d+1}$ given by $p \mapsto (\pi(p), g(p))$. The image of this map is a polytope $\hat{Q}$ in $\mathbb{R}^{d+1}$ which maps onto Q under the forgetful map $\mathbb{R}^{d+1} \to \mathbb{R}^d$. Therefore, the set of *lower faces* of $\hat{Q}$ (those faces whose normal cone contains a vector with negative last coordinate) form a polytopal subdivision of Q. We identify this subdivision of Q with the family of faces $\mathcal{F} = \{F\}$ in P which are the inverse images under $\hat{\pi}$ of the lower faces of $\hat{Q}$. Under this identification, it is not hard to check that the subdivision of Q is exactly the same as the π-coherent subdivision induced by g, described in the previous paragraph. A glance at the definitions shows that this second definition of π-coherence generalizes the ones we gave for coherent subdivisions of a point set $\mathcal{A}$ and for coherent zonotopal subdivisions of a zonotope Z.

Let $\omega_{\mathrm{coh}}(P \xrightarrow{\pi} Q)$ denote the induced subposet of the Baues poset $\omega(P \xrightarrow{\pi} Q)$ on the set of π-coherent subdivisions of Q. The following beautiful result of Billera and Sturmfels which explains all of our pretty polytopal pictures is the following:

THEOREM 2.1 [Billera and Sturmfels 1992, Theorem 3.1]. *Let P be a d'-polytope, Q a d-polytope, and $\pi : P \to Q$ a linear surjection. Then the poset*

$$\omega_{\mathrm{coh}}(P \xrightarrow{\pi} Q)$$

is the face poset of a $(d'-d)$-polytope $\Sigma(P \xrightarrow{\pi} Q)$.

In particular, the tight π-coherent subdivisions of Q correspond to the vertices of $\Sigma(P \xrightarrow{\pi} Q)$.

The $(d'-d)$-polytope $\Sigma(P \xrightarrow{\pi} Q)$ is called the *fiber polytope* of the surjection π. It generalizes the *secondary polytopes* $\Sigma(\mathcal{A})$, *fiber zonotopes*, and *monotone path polytopes* encountered in Section 1. A striking feature of $\Sigma(P \xrightarrow{\pi} Q)$ is that it can also be constructed as the "Minkowski average" over points $q \in Q$ (in a well-defined sense; see [Billera and Sturmfels 1992, § 2]) of all of the polytopal fibers $\pi^{-1}(q)$. For an algebro-geometric interpretation of the fiber polytope $\Sigma(P \xrightarrow{\pi} Q)$ in terms of *Chow quotients* of toric varieties; see [Kapranov et al. 1991; Hu $\geq$ 1999].

As a consequence of Theorem 2.1, if one removes the top element $\hat{1}$ from $\omega_{\mathrm{coh}}(P \xrightarrow{\pi} Q)$, corresponding to the improper π-coherent subdivision $\mathcal{F} = \{P\}$, one obtains the face poset of a polytopal $(d'-d-1)$-sphere, that is the boundary of $\Sigma(P \xrightarrow{\pi} Q)$. The generalized Baues problem asks roughly how close the whole Baues poset $\omega(P \xrightarrow{\pi} Q) - \hat{1}$ is topologically to this sphere. Before phrasing the problem precisely, we must first give the poset $\omega(P \xrightarrow{\pi} Q) - \hat{1}$ a topology. The standard way to do this is to consider its *order complex*, the abstract simplicial complex of chains in the poset [Björner 1995, (9.3)]. From here on, we will abuse notation and use the name of any poset also to refer to the topological space which is the geometric realization of its order complex.

We can now state the *Generalized Baues Problem*, in at least two forms, one stronger than the other. Both of these forms appear, implicitly or explicitly, either in the first mention of the problem by Billera and Sturmfels [1992, p. 545] or in the later formulation of [Billera et al. 1994, § 3].

QUESTION 2.2 (WEAK GBP). *Is $\omega(P \xrightarrow{\pi} Q) - \hat{1}$ homotopy equivalent to a $(d'-d-1)$-sphere ?*

QUESTION 2.3 (STRONG GBP). *Is the inclusion*

$$\omega_{\mathrm{coh}}(P \xrightarrow{\pi} Q) - \hat{1} \quad \hookrightarrow \quad \omega(P \xrightarrow{\pi} Q) - \hat{1}$$

a strong deformation retraction?

The strong GBP captures the sense we had from Figures 2 and 13 that the incoherent subdivisions were nothing more than "warts" attached to the spherical subcomplex indexed by the coherent subdivisions, and that these warts could be retracted onto this subcomplex. We should beware, however, that these pictures of small examples can be deceptive. In particular, we mention a vague metaconjecture that has several examples of empirical evidence; see [Athanasiadis 1999; de Loera et al. 1996; Athanasiadis et al. 1997, Remark 3.6]:

Vague Metaconjecture 2.4. *Let $P_n \to Q_n$ be a "naturally occurring" infinite sequence of polytope surjections in which either*

$$\dim(Q_n) \to \infty \quad \textit{or} \quad \dim(P_n) - \dim(Q_n) \to \infty$$

as n approaches infinity. Assume also that for some value of n there exist π-induced subdivisions of Q_n that are π-incoherent. Then as n approaches infinity, the fraction of the number of π-coherent subdivisions out of the total number of π-induced subdivisions approaches 0.

In other words, the warts take over eventually.

Besides the weak and strong versions, one can imagine other intermediate versions of the GBP. For example, one might ask whether the inclusion referred to in the strong GBP induces only a homotopy equivalence, rather than the stronger property of being a deformation retraction. We will resist naming these other versions, since they seem not to have been addressed in the literature.

Knowing that $\omega_{\text{coh}}(P \xrightarrow{\pi} Q)$ is the poset of faces of a (polytopal) regular cell complex, the reader may be disappointed that we have not defined the entire Baues poset $\omega(P \xrightarrow{\pi} Q)$ to be the poset of faces in some regular cell complex, since it appears to be so in all of our small examples. For example, Figure 15 shows the order complex of $\omega(P \xrightarrow{\pi} Q)-\hat{1}$ for the example in Figure 13, which turns out to be the *barycentric subdivision* of the regular cell complex one would have liked to call "the Baues complex". Whenever such a regular cell complex exists, then of course, the order complex of $\omega(P \xrightarrow{\pi} Q)-\hat{1}$ will be its barycentric subdivision, and hence homeomorphic to the original regular cell complex. Unfortunately,

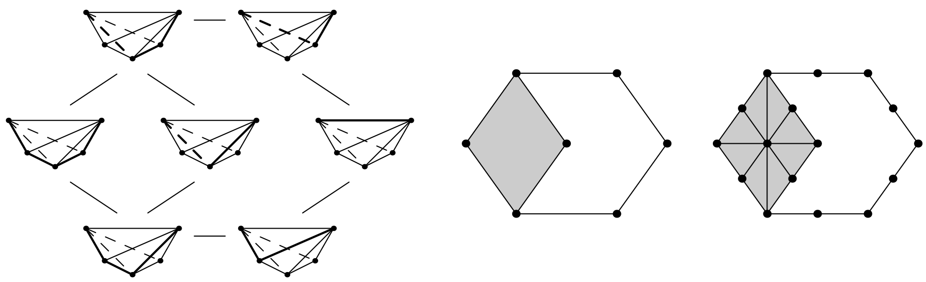

Figure 15. Left: Monotone paths as in Figure 13, defining P and Q. Middle: A regular cell complex that has $\omega(P \xrightarrow{\pi} Q) - \hat{1}$ as its poset of faces. Right: The order complex of $\omega(P \xrightarrow{\pi} Q) - \hat{1}$.

such a regular cell complex does not exist in general; relatively small examples show that lower intervals in $\omega(P \xrightarrow{\pi} Q) - \hat{1}$ need not be *homeomorphic* to spheres, which is the necessary condition for a poset to be the poset of faces of a regular cell complex [Björner 1995, (12.5)]. One way to obtain such an example is to add a seventh point a_0 to the point configuration $\mathcal{A}$ in Figure 2, in any location in the same plane. Then the unique proper subdivision of $\mathcal{A} \cup \{a_0\}$ which leaves the convex hull of $\mathcal{A}$ completely unrefined lies at the top of a lower interval that is not homeomorphic to a sphere.

3. Relations to Other Problems

Having stated the GBP, we can now explain how it relates to some of our previous questions, and to other problems in discrete geometry and topological combinatorics.

Connectivity questions. Questions 1.2, 1.3, and 1.4 are clearly related to the GBP, and appear at first glance to be weaker, in that they only ask for connectivity of a certain graph rather than homotopy sphericity of a complex. However, a positive answer to the strong GBP does not quite imply a positive answer to either of these questions. There are at least two subtleties associated with this conclusion, which we will now attempt to make precise.

For an element of a finite poset, let its *rank* be the length of the shortest saturated chain below it in the poset, so that minimal elements have rank 0. Let A_π, B_π denote the elements at rank 0 and rank 1 respectively in the Baues poset $\omega(P \xrightarrow{\pi} Q)$, and let G_π be the graph on the union $A_\pi \cup B_\pi$ obtained by restricting the Hasse diagram for $\omega(P \xrightarrow{\pi} Q)$ to this union of its bottom two ranks. Given a point set $\mathcal{A}$, let $G_{\mathcal{A}}$ denote its graph of triangulations and bistellar operations. Similarly, for a zonotope Z, let G_Z be its graph of cubical tilings and cube flips, and for a polytope P with a linear functional f, let $G_{P,f}$ be the graph of monotone paths and polygon moves.

The first subtlety we encounter is the relation between the graphs $G_{\mathcal{A}}$, G_Z, $G_{P,f}$ and the graph G_π. It is tempting to say that the barycentric subdivision of $G_{\mathcal{A}}$, G_Z, or $G_{P,f}$ is the same as G_π for the appropriate map π, since the vertex sets of each of these barycentric subdivisions forms a subset of the vertices of the appropriate G_π. However, it takes some work to show that these graphs coincide.

- In the case of point sets $\mathcal{A}$, a slight generalization of the *pulling construction* described by Lee [1991, § 2] can be used to show that every subdivision can be refined to a triangulation, so elements of rank 0 in the Baues poset coincide with the triangulations. Furthermore, the results and ideas of [Santos 1999] can be used to show that the elements of rank 1 coincide with the bistellar operations. Both of these assertions are easy when $\mathcal{A}$ is in general position, but otherwise become subtle.

- In the case of zonotopes Z, it is known that every zonotopal subdivision can be refined to a cubical tiling [Björner et al. 1993, Corollary 7.7.9], so elements of rank 0 in the Baues poset coincide with cubical tilings. For cube flips, one must first define these flips "correctly" for zonotopes in dimensions higher than 2, using the oriented matroid notion of mutations on the single element lifting associated to the tiling. Then a result of Santos [1997a, Theorem 4.14(ii)] combined with our previous assertion about bistellar flips can be used to prove that the elements of rank 1 in the Baues poset correspond to cube flips [Santos 1998].
- In the case of a polytope P and functional f, one can check directly that every cellular string can be refined to a monotone path, so elements of rank 0 in the Baues poset coincide with monotone paths. However, whenever the functional f is not generic in the sense that it is constant on some edge of P, one needs to be careful about how one defines polygon moves. Our previous naive definition will not suffice, as illustrated by Figure 16. Nevertheless, it is possible to correct this definition so that all elements of rank 1 in the Baues poset correspond to these corrected polygon moves [Santos 1998].

The second subtlety arises from the fact that even in cases where the GBP has a positive answer, connectivity of $\omega(P \xrightarrow{\pi} Q)$ does not necessarily imply connectivity of the graph G_π, since the 1-skeleton of $\omega(P \xrightarrow{\pi} Q)$ (or rather its order complex) contains some vertices corresponding to poset elements with ranks higher than $0, 1$. On the other hand, one would like to be able to apply the following easily verified lemma to $\omega(P \xrightarrow{\pi} Q)$:

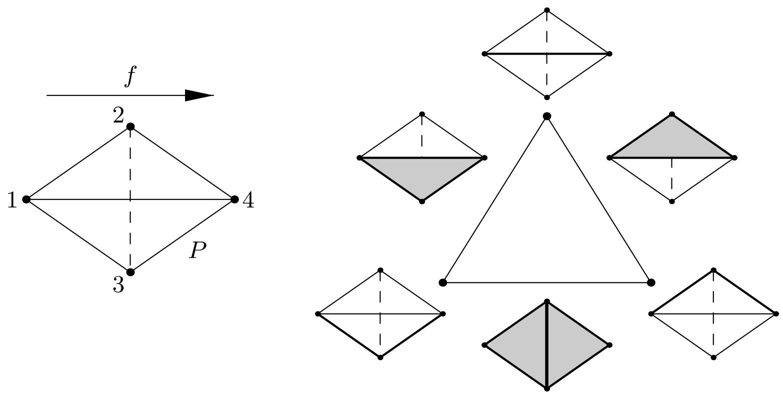

Figure 16. Left: A tetrahedron P with a nongeneric functional f for which vertices $2, 3$ have the same f-value. Right: The boundary complex of the fiber polytope $\Sigma(P \xrightarrow{\pi} Q)$, whose face poset coincides with the poset of all cellular strings (all are coherent). Note that the cellular string $(123, 234)$ labelling the bottom edge of this complex does not correspond to a polygon move as we had earlier defined it.

LEMMA 3.1. *Let X be a finite poset with a top element $\hat{1}$, and assume that X has the property that every strict principal order ideal $X_{<x} := \{x' \in X : x' < x\}$ either is connected, empty, or consists of two incomparable poset elements.*

Then the graph obtained by restricting X to its elements at rank 0 and 1 is connected.

Of course the GBP only implies the hypotheses of this lemma are satisfied with $X = \omega(P \xrightarrow{\pi} Q)$ for the strict principal order ideal $X_{<\hat{1}}$. But there is some hope that if one could prove the weak GBP in some case, then one can also prove homotopy sphericity for the rest of the order ideals $X_{<x}$, and hence can use the lemma. Under the genericity assumptions which were mentioned above for triangulations and monotone paths, one can check that these principal order ideals are Cartesian products of Baues posets for smaller polytopes, and hence their connectivity follows from positive answers to the GBP for these smaller polytopes. In particular, the positive answer for the strong GBP for monotone paths [Billera et al. 1994] (to be discussed in the next section) implies a positive answer to all of Question 1.4 under the assumption that the functional f is generic. Without such genericity assumptions, the structure of these principal order ideals may be more complicated. A specific study of these principal order ideals in the case of triangulations of a point set $\mathcal{A}$ was initiated by Santos [1999].

Flip deficiency. While we are discussing Questions 1.2, 1.3, and 1.4, it is appropriate to mention questions about the number of bistellar neighbors of a triangulation, the number of cube flip neighbors of a tiling, and the number of polygon-move neighbors of a monotone path. In the general setting of $\pi : P \to Q$, every tight π-coherent subdivision of Q represents a vertex of the $(d'-d)$-polytope $\Sigma(P \xrightarrow{\pi} Q)$, and therefore will have at least $d' - d$ neighboring tight π-coherent subdivisions lying along the edges of the polytope. On the other hand, π-induced subdivisions which are not π-coherent may have fewer neighbors, in which case we will say that the subdivision in question has *flip deficiency.* If the subdivision has no neighbors we say that it is *isolated*, which of course gives a negative answer to the GBP if $d' - d > 1$ in that case. Note that the example with all coherent cellular strings in Figure 16 shows that for monotone paths, we must either be careful to restrict ourselves to the case of a generic functional f, or else redefine what is meant by a "polygon-move" in talking about flip-deficiency.

Flip deficiency has been very well-explored for cubical tilings of zonotopes in the guise of counting *simplicial regions* of hyperplane arrangements or *mutations* in oriented matroids; see [Richter-Gebert 1992, Introduction § 3] for a nice summary. For triangulations, flip-deficiency has been explored only more recently; see [de Loera et al. 1999; Santos 1997b]. For monotone paths, the question of flip deficiency appears not to have been considered much at all.

A related question concerns the level of connectivity of the graphs $G_{\mathcal{A}}$, G_Z, $G_{P,f}$ of triangulations and bistellar moves, tilings and cube flips, monotone paths and polygon moves respectively. For each of these graphs, the induced subgraph on the coherent elements is the 1-skeleton of $(d'-d)$-polytope and hence is $(d'-d)$-*vertex-connected* in the graph-theoretic sense by Balinski's Theorem [Ziegler 1995, § 3.5]. One can ask whether the entire graphs $G_{\mathcal{A}}$, G_Z, $G_{P,f}$ share the same level of vertex-connectivity, which is stronger than saying that every vertex has at least $d'-d$ neighbors. Very recently, Azaola and Santos [Azaola and Santos 1999] proved the first nontrivial positive result in this direction, showing that for point sets $\mathcal{A}$ in $\mathbb{R}^d$ with $d+4$ points (so $d'-d=3$), the graph of triangulations and bistellar moves is 3-connected. The question has only been resolved negatively in some cases where flip-deficiency exists [de Loera et al. 1999; Santos 1997b].

Extension spaces, MacPhersonians and OM-Grassmannians. Let Z be a d-dimensional zonotope generated by a set of n vectors V. As mentioned in Section 1, one can associate to V its oriented matroid $\mathcal{M}$. The Bohne–Dress Theorem [Bohne 1992; Richter-Gebert and Ziegler 1994] then implies that the Baues poset $\omega(P \xrightarrow{\pi} Q)$ is isomorphic to the *extension poset* $\mathcal{E}(\mathcal{M}^*)$, consisting of all single-element extensions of the dual oriented matroid $\mathcal{M}^*$ ordered by weak maps. The following *Extension Space Conjecture* [Björner et al. 1993, § 7.2] appears not to be attributable to any single source:

Conjecture 3.2. *For a realizable oriented matroid* $\mathcal{N}$, *the order complex of the extension poset* $\mathcal{E}(\mathcal{N}) - \hat{1}$ *is homotopy equivalent to a* $(\operatorname{rank}(\mathcal{N})-1)$-*sphere*.

Hence the extension space conjecture is equivalent to the special case of the weak GBP dealing with zonotopal subdivisions. We will discuss positive cases of this conjecture, mostly taken from [Sturmfels and Ziegler 1993], in Section 4, but we mention that the results of Mnëv and Richter-Gebert [1993] show that one cannot remove the assumption that $\mathcal{N}$ is realizable. They cleverly construct nonrealizable oriented matroids $\mathcal{N}$ of rank 4 for which $\mathcal{E}(\mathcal{N}) - \hat{1}$ is disconnected!

The extension space $\mathcal{E}(\mathcal{N})$ is also closely related to certain combinatorial models of Grassmannians called *OM-Grassmannians* (see [Anderson 1999a] in this same volume, or [Richter-Gebert 1992, Introduction, § 4; Rambau 1996, § 1.2; Mnëv and Ziegler 1993] for fuller discussions). Briefly, given an oriented matroid $\mathcal{M}$, the *OM-Grassmannian* $\mathbb{G}_k(\mathcal{M})$ is the poset of rank k oriented matroids which are *strong images* [Björner et al. 1993, § 7.7] of $\mathcal{M}$, ordered by weak maps. If $\mathcal{M}$ has rank d, the order complex of $\mathbb{G}_k(\mathcal{M})$ is intended as a combinatorial model for the Grassmannian of k-planes in $\mathbb{R}^d$. In the special case where $\mathcal{M}$ is the *Boolean* or *free* oriented matroid on d elements, $\mathbb{G}_k(\mathcal{M})$ is called the *MacPhersonian* $\operatorname{MacP}(d,k)$, due to its occurrence in the work of Gelfand and MacPherson [1992; [MacPherson 1993]] on combinatorial formulas for characteristic classes. The following was conjectured by MacPherson when $\mathcal{M}$ is Boolean, and for all realizable $\mathcal{M}$ by Mnëv and Ziegler [Richter-Gebert 1992, Conjecture 4.2; Mnëv and Ziegler 1993, Conjecture 2.2].

CONJECTURE 3.3. *If $\mathcal{M}$ is a realizable oriented matroid of rank d, then $\mathbb{G}_k(\mathcal{M})$ is homotopy equivalent to the Grassmannian of k-planes in $\mathbb{R}^d$.*

Babson [1993] showed that Conjecture 3.3 is true for $k \leq 2$, and in the Boolean case, that $\mathrm{MacP}(d, 3)$ is homotopy equivalent to the appropriate Grasmmannian. See [Mnëv and Ziegler 1993].

The relation to extension spaces and the Baues problem is that the extension poset $\mathcal{E}(\mathcal{M})$ is a double cover of $\mathbb{G}_{d-1}(\mathcal{M})$ in the sense that there is a two-to-one order-preserving map $\mathcal{E}(\mathcal{M}) \to \mathbb{G}_{d-1}(\mathcal{M})$. As a consequence, one can view the conjecture that $\mathbb{G}_{d-1}(\mathcal{M})$ is homotopy equivalent to the Grassmannian of $(d-1)$-planes in $\mathbb{R}^d$ (or $(d-1)$-dimensional real projective space) as a projectivized version of the Extension Space Conjecture. This also implies that the positive results of Sturmfels and Ziegler [1993] on the Extension Space Conjecture 3.2 give some special cases of Conjecture 3.3.

4. Positive Results

In this section we review results which give a positive answer to the weak or strong GBP. The methods used tend to segregate into the three paradigms described below, where we have indicated the references whose proofs exemplify these paradigms:

Retraction: A proof of the strong GBP, by exhibiting an explicit homotopy retracting $\omega(P \xrightarrow{\pi} Q)$ onto $\omega_{\mathrm{coh}}(P \xrightarrow{\pi} Q)$. See [Billera et al. 1994, Theorem 2.3; Rambau and Ziegler 1996, Theorem 1.4; Athanasiadis et al. 1997, Theorem 1.2].

Homotopies: A proof of the weak GBP by a short chain of homotopy equivalences from $\omega(P \xrightarrow{\pi} Q)$ to some poset known to have spherical homotopy type. See [Björner 1992, Theorem 2; Edelman et al. 1997, Theorem 1.2].

Deletion-Contraction: An inductive proof of the weak GBP using (sometimes implicitly) the notion of *deletion-contraction* from matroid theory. See [Billera et al. 1994, Theorem 1.2; Sturmfels and Ziegler 1993, Theorem 1.2; Edelman and Reiner 1998, Theorem 3; Rambau and Santos 1997, Theorem 1.1].

Recall the general set-up: we consider a linear surjection of polytopes $\pi : P \to Q$ with P, Q being d', d-dimensional, respectively, and with P having n vertices. We divide our discussion of positive results into the following categories:

- $d = 1$ (monotone paths),
- $d' - d = 2$ (low codimension),
- P=cube (zonotopal tilings),
- $n - d' = 1$ or P=simplex (triangulations),
- cyclic polytopes.

The case $d = 1$: Monotone paths. The original paper of Billera, Kapranov and Sturmfels that posed the GBP [Billera et al. 1994] proves both the weak and strong GBP for monotone paths and cellular strings, under our usual genericity assumption that f is nonconstant along each edge of P. Rambau and Ziegler [1996] claim that the proofs in [Billera et al. 1994] can be adapted to remove this assumption. There are two proofs given in [Billera et al. 1994], one which follows the Retraction paradigm in proving the strong GBP (their Theorem 2.3) and one which implicityly uses the Deletion-Contraction paradigm (their Theorem 1.2) to prove the weak GBP.

This settles the original problem of Baues [1980, Conjecture 7.4], which is the special case in which the polytope P is a permutohedron and f is a generic functional. The weak GBP for cellular strings on zonotopes, as in Baues' special case, also follows from work of Björner [1992, Theorem 2] (motivated by the preprint version of [Billera et al. 1994]), which is a good example of the Homotopies paradigm. Björner observes that cellular strings on a zonotope Z are the same as what he calls the *essential chains* in the *poset of regions* [Edelman 1984] of the hyperplane arrangement which is the polar dual [Ziegler 1995, § 7.3] to Z. An essential chain in a bounded poset is a chain from the bottom element to the top element in which every step corresponds to a noncontractible (open) interval. Björner shows that the subposet of essential chains ordered by refinement is homotopy equivalent to the order complex of all chains in the proper part of the poset. For the poset of regions of a hyperplane arrangement, the homotopy type is known to be spherical by work of Edelman and Walker [1985]. Björner actually proves his result not just for zonotopes or realized oriented matroids, but for an arbitrary oriented matroid, where the notion of a cellular string and the poset of regions still make sense.

The case $d' - d \leq 2$: Low codimension. In very low codimension there is not much to say. If $d' - d = 0$ then $P = Q$ and the only π-induced subdivision of Q is the improper one. In the case $d' - d = 1$, there are exactly two proper π-induced subdivisions of Q, one coming from the "top" faces of P with respect to the projection π, the other coming from the "bottom" faces. Both of these subdivisions are coherent, and hence $\omega_{\mathrm{coh}}(P \xrightarrow{\pi} Q)$, $\omega(P \xrightarrow{\pi} Q)$ are both 0-spheres.

In the case $d' - d = 2$, the fiber polytope $\Sigma(P \xrightarrow{\pi} Q)$ is a polygon, and hence $\omega_{\mathrm{coh}}(P \xrightarrow{\pi} Q)$ is its boundary circle. Rambau and Ziegler [1996] use the Retraction paradigm to prove the strong GBP in this case.

The case P = cube: zonotopal tilings. We saw in Section 2 that the case when P is a d'-cube corresponds to the case of zonotopal subdivisions and tilings of the zonotope $Z = Q = \pi(P)$. Furthermore, if $\mathcal{M}$ denotes the oriented matroid associated to the generating segments V of Z, then we saw that the Baues poset is the same as the poset of *single-element extensions* $\mathcal{E}(\mathcal{M}^*)$ for the dual oriented matroid $\mathcal{M}^*$, and the weak GBP is the same as the Extension Space Conjecture (Conjecture 3.2) for $\mathcal{M}^*$.

The extension space conjecture was investigated by Sturmfels and Ziegler [1993], who proved most of the strongest positive results at present. They showed that an inductively defined technical hypothesis called *strong Euclideanness* on the oriented matroid $\mathcal{M}$ implies that the extension space conjecture holds, using the Deletion-Contraction paradigm. They then showed that an oriented matroid on n elements with rank r is strongly Euclidean under various hypotheses: if $r \leq 3$, or $n - r \leq 2$, or when $\mathcal{M}$ is the *alternating* oriented matroid $C^{n,r}$ that comes from a cyclic arrangement of vectors [Björner et al. 1993, § 9.4]. Since oriented matroid duality exchanges r for $n - r$ and keeps n fixed, and since the alternating oriented matroids satisfy $(C^{n,r})^* \cong C^{n,n-r}$, their results imply the weak GBP when P is a d'-cube and $Q = Z$ is a d-dimensional zonotope under the following conditions:

- $d' - d \leq 3$, or
- $d \leq 2$, or
- Z is a cyclic zonotope.

It was also shown by Bailey [1997] that the hypothesis of strong Euclideanness holds for $\mathcal{M}^*$ when $\mathcal{M}$ is the oriented matroid associated to a d-dimensional zonotope having $d + 1$ generic generating segments, but with arbitrary multiple copies of each segment. Hence the weak GBP also holds for tilings of such zonotopes. We remark that for $d = 2$, the cubical tilings of these zonotopes (hexagons) were enumerated by MacMahon [1915–16, vol. 2, § X] in 1899.

Before closing our discussion of the Baues problem for tilings, we would like to mention an important result of Santos which shows that the GBP for zonotopal tilings is a special case of the GBP for triangulations. To any realized oriented matroid $\mathcal{M}$ one can associate a polytope $\Lambda(\mathcal{M})$ known as its *Lawrence polytope* [Bayer and Sturmfels 1990; Billera and Munson 1984; Santos 1997a, Chapter 4; Björner et al. 1993, § 9.3], using the technique of *Gale transforms*. This construction, due to Jim Lawrence (unpublished; see [Ziegler 1995, p. 183]) gives an encoding of all the information of the oriented matroid $\mathcal{M}$ into the face lattice of the polytope $\Lambda(\mathcal{M})$, and is useful for transferring matroid constructions and examples into the world of polytopes.

THEOREM 4.1 [Santos 1997a, Theorem 4.14; Huber et al. 1998]. *Let Z be a zonotope with associated oriented matroid $\mathcal{M}$. There is a natural bijection between the subdivisions of $\Lambda(\mathcal{M})$ and the zonotopal subdivisions of Z (=single-element liftings of $\mathcal{M}$) which induces an isomorphism between the associated Baues posets.*

Consequently, a negative answer to the GBP for zonotopal tilings produces a negative answer for triangulations. We remark that the Lawrence construction applies more generally to oriented matroids $\mathcal{M}$ which are not necessarily realizable, yielding a *matroid polytope* $\Lambda(\mathcal{M})$ [Björner et al. 1993, § 9.1] rather than a polytope. Santos' result also applies in this situation, where one defines the triangulation of a matroid polytope via his definition of a *triangulation of an* *oriented matroid* [Santos 1997a]. This definition unifies previous notions of such triangulations that had been proposed by Billera and Munson [1984] and Anderson [1999b].

The case $n - d' = 1$ or P = simplex: triangulations. When $n - d' = 1$ the polytope P must be an n-dimensional simplex Δ^{n-1}. We saw that in this case the Baues poset is the poset of subdivisions of the point set $\mathcal{A}$, where $\mathcal{A}$ is the image under π of the vertices of Δ^{n-1}.

When the dimension d of $\mathcal{A}$'s ambient space is very small, as in the case of zonotopal tilings in $\mathbb{R}^1$, there is not much to say. For $d \leq 1$ every subdivision is coherent, and the secondary (or fiber) polytope $\Sigma(\mathcal{A})$ is a Cartesian product of simplices whose dimensions are given by the multiplicities of the interior points in $\mathcal{A}$, and one less than the multiplicities of the end points.

For $d = 2$, things start to get interesting. The fact that the graph of triangulations and bistellar operations is connected follows from work of Lawson [1977], who gave an algorithm which starts with any triangulation and moves it toward a particular coherent triangulation called the *Delaunay triangulation*. Joe [1989] observed that this procedure does *not* work in general for $d = 3$. However, Rajan [1994] and Edelsbrunner and Shah [1992] observed that a generalization of this flipping procedure works to move a particular coherent triangulation to the unique coherent triangulation which is induced by some chosen set of heights. In fact, this procedure amounts to nothing more than linear programming on the secondary polytope $\Sigma(\mathcal{A})$.

It is claimed at the end of [Billera et al. 1994] that one can positively answer the GBP for $\mathcal{A}$ in $\mathbb{R}^2$, and this was justified under the extra assumption that the points lie in general position by Edelman and Reiner [1998] using the Deletion-Contraction paradigm. The idea in their proof is to choose an extreme point v of $\mathcal{A}$, and use the fact that every subdivision of $\mathcal{A}$ gives rise to a lower-dimensional subdivision of the *vertex figure* of $\mathcal{A}$ at v. This gives an order-preserving map of subdivision posets which is shown to induce a homotopy equivalence by a technical argument, akin to the usual Quillen Fiber Lemma [Björner 1995, (10.5)]. The question of flip-deficiency for $\mathcal{A}$ in $\mathbb{R}^2$ was resolved by de Loera, Santos and Urrutia [1999]. They give a clever counting argument involving Euler's formula, showing that every triangulation has at least $|\mathcal{A}| - 3$ bistellar neighbors, so there is no flip deficiency.

For $\mathcal{A}$ in $\mathbb{R}^3$ and higher dimensions, our knowledge of bistellar connectivity and the GBP for triangulations is astoundingly limited. Very recent work of Azaola and Santos [1999] shows that in low codimension, $d' - d = 3$, the graph of triangulations and bistellar operations is 3-connected, so in particular, there is no flip deficiency in this case. De Loera, Santos and Urrutia [de Loera et al. 1999] used a similar counting argument as in the $d = 2$ case to show that for $\mathcal{A}$ in $\mathbb{R}^3$ in general position *and convex position* (i.e., no point of $\mathcal{A}$ is in the convex hull of the rest) there can be no flip deficiency. Both of these positive results are tight, in a sense, since an unpublished example of de Loera, Santos and Urrutia gives a triangulation of a configuration of 8 points in $\mathbb{R}^3$ with one point interior, having only 3 bistellar neighbors. They also exhibit in [de Loera et al. 1999], a triangulation of 9 points in $\mathbb{R}^3$ in general position with on point interior, having only 4 bistellar neighbors, and a triangulation of 10 points in convex general position in $\mathbb{R}^4$ having only 4 bistellar neighbors.

There are relatively few families of polytopes in higher dimensions whose triangulations have been well-studied, other than the cyclic polytopes which will be discussed in the next heading. We mention a few of these other families here.

Triangulations of the d-cube which use few maximal simplices are desirable for the purposes of fixed point algorithms [Todd 1976; Ziegler 1995, Problem 5.10]. Therefore one would be interested in algorithms which enumerate the triangulations, such as the program PUNTOS [de Loera 1995a], which enumerates all the triangulations lying in the same connected component of the graph of bistellar operations as the coherent triangulations. Unfortunately, de Loera [de Loera 1995b, Theorem 2.3.20; 1996] has shown that incoherent triangulations of the d-cube exist for $d \geq 4$ (including some with flip deficiency) so it is not known whether one can produce all triangulations of the cube by this method.

We momentarily digress to point out a (perhaps) surprising fact about the triangulations of a point set $\mathcal{A}$ which are extremal with respect to the number of maximal simplices — they need not be coherent! Such an example comes from work of Ohsugi and Hibi [1997], and was further analyzed by de Loera, Firla and Ziegler; see [Firla and Ziegler 1997]. This example is a point configuration $\mathcal{A}$ having 15 points in $\mathbb{R}^9$ lying in convex position (in fact, having all coordinates 0 or 1), for which the the maximal number of maximal simplices in a regular triangulation is smaller than for an arbitrary triangulation. This example also has the same property for triangulations with the minimal number of maximal simplices.

Cartesian products of simplices $\Delta^m \times \Delta^n$ were conjectured to have only coherent triangulations (see [Ziegler 1995, Problem 5.3]). This is true when m or n is equal to 1, as the secondary polytope in this case is known to be the *permutohedron* [Gel'fand et al. 1994, p. 243]. However, de Loera [1995b, Theorem 2.2.17; 1996] showed that there are incoherent triangulations whenever $m, n \geq 3$, and Sturmfels [1996, Theorem 10.15] showed that they exist when $m = 2$ and $n \geq 5$. A close study of the secondary polytope $\Sigma(\Delta^m \times \Delta^n)$ and its facets was initiated by Billera and Babson [1998], whose point of departure was the fact that a typical fiber of the map $\Delta^{(m+1)(n+1)-1} \to \Delta^m \times \Delta^n$ is a *transportation polytope*, i.e., the polytope of nonnegative $(m+1) \times (n+1)$ matrices with some prescribed row and column sums. The Ph.D. thesis of R. Hastings [1998] contains some interesting ways to view arbitrary triangulations of $\Delta^m \times \Delta^n$, and a few different ways to view incoherence for triangulations of point sets in general.

Another interesting family of polytopes are the (k, n)-*hypersimplices* $\Delta(k, n)$ defined in [Gelfand and MacPherson 1992] as the convex hull of all sums of k distinct standard basis vectors $e_{i_1} + \cdots + e_{i_k}$ in $\mathbb{R}^n$. Particular triangulations of the second hypersimplex $\Delta(2,n)$ were studied by de Loera, Thomas and Sturmfels [de Loera et al. 1995], and by Gelfand, Kapranov, and Zelevinsky (see [de Loera 1995b, § 2.5]). Stanley [1977] gave a triangulation of $\Delta(k,n)$ in general, which recovers a computation of its normalized volume due to Lagrange. In the special case $k = 2$ this triangulation coincides with the one given in [de Loera et al. 1995].

Cyclic polytopes $C(n,d)$. The cyclic polytope $C(n,d)$ is defined to be the convex hull of any n distinct points on the d-dimensional *moment curve*

$$\{(t, t^2, \ldots, t^d); t \in \mathbb{R}\}.$$

Cyclic polytopes play an important role in polytope theory because of the *Upper Bound Theorem* of McMullen [Ziegler 1995, § 8.4]: for any i, the cyclic polytope $C(n,d)$ achieves the maximum number of i-dimensional faces possible for a d-dimensional polytope with n vertices. Although the definition of $C(n,d)$ implicitly depends upon the parameters $t_1 < \cdots < t_n$ which are the x_1-coordinates of the points chosen on the moment curve, much of the combinatorial structure of $C(n,d)$ (including its face lattice, and its set of triangulations and subdivisions) does not depend upon this choice. Therefore we will omit the reference to these parameters except when necessary.

Note that the moment curve in $\mathbb{R}^{d'}$ maps to the moment curve in $\mathbb{R}^d$ under the natural surjection $\pi : \mathbb{R}^{d'} \to \mathbb{R}^d$ which forgets the last $d' - d$ coordinates. This equips the cyclic polytopes with natural surjections $\pi : C(n,d') \to C(n,d)$. Much has been said recently about the fiber polytopes and GBP for these natural maps, which include as special cases the study of triangulations of $C(n,d)$ when $d' = n-1$, and the monotone paths on $C(n,d')$ with respect to the functional $f(\boldsymbol{x}) = x_1$ when $d = 1$. The culmination of much of this work on the GBP was achieved very recently by Athanasiadis, Rambau, and Santos [Athanasiadis et al. 1998]. They use the deletion-contraction paradigm along with the "sliding" technique from [Rambau 1997b; Rambau and Santos 1997] to give a positive answer to the weak GBP for all of the maps $\pi : C(n,d') \to C(n,d)$. The following table gives a chronological summary of the progress on positive answers to the (weak) GBP for $\pi : C(n,d') \to C(n,d)$.

$d' = n-1,\ d = 1$	(folklore)
$d' = n-1,\ d = 2$	[Stasheff 1963]
$d = 1$	[Billera et al. 1994]
$d' - d \le 2$	[Rambau and Ziegler 1996]
$d' = n-1,\ d \le 3$	[Edelman et al. 1997]
$d' = n-1$	[Rambau and Santos 1997]
$d = 2, d' = n-2$	[Athanasiadis et al. 1997]
$d = 2, n < 2d' + 2, d' \ge 9$	[Reiner 1998]
arbitrary n, d', d	[Athanasiadis et al. 1998]

In [Athanasiadis et al. 1997], the authors determine when the fiber polytope $\Sigma(C(n,d') \xrightarrow{\pi} C(n,d))$ is canonical in either of the following two ways:

- all π-induced subdivisions of $C(n,d)$ are π-coherent (this happens only when $d-d' \leq 2$, or $n-d'=1$ and $d=2$, excepting a few sporadic cases), or
- not all π-induced subdivisions are π-coherent, but the subset of π-coherent subdivisions does not depend upon the choice of parameters $t_1 < \cdots < t_n$ (this happens exactly if $d=1$, $d'-d \geq 2$, and $n-d' \geq 2$).

The remaining results about cyclic polytopes deal exclusively with the case of triangulations of $C(n,d)$ and their combinatorics, that is, $n-d'=1$. The philosophy here has been to try and generalize as many things as possible from the case $d=2$, where the cyclic polytope $C(n,2)$ is a convex polygon as in Figure 3. For $d=2$ we know almost everything about the triangulations and subdivisions, as was described in Section 1. All these subdivisions of $C(n,2)$ are coherent, so the poset of subdivisions is the face poset of the secondary polytope $\Sigma(\mathcal{A})$, the $(n-3)$-dimensional associahedron. The 1-skeleton of the associahedron is the Hasse diagram for the Tamari poset (see Figure 3), and this poset turns out to be a lattice (oriented sideways in that figure).

In contrast to the $d=2$ case, not every triangulation of $C(n,d)$ is coherent in general, starting with $C(9,3)$, $C(9,4)$, $C(9,5)$; see [Athanasiadis et al. 1997]. It is also perhaps disappointing that triangulations of $C(n,d)$ can have flip-deficiency [Rambau and Santos 1997], and the higher Stasheff–Tamari posets are not lattices for $d \geq 4$ [Edelman et al. 1997]. On the bright side, Rambau and Santos [1997] prove that even though triangulations of $C(n,d)$ are not always coherent, they do enjoy the somewhat weaker property of being *lifting* triangulations; see [Santos 1997a, Definition 3.4; Björner et al. 1993, p. 410]. Rambau [1997b] also proves the interesting fact that triangulations of $C(n,d)$ are always *shellable* as simplicial complexes (see [Björner 1995, § 11.1] for the definition and significance of shellability).

Kapranov and Voevodsky [1991] suggested a generalization of the Tamari poset on triangulations of $C(n,2)$ to a partial order on triangulations of $C(n,d)$, which they called the *higher Stasheff orders*, and which were studied by Edelman and Reiner [1996] under the name of *higher Stasheff-Tamari orders*. Actually, this latter paper defines two possible such orders which are related to each other, and it is not quite clear (though presumably true) that one of these orders is the same as that considered by Kapranov and Voevodsky. In [Edelman and Reiner 1996] it was proved for $d \leq 3$ that these two partial orders coincide and both are lattices, and also that for $d \leq 5$ the graph of bistellar operations on triangulations of $C(n,d)$ is connected. This last result was greatly improved by Rambau [1997b], who showed that the graph is connected for all d. In this paper, Rambau introduces the important "sliding" idea mentioned earlier: when one slides the n-th vertex on the moment curve down toward the $(n-1)$-st vertex, a subdivision of $C(n,d)$ induces a subdivision of $C(n-1,d)$. This map on subdivisions plays a crucial role in the positive answer to the weak GBP for triangulations of $C(n,d)$ in [Rambau and Santos 1997], and more generally, the weak GBP for $\pi : C(n,d') \to C(n,d)$ in [Athanasiadis et al. 1998]. Previously Edelman, Rambau and Reiner [Edelman et al. 1997] had used the lattice structure on the poset of triangulations and the Homotopies paradigm to positively answer the weak GBP for triangulations of $C(n,d)$ with $d \leq 3$. In that same paper, the authors show that for arbitrary d, both higher Stasheff–Tamari orders on the set of triangulations have proper parts which are homotopy equivalent to $(d-4)$-spheres.

5. Negative Results

Recall from the previous section that we had a positive answer to the strong GBP by Billera, Kapranov, and Sturmfels [Billera et al. 1994] for $d = 1$, and by Rambau and Ziegler [1996] for $d' - d \leq 2$. Together these imply that a negative answer to the weak GBP would require a surjection $\pi : P \to Q$ with $d \geq 2$ and $d' - d \geq 3$ so that $d' \geq 5$. In the paper that includes their positive result, Rambau and Ziegler cleverly construct such a counterexample $\pi : P \to Q$ with the minimum possible dimensions $d' = 5, d = 2$. In fact, they show that the Baues poset $\omega(P \xrightarrow{\pi} Q)$ in this case is not homotopy equivalent to a 2-sphere by showing that it has an isolated element: a π-coherent subdivision of Q which lies below no other element of the poset! Their counterexample is also quite small and uncomplicated, in the sense that P has only 10 vertices, and the point configuration $\mathcal{A}$ in $\mathbb{R}^2$ which is the image of these 10 vertices under π is relatively simple: it consists of a triangle with three copies of each corner vertex, along with one interior point of the triangle. They also give a perturbed version of this same example in which the point configuration $\mathcal{A}$ lies in general position in $\mathbb{R}^2$, and the Baues poset is again disconnected (although it does not have any isolated points). These counterexamples can also be used to produce negative answers to the weak GBP for all d', d with $d \geq 2$ and $d' - d \geq 3$.

In light of this counterexample, attention has shifted to the motivating special cases of the GBP dealing with triangulations of point sets $\mathcal{A}$ and zonotopal tilings of a zonotope Z. Here no counterexamples have been found. The construction closest to a counterexample was provided by the previously mentioned work of Mnëv and Richter-Gebert [1993]. They produce (by two different methods) examples of rank 4 oriented matroids $\mathcal{M}$ whose extension posets $\mathcal{E}(\mathcal{M})$ contain isolated points. These examples do not give a counterexample to the Extension Space Conjecture or to the weak GBP because the oriented matroids in question are not *realizable*, that is they do not come from a zonotope Z. However, they do settle in the negative an earlier extension space conjecture which did not assume realizability of $\mathcal{M}$.

We should also view the instances of flip deficiency for triangulations found in [de Loera et al. 1999; Santos 1997b] as negative results, although they are far from settling the GBP. In particular, Santos' constructions [Santos 1997b] show that the ratio of the number of bistellar flips of a triangulation of $\mathcal{A}$ to the "expected lower bound" $|\mathcal{A}| - d - 1$ can approach zero.

We summarize the main open cases of the (weak) GBP here:

QUESTION 5.1. (i) *Is the poset of zonotopal subdivisions of a* d*-dimensional zonotope with* d' *generators homotopy equivalent to a* $(d' - d - 1)$*-dimensional sphere*?

(ii) *Is the poset of subdivisions of a point set* $\mathcal{A}$ *in* $\mathbb{R}^d$ *homotopy equivalent to a* $(|\mathcal{A}| - d - 2)$*-dimensional sphere*?

As was mentioned earlier, the work of Santos [1997a] shows that the first question is a special case of the second, and therefore a counterexample for the first would also settle the second, as well as the Extension Space Conjecture 3.2 and Conjecture 3.3.

6. Open Questions, Problems, Conjectures

The main open problems related to the GBP are Questions 1.2, 1.3, 5.1. In this section, we collect other problems and questions, some of which address more specifically the expected frontier between the cases of $\pi : P \to Q$ for which the GBP has positive and negative answer. In some cases, we go out on a limb by offering our predictions, but we warn the reader that many of these opinions are not based on very much data, and are only the opinion of this author.

We begin by conjecturing the frontier between good and bad behavior for triangulations, inspired by the positive and negative results contained in [Azaola and Santos 1999; de Loera et al. 1999].

CONJECTURE 6.1. *Let* $\mathcal{A}$ *be a point configuration in* $\mathbb{R}^d$ *with* $d \leq 2$, *or* $d = 3$ *and in convex position*, *or* $|\mathcal{A}| - d \leq 4$. *Then*

(a) *The strong GBP has positive answer for subdivisions of* $\mathcal{A}$ (*without the general position assumption needed in* [Edelman and Reiner 1998]).

(b) Furthermore, the graph of triangulations and bistellar operations is $(|A| - d - 1)$-vertex-connected, so in particular every triangulation of $\mathcal{A}$ has at least $|A| - d - 1$ bistellar neighbors.

(c) On the other hand, there exists a point configuration in convex position in $\mathbb{R}^4$ and also one *not* in convex position in $\mathbb{R}^3$, each of which has an isolated triangulation which refines no other subdivision.

In fact, it would be nice to have a simpler proof of the weak GBP for $\mathcal{A}$ in $\mathbb{R}^2$, even assuming general position, or perhaps a proof of the strong GBP via the Retraction paradigm.

For zonotopal tilings, one wonders whether there are realizable oriented matroids exhibiting the behavior of the counterexamples of Mnëv and Richter-Gebert [1993].

QUESTION 6.2. *Does there exist a zonotope with a cubical tiling that refines no other zonotopal tiling*?

For monotone paths we know that the strong GBP has a positive answer, but an interesting question remains about connectivity via polygon moves. Note that as was mentioned in Section 3, one must be careful to define polygon moves correctly in the case where f is constant on some of the edges of P.

CONJECTURE 6.3. *For a d-dimensional polytope P, the graph of f-monotone paths and polygon moves is $(d-1)$-vertex-connected.*

In particular, every f-monotone path has at least $d-1$ neighbors in the graph $G_{P,f}$ of polygon moves.

Perhaps this can be proven by adapting the proof of the weak GBP for monotone paths given in [Billera et al. 1994, Theorem 1.2]?

One can also ask for bounds on the number of monotone paths. The existence of *neighborly* polytopes [Ziegler 1995, p. 16], in which every pair of vertices forms a boundary edge, shows that the number of f-monotone paths on P can grow exponentially in the number of vertices of P. For *coherent* monotone paths, the story is different. Let $r_d(n)$ denote the maximum number of coherent f-monotone paths for any linear functional f on a d-dimensional polytope P having n vertices. It is shown in [Athanasiadis et al. 1997, Remark 3.8] by a simple geometric argument that for d fixed, $r_d(n)$ grows no faster than $O(n^{3d-6})$. Motivated by McMullen's Upper Bound Theorem [Ziegler 1995, § 8.4], one might expect that the linear functional $f(\boldsymbol{x}) = x_1$ on the cyclic polytope $C(n,d)$ which induces the natural surjection $C(n,d) \to C(n,1)$, also achieves this maximum value $r_d(n)$ for f-monotone paths. However, a counterexample is given in [Athanasiadis et al. 1997, Remark 3.8]. Nevertheless, the results of [Athanasiadis et al. 1997] show that the number of f-monotone paths for $f(\boldsymbol{x}) = x_1$ on $C(n,d)$ grows like a polynomial in n of degree $d-2$, and the authors pose the following question [Athanasiadis et al. 1997, Question 3.10]:

QUESTION 6.4. *For fixed d, does $r_d(n)$ grow no faster than $O(n^{d-2})$?*

Athanasiadis [Athanasiadis 1998] has answered this question positively for $d \leq 4$.

In the special case where P is a d-dimensional zonotope having n generators, counting f-monotone paths turns out to be equivalent to counting the number of different possible linear orderings by linear functionals of a certain affine point configuration with n points in $\mathbb{R}^{d-1}$. Upper and lower bounds for the number of such functionals were addressed recently by Edelman [1998].

Although much is known about cyclic polytopes relating to the GBP, there remain several interesting open questions. One challenge is to count the number of triangulations of $C(n,d)$. Some data is given in [Athanasiadis et al. 1997, Table 4], compiled using PUNTOS [de Loera 1995a] and software of Rambau dedicated to this task. As said in the introduction, almost the only nontrivial known formula counting triangulations is the Catalan number

$$\frac{1}{n-1}\binom{2n-4}{n-2},$$

enumerating triangulations of the n-gon $C(n,2)$. We also have the following mostly trivial results:

- $C(n,1)$ has 2^{n-2} triangulations,
- $C(d+1,d)$ has 1 triangulation,
- $C(d+2,d)$ has 2 triangulations, and
- $C(d+3,d)$ has $d+3$ triangulations by the results of [Lee 1991].

Santos [1998] recently made the following conjecture for $C(d+4,d)$, based on the known data:

Conjecture 6.5. *Let a_d be the number of triangulations of $C(d+4,d)$. Then the second difference $a_d - 2a_{d-1} + a_{d-2}$ has the following form*:

$$\begin{aligned} a_{2k} - 2a_{2k-1} + a_{2k-2} &= 2^k && \text{(proved by Santos)},\\ a_{2k+1} - 2a_{2k} + a_{2k-1} &= 2^{k+1} + k2^{k-1}. \end{aligned}$$

He points out that this conjecture easily leads to simple closed form for a_d. Specifically, one would have

$$a_d = k_d 2^{d/2} - (d+4),$$

where

$$k_d = \begin{cases} d+8 & \text{if } d \text{ is even},\\ (3d+23)/(2^{3/2}) \approx 1.06d + 8.13 & \text{if } d \text{ is odd}. \end{cases}$$

As was mentioned earlier, in [Athanasiadis et al. 1998] the authors give a positive answer to the GBP for all of the projections between cyclic polytopes. The special case of the projection $\pi : C(n,d') \to C(n,2)$, along with the methods used in [Athanasiadis et al. 1997, Theorem 1.2; Reiner 1998] inspire the following conjecture, which would also partly explain the importance of the interior point of Q present in the Rambau–Ziegler counterexample [Rambau and Ziegler 1996].

Conjecture 6.6. *The strong GBP has positive answer for $\pi : P \to Q$ if Q lies in $\mathbb{R}^2$ and all vertices of P project under π to the boundary of Q.*

The proof of the weak GBP for triangulations of cyclic polytopes given by Rambau and Santos [1997] uses the Deletion-Contraction paradigm. We next discuss some other conjectural approaches to this result, involving the relation of cyclic polytopes to *cyclic zonotopes* and *alternating matroids* [Björner et al. 1993, § 9.4].

For any point configuration $\mathcal{A}$ in $\mathbb{R}^d$, the *dual point configuration* or *Gale transform* $\mathcal{A}^*$ lives in $\mathbb{R}^{|\mathcal{A}|-d-1}$ [Ziegler 1995, Lecture 6]. A single element extension of the oriented matroid $\mathcal{M}$ corresponding to $\mathcal{A}^*$ gives rise to a subdivision of $\mathcal{A}$ called a *lifting subdivision* [Björner et al. 1993, p. 410], and hence gives a map from the extension poset $\mathcal{E}(\mathcal{M})$ to the poset of subdivisions of $\mathcal{A}$. In the case where $\mathcal{A}$ is the set of vertices of a cyclic polytope $C(n,d)$, $\mathcal{A}^*$ is a cyclic arrangement of vectors [Ziegler 1995, Problem 6.13] so that $\mathcal{M}$ is the alternating matroid $C^{n,n-d-1}$. Therefore the results of [Sturmfels and Ziegler 1993] imply that $\mathcal{E}(C^{n,n-d-1})$ is homotopy equivalent to an $(n-d-2)$-sphere. Furthermore, [Rambau and Santos 1997] shows that every triangulation of $C(n,d)$ is a lifting triangulation (although it is not known whether this is true for all subdivisions), so that this map has a chance to be surjective.

Conjecture 6.7. *When $\mathcal{A}$ is the set of vertices of a cyclic polytope $C(n,d)$, the map described in the previous paragraph is a homotopy equivalence from the extension space $\mathcal{E}(C^{n,n-d-1})$ to the poset of subdivisions.*

A different approach relates the triangulations of cyclic polytopes to cyclic hyperplane arrangements and the work of Manin and Schechtman [1989], Ziegler [1993], and Kapranov and Voevodsky [1991] on *cyclic hyperplane arrangements and zonotopes* and the *higher Bruhat orders*.

Let $Z(n,d)$ be the d-dimensional *cyclic zonotope* with n generating segments in the directions $\{(1,t_i,t_i^2,\dots,t_i^{d-1})\}_{i=1}^n$ for any n distinct values of the parameters $t_1<\dots<t_n$. The higher Bruhat orders $B(n,d)$ were defined in [Manin and Schechtman 1989], and may be thought of as a natural poset structure on the cubical tilings of $Z(n,d)$. For $d=1$, $B(n,1)$ is the the *weak Bruhat order* [Björner et al. 1993, § 2.3(b)] on the symmetric group. Ziegler [1993] observed that there were actually two natural and related (but different!) definitions for higher Bruhat orders, which he called $B(n,d)$ and $B_{\subseteq}(n,d)$. Among other things, he showed that the homotopy type of the second of these posets $B_{\subseteq}(n,d)$ is spherical. Rambau [1997a] later showed that $B(n,d)$ also has spherical homotopy type.

As was mentioned in Section 4, Kapranov and Voevodsky [1991] define a partial order on triangulations of $C(n,d)$, and Edelman and Reiner [1996] consider two such related partial orders $S_1(n,d)$ and $S_2(n,d)$, generalizing the Tamari poset on triangulations of $C(n,2)$. It is not quite clear, although presumably true, that the order $S_1(n,d)$ coincides with the order defined in [Kapranov and Voevodsky 1991]. In [Edelman et al. 1997], it is shown that both posets $S_1(n,d)$ and $S_2(n,d)$ have spherical homotopy type. Kapranov and Voevodsky also define an order-preserving map $B(n,d)\to S_1(n+2,d+1)$, and a similar map was given two definitions by Rambau in [1997b]. Rambau shows that his two definitions give the same map, but it is not clear that his map is the same as the one in [Kapranov and Voevodsky 1991]. For $d=1$, this map coincides with a map from permutations to triangulations of an n-gon studied by Björner and Wachs [1997, § 9], and by Tonks [Tonks 1997].

Conjecture 6.8. *The maps $B(n,d)\to S_1(n+2,d+1)$ defined by Kapranov–Voevodsky and Rambau are the same map f_{KVR}, and f_{KVR} induces a homotopy equivalence between the proper parts of these posets.* (True for $d=1$, by [Björner and Wachs 1997, § 9].)

The fact that these posets have homotopy equivalent proper parts already follows from the sphericity results previously mentioned. What does this have to do with the GBP? It is easy to see that for any zonotopal subdivision of $Z(n,d)$, the set of all cubical tilings which refine it forms an interval both in $B(n,d)$ and in $B_{\subseteq}(n,d)$. This gives a very natural order-preserving map from the Baues poset

$$\omega(I^n \to Z(n,d))$$

to the *poset of proper intervals* in $B(n,d)$ or $B_{\subseteq}(n,d)$. Similarly, for any subdivision of $C(n,d)$, the set of triangulations which refine it forms an interval both in $S_1(n,d)$ and in $S_2(n,d)$, giving an order-preserving map from the Baues poset $\omega(\Delta^{n-1} \to C(n,d))$ to the poset of proper intervals in $S_1(n,d)$ or $S_2(n,d)$.

CONJECTURE 6.9. (a) *The image of the map from $\omega(I^n \to Z(n,d))$ to the poset of proper intervals in either $B(n,d)$ or $B_{\subseteq}(n,d)$ is exactly the set of noncontractible* (*open*) *intervals.* (True for $d = 1$, by [Björner and Wachs 1997, § 9].)

(b) *The image of the map from $\omega(\Delta^{n-1} \to C(n,d))$ to the poset of proper intervals in either $S_1(n,d)$ or $S_2(n,d)$ is exactly the set of noncontractible intervals.* (True for $d \leq 3$, by [Edelman et al. 1997, Lemma 6.3].)

The previous conjecture would have two nice consequences:

(i) It would completely describe the homotopy type of all intervals (and hence compute the *Möbius function*) in both higher Bruhat orders $B(n,d), B_{\subseteq}(n,d)$ and in both higher Stasheff–Tamari orders $S_1(n,d), S_2(n,d)$. The intervals which are the images of the above maps are always isomorphic to Cartesian products of posets $B(n',d), B_{\subseteq}(n',d)$ or $S_1(n',d), S_2(n',d)$ for smaller values $n' < n$, and hence by the known sphericity results, are also homotopy spherical.

(ii) It would imply that $\omega(I^n \to Z(n,d))$ is homotopy equivalent to the suspension of the proper part of $B(n,d)$ or $B_{\subseteq}(n,d)$, and similarly $\omega(\Delta^{n-1} \to C(n,d))$ is homotopy equivalent to the suspension of the proper part of $S_1(n,d)$ or $S_2(n,d)$. This follows from the fact observed by Walker [Walker 1988] that the poset of proper intervals in a bounded poset P is homeomorphic to the suspension of the proper part of P, and the fact that the poset of proper noncontractible intervals in P is a deformation retract of the poset of all proper intervals in P [Edelman et al. 1997, Lemma 6.5].

Conjectures 6.8 and 6.9 fit into a diagram of conjectural homotopy equivalences (among spaces which are all known to be homotopy equivalent to an $(n-d-1)$-sphere) connecting the Baues posets for triangulations of $C(n,d)$ and zonotopal tilings of $Z(n,d)$ to each other and to the higher Bruhat and higher Stasheff–Tamari orders:

$$\begin{tikzcd}
\mathrm{Susp}(B(n,d)-\{\hat{0},\hat{1}\}) \arrow[r, "\mathrm{Susp}(f_{KVR})"] & \mathrm{Susp}(S_1(n+2,d+1)-\{\hat{0},\hat{1}\}) \\
\omega(I^n \to Z(n,d)) \arrow[u] & \omega(\Delta^{n+1} \to C(n+2,d+1)). \arrow[u]
\end{tikzcd}$$

In addition to the previous specific conjectures, we would also like to describe some more general problems related to the GBP.

The first of these relates to the concept of an *iterated fiber polytope* introduced by Billera and Sturmfels [Billera and Sturmfels 1994]. Given a tower $P \xrightarrow{\pi} Q \xrightarrow{\rho} R$ of linear surjections of polytopes, it was shown in [Billera and Sturmfels 1992] that the map π induces a surjection of the fiber polytopes

$$\pi : \Sigma(P \xrightarrow{\rho\circ\pi} R) \quad \longrightarrow \quad \Sigma(Q \xrightarrow{\rho} R).$$

They called the fiber polytope of this surjection the *iterated fiber polytope*

$$\Sigma(P \xrightarrow{\pi} Q \xrightarrow{\rho} R).$$

It is also clear how one can iterate this construction further, to define higher iterated fiber polytopes associated to longer towers of surjections.

QUESTION 6.10. *Study the iterated fiber polytopes for subsequences of the tower of natural surjections*

$$\Delta^{n-1} = C(n,n-1) \to C(n,n-2) \to \cdots \to C(n,2) \to C(n,1)$$

between cyclic polytopes. Are there any cases (like those classified in [Athanasiadis et al. 1997]) *where the structure of the iterated fiber polytope does not depend upon the choice of points on the moment curve defining* $C(n,d)$?

Another line of inquiry is suggested by the first part of Conjecture 6.9. Ziegler [1993] considers the higher Bruhat orders $B(n,d), B_{\subseteq}(n,d)$ as posets of *uniform extensions* of the affine oriented matroid corresponding to a cyclic hyperplane arrangement of hyperplanes. More generally, he introduces these *uniform extension posets* $\mathcal{U}(\mathcal{M},g), \mathcal{U}_{\subseteq}(\mathcal{M},g)$ for *any* affine oriented matroid $(\mathcal{M},g)$. The map considered in Conjecture 6.9 can be generalized to a map from the extension poset $\mathcal{E}(\mathcal{M})$ to the set of proper intervals in $\mathcal{U}_{\subseteq}(\mathcal{M},g)$: send a single element extension of $\mathcal{M}$ to the set of uniform extensions which lie below it in $\mathcal{E}(\mathcal{M})$.

QUESTION 6.11. *Study the map from the extension space* $\mathcal{E}(\mathcal{M})$ *to the poset of proper intervals in* $\mathcal{U}_{\subseteq}(\mathcal{M},g)$. *Are there any nice classes of affine oriented matroids* $(\mathcal{M},g)$ *where the image of the map is exactly the set of noncontractible proper intervals*?

As was the case in Conjecture 6.9, whenever the above question has a positive answer, the map in question induces a homotopy equivalence between $\mathcal{E}(\mathcal{M})$ and the suspension of the proper part of $\mathcal{U}(\mathcal{M}, g)$. The examples of Mnëv and Richter-Gebert [1993] show that $\mathcal{E}(\mathcal{M})$ does not always have spherical homotopy type, but it is still possible that such a homotopy equivalence may exist even in cases where sphericity fails.

Our last question relates to Stembridge's $q = -1$ *phenomenon* occurring in the context of cubical tilings of zonotopes; see [Stembridge 1994]. A zonotope Z is a centrally-symmetric polytope, and hence the antipodal map induces a natural involution ω on its set of cubical tilings. Say that a tiling of Z is *centrally symmetric* if it is fixed by this involution ω. Consider also the graph G_Z of cubical tilings and cube flips on Z. It can be shown that this graph will always be bipartite. We say that the $q = -1$ *phenomenon holds for* Z if the number of centrally symmetric tilings of Z is the same as the difference in cardinality of the two sides of the bipartition of G_Z.

Stembridge [1994] observed that known formulas counting symmetry classes of plane partitions implied the $q = -1$ phenomenon for zonotopal hexagons in the plane (with multiple copies of the three line segments which generate the hexagon as a zonotope). Further examples involving certain zonotopal octagons were found by Elnitsky [1997] and Bailey [1997]. However, one can check that the phenomenon does not hold for all zonotopes Z, as there are already examples of zonotopal octagons for which it fails.

Question 6.12. *For which zonotopes Z does the $q = -1$ phenomenon hold?*

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Note Added in Proof

The recent paper [Athanasiadis et al. 1999] resolves Conjecture 6.3 affirmatively for simple polytopes and for 3-dimensional polytopes, but negatively in general. Specifically, it is shown that for $d \geq 3$, the graph of f-monotone paths in a d-polytope with respect to a generic functional f is at least 2-connected, but there exist examples for each $d \geq 3$ in which the graph contains a vertex of degree 2.

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