Model Theory, Algebra, and Geometry
MSRI Publications
Volume **39**, 2000

Arithmetic and Geometric Applications of Quantifier Elimination for Valued Fields

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ABSTRACT. We survey applications of quantifier elimination to number theory and algebraic geometry, focusing on results of the last 15 years. We start with the applications of p-adic quantifier elimination to p-adic integration and the rationality of several Poincar series related to congruences $f(x) = 0$ modulo a prime power, where f is a polynomial in several variables. We emphasize the importance of p-adic cell decomposition, not only to avoid resolution of singularities, but especially to obtain much stronger arithmetical results. We survey the theory of p-adic subanalytic sets, which is needed when f is a power series instead of a polynomial. Next we explain the fundamental results of Lipshitz–Robinson and Gardener–Schoutens on subanalytic sets over algebraically closed complete valued fields, and the connection with rigid analytic geometry. Finally we discuss recent geometric applications of quantifier elimination over $\mathbb{C}((t))$, related to the arc space of an algebraic variety.

One of the most striking applications of the model theory of valued fields to arithmetic is the work of Ax and Kochen [1965a; 1965b; 1966; Kochen 1975], and of Ershov [1965; 1966; 1967], which provided for example the first quantifier elimination results for discrete valued fields [Ax and Kochen 1966], and the decidability of the field $\mathbb{Q}_p$ of p-adic numbers. As a corollary of their work, Ax and Kochen [1965a] proved the following celebrated result: For each prime number p, big enough with respect to d, any homogeneous polynomial of degree d over $\mathbb{Q}_p$ in d^2+1 variables has a nontrivial zero in $\mathbb{Q}_p$. However in the present survey we will not discuss this work, but focus on results of the last 15 years.

In Section 1 we explain the applications of p-adic quantifier elimination to p-adic integration and the rationality of several Poincaré series related to a congruence $f(x) \equiv 0 \bmod p^m$, where $f(x)$ is a polynomial in several variables with integer coefficients. We emphasize the importance of p-adic cell decomposition, not only to avoid resolution of singularities, but especially to obtain much stronger results (for example, on local singular series in several variables).

To obtain results similar to those in Section 1, but when f is a power series instead of a polynomial, one needs the theory of p-adic subanalytic sets which we survey in Section 2.

In Section 3 we explain the fundamental results of Lipshitz–Robinson and Gardener–Schoutens on subanalytic sets over algebraically closed nonarchimedean complete valued fields and the connection with rigid analytic geometry. Finally, in Section 4 we discuss recent geometric applications of quantifier elimination over the field $\mathbb{C}((t))$ of Laurent series over $\mathbb{C}$. Here p-adic integration is replaced by "motivic integration", a notion recently introduced by Kontsevich.

1. Integration on Semi-Algebraic Subsets over $\mathbb{Q}_p$

1.1. Motivating Problem. Let $f(x) \in \mathbb{Z}[x], x = (x_1, \dots, x_n)$. Let p be a prime number and $m \in \mathbb{N}$. Denote the ring of p-adic integers by $\mathbb{Z}_p$ and the field of p-adic numbers by $\mathbb{Q}_p$; see [Koblitz 1977], for example. For $a \in \mathbb{Z}_p$, we denote the image of a in $\mathbb{Z}/p^m\mathbb{Z}$ by a mod p^m. We use the notations

$$N_m := \text{number of elements in } \{x \in (\mathbb{Z}/p^m\mathbb{Z})^n \mid f(x) \equiv 0 \equiv p^m\},$$

$$\tilde{N}_m := \text{number of elements in } \{x \equiv p^m \mid x \in \mathbb{Z}_p^n, f(x) = 0\},$$

$$P(T) := \sum_{m \in \mathbb{N}} N_m T^m, \qquad \tilde{P}(T) := \sum_{m \in \mathbb{N}} \tilde{N}_m T^m.$$

Borevich and Shafarevich conjectured that $P(T)$ is a rational function of T. This was proved by Igusa [1974; 1975; 1978] using Hironaka's resolution of singularities. Serre [1981, § 3] and Oesterlé [1982] investigated the behaviour of $\tilde{N}_m$ for $m \to \infty$, and they asked the question whether $\tilde{P}(T)$ is a rational function of T. This was proved by Denef [1984] using resolution of singularities and Macintyre's Theorem [Macintyre 1976] on quantifier elimination for $\mathbb{Q}_p$. Denef [1984] also gave an alternative proof of the rationality of $P(T)$ and $\tilde{P}(T)$, avoiding the use of resolution of singularities, using instead Macintyre's Theorem and a cell decomposition theorem. We will briefly explain these proofs below.

1.2.1. The p-adic measure. There exists a unique ($\mathbb{R}$-valued Borel) measure on $\mathbb{Q}_p^n$ which is invariant under translation such that $\mathbb{Z}_p^n$ has measure 1. We denote this Haar measure by $|dx| = |dx_1| \cdots |dx_n|$. The measure of $a + p^m\mathbb{Z}_p^n$ equals p^{-mn}, for each $a \in \mathbb{Q}_p^n$, because these sets have the same measure (being translates of $p^m\mathbb{Z}_p^n$) and p^{nm} of them form a partition of $\mathbb{Z}_p^n$. For any measurable $A \subset \mathbb{Q}_p^n$ and $\lambda \in \mathbb{Q}_p$, the measure of $\lambda A = \{\lambda a \mid a \in A\}$ equals the measure of A times $|\lambda|^n$, where $|\lambda|$ denotes the p-adic absolute value $|\lambda| := p^{-\operatorname{ord}\lambda}$, with $\operatorname{ord} : \mathbb{Q}_p \to \mathbb{Z} \cup \{+\infty\}$ the p-adic valuation. We recall that each λ in $\mathbb{Q}_p \setminus \{0\}$ can be written as $\lambda = up^{\operatorname{ord}\lambda}$ with u a unit in the ring $\mathbb{Z}_p$. Integration of (integrable) real valued functions on $\mathbb{Q}_p^n$ is defined in the standard way. As an example we calculate the following integral for $n = 1$:

$$\int_{x\in\mathbb{Z}_p,\,\mathrm{ord}\,x\geq m} |x|^s\,|dx| = \sum_{j\geq m} p^{-sj}\int_{\mathrm{ord}\,x=j} |dx| = \sum_{j\geq m} p^{-sj}(p^{-j}-p^{-j-1})$$
$$= (1-p^{-1})p^{-(s+1)m}/(1-p^{-s-1}),$$

for any nonnegative $s \in \mathbb{R}$.

1.2.2. Rationality of $P(T)$ and $\tilde{P}(T)$. The proof of the rationality of $P(T)$ and $\tilde{P}(T)$ is based on the simple formulas

$$N_m = p^{mn} \text{ measure } (\{x \in \mathbb{Z}_p^n \mid \mathrm{ord}\, f(x) \geq m\}),$$
$$\tilde{N}_m = p^{mn} \text{ measure } (\{x \in \mathbb{Z}_p^n \mid \exists y \in \mathbb{Z}_p^n : f(y) = 0, y \equiv x \equiv p^m\}),$$

which are justified by observing that the set in the right-hand side is a union of respectively N_m and $\tilde{N}_m$ residue classes mod p^m, each having measure p^{-nm}.

The set in the first formula is of a very simple type, but the set in the second is more complicated, involving an existential quantifier. We need Macintyre's Theorem (see Section 1.3 below) on elimination of quantifiers to see that this set is not too complicated, so that its measure (as a function of m) can be controlled. To prove the rationality of $P(T)$ and $\tilde{P}(T)$ one has to know how the measures of the above sets vary with m. This is provided by the Basic Theorem 1.5 below.

1.3. Definable Subsets of $\mathbb{Q}_p$. Let $\mathcal{L}_{\mathrm{Pres}}$ be the (first order) language (in the sense of logic) whose variables run over $\mathbb{Z}$ and with symbols to denote $+, \leq, 0, 1$ and with for each $d = 2, 3, 4, \ldots$ a symbol to denote the binary relation $x \equiv y \equiv d$. Note that in $\mathcal{L}_{\mathrm{Pres}}$ there is no symbol for multiplication. As for any (first order) language, the formulas of $\mathcal{L}_{\mathrm{Pres}}$ are built up in the obvious way from the above specified symbols and variables, together with the logical connectives $\wedge$ (and), $\vee$ (or), $\neg$, the quantifiers $\exists, \forall$, brackets, and $=$. A well-known result of Presburger [1930] states that $\mathbb{Z}$ has elimination of quantifiers in the language $\mathcal{L}_{\mathrm{Pres}}$, meaning that each formula in that language is equivalent (in $\mathbb{Z}$) to a formula without quantifiers. (For readers who are not familiar with this terminology from logic, we refer to [Denef and van den Dries 1988, § 0], where these notions are explained for non-logicians.)

Let $\mathcal{L}_{\mathrm{Mac}}$ be the (first order) language whose variables run over $\mathbb{Q}_p$ and with symbols to denote $+, -, \times, 0, 1$ and with for each $d = 2, 3, 4, \ldots$ a symbol P_d to denote the predicate "x is a d-th power in $\mathbb{Q}_p$". Moreover for each element in $\mathbb{Z}_p$ there is a symbol to denote that element. Macintyre's theorem [1976] states that $\mathbb{Q}_p$ has elimination of quantifiers in the language $\mathcal{L}_{\mathrm{Mac}}$, meaning that each formula in that language is equivalent (in $\mathbb{Q}_p$) to a formula without quantifiers.

Let $\mathcal{L}$ be the (first order) language with two sorts of variables: A first sort of variables running over $\mathbb{Q}_p$, and a second sort of variable running over $\mathbb{Z}$. The symbols of $\mathcal{L}$ consist of the symbols of $\mathcal{L}_{\mathrm{Mac}}$ (for the first sort), the symbols of $\mathcal{L}_{\mathrm{Pres}}$ (for the second sort), and a symbol to denote the valuation function ord : $\mathbb{Q}_p \setminus \{0\} \to \mathbb{Z}$ (from the first sort to the second sort). (We use the convention that $\operatorname{ord} 0 = +\infty, (+\infty) + l = +\infty$ and $+\infty \equiv l \bmod d$, for all l in $\mathbb{Z} \cup \{+\infty\}$.) An easy adaptation of Macintyre's proof yields that $\mathbb{Q}_p$ has elimination of quantifiers in the language $\mathcal{L}$; see [Denef 1984, Remark 6.4].

A subset of $\mathbb{Q}_p^n$ is called *semi-algebraic* if it is definable by a quantifier-free formula of $\mathcal{L}_{\mathrm{Mac}}$ (that is, a formula without quantifiers). Every subset of $\mathbb{Q}_p^n$ which is definable in $\mathcal{L}$ is semi-algebraic. This follows from quantifier elimination for $\mathcal{L}$ and the fact that the relation "$\operatorname{ord} x \le \operatorname{ord} y$" can be expressed in terms of the predicate P_2; see [Denef 1984, Lemma 2.1].

1.4. The Cell Decomposition Theorem

THEOREM [Denef 1984; 1986]. *Let $f_i(x,t) \in \mathbb{Q}_p[x,t]$, where $i = 1, \dots, m$, $x = (x_1, \dots, x_{n-1})$, and t is one variable. Fix $d \in \mathbb{N}$ with $d \ge 2$. Then there exists a finite partition of $\mathbb{Q}_p^n$ into subsets (called cells) of the form*

$$A = \{(x,t) \in \mathbb{Q}_p^n \mid x \in C \text{ and } |a_1(x)| \,\square_1\, |t - c(x)| \,\square_2\, |a_2(x)|\},$$

where C is an $\mathcal{L}$-definable subset of $\mathbb{Q}_p^{n-1}$, each of $\square_1$ and $\square_2$ denotes either $\le$, $<$, or no condition, and $a_1(x), a_2(x), c(x)$ are $\mathcal{L}$-definable functions from $\mathbb{Q}_p^{n-1}$ to $\mathbb{Q}_p$, such that, for all $(x,t) \in A$,

$$f_i(x,t) = u_i(x,t)^d h_i(x)(t - c(x))^{\nu_i}, \text{ for } i = 1, \dots, m,$$

with $u_i(x,t)$ a unit in $\mathbb{Z}_p$ for all (x,t) in A, $h_i(x)$ an $\mathcal{L}$-definable function from $\mathbb{Q}_p^{n-1}$ to $\mathbb{Q}_p$, and $\nu_i \in \mathbb{N}$.

We recall that a function is called $\mathcal{L}$-definable if its graph is $\mathcal{L}$-definable, meaning that it can be expressed by a formula in the language $\mathcal{L}$.

REMARK. This was first proved in [Denef 1984] using Macintyre's Theorem. Conversely Macintyre's Theorem follows easily from The Cell Decomposition Theorem which can be proved directly using a method due to Cohen [1969]; see [Denef 1986].

1.5. Basic Theorem on p-adic Integration

THEOREM [Denef 1985]. *Let $(A_{\lambda,l})_{\lambda \in \mathbb{Q}_p^k, l \in \mathbb{Z}^r}$ be an $\mathcal{L}$-definable family of bounded subsets of $\mathbb{Q}_p^n$. Then*

$$I(\lambda, l) := \text{measure of } A_{\lambda,l} := \int_{A_{\lambda,l}} |dx|$$

is a $\mathbb{Q}$-valued function of λ, l belonging to the $\mathbb{Q}$-algebra generated by the functions

$$\theta(\lambda, l) \quad \text{and} \quad p^{\theta(\lambda,l)},$$

where θ is $\mathbb{Z}$-valued $\mathcal{L}$-definable.

(Saying that $(A_{\lambda,l})$ is $\mathcal{L}$-definable means that the relation $x \in A_{\lambda,l}$ can be expressed by a formula in the language $\mathcal{L}$ where x, λ are variables running over $\mathbb{Q}_p$ and l are variables running over $\mathbb{Z}$. Saying that θ is $\mathbb{Z}$-valued $\mathcal{L}$-definable means that the relation $z = \theta(\lambda, l)$ can be expressed by a formula in $\mathcal{L}$, where λ are variables running over $\mathbb{Q}_p$ and z, λ are variables running over $\mathbb{Z}$.)

We call the elements of the algebra mentioned in the theorem *$\mathcal{L}$-simple p-exponential functions*, and if there are no variables λ involved we call them *$\mathcal{L}_{\text{Pres}}$-simple p-exponential functions.* The Basic Theorem and its proof also hold for integrals of the form $\int_{A_{\lambda,l}} p^{-\alpha(x,\lambda,l)}\,|dx|$, with α a *positive* $\mathbb{Z}$-valued $\mathcal{L}$-definable function.

PROOF OF THE BASIC THEOREM. By quantifier elimination $A_{\lambda,l}$ is given by a quantifier-free formula Ψ of $\mathcal{L}$. Let $f_1, f_2, \ldots, f_m$ be the polynomials (in variables of the first sort) which appear in this formula Ψ. We now apply the Cell Decomposition Theorem 1.4 to $f_1, \ldots, f_m$. This enables us to separate off the last variable and integrate first with respect to that variable. The Basic Theorem is obtained by iterating this procedure. For the details we refer to [Denef 1985, § 3], where a similar result is proved. $\square$

1.6. Meaning of the Basic Theorem with No λ. If in Theorem 1.5 there are no variables λ, then the function $I(l)$ is built from Presburger functions (that is, $\mathcal{L}_{\text{Pres}}$-definable functions from $\mathbb{Z}^r$ to $\mathbb{Z}$) by multiplication, exponentiation, and $\mathbb{Q}$-linear combinations. Such functions $I(l)$ are easy to understand because any Presburger function is piecewise $\mathbb{Q}$-linear, the pieces being Presburger subsets of $\mathbb{Z}^r$ (that is, $\mathcal{L}_{\text{Pres}}$-definable subsets). But Presburger subsets are finite unions of convex polyhedrons intersected with residue classes. A completely elementary argument now yields:

THEOREM 1.6.1. *Assume the notation of Theorem 1.5 with no λ involved. Let $T = (T_1, \ldots, T_r)$. Then*

$$\sum_{l \in \mathbb{N}^r} I(l) T^l \in \mathbb{Q}[[T_1, \ldots, T_r]]$$

is a rational function of T.

Actually this holds for any $\mathcal{L}_{\text{Pres}}$-simple p-exponential function $I(l)$.

COROLLARY 1.6.2. *The series $P(T)$ and $\tilde{P}(T)$ from Section 1.1 are rational.*

PROOF. Direct consequence of 1.6.1 and 1.2.2. $\square$

COROLLARY 1.6.3. *Assume the notation of Section 1.1 and let $N_{m,r}$ be the number of solutions in $\mathbb{Z}/p^m\mathbb{Z}$ of $f(x) \equiv 0$ mod p^m that can be lifted to a solution of $f(x) \equiv 0$ mod p^{m+r} in $\mathbb{Z}/p^{m+r}\mathbb{Z}$. Then $\sum_{m,r \in \mathbb{N}} N_{m,r} T^m U^r$ is a rational function of T, U.*

PROOF. This is a direct consequence of 1.6.1 and the obvious fact that $p^{-mn}N_{m,r}$ equals the measure of the set

$$\{x \in \mathbb{Z}_p^n \mid \exists y \in \mathbb{Z}_p^n : f(y) \equiv 0 \bmod p^{m+r},\ y \equiv x \bmod p^m\}. \qquad \square$$

The Basic Theorem 1.5 with no λ involved can also be proved without using the Cell Decomposition Theorem, using instead resolution of singularities. Indeed, by the p-adic Analytic Resolution Theorem of Section 1.7 below (applied to the polynomials $f_1, \ldots, f_m$ appearing in a quantifier-free formula Ψ describing A_l), we can pull back the integral $I(l)$ to the p-adic manifold M. The so obtained integral on M can be easily evaluated by an elementary local calculation, using [Denef 1985, Lemma 3.2]. A special case of such a calculation is given in the example of Section 1.2.1. However when there are at least two variables λ involved (meaning that $k \geq 2$) then I do not know how to prove the Basic Theorem 1.5 without using the Cell Decomposition Theorem (even when $r = 0$).

1.7. Resolution of Singularities (p-adic Analytic Case). Let $U \subset \mathbb{Q}_p^n$ be open and $f : U \to \mathbb{Q}_p$ a map. We call f analytic if each $a \in U$ has an open neighbourhood V_a on which f can be written as a power series in $x - a$, with coefficients in $\mathbb{Q}_p$, which converges for all $x \in V_a$.

By a *p-adic manifold* we mean a p-adic analytic manifold (defined in the same way as a complex analytic manifold) which is Hausdorff and everywhere of the same dimension (see [Bourbaki 1967], for example). *Analytic functions* from a p-adic manifold M_1 to a p-adic manifold M_2 are defined in the obvious way by working locally. Also the notion of *isomorphic* p-adic manifolds is defined straightforwardly.

It is easy to verify that each compact p-adic manifold of dimension n is a disjoint union of a finite number of open compact submanifolds which are isomorphic to $\mathbb{Z}_p^n$.

Let M be a compact p-adic manifold and C a closed submanifold of codimension r at least 2. We refer to [Denef and van den Dries 1988, § 2.1] for the definition of the *blowing-up of M with respect to C*. This is an analytic map $h : \tilde{M} \to M$, with $\tilde{M}$ a compact p-adic manifold of the same dimension as M, such that the restriction $\tilde{M} \setminus h^{-1}(C) \to M \setminus C$ of h is an isomorphism, and which is constructed in a special way (well-known to geometers). In particular, using suitable local coordinates, the map h is locally given by

$$(x_1, \ldots, x_n) \mapsto (x_1x_r, x_2x_r, \ldots, x_{r-1}x_r, x_r, \ldots, x_n).$$

(In these local coordinates, the submanifold C is locally given by $x_1 = x_2 = \cdots = x_r = 0$.)

p-ADIC ANALYTIC RESOLUTION THEOREM. *Let $f_1, \ldots, f_m : \mathbb{Z}_p^n \to \mathbb{Q}_p$ be analytic functions. Then there exists a compact p-adic manifold M of dimension n and an analytic map $\pi : M \to \mathbb{Z}_p^n$ such that*

(i) *M is the disjoint union of a finite number of clopens $U_i = \mathbb{Z}_p^n$, such that on each U_i, the jacobian of π and all $f_j \circ \pi$ are monomials times analytic functions with constant absolute value.*
(ii) *π is a composition of finitely many blowing-up maps with respect to closed submanifolds of codimension ≥ 2. In particular π is an isomorphism outside closed sets of measure zero.*

This is an easy consequence of Hironaka's embedded resolution of singularities [Hironaka 1964]; see, for example, [Denef and van den Dries 1988, Theorem 2.2].

1.8. Meaning of the Basic Theorem with No l. If in Theorem 1.5 there are no variables l, then the function $I(\lambda)$ is built from $\mathfrak{L}$-definable functions $\mathbb{Q}_p^k \to \mathbb{Z}$, by multiplication, exponentiation and $\mathbb{Q}$-linear combinations. Such functions $I(\lambda)$ are easy to understand. Indeed, by [Denef 1984, Theorem 6.3], for any $\mathfrak{L}$-definable function $\theta : \mathbb{Q}_p^k \to \mathbb{Z}$ there exists a finite partition of $\mathbb{Q}_p^k$ in semi-algebraic subsets S such that on each such S the function θ is a $\mathbb{Q}$-linear combination of the ord of polynomials over $\mathbb{Q}_p$ with no zeros on S. Applying the Analytic Resolution Theorem (Section 1.7) to the polynomials appearing in the linear combinations and formulas for S above mentioned, and expressing any locally constant function on $\mathbb{Z}_p^\times$ as a $\mathbb{C}$-linear combination of characters (i.e., homomorphisms $\chi : \mathbb{Z}_p^\times \to \mathbb{C}^\times$ with finite image, where $\mathbb{Z}_p^\times$ and $\mathbb{C}^\times$ are the groups of units in the rings $\mathbb{Z}_p$ and $\mathbb{C}$), we obtain:

THEOREM 1.8.1. *Let $I : \mathbb{Z}_p^k \to \mathbb{Q}$ be an $\mathfrak{L}$-simple p-exponential function (e.g., the function I (restricted to $\mathbb{Z}_p^k$) in the Basic Theorem 1.5, when there is no l involved). Then there exists a p-adic manifold M of dimension k and an analytic map $\pi : M \to \mathbb{Z}_p^k$, which is the composition of finitely many blowing-up maps with respect to closed submanifolds of codimension ≥ 2, such that locally at each $b \in M$ there exist local coordinates $y_1, \dots, y_k$ centered at b such that $I \circ \pi$ is a finite $\mathbb{C}$-linear combination of functions of the form*

$$\prod_{i=1}^{k} \chi_i(\mathrm{ac}(y_i))(\mathrm{ord}\, y_i)^{n_i} |y_i|^{\gamma_i}, \tag{$*$}$$

where the χ_i are characters on $\mathbb{Z}_p^\times$, $\mathrm{ac}(y_i) := y_i p^{-\mathrm{ord}\, y_i}$ denotes the angular component of $y_i \in \mathbb{Q}_p$, the n_i are in $\mathbb{N}$, and the γ_i are in $\mathbb{C}$. (Here we use the following conventions: $\chi(\mathrm{ac}(0)) = 0$ if χ is a nontrivial character, $\chi(\mathrm{ac}(0)) = 1$ if χ is the trivial character 1; and $(\mathrm{ord}\, 0)^{n_i}|0|^{\gamma_i} = 0$, unless $n_i = \gamma_i = 0$ in which case it equals 1.)

REMARK. Working with complex exponents in $(*)$ we are able to express, for example, the function $g : \mathbb{Z}_p \setminus \{0\} \to \mathbb{Q}$ with $g(x) = 1$ if $(\mathrm{ord}\, x) \equiv 0 \equiv d$ and $g(x) = 0$ otherwise.

Application to the Local Singular Series in Several Variables. Let

$$f = (f_1, \dots, f_k) \in (\mathbb{Z}_p[x])^k,$$

with $x = (x_1, \dots, x_n)$. Let $a = (a_1, \dots, a_k) \in \mathbb{Z}_p^k$ be a regular value of $f : \mathbb{Z}_p^n \to \mathbb{Z}_p^k$, this means that a belongs to the image $f(\mathbb{Z}_p^n)$ but is not the image of any point in $\mathbb{Z}_p^n$ where the Jacobian of f has rank $< k$. Then it is known [Igusa 1978] that

$$p^{-m(n-k)} \#\{x \in (\mathbb{Z}/p^m)^n \mid f_i(x) \equiv a_i \bmod p^m \text{ for } i = 1, \dots, k\}$$

is constant for m big enough. (Here # stands for the number of elements.) We denote this constant value by $F(a)$. The function $\lambda \mapsto F(\lambda)$, for λ a regular value of f, is called the local singular series of $f : \mathbb{Z}_p^n \to \mathbb{Z}_p^k$ and plays an important role in number theory (for example, for the circle method). We put $F(\lambda) = 0$ if λ is not a regular value; thus F is a $\mathbb{Q}$-valued function on $\mathbb{Z}_p^k$. It is easy to see that $F(\lambda)$ is a locally constant function in the neighbourhood of any regular value a of f. But if λ tends to a nonregular value c, then $F(\lambda)$ has a nontrivial singular behavior. For $k = 1$, this has been studied in depth by Igusa [1974; 1975; 1978], who obtained an asymptotic expansion of $F(\lambda)$ for $\lambda \to c$. His work is based on Mellin inversion over p-adic fields and the study of local zeta functions using resolution of singularities. Igusa [1978, p. 32] asked how one could extend his result to the general case $k > 1$. A contribution to Igusa's question is given by

Corollary 1.8.2. *The local singular series $F(\lambda)$ is an $\mathfrak{L}$-simple p-exponential function of $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{Z}_p^k$. Hence Theorem* 1.8.1 *applies to $I(\lambda) := F(\lambda)$.*

Proof. This follows from Theorem 1.5 and the simple fact that

$$F(\lambda) = \int_{x \in \mathbb{Z}_p^n,\, f(x) = \lambda} |dx/(df_1 \wedge \cdots \wedge df_k)|,$$

whenever λ is a regular value of f. □

Some first results on local singular series in several variables were obtained by Loeser [1989], who conjectured that Theorem 1.8.1 holds for $I(\lambda) := F(\lambda)$ with π being an isomorphism above the set of regular values, when $f = (f_1, \dots, f_k)$ satisfies some nice geometric conditions (for example when the fibers of $f : \bar{\mathbb{Q}}_p^n \to \bar{\mathbb{Q}}_p^k$ are $(n-k)$-dimensional complete intersections with only isolated singularities, where $\bar{\mathbb{Q}}_p$ denotes the algebraic closure of $\mathbb{Q}_p$). Loeser's conjecture has several important implications and is still wide open. Indeed Corollary 1.8.2 does not yield any information about where π is locally an isomorphism. Very recently Lichtin [$\geq$ 2000a; $\geq$ 2000b] obtained explicit results assuming $k = 2$ together with some other hypothesises. It was only after seeing Lichtin's results that I obtained Theorem 1.8.1 and Corollary 1.8.2. I do not know how to prove Corollary 1.8.2 (for $k \geq 2$) without using the Cell Decomposition Theorem. The problem of relating the γ_i in Theorem 1.8.1 to geometric invariants remains open, although Lichtin [$\geq$ 2000a; $\geq$ 2000b] achieved a first breakthrough. Much remains to be done. Moreover Lichtin's method also has important applications in analysis and geometry.

Applications to Ax–Kochen-Definable Subsets. Let A be an $\mathcal{L}$-definable subset of $\mathbb{Q}_p^n$, then

$$\tilde{P}_A(T) := \sum_{m\in\mathbb{N}} (\#\{x \bmod p^m \mid x \in A\})T^m$$

is a rational function of T, the proof being the same as for $\tilde{P}(T)$. This can be proved without the Cell Decomposition Theorem (using instead resolution of singularities and quantifier elimination; compare Section 1.6). By contrast, it was proved in [Denef 1985] that, if we take for A a subset definable in the language of Ax and Kochen [1966], then $\tilde{P}_A(T)$ is still rational, but in this case the Cell Decomposition Theorem seems to be essential. (The language of Ax and Kochen is equivalent to the language obtained from $\mathcal{L}$ by adjoining a symbol for the function $\mathbb{Z} \to \mathbb{Q}_p : m \mapsto p^m$ from the second sort to the first sort.)

1.9. Dependence on p. It is well kown that $\mathbb{Q}_p$ does not have a quantifier elimination in $\mathcal{L}_{\mathrm{Mac}}$ or $\mathcal{L}$ which holds for all p (or for almost all p). To have a uniform quantifier elimination one has to work in a more complicated language (and here it becomes tedious to avoid the logical terminology of languages.) For such a quantifier elimination and its applications to integration we refer to [Pas 1989; 1990; 1991; Macintyre 1990].

1.10. Igusa's Local Zeta Function. Let $f(x) \in \mathbb{Z}[x], x = (x_1, \ldots, x_n)$. Igusa's local zeta function (for the trivial character) is the function

$$Z(s) := \int_{\mathbb{Z}_p^n} |f(x)|^s \, |dx|,$$

for $s \in \mathbb{C}$ with $Re(s) \geq 0$. It is an easy excercise to verify that $P(p^{-n-s}) = (1 - p^{-s}Z(s))/(1 - p^{-s})$. The rationality of $P(T)$ is equivalent to $Z(s)$ being a rational function of p^{-s}. It was in this way that Igusa [1974; 1975; 1978] proved that rationality of $P(T)$, by applying a resolution of singularities $\pi : M \to \mathbb{Z}_p^n$ as in Section 1.7, and pulling back the integral $Z(s)$ through π, so obtaining a very simple integral on M whose calculation is a straightforward exercise (compare the example in Section 1.2.1). There are fascinating conjectures about $Z(s)$, such as the monodromy and holomorphy conjectures, which relate the poles of $Z(s)$ (and hence the poles of $P(T)$) to topological invariants of the singularities of $\{x \in \mathbb{C}^n \mid f(x) = 0\}$. For all these and the many geometric and arithmetic results related to this we refer to the survey papers [Denef 1991; Igusa 1987; 1996; Veys 1996], and to the articles [Veys 1993; 1997].

1.11. Integration on Orbits. Let G be an algebraic group (defined over $\mathbb{Q}_p$) acting (algebraically) on the affine n-space (over $\mathbb{Q}_p$). Let $U \subset \mathbb{Q}_p^n$ be a $G(\mathbb{Q}_p)$-orbit (where $G(\mathbb{Q}_p)$ denotes the group of $\mathbb{Q}_p$-rational points on G). Igusa [1984] considered the orbital integral $Z_U(s) = \int_{U\cap\mathbb{Z}_p^n} |f(x)|^s \, |dx|$ which plays an essential role in several investigations (for example, study of the Γ-matrix of a prehomogeneous vectorspace [Sato 1989]). For this work it is essential to know that $Z_U(s)$ is a rational function of p^{-s}. The rationality is proved by using quantifier elimination: Indeed,

$$Z_U(s) = \sum_{m\in\mathbb{N}} \Bigg(\int_{\substack{U\cap\mathbb{Z}_p^n \\ \operatorname{ord} f(x)=m}} |dx| \Bigg) (p^{-s})^m,$$

so that we can apply Theorem 1.6.1, since the orbit U is definable by an existential $\mathcal{L}$-formula.

2. Integration on Subanalytic Sets over $\mathbb{Q}_p$

2.1. Motivating Problem. Let $P(T)$ and $\tilde{P}(T)$ be as in Section 1.1, but now with $f(x)$ a power series over $\mathbb{Z}_p$ which converges on $\mathbb{Z}_p^n$. Again we can ask whether $P(T)$ and $\tilde{P}(T)$ are rational. And indeed they are rational. For $P(T)$ this can be proved by adapting Igusa's method in a straightforward way; compare Section 1.10. Concerning $\tilde{P}(T)$, we have a problem in adapting the proof in § 1: the set $\{x \in \mathbb{Z}_p^n \mid \exists y \in \mathbb{Z}_p^n : f(y) = 0, y \equiv x \equiv p^m\}$ is in general not $\mathcal{L}$-definable when f is a power series. For this reason we have to introduce analytic functions in our language.

2.2. The Languages $\mathcal{L}_{\text{an}}$ and $\mathcal{L}_{\text{an}}^D$. We continue to use the language $\mathcal{L}$ introduced in 1.3, but from now on the variables of the first sort will run over $\mathbb{Z}_p$ (instead of over $\mathbb{Q}_p$ in § 1). Thus quantifiers with respect to variables of the first sort will always run over $\mathbb{Z}_p$ instead of over $\mathbb{Q}_p$. (Otherwise existential formulas in $\mathcal{L}_{\text{an}}$ could define very pathological sets, if we also allowed symbols for analytic functions in these variables.)

Let $\mathcal{L}_{\text{an}}$ be the (first order) language (in the sense of logic) obtained from $\mathcal{L}$ by adding a symbol for each analytic function $g : \mathbb{Z}_p^n \to \mathbb{Z}_p$.

Let $\mathcal{L}_{\text{an}}^D$ be the language obtained from $\mathcal{L}_{\text{an}}$ by adding a symbol D for the function (truncated division)

$$D : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p : (x, y) \mapsto \begin{cases} x/y & \text{if } y \neq 0 \text{ and } |x| \leq |y|, \\ 0 & \text{otherwise.} \end{cases}$$

Let S be a subset of $\mathbb{Z}_p^n$. We call S *semi-analytic* in $\mathbb{Z}_p^n$, if S is definable by a quantifier-free formula of $\mathcal{L}_{\text{an}}$. We say that S is *D-semi-analytic in* $\mathbb{Z}_p^n$ if it is definable by a quantifier-free formula of $\mathcal{L}_{\text{an}}^D$. Finally, we call S *subanalytic in* $\mathbb{Z}_p^n$ if it is definable by an existential formula of $\mathcal{L}_{\text{an}}$. (A formula is called existential if it is obtained from a quantifier-free formula by putting some existential quantifiers in front of it.)

Let S be a subset of a p-adic manifold M, and $a \in M$. We say that S is *blue in* M *at* a, where "blue" is one of the three above properties, if a has an open neighbourhood $U \cong \mathbb{Z}_p^n$ in M such that $S \cap U$ is blue. We call S *blue in* M if S is blue in M at each $a \in M$. Note that the subanalytic subsets of M are precisely the images of semi-analytic sets under proper analytic maps.

2.3. The p-adic Analytic Elimination Theorem

THEOREM [Denef and van den Dries 1988; Denef 1988]. *$\mathbb{Z}_p$ has elimination of quantifiers in $\mathcal{L}_{\text{an}}^D$.*

Easy examples show that $\mathbb{Z}_p$ has no quantifier elimination in $\mathcal{L}_{\text{an}}$.

2.4. COROLLARY. (i) *A subset of $\mathbb{Z}_p^n$ is subanalytic in $\mathbb{Z}_p^n$ if and only if it is D-semi-analytic in $\mathbb{Z}_p^n$.*
(ii) *Each $\mathcal{L}_{\text{an}}^D$-definable subset of $\mathbb{Z}_p^n$ is subanalytic in $\mathbb{Z}_p^n$.*
(iii) *The complement and the closure of a subanalytic subset in a p-adic manifold are again subanalytic.*

2.5. About the Proof of Theorem 2.3. It suffices to prove that every subanalytic subset of $\mathbb{Z}_p^n$ is D-semi-analytic. Consider for *example* a subanalytic set $S \subset \mathbb{Z}_p^n$ of the form

$$S = \{x = (x_1, \dots, x_n) \in \mathbb{Z}_p^n \mid \exists y = (y_1, \dots, y_m) \in \mathbb{Z}_p^m : f(x, y) = 0\},$$

with $f = \sum_{i \in \mathbb{N}^m} a_i(x) y^i$ a power series over $\mathbb{Z}_p$ which converges on $\mathbb{Z}_p^{n+m}$, and $f \not\equiv 0 \bmod p$. If f where regular in y_m (meaning that $f \equiv p$ is a monic polynomial in y_m over $\mathbb{Z}/p\mathbb{Z}[[x, y_1, \dots, y_{m-1}]]$), then, by a well-known p-adic version of the Weierstrass Preparation Theorem, we could write $f = ug$, with u having no zeros in $\mathbb{Z}_p^{n+m}$ and g a polynomial with respect to the last variable y_m. (Both u and g are power series over $\mathbb{Z}_p$ which converge on $\mathbb{Z}_p^{n+m}$.) Hence we could apply quantifier elimination in the language $\mathcal{L}_{\text{Mac}}$ to get rid of the quantifier $\exists y_m$. Although there exists an invertible change of the variables (x, y) which makes f regular in y_m, this is of no help because we are not allowed to mix the variables x and y. However, dividing f by a coefficient $a_j(x)$ with maximal absolute value (depending on $x \in \mathbb{Z}_p^n$, using case distinction), we can nevertheless apply the Weierstrass Preparation Theorem after an invertible transformation of only the y variables (which is certainly permitted). Divisions by $a_j(x)$ introduce the D-functions. In order to apply the D-function only a finite number of times, one has to express all the $a_i(x)$ as linear combinations of only finitely many of them, which is possible by Noetherianness. See [Denef and van den Dries 1988] for the details of the proof, which are somewhat lengthy. □

2.6. Basic Theorem on p-adic integration (analytic case)

THEOREM. *Theorem* 1.5 *with no λ involved* (*and hence also Theorem* 1.6.1) *still holds if we replace $\mathbb{Q}_p$ by $\mathbb{Z}_p$ and "$\mathcal{L}$-definable" by "$\mathcal{L}_{\text{an}}$-definable.*

An easy adaptation of the proof of this theorem shows that $\int_{A_l} p^{-\theta(x,l)} |dx|$ is an $\mathcal{L}_{\text{Pres}}$-simple p-exponential function, whenever $A_l \subset \mathbb{Z}_p^n$ and $\theta : \mathbb{Z}_p^n \times \mathbb{Z}^r \to \mathbb{N}$ are $\mathcal{L}_{\text{an}}$-definable. (Here $l = (l_1, \dots, l_r)$ are $\mathbb{Z}$-variables).

COROLLARY [Denef and van den Dries 1988]. *$\tilde{P}(T)$ is rational.*

Proof of Theorem 2.6. The next Theorem reduces it to Theorem 1.5, by pulling back the integral through π. □

Remark. We expect that Theorem 2.6 remains true when there are variables λ involved as in 1.5, but the above proof collapses in this case. Probably a proof can be obtained using the Cell Decomposition Theorem 1.5 and the method in Section 2.5 of [van den Dries 1992].

2.7. Uniformization Theorem for Subanalytic Sets

Theorem [Denef and van den Dries 1988]. *Let $A \subset \mathbb{Z}_p^n$ be subanalytic in $\mathbb{Z}_p^n$. Then there exists a compact p-adic manifold M of dimension n and an analytic map $\pi : M \to \mathbb{Z}_p^n$ satisfying these conditions*:

(i) *$\pi^{-1}(A)$ is semi-analytic, and actually semi-algebraic on each $U_i = \mathbb{Z}_p^n$ in a suitable decomposition of M as disjoint union of compact open subsets U_i.*
(ii) *π is a composition of finitely many blowing-up maps with respect to closed submanifolds of codimension ≥ 2. In particular π is an isomorphism outside closed sets of measure zero.*

Moreover the same holds if A depends in an $\mathcal{L}_{\mathrm{an}}$-definable way on a parameter $l \in \mathbb{Z}^r$ (replacing "semi-analytic", resp. "semi-algebraic", by "definable by a quantifier-free formula in $\mathcal{L}_{\mathrm{an}}$, resp. $\mathcal{L}$, involving the parameter l). We can also require that on each U_i the Jacobian of π equals a monomial times an analytic function with constant absolute value.

The proof of Theorem 2.7 is based on the fact that A is D-semi-analytic and on an induction on the number of occurences of D in the description of A, using p-adic analytic resolution (Section 1.7).

2.8. Theorem [Denef and van den Dries 1988]. *A subanalytic subset of $\mathbb{Z}_p^2$ is semi-analytic.*

Proof. Follows from Theorem 2.7 taking advantage of the simple nature of blowing-ups of $\mathbb{Z}_p^2$. □

2.9. Further Results. Using the above theorems one can prove (see [Denef and van den Dries 1988]) that subanalytic sets have many good properties: finite stratification in subanalytic manifolds, good dimension theory, Lojasiewicz inequalities, rationality of Lojasiewicz exponents, existence of a uniform bound for the cardinality of the finite members of a subanalytic family of subanalytic sets, semi-analytic nature of one-dimensional subanalytic sets, etc. Finally we mention the result of Z. Robinson [1993] that the singular locus of a subanalytic set is subanalytic.

To make the Analytic Elimination Theorem 2.3 uniform in p, one has to work in a more complicated language; see [van den Dries 1992].

2.10. Application to Counting Subgroups. For a group G and an integer $n \geq 1$, let $a_n(G)$ be the number of subgroups of index n in G. For a finitely generated group or for compact p-adic analytic group this number $a_n(G)$ is always finite (see [Grunewald, Segal, and Smith 1988; du Sautoy 1993], for example).

THEOREM 2.10.1 [Grunewald, Segal, and Smith 1988]. *If G is a torsion-free finitely generated nilpotent group, then $\sum_m a_{p^m}(G)T^m$ is rational, for each prime number p.*

THEOREM 2.10.2 [du Sautoy 1993]. *If G is a compact p-adic analytic group then $\sum_m a_{p^m}(G)T^m$ is rational.*

Theorem 2.10.1 is proved by expressing $a_{p^m}(G)$ in terms of a p-adic integral $\int_{A_m} p^{-\theta(x)}\,|dx|$ with $(A_m)_{m\in\mathbb{N}}$ and θ definable in $\mathcal{L}$. The proof of 2.10.2 is based on the same idea, with $\mathcal{L}_{\text{an}}$ replacing $\mathcal{L}$.

3. Subanalytic Sets over $\mathbb{C}_p$ and Rigid Analytic Geometry

3.1. Definition of $\mathbb{C}_p$. $\mathbb{C}_p$ is the completion of the algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$: The valuation ord on $\mathbb{Q}_p$ extends to a valuation ord on $\bar{\mathbb{Q}}_p$, taking values in $\mathbb{Q}$. This yields a norm $|\cdot| = p^{-\operatorname{ord}(\cdot)}$ on $\bar{\mathbb{Q}}_p$, and we can take the completion $\mathbb{C}_p$ of $\bar{\mathbb{Q}}_p$ with respect to this norm. One verifies that $\mathbb{C}_p$ is a nonarchimedean normed field and that $\mathbb{C}_p$ is algebraically closed. Most of what follows holds for any algebraically closed nonarchimedean complete normed field, except possibly Theorem 3.9 where we have to assume at this moment that the characteristic is zero to apply resolution of singularities.

NOTATION. Put $R = \{x \in \mathbb{C}_p \mid \operatorname{ord} x \geq 0\}$.

3.2. Motivating Problem. Let $f : R^m \to R^n$ be "analytic" (we will discuss in 3.3 below what we mean by "analytic"). What can be said about the image $f(R^m)$ of f? Can one make $f(R^m)$ semi-analytic by blowing-ups? The work of Lipshitz, Robinson, Gardener and Schoutens yields analogiess over $\mathbb{C}_p$ for most of the p-adic results in § 2, but the proofs are much more complicated.

3.3. First Motivation for Rigid Analysis. If in 3.2 we define "analytic" in the local sense (namely, that each point $a \in R^m$ has an open neighbourhood U_a in R^m on which f can be written as a converging power series), then any non-empty countable subset of R^n can be obtained as the image of a suitable "analytic" map $f : R^m \to R^n$. (The reason is that R^m is the disjoint union of infinitely many clopen subsets.) With this definition of "analytic" we obtain very "pathological" sets as images. To avoid this we will require that f is rigid analytic.

A rigid analytic function $h : R^m \to \mathbb{C}_p$ is a function which is given by a power series over $\mathbb{C}_p$ which converges on R^m. We denote the ring consisting of these functions by

$$\mathbb{C}_p\langle X_1,\dots,X_m\rangle := \{h : R^m \to \mathbb{C}_p \mid h \text{ is rigid analytic}\}.$$

This is called a Tate algebra, and is a Noetherian unique factorization domain (see [Bosch et al. 1984], for example).

3.4. The Languages L_{an} and L^D_{an}. Let L be the (first order) language (in the sense of logic) whose variables run over R, and with symbols to denote $+\,,-\,,\times\,,0,1$ and the binary relation $|x| \le |y|$. It follows from a well-known result of A. Robinson [1956] that R has quantifier elimination in the language L. (In that paper Robinson only proves model completeness for the theory of algebraically closed valued fields. But since this theory satisfies the prime extension property, its model completeness actually implies elimination of quantifiers; see [van den Dries 1978], for example.)

Let L_{an} be the (first order) language obtained from L by adding a symbol for each rigid analytic function $f : R^m \to R$. Easy examples show that R has no quantifier elimination in L_{an}.

Let L^D_{an} be the language obtained from L_{an} by adding a symbol D for the function (truncated division)

$$D : R \times R \to R : (x,y) \mapsto \begin{cases} x/y & \text{if } y \ne 0 \text{ and } |x| \le |y|,\\ 0 & \text{otherwise.}\end{cases}$$

Let A be a subset of R^n. We call A *globally semi-analytic in* R^n, resp. *D-semi-analytic in* R^n, if A is definable by a quantifier-free formula of L_{an}, resp. L^D_{an}. We call A (rigid) *subanalytic* in R^n, if it is definable by an existential formula of L_{an}.

3.5. The Main Theorems

THEOREM 3.5.1 (MODEL COMPLETENESS THEOREM [Lipshitz and Robinson 1996a]). *R is model complete in L_{an}, meaning that any formula in L_{an} is equivalent (for R) to an existential formula in L_{an}.*

Some of the ingredients in the proof of this theorem are discussed in Section 3.10 below.

COROLLARY 3.5.2. (i) *Each L_{an}-definable subset of R^n is subanalytic in R^n.*
(ii) *The complement and the closure (with respect to the norm topology) of a subanalytic subset of R^n are again subanalytic.*

THEOREM 3.5.3 (RIGID ANALYTIC ELIMINATION THEOREM [Gardener and Schoutens $\ge$ 2000]). *R has quantifier elimination in L^D_{an}.*

Some of the ideas in the proof of this theorem are discussed in Section 3.11 below.

REMARKS 3.5.4. Theorem 3.5.1 is a direct consequence of Theorem 3.5.3, but 3.5.3 uses 3.5.1 in its proof. Lipshitz [1993] proved already much earlier that R

has quantifier elimination in the language L^D_{sep} (see Section 3.10 below), which is richer than L^D_{an}. This important result of Lipshitz is at the same time stronger and weaker than Theorem 3.5.3.

COROLLARY 3.5.5. (i) *A subset of R^n is subanalytic in R^n if and only if it is D-semi-analytic in R^n.*

(ii) *The image of a rigid analytic map $R^m \to R^n$ is D-semi-analytic.*

REMARK 3.5.6. Using the theorems above one proves [Lipshitz 1993; Lipshitz and Robinson 1996a; Lipshitz and Robinson 1999] that subanalytic sets in R^n have many good properties. In particular all the results mentioned in Section 2.9 remain valid.

3.6. Further Motivation for Rigid Analysis. In the p-adic case any subanalytic subset of $\mathbb{Z}_p^2$ is semi-analytic. It is not true that any subanalytic subset of R^2 is globally semi-analytic. The reason is that the definition of "global semi-analytic" is too rigid. We need a more local definition. If we make the definition completely local, then we lose information and projections of semi-analytic sets would become pathological in some cases. Therefore we define a subset A of R^n to be (rigid) semi-analytic if there exists a finite covering of R^n by *admissible* open sets U in R^n, such that on each such U, $A \cap U$ is a finite boolean combination of sets of the form $\{x \in R^n \mid |f(x)| \leq |g(x)|\}$ with f, g rigid analytic on U. We still have to define the notions "admissible open in R^n" and "rigid analytic function on an admissible open". We give these definitions in Section 3.7 below. They are the key notions of rigid analysis and rigid analytic geometry. With these definitions, subanalytic subsets of R^2 are indeed semi-analytic; see Theorem 3.8.

3.7. First Steps in Rigid Analysis. A (reduced) *affinoid variety* V is a subset of some R^n of the form

$$V = \{x \in R^n \mid f_1(x) = \cdots = f_r(x) = 0\},$$

where the f_i are rigid analytic functions on R^n. The elements of V are in one-one correspondence with the maximal ideals of the affinoid algebra $A := \mathbb{C}_p\langle X_1, \ldots, X_n\rangle/(f_1, \ldots, f_r)$.

A *rigid analytic function* on V is the restriction to V of a rigid analytic function on R^n. A *morphism* $f : W \to V$ of affinoid varieties is a map $f = (f_1, \ldots, f_n)$ with each f_i rigid analytic.

A *rational subdomain* U of an affinoid variety V is a subset of V of the form

$$U = \{x \in V \mid |\, p_i(x)| \leq |p_0(x)|, \quad \text{for } i = 1, \cdots, s\}, \tag{$*$}$$

where $p_0, p_1, \ldots, p_s$ are rigid analytic functions on V with no common zero in V. Note that U is open and closed in V. Moreover U is actually an affinoid variety, its points being in 1-1 correspondence with $\{(x, t_1, \cdots, t_s) \mid x \in V, p_i - p_0 t_i = 0\}$.

Note that a "closed" disc $\{x \in V \mid |x-a| \le |r|\}$, and the complement $\{x \in V \mid |x-a| \ge |r|\}$ of an "open" disc, with $a, r \in \mathbb{C}_p$, are rational subdomains of V. Moreover the intersection of two rational subdomains is again a rational subdomain.

A *rigid analytic function* on the rational subdomain U of V is a function of the form

$$f\left(x, p_1/p_0, \ldots, p_s/p_0\right),$$

with $f(x, t_1, \ldots, t_s) \in \mathbb{C}_p\langle x, t_1, \ldots, t_s\rangle$ and $p_0, p_1, \ldots, p_s$ as in $(*)$.

An *admissible open* of an affinoid variety V is a rational subdomain of V, and an *admissible covering* of an admissible open U is a finite covering of U consisting of rational subdomains of V.

THEOREM (TATE). *Let $U_1, U_2, \ldots, U_k$ be an admissible cover of an affinoid variety V. Let $f : V \to \mathbb{C}_p$ be a function whose restriction to each U_i is rigid analytic. Then f is rigid analytic.*

A *quasi-compact rigid analytic variety* is obtained by "gluing together" a finite number of affinoid varieties (see [Bosch et al. 1984] for the details).

The preceding notions are the cornerstones of rigid analysis and rigid analytic geometry, founded by J. Tate. Basic references are [Bosch et al. 1984; Fresnel and van der Put 1981].

The definition of semi-analytic subsets of R^n given in Section 3.6 (based on the notion of admissible open given above) extend in the obvious way to the notion of semi-analytic subsets of an affinoid variety V.

3.8. THEOREM [Gardener and Schoutens $\ge$ 2000]. *Let $A \subset R^2$ be subanalytic in R^2. Then A is semi-analytic.*

The proof is based on the following theorem of Gardener and Schoutens [$\ge$ 2000], and on the simple nature of blowing-ups of R^2.

3.9. Uniformization Theorem for Rigid Subanalytic Sets

THEOREM. *Let $A \subset R^n$ be subanalytic in R^n. Then there exist a finite number of morphisms $f_i : V_i \to R^n$ with the following properties*:

(i) *V_i is an affinoid variety and f_i is a composition of smooth local blowing-up maps.* (By a smooth local blowing-up map we mean the restriction to an open affinoid subvariety of a blowing-up map (in the sense of rigid analytic geometry) with respect to a smooth center of codimension at least 2.)
(ii) $\bigcup_i f_i(V_i) = R^n$.
(iii) *$f_i^{-1}(A)$ is semi-analytic in V_i.*

The proof (see [Gardener and Schoutens $\ge$ 2000]) is not difficult, since we know already by Corollary 3.5.5 that A is D-semi-analytic, and by resolution of singularities it can be proved that D-semi-analytic sets can be made semi-analytic by smooth local blowing-ups (compare the proof of Theorem 2.7).

3.10. Ideas in the Proof of the Model Completeness Theorem 3.5.1

3.10.1. The Languages L_{sep} and L_{sep}^D. Let L_{sep} be the (first order) language obtained from L by introducing a second sort of variables running over $\mathcal{P} := \{x \in R \mid |x| < 1\}$ and by adding a symbol for each function $f : R^n \times \mathcal{P}^m \to R$ with $f \in \mathbb{C}_p\langle X_1, \ldots, X_n\rangle[[Y_1, \ldots, Y_m]]_s$. Here $\mathbb{C}_p\langle X\rangle[[Y]]_s$ is the ring of *separated power series*, which is a Noetherian subring of $R < X > [[Y]] \otimes_R \mathbb{C}_p$, where $R\langle X\rangle$ denotes the ring of power series over R which converge on R^n. We refer to [Lipshitz 1993; Lipshitz and Robinson $\geq$ 2000] for the exact definition. The restriction to separated power series is essential to avoid pathologies. At any rate we have $\mathbb{Z}_p\langle X\rangle[[Y]] \subset \mathbb{C}_p\langle X\rangle[[Y]]_s$. A nonzero separated power series $f(Y_1)$ in one variable has only a finite number of zeroes in R. (This can fail when $f(Y_1)$ is not separated.) A systematic study of the rings of separated power series has been made by Lipshitz and Robinson in their fundamental paper [Lipshitz and Robinson $\geq$ 2000].

Let L_{sep}^D be the (first order) language obtained from L_{sep} by adding a symbol for the function D defined in 3.4, and a symbol for the function

$$D_0 : R \times R \to \mathcal{P} : (x, y) \mapsto \begin{cases} x/y & \text{if } y \neq 0 \text{ and } |x| \leq |y|, \\ 0 & \text{otherwise.} \end{cases}$$

3.10.2. Theorem [Lipshitz 1993]. *R has elimination of quantifiers in L_{sep}^D.*

The proof uses ideas from the proof of the p-adic Analytic Elimination Theorem 2.3, but is much more complicated. Variants of the Weierstrass Preparation Theorem play an important role.

3.10.3. The language L_E^D. Let L_E^D be the sublanguage of L_{sep}^D having a symbol for $f : R^n \times \mathcal{P}^m \to R$ only if f and all its partial derivatives are definable by existential formulas of L_{an}. The set of these functions is denoted by E. To be fully correct one should include more local functions as well, which only converge on $U \times \mathcal{P}^m$, with U a rational subdomain of R^n. (When $\mathbb{C}_p$ is replaced by an algebraically closed nonarchimedean normed field of nonzero characteristic one has to modify the definition of L_E^D slightly.)

3.10.4. Theorem [Lipshitz and Robinson $\geq$ 2000; 1996a]. *R has quantifier elimination in L_E^D.*

The proof is based on the verification that in the proof of the quantifier elimination for L_{sep}^D one only needs functions in E. For this, one has (among other things) to prove a Weierstrass Preparation Theorem for E.

3.10.5. Note now that the Model Completeness Theorem 3.5.1 is a direct consequence of the above Theorem 3.10.4.

3.11. Some ideas in the proof of the Rigid Analytic Elimination Theorem 3.5.3. We have to prove that any subanalytic subset A of R^n is D-semi-analytic. By Corollary 3.5.2 of the Model Completeness Theorem, we may suppose that A is closed in R^n. Indeed the closure $\bar{A}$ of A and $\bar{A} \setminus A$ are subanalytic and $\dim(\bar{A} \setminus A) < \dim A$ (see [Lipshitz and Robinson 1996a]), so that we can use induction.

An easy argument shows that a closed subanalytic subset of R^n is "almost" the image $f(X)$ of a morphism $f : X \to R^n$ with X an affinoid variety.

Recall that a morphism $f : X \to Y$ is called flat if, for each point x in X, the local ring of X at x is flat over the local ring of Y at $f(x)$. When $f : X \to Y$ is a flat morphism of affinoid varieties, a theorem of Raynaud and Mehlmann states that $f(X)$ is a finite union of rational subdomains of Y, hence D-semi-analytic.

When f is not flat, Gardener and Schoutens [$\geq$ 2000] proved using results from [Gardener 2000; Schoutens 1999] that one can make f flat by taking its strict transform under a suitable finite sequence of local blowing-ups. This Flattening Theorem is an analogy of a difficult result of Hironaka in real analytic geometry. The adaptation to the rigid analytic case is difficult and is based on Berkovich's approach [1990] to rigid analytic geometry. Since the image of a D-semi-analytic set under a local blowing-up map is D-semi-analytic (up to a subanalytic subset in an affinoid variety of smaller dimension), the Flattening Theorem (and some extra work) reduces us to the case that f itself is flat, which we considered already.

4. Semi-algebraic Sets over $\mathbb{C}((t))$ and Motivic Integration

4.1. Motivating Problem. Let $f(x) \in \mathbb{C}[x]$, with $x = (x_1, \ldots, x_n)$. We use the notations

$$X := \{x \in \mathbb{C}^n \mid f(x) = 0\}, \text{ the hypersurface defined by } f,$$

$$\mathcal{A} := \mathcal{A}(X) := \{\gamma \in (\mathbb{C}[[t]])^n \mid f(\gamma) = 0\} = \text{ the arc space of } X,$$

$$\mathcal{A}_m := \mathcal{A}_m(X) := \{\gamma \in (\mathbb{C}[t]/t^m)^n \mid f(\gamma) \equiv 0 \rangle (t^m)\},$$

$$\pi_m : \mathcal{A} \to \mathcal{A}_m \text{ the natural projection},$$

$$\tilde{\mathcal{A}}_m := \tilde{\mathcal{A}}_m(X) := \pi_m(\mathcal{A}) = \text{ the set of truncations mod } t^m \text{ of arcs on } X.$$

Note that $\mathcal{A}_m$ is an algebraic variety over $\mathbb{C}$ in a natural way. Indeed, we identify it with

$$\{(a_{1,0}, a_{1,1}, \ldots, a_{1,m-1}, a_{2,0}, \ldots, a_{n,m-1}) \in \mathbb{C}^{nm} \mid \\ f(a_{1,0} + a_{1,1}t + \cdots, \ldots, a_{n,0} + a_{n,1}t + \cdots) \equiv 0 \rangle (t^m)\}.$$

Proposition 4.1.1 [Nash 1995]. *$\tilde{\mathcal{A}}_m$ is a constructible subset of the algebraic variety $\mathcal{A}_m$, meaning that it is a finite union of (Zariski) locally closed subvarieties of $\mathcal{A}_m$.*

Proof. By a theorem of Greenberg [Greenberg 1966], for each m there exists $m' \geq m$ such that $\tilde{\mathcal{A}}_m$ equals the image of $\mathcal{A}_{m'}$, under the natural map $\mathcal{A}_{m'} \to \mathcal{A}_m$. The Proposition follows now from quantifier elimination for $\mathbb{C}$ (Chevalley's

Theorem asserting that the image of a constructible subset under a morphism of algebraic varieties is again constructible). $\square$

REMARK. The notions above can be defined for any algebraic variety over $\mathbb{C}$, and all results of the present § 4 hold in this more general case.

REMARK. The $\tilde{\mathcal{A}}_m$ were first studied by J. Nash [1995], in relation with Hironaka's resolution of singularities. In the same paper, Nash formulated a very intriging conjecture about the $\tilde{A}_m$, which is still open. For related work see [Gonzalez-Sprinberg and Lejeune Jalabert 1996; Lejeune-Jalabert 1990].

FORMULATION OF THE PROBLEM. How does $\tilde{A}_m$ vary with m? We will give an answer (Theorem 4.3) to this problem, modulo the equivalence relation which calls two algebraic varieties equivalent if they can be cut in a finite number of (Zariski locally closed) pieces, the pieces of the first variety being isomorphic (as algebraic varieties) with the pieces of the second, or if this can be done after replacing the two varieties by the disjoint union with a third variety. The set of all varieties modulo this equivalence relation generates a ring:

4.2. The Grothendieck Ring $\mathcal{M}$ of Algebraic Varieties over $\mathbb{C}$. This ring $\mathcal{M}$ is generated by symbols $[V]$, for V running over all algebraic varieties $\mathbb{C}$ (reduced and separated schemes of finite type over $\mathbb{C}$), with relations

$$
\begin{aligned}
[V] &= [V'] \quad \text{if } V \text{ is isomorphic with } V' \;, \\
[V] &= [V \setminus V'] + [V'] \quad \text{if } V' \text{ is Zariski closed in } V, \\
[V \times V'] &= [V][V'].
\end{aligned}
$$

Note that for V any algebraic variety over $\mathbb{C}$, the map $V' \mapsto [V']$, for V' Zariski locally closed in V, extends uniquely to the map $W \mapsto [W]$, for W any constructible subset of V, satisfying $[W \cup W'] = [W] + [W'] - [W \cap W']$.

Set $\mathbb{L} := [\mathbb{A}^1] \in \mathcal{M}$, where $\mathbb{A}^1$ denotes the affine line over $\mathbb{C}$.

Set $\mathcal{M}_{\mathrm{loc}} := M[\mathbb{L}^{-1}]$, the localization of the ring M obtained by inverting $\mathbb{L}$.

4.3. Rationality Theorem

THEOREM [Denef and Loeser 1999a]. $\tilde{P}(T) := \sum_{m=1}^{\infty}[\tilde{A}_m]T^m$, *considered as a power series over* $\mathcal{M}_{\mathrm{loc}}$, *is rational and belongs to the subring of* $\mathcal{M}_{\mathrm{loc}}[[T]]$ *generated by* $\mathcal{M}_{\mathrm{loc}}[T]$ *and the series* $(1 - \mathbb{L}^a T^b)^{-1}$ *with* $a \in \mathbb{Z}$ *and* $b \in \mathbb{N} \setminus \{0\}$.

4.4. Analogy with the p-adic Case. Note the analogy with the series $\tilde{P}(T)$ in Section 1.1, considering $\mathbb{Z}_p$ as an analogue of $\mathbb{C}[[t]]$. The proof of the rationality Theorem 4.3 runs along the same lines as in Section 1 and is based on the quantifier elimination for $\mathbb{C}((t))$ due to Pas and integration on the arc space $(\mathbb{C}[[t]])^n$ of the affine n-space $\mathbb{A}^n$. Integration on $(\mathbb{C}[[t]])^n$ is called motivic integration and was recently introduced by Kontsevich [1995] and refined by Denef and Loeser [1999a]. We briefly discuss motivic integration in Section 4.7 and in 4.8 we present some ideas of the proof of Theorem 4.3. These integrals take values in a certain completion $\hat{\mathcal{M}}$ of $\mathcal{M}$, unlike the p-adic integrals of Section 1 which take values in $\mathbb{R}$.

4.5. The Completion $\hat{\mathcal{M}}$ of $\mathcal{M}_{\mathrm{loc}}$. Define $F^m(\mathcal{M}_{\mathrm{loc}})$ as the subgroup of $\mathcal{M}_{\mathrm{loc}}$ generated by

$$\{[V]\mathbb{L}^{-i} \mid V \text{ is algebraic variety and } i \geq m + \dim V\}.$$

These form a filtration of $\mathcal{M}_{\mathrm{loc}}$. Let $\hat{\mathcal{M}}$ be the completion of $\mathcal{M}_{\mathrm{loc}}$ with respect to this filtration. (An element of $\mathcal{M}_{\mathrm{loc}}$ is "small" if it belongs to $F^m(\mathcal{M}_{\mathrm{loc}})$ for m big.) In comparison with p-adic integration, consider $\hat{\mathcal{M}}$ as the analogue of $\mathbb{R}$ (the target of integration), and $\mathbb{L}$ as the analogue of $p \in \mathbb{R}$. The ring structure on $\mathcal{M}_{\mathrm{loc}}$ induces a ring structure on $\hat{\mathcal{M}}$. The ring $\hat{\mathcal{M}}$ was first introduced by Kontsevich [Kontsevich 1995].

Remark. We do not know whether the natural map $\mathcal{M}_{\mathrm{loc}} \to \hat{\mathcal{M}}$ is injective. But many geometric invariants, such as the topological Euler characteristic factor through the image $\overline{\mathcal{M}}_{\mathrm{loc}}$ of $\mathcal{M}_{\mathrm{loc}}$ in $\hat{\mathcal{M}}$.

4.6. Semi-Algebraic Sets over $\mathbb{C}[[t]]$. Let $\mathcal{L}_{\mathrm{Pas}}$ be the (first order) language (in the sense of logic) with three sorts of variables: variables running over the valued field $\mathbb{C}((t))$ (= the fraction field of $\mathbb{C}[[t]]$), variables running over the value group $\mathbb{Z}$, and variables running over the residue field $\mathbb{C}$. The symbols of $\mathcal{L}_{\mathrm{Pas}}$ consist of the symbols of Presburger's language $\mathcal{L}_{\mathrm{Pres}}$ for $\mathbb{Z}$ (see 1.3), symbols to denote $+, -, \times, 0, 1$ in $\mathbb{C}((t))$ and in $\mathbb{C}$, and symbols for the valuation $\mathrm{ord} : \mathbb{C}((t)) \setminus \{0\} \to \mathbb{Z}$ and for the function $\overline{\mathrm{ac}} : \mathbb{C}((t)) \to \mathbb{C} : \gamma \mapsto$ the leading coefficient of the series γ. (We use the convention that $\overline{\mathrm{ac}}(0) = 0$, $\mathrm{ord}\, 0 = +\infty, (+\infty) + l = +\infty$ and $+\infty \equiv l \equiv d$, for all l in $\mathbb{Z} \cup \{+\infty\}$.)

A theorem of Pas [1989] states that $\mathbb{C}((t))$ has quantifier elimination in $\mathcal{L}_{\mathrm{Pas}}$.

A subset of $\mathbb{C}((t))^n$ which is definable by a formula without quantifiers in $\mathcal{L}_{\mathrm{Pas}}$ is called *semi-algebraic*.

Proposition 4.6.1. *Let X be as in* 4.1, *and let $S \subset \mathcal{A}(X) \subset (\mathbb{C}[[t]])^n$ be semi-algebraic. Then $\pi_m(S)$ is a constructible subset of the algebraic variety $\mathcal{A}_m(X)$.*

Proof. An easy application of the Theorem of Pas. □

More generally one defines (in the obvious way) semi-algebraic subsets of $\mathcal{A}(X)$, for any algebraic variety X over $\mathbb{C}$. Obviously Proposition 4.6.1 remains valid.

4.7. Motivic integration on the arc space $\mathcal{A}(X)$. Motivic integration was recently introduced by Kontsevich [1995] and further developed and refined by Denef and Loeser [1999a].

Theorem 4.7.1 [Denef and Loeser 1999a]. *Let X be as in Section* 4.1 (*or more generally any algebraic variety over* $\mathbb{C}$). *Let $S \subset \mathcal{A}(X)$ be semi-algebraic and $d = \dim X$. Then*

$$\mu(S) := \lim_{m \to +\infty} [\pi_m(S)]\mathbb{L}^{-md} \in \hat{\mathcal{M}}$$

exists in $\hat{\mathcal{M}}$. *Moreover* $S \mapsto \mu(S)$ *is an* $\hat{\mathcal{M}}$*-valued* σ*-additive measure on the boolean algebra of semi-algebraic subsets of* $\mathcal{A}(X)$.

We call μ the *motivic measure* on the arc space $\mathcal{A}(X)$ of X. This allows us to define

$$\int_S \mathbb{L}^{-\theta} d\mu := \sum_{m \in \mathbb{N}} \mathbb{L}^{-m} \mu(\theta^{-1}(m)) \in \hat{\mathcal{M}},$$

for any $\theta : \mathcal{A}(X) \to \mathbb{N}$ which is definable in $\mathcal{L}_{\mathrm{Pas}}$. These motivic integrals have nice properties, such as an analogue of the classical change of variables formula; see [Denef and Loeser 1999a].

4.8. Some Ideas in the Proof of the Rationality Theorem. We only consider the weaker assertion that the image of $\tilde{P}(T)$ in $\hat{\mathcal{M}}[[T]]$ is rational. (The proof of the original statement is more difficult.) To prove this weaker assertion, we consider the motivic measure μ on the arc space $\mathcal{A}(\mathbb{A}^m) = (\mathbb{C}[[t]])^n$. For $m \in \mathbb{N} \setminus \{0\}$, put

$$S_m := \{\gamma \in (\mathbb{C}[[t]])^n \mid \exists \gamma' \in (\mathbb{C}[[t]])^n : f(\gamma') = 0,\ \gamma \equiv \gamma' \equiv t^m\}.$$

Then

$$[\tilde{A}_m(X)] = \mu(S_m)\mathbb{L}^{mn} \quad \text{in } \hat{\mathcal{M}},$$

where X is the locus of $f = 0$. The proof of the weaker assertion above proceeds now in close analogy with the proof of the rationality of $\tilde{P}(T)$ in the p-adic case (using resolution of singularities, but no cell decomposition). $\square$

4.9. Construction of New Invariants of Algebraic Varieties. p-adic integration was used by Denef and Loeser [1992] (see also [Denef 1991]) to obtain new geometric invariants, such as the topological zeta functions. These are calculated from a resolution of singularities using Euler characteristics and multiplicities. (Independence from the chosen resolution is proved by p-adic integration and use of the Grotendieck–Lefschetz trace formula.) See [Veys 1999] for related work.

Kontsevich [1995] obtained many more geometric invariants using motivic integration instead of p-adic integration. (This makes it possible to work with Hodge polynomials instead of Euler characteristics.) In the same paper he also used motivic integration to prove the conjecture that birationally isomorphic Calabi–Yau manifolds have the same Hodge numbers. (That they have the same Betti numbers was proved before by Batyrev [1997a] using p-adic integration.)

Denef and Loeser [1998; 1999a] have obtained some more geometric invariants by motivic integration. For example, if X is an algebraic variety over $\mathbb{C}$, one can consider

$$\chi(\mu(\mathcal{A}(X))) \in \mathbb{Q},$$

where χ denotes the Euler characteristic, since

$$\mu(\mathcal{A}(X)) \in \overline{M}_{\mathrm{loc}}[((1 + \mathbb{L} + \mathbb{L}^2 + \cdots + \mathbb{L}^i)^{-1})_{i \in \mathbb{N}}].$$

(Recall that $\overline{\mathcal{M}}_{\mathrm{loc}}$ is the image of $\mathcal{M}_{\mathrm{loc}}$ in $\hat{\mathcal{M}}$.) When X is nonsingular we have $\chi(\mu(\mathcal{A}(X))) = \chi(X)$, but for singular X one gets something new which can be expressed in terms of Euler characteristics of the exceptional divisors (and their intersections) of a suitable resolution of singularities of X. Similar invariants can be defined using Hodge polynomials.

Very recently Batyrev [1998; 1997b] constructed some related new invariants (for algebraic varieties with "mild" singularities) which he calls the "string Euler characteristic" and the "string Hodge numbers". They play a role in quantum cohomology and mirror symmetry.

For connections with the theory of motives, see [Denef and Loeser 1998; 1999b].

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