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Notes on o-Minimality and Variations

DUGALD MACPHERSON

Abstract. The article surveys some topics related to o-minimality, and is based on three lectures. The emphasis is on o-minimality as an analogue of strong minimality, rather than as a setting for the model theory of expansions of the reals. Section 2 gives some basics (the Monotonicity and Cell Decomposition Theorems) together with a discussion of dimension. Section 3 concerns the Peterzil–Starchenko Trichotomy Theorem (an o-minimal analogue of Zil'ber Trichotomy). There follows some material on definable groups, with powerful applications of the Trichotomy Theorem in work by Peterzil, Pillay and Starchenko. The final section introduces weak o-minimality, P-minimality, and C-minimality. These are analogues of o-minimality intended as settings for certain henselian valued fields with extra structure.

1. Introduction

This paper is a survey of selected topics in and around o-minimality. The emphasis is on analogies with stability theory, and there is little here on analytic expansions of the reals. On the other hand, there is quite a lot on dimension in o-minimal and related structures. I have concentrated on algebraic examples.

Section 2 is introductory, and covers definable functions, cell decomposition, dimension for definable sets, prime models, and definable types. Section 3 is on groups definable in o-minimal theories: there I describe a trichotomy theorem due to Peterzil and Starchenko, and consequences, due to them and Pillay, for definable groups. In Section 4, I leave o-minimality and turn to other settings (weak o-minimality, C-minimality, P-minimality) which are superficially similar, and survey some of the main examples and results.

Generally I have omitted proofs, but where possible try to give the idea of a proof. As a general source for o-minimality, I recommend [van den Dries 1998; 1996]. Much of the material from Section 2 comes from [Knight et al. 1986] and [Pillay and Steinhorn 1986], and the latter paper gives an excellent introduction to the subject.

2. Basics of o-minimality

We consider first-order structures $\mathcal{M} = (M, <, \ldots)$, where M is the domain, $<$ is a binary relation symbol interpreted by a dense total order on M, and there may be other symbols for relations, functions or constants in the language. The assumption that $<$ is dense is not necessary for all the theory, but holds in the examples of interest to us. Indeed, by results from [Pillay and Steinhorn 1987; 1988], any discrete o-minimal structure is essentially trivial, in the sense that definable functions are given piecewise by translations.

DEFINITION 2.0.1. The above structure $\mathcal{M}$ is *o-minimal* if every definable subset of M is a finite union of singletons and open intervals (with endpoints in $M \cup \{\infty, -\infty\}$).

REMARKS. 1. Here, as throughout these notes, 'definable' means 'definable with parameters'.
2. It is crucial that the intervals are not just convex sets, but have endpoints in $M \cup \{+\infty, -\infty\}$. Without this we have weak o-minimality, with a much weaker structure theory (see Section 4).
3. There is an obvious question whether, if $\mathcal{M}$ is o-minimal and $\mathcal{N}$ is elementarily equivalent to $\mathcal{M}$, then $\mathcal{N}$ must also be o-minimal. The answer is positive (see Remark 2 after Theorem 2.1.3 below).
4. The class of o-minimal structures is closed under reducts (so long as $<$ stays in the language). Frequently a structure in a rich language is proved to be o-minimal by quantifier-elimination, and it follows that all reducts (with the ordering still in the language) are o-minimal. Also, o-minimality is closed under expansions by constants.
5. The definition says that every definable subset of M is quantifier-free definable just using the symbols $=$ and $<$. This suggests an analogy with strong minimality, which says that *in all models of the theory*, every definable set is quantifier-free definable just from $=$.
6. The order topology on M has a uniformly definable basis (of intervals). Likewise, the induced topology on M^n has a uniformly definable basis. Hence, given a definable function $f : U \to M^n$, say, where $U \subset M^m$, the condition 'f is continuous at $\bar{a}$' is first-order expressible, uniformly in $\bar{a}$.
7. Definable continuous partial functions $M \to M$ satisfy the intermediate value property.

EXAMPLES. The following structures are o-minimal. I emphasise that there is a large literature now on the rich supply of o-minimal expansions of the reals, not touched on here.

1. $(\mathbb{Q}, <)$.
2. $(\mathbb{Q}, <, +)$.
3. $\mathcal{R} = (\mathbb{R}, <, +, -, \cdot, 0, 1)$. By Tarski's quantifier elimination, we need only check that atomic formulas with parameters define finite unions of intervals.

This is clear, for one need only consider the formulas $\sum_{i=0}^{n} a_i x^i < 0$ and $\sum_{i=0}^{n} a_i x^i = 0$.

4. $(\mathcal{R}, \exp)$. Here, o-minimality follows from Wilkie's model-completeness result [1996].

The following result is at the root of most of the theory of o-minimality.

THEOREM 2.0.2 [Pillay and Steinhorn 1986]. *Let $\mathcal{M}$ be o-minimal, and $f : (a,b) \to M$ be a definable function with domain (a,b) (possibly $a = -\infty$, or $b = \infty$). Then there are points $a = a_0 < \cdots < a_{k+1} = b$ such that for each $j = 0, \ldots, k$, the restriction $f|_{(a_j, a_{j+1})}$ is either constant, or a strictly monotonic and continuous bijection to an interval.*

SKETCH PROOF. It suffices to prove, for any definable function $f : I \to M$, where I is an interval, that

(i) there is an infinite subinterval of I on which f is constant or injective;
(ii) if f is injective, then it is strictly monotonic on a subinterval,
(iii) if f is strictly monotonic, then f is continuous on a subinterval.

For given (i)–(iii), let X be the set of $x \in (a,b)$ such that on some open interval containing x, f is constant, or strictly monotonic and continuous. By (i)-(iii) above, $(a,b) \setminus X$ is finite, so we may assume (by throwing away finitely many points and replacing (a,b) by subintervals) that $(a,b) \setminus X = \varnothing$, so f is continuous on (a,b). There are finitely many possible kinds of local behaviour, so, after partitioning (a,b) further we may suppose that f has the same local behaviour throughout (a,b). If, for example, f is locally strictly increasing everywhere, then it follows easily by o-minimality that f is strictly increasing everywhere.

I sketch a proof of (i) above. We may assume that all sets $f^{-1}(x)$ are finite (for otherwise by o-minimality some set $f^{-1}(x)$ contains an interval, and f is constant on this interval). Hence $f(I)$ is infinite, so contains an interval J. Define $g : J \to I$ by $g(y) := \mathrm{Min}\{x \in I : f(x) = y\}$, find an infinite interval $K \subset g(J)$, and observe that $f|_K$ is injective. □

REMARKS. 1. If $\mathcal{M}$ is an expansion of an ordered field, then the notion of differentiability makes sense, and we can sharpen the above theorem to arrange that f is continuously differentiable on each (a_j, a_{j+1}) (see Chapter VII of [van den Dries 1998]). In fact, for any n, we can arrange that f is $C^{(n)}$ (that is, n times continuously differentiable) on each (a_j, a_{j+1}). However, as n increases we may need more and more intervals, so we cannot expect to arrange that f is $C^{(\infty)}$ on each interval.

2. Here, as elsewhere in the theory, we have good control over parameters. In particular, we may choose the a_i so that they are definable over the parameters used to define f.

2.1. Cell decomposition. The notion of o-minimality tells us about definable sets in one variable. The cell decomposition theorem (and its variants) carry such information to definable sets in several variables. I follow the treatment from [van den Dries 1998].

Given a definable $X \subseteq M^n$, let

$$C(X) := \{f : X \to M, f \text{ is definable and continuous}\},$$

and let $C_\infty(X) := C(X) \cup \{-\infty, +\infty\}$ (here, ∞ denotes the 'function' on X taking value ∞ everywhere, and $-\infty$ is defined similarly.) Suppose that $f, g \in C_\infty(X)$ and that $(\forall \bar{x} \in X)(f(\bar{x}) < g(\bar{x}))$. Then

$$(f, g)_X := \{(\bar{x}, y) \in X \times M : f(\bar{x}) < y < g(\bar{x})\}.$$

Definition 2.1.1. Let $(i_1, \ldots, i_m)$ be a sequence of zeros and ones. Then an $(i_1, \ldots, i_m)$-*cell* is a definable subset of M^m, defined as follows by induction on m.

(i) A (0)-cell is a singleton of M, and a (1)-cell is a non-empty open interval, possibly unbounded.
(ii) Suppose $(i_1, \ldots, i_m)$-cells have been defined. Then an $(i_1, \ldots, i_m, 0)$-cell is the graph of a function $f \in C(X)$, where X is an $(i_1, \ldots, i_m)$-cell. An $(i_1, \ldots, i_m, 1)$-cell is a set $(f, g)_X$, with X some $(i_1, \ldots, i_m)$-cell and $f, g \in C_\infty(X)$ with $f(\bar{x}) < g(\bar{x})$ for all $\bar{x} \in X$.

A *cell* is an $(i_1, \ldots, i_m)$-cell for some $i_1, \ldots, i_m \in \{0, 1\}$.

Remarks. 1. The numbers $i_1, \ldots, i_m$ are uniquely determined by the cell.
2. A cell in M^m is open if and only if it is a $(1, \ldots, 1)$-cell.
3. More generally, let X be an $(i_1, \ldots, i_m)$-cell, let $k := i_1 + \cdots + i_m$, and suppose we have $\lambda(1) < \cdots < \lambda(k)$ and $i_{\lambda(1)} = \ldots = i_{\lambda(k)} = 1$. Let $\pi : M^m \to M^k$ project to the $\lambda(1), \ldots, \lambda(k)$ coordinates. Then π is a homeomorphism onto an open cell in M^k.
4. Cells are *definably connected*; that is, a cell X cannot be expressed as the disjoint union of two non-empty definable sets which are open in X. If $M = \mathbb{R}$, then cells are even connected.

Definition 2.1.2. A *decomposition* of M^m is a partition of M^m into finitely many cells, defined as follows by induction.

(i) Any partition of M into finitely many disjoint cells is a decomposition.
(ii) A decomposition of M^{m+1} is a finite partition of M^{m+1} into cells, such that if $\pi : M^{m+1} \to M^m$ is the projection onto the first m coordinates, then the set of π-projections of the cells forms a decomposition of M^m.

Theorem 2.1.3 [Knight et al. 1986]. *For each $n > 0$, the following statements hold.*

$(\mathrm{I})_n$ *Given definable $A_1, \ldots, A_k \subseteq M^n$, there is a decomposition of M^n which partitions each of the A_i.*

$(\mathrm{II})_n$ *Given definable $A \subseteq M^n$ and a definable function $f : A \to M$, there is a decomposition $\mathcal{D}$ of M^n partitioning A, such that for each $B \in \mathcal{D}$ with $B \subseteq A$, $f|_B : B \to M$ is continuous.*

$(\mathrm{III})_n$ *Suppose that $Y \subseteq M^{n+1}$ is definable. For any $\bar{a} \in M^n$, let*

$$Y_{\bar{a}} := \{x \in M : (\bar{a}, x) \in Y\}.$$

Then there is a number N (depending on Y) such that any finite set of the form $Y_{\bar{a}}$ has size at most N.

REMARKS. 1. If condition $(\mathrm{III})_n$ holds for all n for a structure $\mathcal{M}$ (not necessarily o-minimal), then we say that $\mathcal{M}$ is *uniformly bounded.* This is a property of $\mathrm{Th}(\mathcal{M})$, that is, it is preserved by elementary equivalence.

2. Uniform boundedness ensures that if $\mathcal{M}$ is o-minimal, and $\mathcal{N}$ is elementarily equivalent to $\mathcal{M}$, then $\mathcal{N}$ is o-minimal. For suppose that $Y \subset N$ is definable by a formula $\phi(x, \bar{a})$, and let $\psi(x, \bar{a})$ define (uniformly in $\bar{a}$) the boundary of Y. By uniform boundedness, there is a natural number K such that for any $\bar{b}$ from M, $\psi(x, \bar{b})$ has at most K realisations. Now $\mathrm{Th}(\mathcal{M})$ says this, so it holds in $\mathcal{N}$, so $\psi(x, \bar{a})$ has at most K realisations, and hence (as $\mathrm{Th}(\mathcal{M})$ says that any maximal convex definable set has a supremum and infimum in $M \cup \{\infty, -\infty\}$) $\phi(x, \bar{a})$ is a union of finitely many singletons and intervals.
3. If $\mathcal{M}$ is an expansion of an ordered field, then for any $p > 0$ one can require that the definable functions in the cell decomposition are all $C^{(p)}$.
4. Because of the inductive definition of cells, the cell decomposition theorem makes possible many proofs by induction on the dimension of a definable set.
5. In $(\mathrm{I})_n$ the cells in the decomposition can be chosen definable over the parameters used to define the A_i (and a similar statement holds for $(\mathrm{II})_n$).

SKETCH PROOF. The proof of Theorem 2.1.3 is by simultaneous induction on n. $(\mathrm{I})_1$ holds by definition of o-minimality, and $(\mathrm{II})_1$ follows by the Monotonicity Theorem, whilst $(\mathrm{III})_1$ requires a direct argument which is really the crux of the whole proof, but which we omit. At the inductive step, we prove:

$(\mathrm{I})_m$, $(\mathrm{II})_m$, $(\mathrm{III})_m$ (for $m < n$) $\Longrightarrow (\mathrm{I})_n$.

$(\mathrm{I})_m$ (for $m \leq n$), $(\mathrm{II})_m$ (for $m < n$) $\Longrightarrow (\mathrm{II})_n$.

$(\mathrm{I})_m$, $(\mathrm{II})_m$ (for $m \leq n$) and $(\mathrm{III})_m$ (for $m < n$) $\Longrightarrow (\mathrm{III})_n$.

I sketch the proof of $(\mathrm{I})_n$. For simplicity we suppose that $k = 1$ and $A := A_1$ (as the general case is similar). Let $\pi : M^n \to M^{n-1}$ drop the last coordinate, and for each $\bar{a} \in \pi(A)$ consider the fibre $A_{\bar{a}} := \{y \in M : (\bar{a}, y) \in A\}$, a definable subset of M. By o-minimality, $A_{\bar{a}}$ is a finite union of singletons and intervals, and by $(\mathrm{III})_{n-1}$ there is a bounded number of these. Inductively, we use $(\mathrm{I})_{n-1}$ to decompose the base, partitioning $\pi(A)$, to ensure that, for each cell of the base, all fibres $A_{\bar{a}}$ look the same as $\bar{a}$ ranges through the cell. (Essentially, this means that all these fibres must have the same number of singletons and intervals, occurring in the same order.) Hence there are finitely many definable partial functions $M^{n-1} \to M$ picking out these singletons and the endpoints of the intervals, and by $(\mathrm{II})_{n-1}$ we can ensure that these functions are continuous, partitioning the base further if necessary. Now piece this information together.

The proof of $(\mathrm{II})_n$ is also relatively straightforward, but requires the Monotonicity Theorem (not just piecewise continuity of unary functions). The idea is to reduce to the situation where for any $\bar{a} \in M^{n-1}$, the partial function $f(\bar{a}, y) : M \to M$ is continuous and monotonic (where defined), and for any $b \in M$, the function $f(\bar{x}, b) : M^{n-1} \to M$ is continuous where defined.

The proof of $(\mathrm{III})_n$ is intricate. $\square$

2.2. Definable closure and dimension. The next task is to describe dimension in o-minimal structures, in a way relevant also to Section 4. For an alternative treatment of model-theoretic dimension, see [van den Dries 1989].

Recall that if $A \subset M$ then the *algebraic closure* $\mathrm{acl}(A)$ of A is the union of the finite A-definable sets, and the *definable closure* $\mathrm{dcl}(A)$ is the union of the finite A-definable singletons. In general, $\mathrm{dcl}(A) \subseteq \mathrm{acl}(A)$, but in an o-minimal structure $\mathcal{M}$, they are equal (because in a finite set, we can define the least element, the next least element, and so on).

Notation. If $A \subseteq M$ and $\bar{a} \in M^n$ with $\bar{a} = (a_1, \ldots, a_n)$, I abuse notation by writing $A\bar{a}$ for $A \cup \{a_1, \ldots, a_n\}$.

Definition 2.2.1. A *pregeometry* on a set X is a function $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ (where $\mathcal{P}(X)$ denotes the power set of X) which satisfies the following conditions:

(i) for all $A \subseteq X$, $A \subseteq \mathrm{cl}(A)$;
(ii) for all $A \subseteq X$, $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$;
(iii) for all $A \subseteq X$, $\mathrm{cl}(A) = \bigcup\{\mathrm{cl}(F) : F \subseteq A, F \text{ finite}\}$;
(iv) (exchange) if $A \subseteq X$ and $b, c \in X$ with $b \in \mathrm{cl}(Ac) \setminus \mathrm{cl}(A)$, then $c \in \mathrm{cl}(Ab)$.

In any structure, not necessarily o-minimal, algebraic closure satisfies (i)–(iii), and we show next that in the o-minimal case (iv) holds also. The exchange property holds in many other nice model-theoretic classes (for example strongly minimal sets, and some of the structures discussed in Section 4 below).

Theorem 2.2.2 [Pillay and Steinhorn 1986]. *Let $\mathcal{M}$ be o-minimal, $A \subseteq M$, and $b, c \in M$. If $b \in \mathrm{dcl}(Ac) \setminus \mathrm{dcl}(A)$, then $c \in \mathrm{dcl}(Ab)$.*

Proof. We may suppose the base set $A = \varnothing$ (by adding constants for elements of A to the language), so that $b \in \mathrm{dcl}(c) \setminus \mathrm{dcl}(\varnothing)$. There is a 0-definable (partial) function $f : M \to M$ with $b = f(c)$. We apply the Monotonicity Theorem to f. Since $b \notin \mathrm{dcl}(\varnothing)$, c lies in the interior of an open 0-definable interval I on which f is strictly monotonic. Now $J := f(I)$ is 0-definable, and since $c := f|_J^{-1}(b)$, $c \in \mathrm{dcl}(b)$. $\square$

Theorem 2.2.2 gives us an important notion of dimension in o-minimal structures. Much of what follows is folklore, and possible sources are [Pillay 1988; Hrushovski and Pillay 1994]. We do not restrict to o-minimal structures in 2.2.3 and 2.2.5–2.2.8 below.

Definition 2.2.3. A first-order structure $\mathcal{M}$ is *geometric* if algebraic closure has the exchange property (so defines a pregeometry) in all models of $\mathrm{Th}(\mathcal{M})$, and $\mathcal{M}$ is uniformly bounded.

From Theorems 2.1.3 (III) and 2.2.2, we now have

Corollary 2.2.4. *Every o-minimal structure is geometric.*

There is a general dimension theory for geometric structures which I now sketch (some of it does not require uniform boundedness). First, observe that in a geometric structure $\mathcal{M}$, there is a general notion of independence: $I \subseteq M$ is *independent* if, for all $x \in I$, $x \notin \mathrm{acl}(I \setminus \{x\})$. If $A \subseteq M$, then we can also talk of I being 'independent over A' (regard the elements of A as being interpreted by new constants). If $A \subseteq M$ is algebraically closed, then any two maximal independent subsets of A have the same size (by the proof that any two bases of a vector space have the same size), and we may call this size the *rank* of A (but we will not use this).

Definition 2.2.5. Let $\mathcal{M}$ be geometric, $A \subseteq M$, and $\bar{a} \in M^n$. Then $\dim(\bar{a}/A)$ is the least cardinality of a subtuple $\bar{a}'$ of $\bar{a}$ such that $\bar{a} \subseteq \mathrm{acl}(A\bar{a}')$. If $p(\bar{x}) \in S_n(A)$ (the set of complete n-types over A), then $\dim(p) = \dim(\bar{a}/A)$, for any $\bar{a}$ realising p in an elementary extension of $\mathcal{M}$.

Lemma 2.2.6 [Pillay 1988]. *Let $\mathcal{M}$ be a geometric structure.*

(i) $\dim(\bar{a}/A)$ *is the cardinality of any maximal independent* (*over* A) *subtuple of* $\bar{a}$.
(ii) *If* $A \subseteq B$ *then* $\dim(\bar{a}/A) \geq \dim(\bar{a}/B)$;
(iii) $\dim(\bar{a}\bar{b}/A) = \dim(\bar{a}/A\bar{b}) + \dim(\bar{b}/A)$;
(iv) *If* $p(\bar{x}) \in S_n(A)$ *and* $A \subseteq B$ *then there is* $p'(\bar{x}) \in S_n(B)$ *such that* $p \subseteq p'$ *and* $\dim(p) = \dim(p')$.

Remark. In the above, we can think of p' as a kind of 'non-forking extension of p', but we cannot control the number of non-forking extensions. For example, in the o-minimal structure $(\mathbb{Q}, <)$, if p is the unique 1-type over $\varnothing$, then any of the $2^{\aleph_0}$ non-algebraic extensions of p over $\mathbb{Q}$ is non-forking in this sense. Furthermore, the 'type amalgamation condition' or 'independence theorem' of simple theories (see Theorems 3.5 and 5.8 of [Kim and Pillay 1997]) cannot hold in any o-minimal structure.

We can also use algebraic closure to obtain a notion of dimension for definable sets, mimicking Zariski dimension for constructible sets in algebraically closed fields. In 2.2.7–2.2.10, we shall assume that $\mathcal{M}$ is *sufficiently saturated*; that is, $|A|^+$-saturated for any parameter set A which we might care to mention. This will ensure that certain tuples in a definable set exist *in our model.* Without the saturation assumption, we can still define dimension for definable sets, but have to quantify over elementary extensions of the model, as a realisation in the definable set of the appropriate dimension may not exist *in the model*. It is simplest, whenever talking of generics and dimension in definable sets, to assume enough saturation.

DEFINITION 2.2.7. Let $\mathcal{M}$ be geometric and sufficiently saturated, and $X \subseteq M^n$ be A-definable. Then $\dim(X) := \mathrm{Max}\{\dim(\bar{a}/A) : \bar{a} \in X\}$ (so $\dim(X) = \mathrm{Max}\{\dim(p) : p \in S_n(A), p \text{ realised in } X\}$). In particular, if $\bar{a} \in X$, then $\bar{a}$ is a *generic of* X *over* A (and $\mathrm{tp}(\bar{a}/A)$ is a *generic type in* X *over* A), if $\dim(\bar{a}/A) = \dim(X)$.

There is a possible confusion here, since the notion of *rank* defined after Corollary 2.2.4 is sometimes called dimension. The ordered field of reals has rank $2^{\aleph_0}$ (its transcendence degree) but dimension 1 (as a set defined by the formula $x = x$).

EXAMPLE. In the ordered field $\mathbb{R}$, if A is a finite subset of $\mathbb{R}$, and $\bar{a} \in \mathbb{R}^n$, then $\dim(\bar{a}/A)$ is the transcendence degree of $\mathbb{Q}(A)(\bar{a})$ over $\mathbb{Q}(A)$. Hence, if X is a definable subset of $\mathbb{R}^n$, then $\dim(X)$ is the algebraic-geometric dimension of the Zariski closure of X in $\mathbb{R}^n$.

We sometimes call the above notion of dimension *geometric* dimension, to distinguish it from another (topological) notion defined after Lemma 2.2.8. It is easily checked that this definition is independent of the choice of the defining set A, provided $|A|$ is not too large. The following lemma lists some properties of this dimension. Uniform boundedness is used essentially in (iv).

LEMMA 2.2.8. *Let* $\mathcal{M}$ *be a sufficiently saturated geometric structure.*

(i) $\dim(\{a\}) = 0$ (*for* $a \in M$) *and* $\dim(M) = 1$.
(ii) *If* $X, Y \subseteq M^n$ *are definable, then* $\dim(X \cup Y) = \mathrm{Max}\{\dim(X), \dim(Y)\}$.
(iii) $\dim$ *is invariant under permutation of coordinates.*
(iv) (*Definability of dimension.*) *Let* $X \subseteq M^{m+n}$ *be definable, and for each* $\bar{a} \in M^m$ *let* $X_{\bar{a}} := \{\bar{y} \in M^n : (\bar{a}, \bar{y}) \in X\}$. *For each* $i = 0, \dots, n$ *let* $X(i) := \{\bar{x} \in M^m : \dim(X_{\bar{x}}) = i\}$. *Then each set* $X(i)$ *is definable, and for each* i,
$$\dim(\{(\bar{x}, \bar{y}) \in X : \bar{x} \in X(i)\}) = \dim(X(i)) + i.$$
(v) *If* $f : M^n \to M^m$ *is a definable partial function, and* $A \subseteq M^n$ *is definable, then* $\dim(f(A)) \leq \dim(A)$, *with equality if* A *is injective. In particular, definable bijections preserve dimension.*

In a strongly minimal structure $\mathcal{M}$ (which is geometric), a definable subset of X^n has (geometric) dimension equal to its Morley rank.

In a structure $\mathcal{M}$ which carries a topology with a uniformly definable basis, there is also a notion of *topological dimension* for definable sets. If $X \subseteq M^n$ is definable, then the *topological dimension* $\mathrm{tdim}(X)$ is the greatest $k \leq n$ such that for some projection $\pi : M^n \to M^k$, $\pi(X)$ has non-empty interior in M^k. The following is quite easy to prove.

THEOREM 2.2.9. *Let $\mathcal{M}$ be o-minimal, and $X \subseteq M^n$ a definable set. Then* $\dim(X) = \mathrm{tdim}(X)$.

If X is A-definable, and $\bar{b} \in X$, then $\bar{b}$ is a generic of X over A if and only if $\bar{b}$ does not lie in any A-definable set of dimension less than $\dim(X)$. The last theorem gives the following useful topological characterisation of genericity. It says, very roughly, that if an A-definable property holds of a generic over A, then it holds throughout a neighbourhood of it in the definable set.

LEMMA 2.2.10. *Let $\mathcal{M}$ be o-minimal, $X \subset M^n$ be A-definable, and $\bar{b}$ be a generic of X over A. Then $\dim(X) = k$ if and only if there is an open rectangular neighbourhood $Y \subset M^n$ of $\bar{b}$ and a projection $\pi : M^n \to M^k$ inducing a homeomorphism from $X \cap Y$ onto an open subset of M^k.*

In any o-minimal expansion of the ordered set of reals (in a countable language), any definable set X has a generic *in the model* over any countable set of parameters, even though $(\mathbb{R}, <)$ is not ω_1-saturated. This is easy to prove using the Baire Category Theorem and the fact that $\dim(X) = \mathrm{tdim}(X)$ (see Lemma 2.17 of [Hrushovski and Pillay 1994]).

Finally, I give a rapid consequence of the cell decomposition theorem. If $Y \subseteq X \subseteq M^n$, we say that Y is *large in* X if $\dim(X \setminus Y) < \dim(X)$.

LEMMA 2.2.11. *Let $\mathcal{M}$ be o-minimal, D be a subset of M^k with* $\dim(D) = k$, *and $f : D \to M^n$ be a definable function. Assume that both D and f are definable over a set A. Then there is an A-definable large subset S of D, open in M^k, such that $f|_S$ is continuous. In addition, if $\mathcal{M}$ expands a real closed field, then for any $k > 0$, we can choose S so that f is $C^{(k)}$ on S.*

2.3. Prime models.

DEFINITION 2.3.1. A model M is *prime* over a subset A if for every $N \models \mathrm{Th}(M, a)_{a \in A}$, there is an elementary embedding $f : M \to N$ over A.

By a result of Shelah, prime models exist (and are unique up to isomorphism over A) over arbitrary sets A in ω-stable theories. In particular, in an algebraically closed field, the prime model over a set A is just its field-theoretic algebraic closure. Prime models are a tool for classification of the models in certain classes of ω-stable theories (for example, uncountably categorical theories). The following result draws out the analogy between o-minimality and stability.

THEOREM 2.3.2 [Pillay and Steinhorn 1986]. *If $\mathcal{M}$ is o-minimal, and $A \subseteq M$, then* $\mathrm{Th}(\mathcal{M})$ *has a prime model over A, unique up to A-isomorphism.*

We shall denote the prime model over A by $\mathcal{M}(A)$, or $\mathcal{M}(\bar{a})$ if A is a tuple $\bar{a}$.

SKETCH PROOF. For existence, it suffices, by general model theory, to show that isolated types are dense in the Stone space $S_1(A)$; that is, for any formula $\phi(x)$ over A there is a formula $\psi(x)$ over A such that

(i) $\mathcal{M} \models \forall x\,(\psi(x) \to \phi(x))$, and
(ii) $\psi(x)$ isolates a complete type over A.

But this is straightforward: we may suppose that $\phi(x)$ defines an interval I; either this interval is already a complete type over A, or some A-formula defines a proper subinterval, in which case an endpoint in I of that subinterval will be A-definable, so realise a complete type over A.

The proof of uniqueness of prime models is much harder, and is omitted. An easy back-and-forth argument ensures that any two *countable* prime models over A are A-isomorphic. □

The o-minimal structures most commonly considered are expansions of ordered groups. If $\mathcal{M}$ is an expansion of an ordered group with at least two 0-definable elements (for example, if $\mathcal{M}$ expands an ordered field), then we may uniformly pick the midpoint of a bounded interval. Likewise, since there is a positive 0-definable element a, in any unbounded interval (x, ∞) we may uniformly pick out $x + a$, and in $(-\infty, x)$ we may pick out $x - a$. This means that $\mathrm{Th}(\mathcal{M})$ has definable Skolem functions; for in an $\bar{a}$-definable set we may pick out a cell, then a midpoint b_1 of its projection onto the first coordinate, then a midpoint b_2 of the first coordinate of the fibre above b_1, and so on (only using the parameters $\bar{a}$). The existence of definable Skolem functions ensures that the prime model over A is precisely $\mathrm{dcl}(A)$. Thus for example, in the ordered field $\mathbb{R}$, the prime model over a set A is precisely its real closure. Incidentally, the above hypotheses on $\mathcal{M}$ also ensure that $\mathrm{Th}(\mathcal{M})$ has elimination of imaginaries. This is because, given a 0-definable equivalence relation, by the above we can uniformly pick out an element of each equivalence class.

There are also good prime model theorems in some other classes of unstable algebraic structures. For example, in $\mathrm{Th}(\mathbb{Q}_p)$, the prime model over a set is just its p-adic closure.

2.4. Definable types

Definition 2.4.1. Let $\mathcal{M}$ be an ordered structure. Then a *cut* of M is a maximal consistent set of formulas each of the form $a < x$ or $x < a$ (where $a \in M$).

Lemma 2.4.2 [Pillay and Steinhorn 1986]. *Suppose $\mathcal{M}$ is o-minimal. Then for each cut of M there is a unique* 1*-type over M extending it.*

Proof. Let C be a cut of M, and let $\phi(x)$ be a formula over M such that ϕ is consistent with C. Now ϕ partitions M into intervals, and by o-minimality, only one of these intervals is consistent with C. Hence, just one of ϕ, $\neg\phi$ is consistent with C. □

The following definition is usually associated with stability theory.

DEFINITION 2.4.3. A type $p(\bar{x}) \in S_n(A)$ (where A is a subset of an ambient structure $\mathcal{M}$) is *definable* if, for every formula $\phi(\bar{x},\bar{y})$ over $\varnothing$, there is a formula $\psi(\bar{y})$ over A, such that for any tuple $\bar{a}$ from A, $\phi(\bar{x},\bar{a}) \in p$ if and only if $\mathcal{M} \models \psi(\bar{a})$.

Observe that if A is $\bar{b}$-definable, and all types over A realised in $\mathcal{M}$ are definable, then for each n every $\mathcal{M}$-definable subset of A^n is $A\bar{b}$-definable.

If C is a cut of M, then $a \in M$ is a *standard part of* C (written $a = \mathrm{st}(C)$, or $a = \mathrm{st}(b)$, if b realises C) if, for any b realising C (in an elementary extension), there is no element of M between a and b. Now we easily obtain the following.

LEMMA 2.4.4. *If $\mathcal{R}$ is an o-minimal expansion of $\mathbb{R}$, then every 1-type over $\mathbb{R}$ is definable.*

PROOF. If the 1-type p over $\mathbb{R}$ is algebraic (i.e. realised in $\mathbb{R}$) this is obvious, so suppose that p is non-algebraic. Then either p is the unique type consisting of infinitely large or infinitely small elements, or determines a cut bounded above and below by elements of $\mathbb{R}$. If say p is the type consisting of infinitely large elements, then for any formula $\phi(x,\bar{y})$, and any $\bar{a}$ from $\mathbb{R}$, $\phi(x,\bar{a}) \in p$ if and only if $\phi(x,\bar{a})$ holds on a cofinal subset of $\mathbb{R}$, so the set of such $\bar{a}$ is $\mathbb{R}$-definable. In p determines a bounded cut C, then, as $\mathbb{R}$ is Dedekind complete, C has a standard part, and we may use this to define p. ☐

We can extend this slightly. Suppose $\mathcal{M} \preceq \mathcal{N}$, both o-minimal, and $b \in N \setminus M$ with $\mathrm{tp}(b/M)$ definable. Then $\{x \in M : x < b\}$ and $\{x \in M : b < x\}$ are M-definable. It follows by o-minimality that b has a standard part in M. The most general statement of this form is the following, proved in [Marker and Steinhorn 1994], and given a different treatment in [Pillay 1994].

THEOREM 2.4.5 [Marker and Steinhorn 1994]. *Let $\mathcal{M}$ be o-minimal, and let $p(\bar{x}) \in S_n(M)$. Then p is definable if and only if, for every $\bar{a}$ realising p over M, M is Dedekind-complete in $M(\bar{a})$, that is, every non-algebraic 1-type over M which is realised in the prime model $M(\bar{a})$ over $M\bar{a}$ has a standard part in M.*

3. Definable groups and fields, and a trichotomy theorem

3.1. Ordered groups and fields, and a Trichotomy Theorem. The main goal of this section is to describe a Trichotomy Theorem of Peterzil and Starchenko, and some results of them and Pillay on definably simple groups in o-minimal structures. I begin with two elementary results.

PROPOSITION 3.1.1 [Pillay and Steinhorn 1986]. *Let G be an o-minimal ordered group (so the group operation is part of the language). Then G is divisible abelian.*

PROOF. We first claim that G has no proper non-trivial definable subgroups. For let H be such a subgroup. As G is torsion-free, H is infinite, so contains an infinite interval J, and hence has a maximal non-trivial interval about 0, of the form $[-h, h]$ or $(-h, h)$. Both cases are easily eliminated. For example, if $J = (-h, h)$, pick h' such that $0 < h' < h$, and observe that both h' and $h - h' \in H$, so $h \in H$, a contradiction.

Given the claim, G is abelian (for the centraliser of any element is a non-trivial and definable subgroup, so equal to G). For any $n > 0$, nG is a definable subgroup of G, which cannot be $\{0\}$ (as G is torsion-free), so equals G. Hence G is divisible. □

PROPOSITION 3.1.2 [Pillay and Steinhorn 1986]. *Let R be an o-minimal ordered field. Then R is real closed, that is, elementarily equivalent to the reals.*

PROOF. It suffices to show that polynomials satisfy the intermediate value theorem. But this follows immediately from o-minimality. □

QUESTION 1. Is there a sense in which an o-minimal structure is either 'trivial' (like $(\mathbb{Q}, <)$), or grouplike, or fieldlike?

QUESTION 2. What can we say about definable groups, or fields, in an o-minimal structure? (Note that by a remark after Theorem 2.3.2, many o-minimal structures, such as expansions of ordered fields, admit elimination of imaginaries, and in such structures definability is equivalent to interpretability.)

Question 1 suggests the Zil'ber Conjectures for strongly minimal sets, another class of geometric structures — see the discussion before Theorem 3.1.6 for more on these conjectures. However, there is a major difference. For among other things these conjectures asserted that a (strongly minimal) algebraically closed field has no proper expansions other than those obtained by naming constants. Under extra hypotheses, positive results in this direction are obtained in work on Zariski structures [Hrushovski and Zilber 1996], and the only known counterexamples are artificial [Hrushovski 1992]. On the other hand, the field of reals has many natural o-minimal proper expansions, such as the real exponential field.

Any answer to Question 1 has to be *local*. For example, one could form an o-minimal structure with three parts, L (the leftmost part), M (the middle part), and R (the right part), with L carrying the structure of a pure dense linear order, M that of a divisible ordered abelian group, and R that of a real closed field (it is necessary to put a point between L and M, and one between M and R, to ensure o-minimality). And indeed, M might just be an *interval* of an ordered group, with the induced structure.

I will describe a beautiful answer to Question 1. It is the main theorem of [Peterzil and Starchenko 1998].

DEFINITION 3.1.3. Let $\mathcal{M}$ be o-minimal, and $a \in M$.

(i) The point a is *non-trivial* if there is an open interval $I \subset M$ such that $a \in I$, and a definable continuous function $I \times I \to M$ which is strictly monotonic in each variable.
(ii) A *convex* $\bigwedge$*-definable group* in M is a group $(G, *)$, where $G \subseteq M$ is convex, and the group operation $*$ (regarded as a ternary relation) is the intersection of a definable set with G^3.
(iii) If $(G, <, +, 0)$ is a convex $\bigwedge$-definable group, and $p \in G$ with $p > 0$, then a *group interval* is a structure $([-p,p], <, +, 0)$, where $+$ is the induced *partial* function $[-p,p] \times [-p,p] \to [-p,p]$.
(iv) If I is an interval of M, then $\mathcal{M}|I$ is the structure with domain I, whose 0-definable sets are those of the form $I^k \cap U$ for definable $U \subseteq M^k$ (in fact, by Lemma 2.5 of [Peterzil and Starchenko 1998], if I is closed then such sets are I-definable).

Note that the domain of a convex infinitely definable group is an intersection of intervals, but may not be definable. The model theory of group intervals in o-minimal structures was described in [Loveys and Peterzil 1993]. They are all elementarily equivalent, essentially, because in $(\mathbb{Q}, <, +)$, all positive elements have the same type.

It is easy to see that if $(G, *)$ is a convex $\bigwedge$-definable ordered group in M, and $a \in G$, then $*$ witnesses that a is non-trivial. The following converse is much deeper.

THEOREM 3.1.4 [Peterzil and Starchenko 1998]. *Let $\mathcal{M}$ be ω^+-saturated, and $a \in M$ be non-trivial. Then there is a convex $\bigwedge$-definable infinite group $G \subseteq M$, such that $a \in G$ and G is a divisible ordered abelian group.*

In particular, even without the saturation assumption, there is a closed interval I with $a \in I \subset M$, and a definable group interval structure induced on I. Saturation enables us to find the whole domain of a group, on say an infinitesimal neighbourhood (with respect to an elementary substructure) about a, but this infinitesimal neighbourhood may not be definable.

THEOREM 3.1.5 [Peterzil and Starchenko 1998]. *Suppose that $(I, <, +, 0)$ is a 0-definable group interval in a sufficiently saturated o-minimal structure $\mathcal{M}$. Then precisely one of the following statements holds.*

(i) *There is an ordered vector space $\mathcal{V} = (V, +, c, d(x))_{c \in C, d \in D}$ (with C a set of constants) over an ordered division ring D, an interval $[-p,p]$ in V, and an order-preserving isomorphism of group-intervals $\sigma : I \to [-p,p]$, such that $\sigma(S)$ is 0-definable in $\mathcal{V}$ for every 0-definable $S \subseteq I^n$.*
(ii) *There is a real closed field $\mathcal{R}$ definable in $\mathcal{M}$ with its domain a subinterval of I and its order compatible with $<$.*

The point is that in (i), the division ring D is not a definable structure, so $\mathcal{V}$ is essentially a 'pure' linear structure. I emphasise that in (ii) we get the whole of a field, not just a field interval. Incidentally, if *every* point of M is trivial, then by [Mekler et al. 1992], every definable set is a boolean combination of binary relations.

I describe next an alternative, more algebraic-geometric, treatment of this theorem, given in the introduction to [Peterzil and Starchenko 1998].

Let $\mathcal{M}$ be a geometric structure (see Definition 2.2.3). A *curve* is a 1-dimensional subset of M^2. A set $\mathcal{F}$ of curves is said to be *definable* if there are definable $U \subseteq M^k$ and $F \subseteq U \times M^2$ such that $\mathcal{F} = \{C_{\bar{u}} : \bar{u} \in U\}$, where $C_{\bar{u}} := \{(x, y) : (\bar{u}, x, y) \in F\}$. We say $C_{\bar{u}}$ is generic in $\mathcal{F}$ if $\bar{u}$ is generic in U over any relevant parameters. Also, $\mathcal{F}$ is *normal of dimension n* if $\dim(U) = n$ and $C_{\bar{u}} \cap C_{\bar{v}}$ is finite for any distinct $\bar{u}, \bar{v}$ in U. There is a similar notion of *interpretable* normal family of curves.

In a geometric structure $\mathcal{M}$, one of the following must hold.

(Z1) If $\mathcal{F}$ is an infinite interpretable normal family of curves, and $C \in \mathcal{F}$ is generic, and (a, b) is generic in C, then either $\dim(C \cap (\{a\} \times M)) = 1$ or $\dim(C \cap (M \times \{b\})) = 1$.

(Z2) (Z1) fails, but every interpretable normal family of curves has dimension at most 1.

(Z3) There is an interpretable normal family of curves of dimension greater than 1.

Zil'ber's Conjecture (which is false in general, though parts are true) was that in the strongly minimal case, (Z1) should correspond to the case when there are no interpretable groups, (Z2) to the case when definable sets arise from a module, and the structures satisfying (Z3) should be bi-interpretable with (or at least interpret) an algebraically closed field. If $\mathcal{M}$ is a group, then (Z1) is clearly false (consider for each $g \in G$ the curve $C_g := \{(x, y) : xg = y\}$). Likewise, if $\mathcal{M}$ is a field, then for each $a, b \in M$ we have a curve $\{(x, y) : y = ax + b\}$, and this family of curves has dimension 2.

Suppose now that $\mathcal{M}$ is o-minimal. It is quite easy to see that $a \in M$ is trivial if and only if some induced structure $\mathcal{M}|I$ on a neighbourhood of a has type (Z1). If $a \in M$ is non-trivial, we say a has type (Z2) if there is an open interval I containing a such that the induced structure $\mathcal{M}|I$ satisfies (Z2), and that a has type (Z3) otherwise. Now we can state the following version of the trichotomy theorem (which gives more of a guide to its proof).

THEOREM 3.1.6 [Peterzil and Starchenko 1998]. *Let $\mathcal{M}$ be sufficiently saturated and o-minimal, and $a \in M$ be non-trivial. Then either*

(a) *a has type* (Z2) *and the structure induced on some closed interval I* (*whose interior contains a*) *satisfies Theorem* 3.1.5(i), *or*

(b) *a has type* (Z3), *and some open interval containing a satisfies Theorem* 3.1.5(ii).

REMARKS ON THE PROOF OF THEOREM 3.1.4 AND 3.1.5. First of all, a general mechanism for defining groups is given. Given a definable normal family of dimension greater than one of functions on an interval I, one can obtain, for some open interval $J \subset I$, a very well-behaved 'nice' family of functions of dimension 2, parametrised by an open subset of I^2. It turns out that this is a powerful abstract notion: in the final section of the paper, it is shown that without any algebraic assumptions one can use a nice family of functions to define tangency, and develop elementary differential calculus - the nice family replaces the family of functions of form $f(x) = ax + b$ in the usual definition of differentiation. For such a nice family, there is a technical device for defining a certain quaternary relation (a 'q-relation') on a convex subset of I, and from this a convex $\bigwedge$-definable ordered group (assuming enough saturation). The arguments here are similar to familiar group constructions in stable theories. One starts with a family $\mathcal{F}$ of functions from a set A to a set B, and by taking compositions fg^{-1} obtains a family of partial definable functions $B \to B$. The defining parameters for these compositions (taken up to some equivalence relation corresponding to tangency at a 'fixed' generic point) are projected to an interval to obtain the convex $\bigwedge$-definable ordered group, and the group operation is essentially composition of functions. To prove Theorem 3.1.4, one takes a definable function

$$F : I \times I \to M$$

which witnesses non-triviality of a (here, $a \in I$). Massaging this function using composition, a function $G(x, y, z)$ is obtained, defined on an open subset of I^3, continuous, and strictly increasing in each variable. This gives either a situation where methods from [Peterzil 1994] apply, or a normal family of functions of dimension 2. Either way, a group interval is obtained.

Assume now that a is non-trivial but the conclusion of Theorem 3.1.5(i) does not hold around a. By the last paragraph, there is an interval I containing a such that the structure induced on I is an o-minimal expansion of a group interval. Furthermore, by our case assumption and Proposition 4.2 of [Loveys and Peterzil 1993], there is a definable function on a subinterval J of I which is not 'linear' on any subinterval of J. One can use this to construct a new nice family of curves. From a nice family living on a group interval, one can construct two different q-relations, one using the operation of the group interval, the other using composition. From this, it is possible to define on a convex subset of I two new groups G, H, corresponding to field addition and multiplication. It is also possible to define a continuous faithful action H on G. (Care is needed with the notion of definability, as G and H are convex $\bigwedge$-definable ordered groups, but are not in general definable.) This gives a certain convex $\bigwedge$-definable ring of definable endomorphisms. The fraction field is a real closed field, which turns out to be *definable*, essentially, because any element of the fraction field is represented by infinitely many pairs from the ring, and some of these pairs lie in arbitrary small boxes $J \times J$, where J is an interval about a. □

I conclude this subsection by stating a global dichotomy theorem of Miller and Starchenko which looks similar to Theorem 3.1.5, but is global rather than local. If $\mathcal{G} = (G, <, +, \dots)$ is an expansion of an ordered group, we say that $\mathcal{G}$ is *linearly bounded* if for any definably function $f : G \to G$ there is a definable endomorphism λ of G such that $|f(x)| \le \lambda(x)$ for sufficiently large x.

THEOREM 3.1.7 [Miller and Starchenko 1998]. *Let $\mathcal{R} = (R, <, +, \dots)$ be an o-minimal expansion of an ordered group. Then if $\mathcal{R}$ is not linearly bounded, there is a definable binary operation $\cdot$ such that $(R, <, +, \cdot)$ is an ordered field.*

3.2. Definable groups and fields. I first describe some results from [Pillay 1988], based on the dimension theory of Section 2. I begin with Proposition 1.8 of that reference, which is based on Proposition 2.1 of [Pillay 1986].

LEMMA 3.2.1. *Let $\mathcal{M}$ be o-minimal, and $X \subseteq M^n$. Then $\dim(X) \ge k+1$ if and only if there is a definable equivalence relation E on X with infinitely many classes of dimension at least k.*

Here, the left-to-right direction is trivial and holds for topological dimension in any reasonable class of structures. The right-to-left direction, however, is specific to o-minimality, and fails for the classes considered in Section 4.

COROLLARY 3.2.2 [Pillay 1988]. *Let $G \subseteq M^n$ be a group definable in an o-minimal structure, that is, the group and its operation are definable. Then if $H \le G$ is definable, then $\dim(H) = \dim(G)$ if and only if $|G : H| < \infty$.*

PROOF. If $|G : H| = \infty$, then as the cosets of H form the classes of a definable equivalence relation on G, $\dim(H) < \dim(G)$ by Lemma 3.2.1. The other direction follows from Corollary 2.2.8, parts (ii) and (v). □

I now describe some results from [Pillay 1988], extended slightly in [Otero et al. 1996] and [Peterzil et al. 2000] (whose terminology I follow). Fix an o-minimal structure $\mathcal{M}$, and $p \ge 0$ (if $p > 0$, then we assume that $\mathcal{M}$ expands a real closed field). Let X be a definable set. We wish to endow X with a kind of manifold structure.

A *definable chart* on X is a triple $\mathbf{c} = \langle U, \phi, n \rangle$, where U is a definable subset of X, $n \ge 0$, and ϕ is a definable bijection from U to an open subset of M^n. Two charts $\mathbf{c} = \langle U, \phi, n \rangle$ and $\mathbf{c}' = \langle U', \phi', n' \rangle$ are $C^{(p)}$*-compatible* if either $U \cap U' = \varnothing$, or $\phi(U \cap U'), \phi'(U \cap U')$ are open, and the transition mappings $\phi \circ \phi'^{-1}, \phi' \circ \phi^{-1}$ are $C^{(p)}$ on their domains. A *definable $C^{(p)}$-atlas* on X is a finite pairwise $C^{(p)}$-compatible set of definable charts covering X. A *definable $C^{(p)}$-manifold* is a pair consisting of X, and a definable $C^{(p)}$-atlas on X. Note that given a definable manifold on X, we can talk about the dimension at any point of X.

Using the charts as coordinate systems, one can talk about a definable $C^{(p)}$-function between manifolds, and (if $p > 0$) its *differential* at a point (a linear map given by a matrix of partial derivatives). The following result is essentially Proposition 2.5 of [Pillay 1988] (for the case $p > 0$, it was stated in [Otero et al.

1996], but the proof is essentially that of [Pillay 1988]). Behind the proof lies Lemma 2.2.11.

THEOREM 3.2.3. *If G is a group definable in an o-minimal structure $\mathcal{M}$, then there is an atlas on G making G into a definable $C^{(p)}$-group, i.e., the group operation $G \times G \to G$, and inversion, are $C^{(p)}$.*

As a corollary in the case $p = 0$, it follows (see [Pillay 1988]) that if G is as in the theorem, then G has a unique smallest definable subgroup G^o of finite index, and that this is also its connected component in the above manifold topology. Furthermore, by this and Lemma 3.2.1 again, G has the descending chain condition on definable subgroups; in particular, if $X \subseteq G$ then there is finite $X_0 \subseteq X$ such that $C_G(X) = C_G(X_0)$, so centralisers are definable. Thus, definable groups in o-minimal structures share many properties of groups of finite Morley rank.

Analogous results are shown in [Pillay 1988] to hold for definable fields, and yield (quite rapidly) that any definable infinite field in an o-minimal structure is real closed of dimension 1 or algebraically closed. In the algebraically closed case, by an easy Euler characteristic argument the characteristic is 0, and in fact the field has dimension 2 [Peterzil and Steinhorn 1999]. (The Euler characteristic argument is suggested by the beautiful paper [Strzebonski 1994], which develops a Sylow theory based on Euler characteristic for groups definable in o-minimal structures.) More precisely, it is shown in Theorem 4.1 of [Peterzil and Steinhorn 1999] that if K is any infinite definable ring without zero-divisors in an o-minimal structure $\mathcal{M}$, then K is a division ring and there is a one-dimensional $\mathcal{M}$-definable subring R of K which is a real closed field such that K is either R, or $R(i)$ (where i denotes $\sqrt{-1}$)), or the ring of quaternions over R. Earlier, under the additional assumption that $\mathcal{M}$ is an o-minimal expansion of a real closed field R_0, it was shown in [Otero et al. 1996] that such a ring K is *definably* isomorphic to R_0, or $R_0(i)$, or the quaternions over R_0.

All this suggests that an o-minimal analogue of Cherlin's Conjecture might hold. This conjecture states that any infinite simple group of finite Morley rank is definably isomorphic to a simple algebraic group over an algebraically closed field. Such a result has now been proved, by Peterzil, Pillay and Starchenko [Peterzil et al. 2000], and for the rest of this section I sketch it and the finer structure theory from [Peterzil et al. 1997]. These notes also use the summary [Peterzil et al. 1998].

Before stating the main theorem of [Peterzil et al. 2000], we need some definitions. Let $\mathcal{R}$ be an o-minimal expansion of a real closed field. In the context of $\mathcal{R}$, a *semialgebraic* set or group is just a set or group definable in the *pure* field $\mathcal{R}_0 := (R, <, +, \cdot)$. A semialgebraic linear group over R is a subgroup of $\mathrm{GL}(n, R)$, for some n, which is definable in $\mathcal{R}_0$. It is semialgebraically connected if it has no proper semialgebraic subgroup of finite index. If $H \le \mathrm{GL}(n, R)$ is a semialgebraic linear group over R, then H has Zariski closure in $\mathrm{GL}(n, R(i))$ defined over R by the vanishing of polynomial equations. We let $\bar{H}$ denote the subgroup of this Zariski closure consisting of matrices with entries in R. Clearly we have $\bar{H} \leq \mathrm{GL}(n,R)$, and H and $\bar{H}$ both have a dimension in the sense of Definition 2.2.7. These dimensions are equal to the algebraic-geometric dimension of $\bar{H}$, and so equal to each other. Hence, by Corollary 3.2.2, $|\bar{H} : H| < \infty$.

If $\mathbf{G} = (G, \cdot)$ is a group definable in a structure $\mathcal{M}$, then we say that G is *definably simple* if G has no proper non-trivial $\mathcal{M}$-definable normal subgroups, and that G is $\mathbf{G}$-definably simple if it has no proper non-trivial normal subgroups which are definable just in the structure $\mathbf{G}$ (so $\mathbf{G}$-definable simplicity is the weaker condition). Likewise, G is *definably connected* if it has no proper definable subgroups of finite index, and $\mathbf{G}$-*definably connected* if it has no such subgroups definable in the structure $\mathbf{G}$. In general, 'definable' means 'definable in the sense of $\mathcal{M}$'.

Now I state the main theorems of [Peterzil et al. 2000].

THEOREM 3.2.4. *Assume that G is an infinite $\mathbf{G}$-definably connected group definable in an o-minimal structure $\mathcal{M}$, with no non-trivial abelian normal subgroup. Then there is $k > 0$, and for each $i = 1, \ldots, k$ a definable real closed field R_i, a $\mathbf{G}$-definable subgroup $H_i \leq G$, and a definable isomorphism between H_i and a semialgebraic subgroup of $\mathrm{GL}(n, R_i)$, such that $G = H_1 \times \cdots \times H_k$. Each H_i is $\mathbf{H_i}$-definably simple, and its definably connected component in the sense of $\mathcal{M}$ is definably simple.*

THEOREM 3.2.5. *Let G be a non-abelian infinite $\mathbf{G}$-definably simple group definable in an o-minimal structure $\mathcal{M}$. Then there is a definable real closed field $\mathcal{R}$ such that G is definably isomorphic to a semialgebraic linear group over R.*

REMARK. In the finite Morley rank context, any non-abelian definably simple group is simple, by Zil'ber Indecomposability. In the o-minimal context, there is only an infinitesimal version of Zil'ber Indecomposability, given in Section 2 of [Peterzil et al. 1997]. There are examples of non-abelian groups which are definably simple but not simple. For example, if R is a non-archimedean real closed field, then $SO(3,R)$ is definably simple (in the sense of R) but not simple: it has a normal subgroup consisting of matrices $A + I$, where A has infinitesimal entries.

SKETCH OF THE PROOF OF THEOREM 3.2.5. The first step, given an o-minimal expansion $\mathcal{R}$ of a real closed field, is to develop a general Lie theory. If X is a definable $C^{(1)}$-manifold, with a chart $\langle U, \phi, n\rangle$, and $m \in U$, then the *tangent space at m* is the set of definable $C^{(1)}$-functions $f : R \to X$ with $f(0) = m$, modulo an equivalence relation $f_1 \sim f_2$, which holds if $\phi \circ f_1$ and $\phi \circ f_2$ have the same differential at 0; it is in canonical bijection with R^n, so has a vector space structure. Formally, $T_m(X)$ is not a definable object, but the vector space R^n is, and we often identify $T_m(X)$ with R^n. Given a definable $C^{(1)}$-mapping between definable $C^{(p)}$-manifolds $f : X \to Y$, and $m \in X$, there is a (linear) differential

$d_m(f) : T_m(X) \to T_{f(m)}(Y)$. The rank of $d_m(f)$ equals the dimension of $f(X)$ at $f(m)$.

Suppose that G is a group definable in $\mathcal{R}$, so G carries a definable manifold structure. We consider the tangent space $T_e(G)$ at the identity e of G. Let $n := \dim(G)$. Then $\dim(T_e(G)) = n$. For any $g \in G$, conjugation by g induces an inner automorphism of G, whose differential on $T_e(G)$ is non-singular, so (identifying $T_e(G)$ with R^n), induces an element of $\mathrm{GL}(n, R)$. The induced map $G \to \mathrm{GL}(n, R)$ is a homomorphism. Thus, if G is centreless, we have a definable embedding

$$f : G \to \mathrm{GL}(n, R)$$

(the *adjoint representation*: see [Otero et al. 1996]), but we do not yet know that its image is semialgebraic in the sense of $\mathcal{R}$, that is, that the image is definable in the *pure* field R.

There is a Lie algebra structure on the tangent space $T_e(G)$, given in a standard way. Since the Lie operation is bilinear, this gives an *R-definable* Lie algebra structure $L := L(G)$ on R^n, and $\mathrm{Aut}(L)$ is an *algebraic* subgroup of $\mathrm{GL}(n, R)$. If G is assumed to be semisimple, that is, it has no infinite abelian normal subgroup, then L is semisimple, that is, its only abelian ideal is $\{0\}$. In this case an easy argument shows that $\dim(\mathrm{Aut}(L)) = \dim(L)$ (which equals n). Also the adjoint representation gives a definable embedding $f : G \to \mathrm{Aut}(L)$, so, as the dimensions are equal, by Corollary 3.2.2 $f(G)$ has finite index in $\mathrm{Aut}(L)$.

A useful fact (given in a more general context in Claim 1.3 of [Peterzil et al. 2000]) is that the connected component $\mathrm{Aut}(L)^o$ of $\mathrm{Aut}(L)$ in the sense of the *expansion* $\mathcal{R}$ is the same as the semialgebraic connected component in the sense of the *pure field* R, that is, $\mathrm{Aut}(L)^o$ is semialgebraic. Hence, since $f(G)$ is definable in $\mathcal{R}$, it is a union of finitely many cosets of $\mathrm{Aut}(L)^o$, so $f(G)$ is semialgebraic. This argument shows that if G is centreless and definable, with semisimple connected component, then G is definably isomorphic to a linear semialgebraic group over $\mathcal{R}$.

In the proof of Theorem 3.2.5, one first uses the Trichotomy Theorem 3.1.5 (and the existence of the group G) to find a real closed field $\mathcal{R}$ on a definable interval I of $\mathcal{M}$. This can be used to define a chart on G at e with image in I^n, such that the group multiplication and inversion are $C^{(1)}$ near e. This enables us to use the adjoint representation to embed G definably into $\mathrm{GL}(n, R)$. Thus, the image $h(G)$ of this embedding in $\mathrm{GL}(n, R)$ is a definable group in an o-minimal expansion of $\mathcal{R}$. Since $h(G)$ is definably simple, we can apply the previous paragraph to it to obtain Theorem 3.2.5. For Theorem 3.2.4, it is also necessary to develop an orthogonality theory between intervals, and a notion of a *unidimensional* group. □

I now turn to the finer structure theory for definable groups, from [Peterzil et al. 1997]. So far, we have an o-minimal structure $\mathcal{M}$, and a **G**-definably simple group G, definable in $\mathcal{M}$. We know about the abstract group structure of G, but not much about the model theory of $\mathbf{G}$. In [Peterzil et al. 1997], the structure $\mathbf{G}$ is identified up to bi-interpretability.

A structure $\mathcal{M}$ is *interpretable* in $\mathcal{N}$ if there is an isomorphic copy $f(\mathcal{M})$ definable in $\mathcal{N}^{\text{eq}}$. If $\mathcal{M}$ is interpretable in $\mathcal{N}$ by f, and $\mathcal{N}$ is interpretable in $\mathcal{M}$ by g, then in $\mathcal{N}^{\text{eq}}$ there is an isomorphic copy $f(g(\mathcal{N}))$ of $\mathcal{N}$, and in $\mathcal{M}^{\text{eq}}$ there is an isomorphic copy $g(f(\mathcal{M}))$ of $\mathcal{M}$. In general, the isomorphism $f \circ g$ will not be definable in $\mathcal{N}$. To illustrate this, consider a group G, with a definable subgroup H, which itself has a definable subgroup G^* isomorphic to G; then the domain of G^* is G-definable, but there is no reason why there should be a *G-definable* isomorphism $G \to G^*$. We say that $\mathcal{M}$ and $\mathcal{N}$ are *bi-interpretable* if the isomorphism $f \circ g$ is definable in $\mathcal{N}$ and $g \circ f$ is definable in $\mathcal{M}$. We can now state the main theorem of [Peterzil et al. 1997].

THEOREM 3.2.6. *Let G be a non-abelian infinite group which is $\mathbf{G}$-definably simple, and is definable in an o-minimal structure. Then there is a real closed field $\mathcal{R} = (R, +, \cdot)$ such that $\mathbf{G}$ is bi-interpretable either with $\mathcal{R}$ or with its degree 2 algebraically closed extension $(R(i), +, \cdot)$.*

As a curious corollary, one can bypass Cherlin's Conjecture for groups of finite Morley rank to obtain a model-theoretic characterisation (among infinite simple groups) of algebraic groups over algebraically closed fields of characteristic 0: these are precisely the stable groups definable in some o-minimal structure.

The following further corollary is one of many results on the relationship between abstract homomorphisms and algebraic (or analytic) homomorphisms between algebraic groups. It was already known, by a combination of results in [Borel and Tits 1973] and [Weisfeiler 1979].

COROLLARY 3.2.7 [Peterzil et al. 1997]. *Let G_1 be an unstable semialgebraic $\mathbf{G}_1$-definably simple group over an real closed field R_1, and G_2 another semialgebraic group over a real closed field R_2. Then any abstract group isomorphism $f : G_1 \to G_2$ has the form $f = g \circ h$, where h is induced by an (abstract) field isomorphism $R_1 \to R_2$, and g is an R_2-semialgebraic group isomorphism.*

REMARKS ON THE PROOF OF THEOREM 3.2.6.. The first step is to find an infinite field interpretable in $\mathbf{G}$. There is a real closed field R_1 provided by Theorem 3.2.5, such that G is definably isomorphic to a semialgebraic linear group over R_1. Let $K_1 := R_1(i)$, let $\bar{G}$ be the Zariski closure of G in K_1, and let H be the minimal *algebraic* subgroup of $\bar{G}$ of finite index (so H is a connected linear algebraic group defined over R_1). We say that H is *R_1-isotropic* if it has an R_1-defined algebraic subgroup T which is rationally R_1-isomorphic to a direct product of at least one copy of the multiplicative group of the field K_1, and *R_1-anisotropic* otherwise.

The argument splits into two cases, according to whether H is $\mathcal{R}_1$-isotropic or $\mathcal{R}_1$-anisotropic. If H is R_1-anisotropic, then $H(R_1)$ is closed and bounded in $\mathrm{GL}(n, R_1)$, and this makes possible a model-theoretic transfer of results from

[Nesin and Pillay 1991] on compact Lie groups, which give an interpretable real closed field K.

In the other case, H is R_1-isotropic. Now, arguments familiar from the finite Morley rank case are applicable. Inside an 'R_1-parabolic subgroup of H', using the Levi decomposition one can define in H a connected soluble non-nilpotent group. From this it is is possible to find an infinite definable abelian group M acting faithfully and definably on a definable abelian group A, such that A has no infinite definable M-invariant proper subgroups. Using a local version of Zil'ber Indecomposability proved in [Peterzil et al. 1997], one can interpret an infinite field K in $\mathbf{G}$ (which may be real closed or algebraically closed). The field K is definable in the original field R_1, so, by results of [Otero et al. 1996], is semialgebraically (in the sense of $\mathcal{R}_1$) isomorphic to R_1 or $R_1(i)$. A short argument shows that every $\mathbf{G}$-definable subset of K^n, for any n, is definable in $(K,+,\cdot)$. Here, one may have first to replace $R_1(i)$ by R_1 and apply [Marker 1990].

By the last two paragraphs, there is a real closed field R and an interpretable field K which is equal to R or $R(i)$. By a general model-theoretic argument, to show that $\mathbf{G}$ and $(K,+,\cdot)$ are bi-interpretable it is now necessary to show that G is K-internal, that is, there is a G-definable surjection $K^r \to G$ for some r. By an application of the infinitesimal Zil'ber Indecomposability Theorem and a Lie algebra argument, there is definable and K-internal $U \subseteq G$ with $\dim(U) = \dim(G)$. By elimination of imaginaries in $(K,+,\cdot)$, U is in definable bijection with a subset of K^r for some r. We may suppose that U contains an open neighbourhood of the identity e of G. If K is real closed, we obtain a chart with differential structure on U, and the adjoint representation gives a definable embedding $G \to \mathrm{GL}(r,K)$, so G is K-internal. If K is algebraically closed, then an argument with Morley rank and degree shows that $G = U \cdot U$, so again G is K-internal. □

4. Variants of o-minimality

There are several structures related to topological fields whose model theory is similar to that of $\mathbb{R}$, but which are not o-minimal. Examples include the p-adic field $\mathbb{Q}_p$, any algebraically closed valued field, and any real closed field with a definable convex valuation ring. In each case, the model theory is made manageable by a quantifier elimination theorem. All these structures are geometric in the sense of Section 2, and there is also a good dimension theory for expansions of them by subanalytic sets. (In the real case, one expands by *finitely* subanalytic sets: see [van den Dries 1986], [van den Dries and Miller 1996], or [van den Dries 1996].) I describe here attempts to set their model theory in a general context.

We consider here some model theoretic notions, akin to o-minimality, for some classes of structures with a topology. These are *weak o-minimality*, *C*-

minimality, and *P-minimality*. In each case, the general theory is not nearly as well-developed as o-minimality, and the results obtained are still rather haphazard. Unfortunately, the definable connectedness properties of the topology are not nearly as good as in the o-minimal case. This leads to a number of problems: for example, we cannot expect analogues of the theorem that if a structure is o-minimal then so is any structure elementarily equivalent to it.

4.1. Weak o-minimality

Definition 4.1.1 (Dickmann). A totally ordered structure $\mathcal{M} = (M, <, \dots)$ is *weakly o-minimal* (weakly o-minimal) if every parameter-definable subset of M is a finite union of convex sets. We say that a complete theory T is *w.o.m* if every model of T is weakly o-minimal.

There is an example in [Macpherson et al. 1999] of a weakly o-minimal structure whose theory is not weakly o-minimal. The following is the main motivating example for weak o-minimality.

Example 4.1.2. Let R be a real closed field with a proper convex subring V which is a valuation ring, and induced valuation map v to the value group. Let L_{rcvf} denote the language $(<, +, -, \cdot, 0, 1, D)$ of ordered rings with an additional binary relation symbol D. We interpret D by putting Dxy whenever $v(x) \leq v(y)$. Then, by results from [Cherlin and Dickmann 1983], the theory of all such structures in the language L_{rcvf} is complete and has quantifier elimination, and (by [Dickmann 1985]), is weakly o-minimal.

As a concrete example of such a structure, let R be any non-archimedean real closed field, and V the set of its finite elements, that is, elements bounded in absolute value by a natural number. The unique maximal ideal of V consists of the infinitesimals.

In Example 4.1.2, a weakly o-minimal structure is obtained from an o-minimal one essentially by adding a unary predicate interpreted by a convex set, namely, the valuation ring. In [Macpherson et al. 1999], another example is given of this phenomenon. Let R be the field of real algebraic numbers, and let $\mathcal{R}$ be the structure $(R, <, +, -, \cdot, 0, 1, P)$, where P is a unary predicate interpreted by the convex set $(-\pi, \pi) \cap R$. Then $\mathcal{R}$ has weakly o-minimal theory (in fact, here, π could be any real transcendental).

Cherlin asked whether this phenomenon holds generally, that is, whether *any* expansion of any o-minimal structure by a predicate for a convex set is weakly o-minimal. The following positive answer is given in [Baizhanov 1999] and in [Baisalov and Poizat 1998] (in which the proof uses Theorem 2.4.5).

Theorem 4.1.3. *Let $\mathcal{M}$ be an o-minimal structure, let $\{C_i : i \in I\}$ be a family of convex subsets of M, and let $\mathcal{M}^*$ be the expansion of $\mathcal{M}$ obtained by adding unary predicates interpreted by the C_i. Then* $\mathrm{Th}(\mathcal{M}^*)$ *is weakly o-minimal.*

By this theorem, if $\mathcal{R}$ is an o-minimal expansion of $\mathbb{R}$ and $\mathcal{R}'$ is a non-archimedean elementary extension, then the structure $(\mathcal{R}', V)$ has weakly o-minimal theory, where V denotes the convex valuation ring of finite elements of R'. Structures of this sort have been investigated further in [van den Dries and Lewenberg 1995; van den Dries 1997]. For example, the latter paper shows that if $\mathcal{R}$ is an o-minimal expansion of a real closed field and V is a proper non-empty convex subring closed under 0-definable continuous functions, then $(\mathcal{R}, V)$ is weakly o-minimal (as follows also from Theorem 4.1.3), with weakly o-minimal value group and residue field (as is fairly clear). Furthermore the theory T_{convex} of such expansions of models of T is complete, and has a relative quantifier-elimination (relative to quantifier-elimination for T, and assuming T is universally axiomatised). In addition, the structure induced on the value group is o-minimal precisely if T is *power-bounded* (a generalisation of 'polynomially bounded' for arbitrary o-minimal expansions of real closed fields), precisely if $\mathcal{R}$ has no definable order-preserving isomorphism from $(R, +)$ to the multiplicative group $R^{>0}$.

The structure theory for o-minimal structures begins with the Monotonicity Theorem. In the weakly o-minimal case, we can only hope for a local version of this. For example, let $\mathcal{M} = (M, <, f)$, where $(M, <)$ is naturally identified with $\mathbf{Z} \times \mathbb{Q}$, ordered lexicographically, and, for all $(z, q) \in M$, $f((z, q)) = (-z, q)$. Then $\mathrm{Th}(\mathcal{M})$ is weakly o-minimal, and the function f is locally strictly monotonic, but not piecewise monotonic in the sense of Theorem 2.0.2.

In order to obtain a reasonable cell decomposition theorem, we need to consider not merely definable functions $M \to M$, but definable functions $M \to \bar{M}$, where $\bar{M}$ denotes the Dedekind completion of M. There is a natural notion of *definable sort* in $\bar{M}$. Let $Y \subset M^{n+1}$ be 0-definable, let $\pi : M^{n+1} \to M^n$ be the projection dropping the last coordinate, let $Z := \pi(Y)$, and for each $\bar{a} \in Z$ let $Y_{\bar{a}} := \{y : (\bar{a}, y) \in Y\}$. Suppose that each set $Y_{\bar{a}}$ is bounded above but does not have a supremum in M. Define an equivalence relation $\sim$ on Z, putting $\bar{a} \sim \bar{b}$ if $Y_{\bar{a}}$, $Y_{\bar{b}}$, have a common final segment. Then $Z/\sim$ (which is a sort in $\mathcal{M}^{\mathrm{eq}}$) is naturally identified with a subset of $\bar{M}$.

If I, K are sets each endowed with a dense total order, then we say that a function $f : I \to K$ is *tidy* if each element of I lies in the interior of an open interval on which f is strictly monotonic or constant, with the same possibility (i.e., increasing, decreasing, or constant) holding for each $x \in I$. The following result was proved under extra hypotheses in [Macpherson et al. 1999], and in general by Arefiev [1997].

THEOREM 4.1.4. *Let $\mathcal{M}$ be weakly o-minimal, and $f : M \to \bar{M}$ a definable partial function (to a definable sort). Then there is a partition* $\mathrm{dom}(f) = X \cup I_1 \cup \cdots \cup I_m$, *where X is finite, and for each j the set I_j is definable and convex, and $f|_{I_j}$ is tidy.*

Recall from Section 2 the definition of the *topological dimension* $\mathrm{tdim}(X)$ of a definable set X. We say that *topological dimension is well-behaved in a class $\mathcal{K}$*

of structures, if for all $\mathcal{M} \in \mathcal{K}$, $m, n > 0$, and definable $X_1, \ldots, X_m \in M^n$, $\operatorname{tdim}(X_1 \cup \ldots \cup X_m) = \max(\operatorname{tdim}(X_1), \ldots, \operatorname{tdim}(X_m))$. By Corollary 2.2.8 and Theorem 2.2.9, topological dimension is well-behaved in the class of o-minimal structures. By results from [Macpherson et al. 1999] together with Theorem 4.1.4, we can extend this to obtain

THEOREM 4.1.5. *Topological dimension is well-behaved in the class of weakly o-minimal structures.*

From this (and related results in Section 4 of [Macpherson et al. 1999]) there follows a rather weak cell decomposition theorem for weakly o-minimal theories. It is weak in the sense that the boundary functions defining the cells may be functions to definable sorts in $\bar{M}$, which are not assumed to be continuous. If the *theory* of $\mathcal{M}$ is weakly o-minimal, then we can arrange in addition that each cell has a homeomorphic projection to an open set. Furthermore, in this case, definable bijections preserve topological dimension. There are weakly o-minimal theories in which algebraic closure does not have the exchange property (for example, the *contraction groups* of F.-V. Kuhlmann [1995]). However, models of any weakly o-minimal *theory* are uniformly bounded, so if algebraic closure does have the exchange property in such a theory, then its models are geometric structures in the sense of Section 2. If algebraic closure has the exchange property in a weakly o-minimal theory, then by Theorem 4.12 of [Macpherson et al. 1999], geometric dimension for definable sets is equal to topological dimension as in the o-minimal case (see Theorem 2.2.9).

In the o-minimal case, it is easy to show that any o-minimal ordered group is divisible abelian, and any o-minimal ordered field is real closed (see Propositions 3.1.1 and 3.1.2). The same theorems hold in the weakly o-minimal case:

THEOREM 4.1.6. (i) *Any weakly o-minimal ordered group is divisible abelian.*
(ii) *Any weakly o-minimal ordered field is real closed.*

These results are proved in [Macpherson et al. 1999]. In the group case it is easy, but the argument in the field case is substantial, and uses the fact that topological dimension is well-behaved, a kind of inverse function theorem, and some valuation theory. Observe that it is only assumed that the *structure* is weakly o-minimal, not the *theory*. A slight extension of Theorem 4.1.6, Corollary 5.13 of [Macpherson et al. 1999], states that any weakly o-minimal ordered commutative ring with a unit is a real closed ring, that is, a convex valuation ring of a real closed field (or possibly the whole field).

For a weakly o-minimal expansion $\mathcal{M}$ of an ordered field, quite a nice dichotomy emerges from results in Section 6 of [Macpherson et al. 1999]. Either there is a definable convex valuation ring in $\mathcal{M}$, or $\mathcal{M}$ shares many properties of o-minimality: the Monotonicity Theorem is piecewise, and not just local; there is a cell decomposition theorem in which the boundary functions of the cells are continuous (to a sort in $\bar{M}$); algebraic closure has the exchange property; and the *theory* of $\mathcal{M}$ is weakly o-minimal (and in particular, $\mathcal{M}$ is uniformly bounded). The case when there *is* a definable convex valuation ring also has a combinatorial characterisation: it occurs precisely when there is a definable equivalence relation on M with infinitely many infinite classes.

In another direction, there is a structure theory (with several interesting examples) for ω-categorical weakly o-minimal structures [Herwig et al. 2000]. In particular, it is shown that if such a structure is *3-indiscernible* (that is, there is a unique type of strictly increasing triple) then it is indiscernible. For a typical example of the structures which arise, consider a countable non-archimedean real closed field $\mathcal{R}$ with the archimedean valuation v corresponding to the valuation ring of finite elements, and define a ternary relation C on R, putting $C(x; y, z)$ whenever

$$v(y-x) < v(y-z).$$

Then $(R, <, C)$ is ω-categorical and weakly o-minimal (and C is a C-relation in the sense of the next subsection). In contrast, there are no interesting ω-categorical *o-minimal* structures [Pillay and Steinhorn 1986].

Finally, I comment that, if M is a weakly o-minimal theory which is a geometric structure, then by Chapter 8 of [Mosley 1996], the analogue of Lemma 3.2.3 holds (that is, any definable group is definably topologisable). Analogues of this also hold for C- and P-minimal structures discussed below (but the statement in the P-minimal case is weaker).

The theory of weak o-minimality has been developed further by several authors (Aref'ev, Baizhanov, Baisalov, Kulpeshov, Nurtazyn, Verbovsky) in Almaty.

4.2. C and P-minimality. We now consider a different generalisation of o-minimality, from [Macpherson and Steinhorn 1996], and again obtain general settings for a model theory of certain valued fields.

Suppose $L \subset L^+$ are languages, and $\mathcal{K}$ is an elementary class of L-structures. We say that an L^+-structure $\mathcal{M}$ is $\mathcal{K}$-*minimal* if the reduct $\mathcal{M}|_L \in \mathcal{K}$ and every L^+-definable subset of M is definable by a quantifier-free L-formula. A complete L^+-theory is $\mathcal{K}$-minimal if all its models are. It is easily checked that if $\mathcal{M}$ is $\mathcal{K}$-minimal then its theory is $\mathcal{K}$-minimal if and only if the following condition holds: for any $m > 0$ and L^+-definable subset S of M^{m+1}, there is $m' > 0$ and a quantifier-free L-definable subset $S' \subseteq M^{m'+1}$ such that for each $\bar{a} \in M^m$ there is $\bar{a}' \in M^{m'}$ such that $S_{\bar{a}} = S'_{\bar{a}'}$ (here, as usual, $S_{\bar{x}} := \{y : (\bar{x}, y) \in S\}$).

This setting includes o-minimality as a special case (where L has just a single binary relation, and $\mathcal{K}$ is the class of all dense total orders). If L has no relation, function, or constant symbols (apart, of course, from $=$), and $\mathcal{K}$ is the class of all infinite L-structures, then a theory is $\mathcal{K}$-minimal if and only if it is strongly minimal. However, weak o-minimality does not quite fit into this setting. Like o-minimality, $\mathcal{K}$-minimality is closed under reducts to languages containing L, and under expansions by constants. It is a condition on definable sets in one variable, which makes it easy to verify if, for example, one has quantifier elimination. Like o-minimality, it often gives strong information for definable sets in several variables, such as a dimension theory. It is observed in Theorem 3.2 of [Macpherson and Steinhorn 1996] that if a theory in L^+ is $\mathcal{K}$-minimal, then various stability properties lift from the L-reducts.

Our task is to find classes $\mathcal{K}$ such that there is a reasonable model theory of $\mathcal{K}$-minimality and there are interesting $\mathcal{K}$-minimal structures. This was initiated in [Macpherson and Steinhorn 1996], and developed in [Haskell and Macpherson 1994; 1997; van den Dries et al. 1999; Lipshitz and Robinson 1998].

C-minimality. The symbol C here denotes a ternary relation $C(x;y,z)$ (the semicolon indicates that the first variable is distinguished). We let $L = \{C\}$, and let $\mathcal{K}_C$ be the class of L-structures which satisfy the following axioms, where the free variables are universally quantified. The axioms were isolated by Adeleke and Neumann in work on Jordan permutation groups, and much more information on them can be found in [Adeleke and Neumann 1998].

(C1) $C(x;y,z) \to C(x;z,y)$
(C2) $C(x;y,z) \to \neg C(y;x,z)$
(C3) $C(x;y,z) \to (C(w;y,z) \vee C(x;w,z))$
(C4) $x \neq y \to (\exists z \neq y)\, C(x;y,z)$
(C5) $\exists x\, \exists y\, (x \neq y)$.

As an example, let $(T,\leq)$ be a semilinearly ordered set, that is, a partial order such that any two elements have a common lower bound, but the set of all lower bounds of an element is totally ordered. Suppose that T is infinite there is branching arbitrarily far up every maximal chain of T. Let M be the set of maximal chains of $(T,\leq)$, and interpret $C(x;y,z)$ to hold if either $y = z \neq x$, or x, y, z are distinct and x branches below where y and z branch (that is, $y \cap x \subset y \cap z$, where we regard elements of M as subsets of T). Then (M,C) satisfies (C1)–(C5). A converse to this was shown in [Adeleke and Neumann 1998]: namely, if $(M,C) \in \mathcal{K}_C$, then there is a semilinear order $(T,\leq)$ interpretable in (M,C), living on a quotient of M^2, such that M consists of a set of maximal chains of $(T,\leq)$ with union T, and C is interpreted as above. Since we think of $\mathcal{K}_C$-structures in this way as sets of chains in a semilinear order, we often talk of *nodes* of (M,C), meaning internal nodes of the underlying semilinear order. If $(M,C) \in \mathcal{K}_C$ then there is a Hausdorff topology on M with a uniformly definable basis: each internal node a determines the basic open set consisting of maximal chains which pass through it. This gives an analogy between $\mathcal{K}_C$ and the class of infinite dense total orders, but observe that unlike in the totally ordered case, the above basis consists of clopen sets and so in particular the topology is totally disconnected.

With this class $\mathcal{K}_C$, we now say that a structure $\mathcal{M} = (M, C, \dots)$ is *C-minimal* if its theory is $\mathcal{K}_C$-minimal (unlike 0-minimality, we choose *by definition* to close the condition under elementary equivalence). This notion was introduced in [Macpherson and Steinhorn 1996], where a number of examples were given, and a reasonable structure theory found for C-minimal groups. (A *C-minimal group* is the C-analogue of an o-minimal ordered group; it is a C-minimal structure $\mathcal{M} = (M, C, *)$, where $(M, *)$ is a group, $(M, C) \in \mathcal{K}_C$, and the C-relation is preserved by left and right multiplication.) For example, such a group must have a definable abelian normal subgroup with quotient of finite exponent.

There are nice connections between C-minimality and strong minimality and o-minimality. If $\mathcal{M}$ is C-minimal with underlying semilinear order $(T, \leq)$, then any element of M corresponds to a subset of $S \subset T$, that is, the set of nodes on the chain. Such a set S is interpretable in $\mathcal{M}$ and S together with the induced structure on it is o-minimal. Likewise, if $a \in T$, there is an equivalence relation E_a on the set $\{x \in M : a \in x\}$: put $E_a xy$ if $x \cap y$ contains a node strictly greater than a. The E_a-classes are called *cones at a*, and the set of cones at a is interpretable in $\mathcal{M}$, and, if infinite, is strongly minimal.

More general results were obtained in [Haskell and Macpherson 1994]. We have a notion of topological dimension as in Section 2, and, as in Theorem 4.1.5, obtain:

THEOREM 4.2.1. *Topological dimension is well-behaved in any C-minimal structure.*

The proof of this is by induction, where simultaneously a cell decomposition theorem is proved (as in Theorem 2.1.3). However, the notion of 'cell' is very cumbersome. With this notion of 'cell' one also proves in the induction the following result.

THEOREM 4.2.2. *Let n be a positive integer, X a definable subset of M^n, and $f : M^n \to M$ a definable partial function. Then X can be expressed as the disjoint union of finitely many cells on each of which f is continuous.*

The proof also uses a local version of the o-minimal Monotonicity Theorem, where 'monotonic function' is replaced by 'isomorphism' (of neighbourhoods, endowed with the relation C).

It is shown in [Macpherson and Steinhorn 1996] that in a C-minimal structure $\mathcal{M}$, algebraic closure need not have the exchange property. However, by Proposition 6.1 of [Haskell and Macpherson 1994], the exchange property can only fail in one way, namely if there is a certain kind of definable 'bad' function, between M and the set of internal nodes. Furthermore, any C-minimal structure $\mathcal{M}$ is uniformly bounded, as otherwise, in an elementary extension, there would be an infinite definable set with empty interior, contrary to C-minimality. Hence, if algebraic closure in $\mathcal{M}$ does have the exchange property, then $\mathcal{M}$ is a geometric structure in the sense of Section 2, so algebraic closure in this case also provides a notion of dimension for definable sets. By Proposition 6.3 of [Haskell and Macpherson 1994], in this situation the topological and geometric notions of dimension coincide.

There is a natural class of C-minimal structures. Let

$$\mathcal{F} := (F, V, +, \cdot)$$

be a non-trivially valued field, where V is a valuation ring with corresponding valuation map v to the value group. Define C on F by putting $C(x; y, z)$ if and only if $v(y-x) < v(y-z)$. Then the relation C is invariant under addition, and under multiplication by non-zero elements, and $(F, C) \in \mathcal{K}_C$. In fact, as noted in [Macpherson and Steinhorn 1996], the converse holds: if $\mathcal{F} = (F, +, \cdot)$ is a field, $(F, C) \in \mathcal{K}_C$, and C is preserved by the field operations in the above sense, then C comes from a valuation as above, and the valuation is definable in (F, C). In this situation, the C-relation and the semilinear order provide a natural way of viewing the value group and residue field. For let $(T, \leq)$ be the semilinear order underlying (F, C) (so members of F are maximal chains in T). Now the value group Γ of F is identified with the set of nodes on the chain 0_F (the zero of F), with the natural induced order. If $x \in F$, then $v(x) = \max(x \cap 0_F)$, that is, the node at which the chains x and 0_F meet (there will be such a node). In particular, the zero 0_Γ of the value group is the node at which the chains 1_F and 0_F meet, the valuation ring V is the set of chains in F which pass through this node, and the maximal ideal consists of those chains lying in the cone at 0_Γ which contains 0_F. The residue field consists of the set of cones at 0_Γ. The picture is fairly clear in say the valued power series field $F_p[[t]]$, where we may think of the internal nodes as given by polynomials in t, t^{-1}.

By a quantifier-elimination result of A. Robinson, if $\mathcal{F}$ is an algebraically closed valued field, then $(\mathcal{F}, C)$ is C-minimal [Macpherson and Steinhorn 1996]. In [Haskell and Macpherson 1994], the converse was proved, namely:

THEOREM 4.2.3. *Every C-minimal field F is an algebraically closed non-trivially valued field.*

In the proof it follows from the above identification of the value group Γ and residue field $\bar{F}$ that Γ is an o-minimal ordered group, so is divisible abelian, and $\bar{F}$ is finite or strongly minimal. By the divisibility of the value group, F is closed under Kummer and Artin–Schreier extensions, and this forces $\bar{F}$ to be infinite, so algebraically closed. The proof then uses valuation theory, together with Theorem 4.2.1.

Given the rich supply of o-minimal expansions of the field of reals, it is natural to ask for C-minimal *expansions* of algebraically closed valued fields. The model theory of rigid analytic geometry was first developed by Lipshitz [1993], who proved quantifier elimination for an expansion of an algebraically closed valued field (complete with respect to a definable non-archimedean norm), by a rich and rather complicated non-archimedean structure. Lipshitz and Robinson [1998] have shown that this expansion is C-minimal. This means that subanalytic sets in one variable are uniformly (in the parameters) definable in the pure valued field.

P-minimality. In this subsection I use 'semialgebraic' to mean 'definable in the pure field $\mathbb{Q}_p$.' (It is well-known that the natural valuation on $\mathbb{Q}_p$ is definable in the pure field.) As discussed in the survey [Macintyre 1986], there are many model-theoretic analogues between $\mathbb{Q}_p$ and $\mathbb{R}$, at both the semialgebraic and the subanalytic level (see [Denef and van den Dries 1988] for the latter). It is therefore natural to look for versions of o-minimality which support $\mathbb{Q}_p$, and one such, P-minimality, was proposed in [Haskell and Macpherson 1997].

DEFINITION 4.2.4. Let L be the language $(+, -, \cdot, 0, 1, P_n)_{n>1}$ (where the P_n are unary predicates). Regard $\mathbb{Q}_p$ as an L-structure, letting P_n pick out the n^{th} powers in $\mathbf{Q_p}$. Let $\mathcal{K}_P$ be the class of L-structures elementarily equivalent to $\mathbb{Q}_p$. Then if $L^+ \supseteq L$, an L^+-structure is *P-minimal* if all models of its theory are $\mathcal{K}_P$-minimal.

By [Macintyre 1976], $\mathbb{Q}_p$ has quantifier elimination in the above language L. The version of P-minimality described above differs slightly from that in [Haskell and Macpherson 1997], where, for example, p-adically closed fields in the more general sense of [Prestel and Roquette 1984] are considered P-minimal. I emphasise that unlike the other model-theoretic classes considered in this paper, a P-minimal structure is *always* an expansion of a field.

The model theory of P-minimality has not been developed far, but I sketch some results. In any P-minimal structure F with value group Γ, the valuation topology has a uniformly definable basis of clopen sets, namely, sets of the form $B_\gamma(a) = \{x : v(x-a) \geq \gamma\}$ where $a \in F$ and $\gamma \in \Gamma$. We again obtain that topological dimension is well-behaved in P-minimal structures. As in the C-minimal case, P-minimal structures are uniformly bounded. Furthermore, unlike in the weakly o-minimal and C-minimal cases, algebraic closure has the exchange property in *any* P-minimal structure, so such structures are geometric. The resulting geometric notion of dimension coincides with topological dimension. There are also theorems about continuity of definable functions. No cell decomposition theorem for P-minimal structures has been proved (though it might not be difficult, and a cell decomposition and related results for $\mathbb{Q}_p$ are developed in [Denef 1986; Scowcroft and van den Dries 1988]).

Let L^D_{an} be the language introduced in [Denef and van den Dries 1988] to describe the subanalytic structure on the p-adic integers $\mathbf{Z}_p$. The following is an analogue of the theorem of Lipshitz and Robinson mentioned above, and is the main theorem of [van den Dries et al. 1999].

THEOREM 4.2.5. *The L^D_{an}-structure $\mathbb{Q}_p$ is P-minimal.*

It was shown earlier in [Denef and van den Dries 1988] (Corollary 3.32) that any L^D_{an}-definable subset of $\mathbb{Q}_p$ is semialgebraic. Theorem 4.2.5 shows that such sets are semialgebraic *uniformly* in the defining parameters. The point is that

P-minimality is a property of the *theory*; however, the uniformity provides new information also in the standard model $\mathbb{Q}_p$.

The proof of Theorem 4.2.5 uses the quantifier elimination of [Denef and van den Dries 1988]. One has a quantifier-free L_{an}^D-formula $\phi(x, \bar{y})$, and has to show that the set defined by $\phi(x, \bar{a})$ is semialgebraic, uniformly in $\bar{a}$. The formula $\phi(x, \bar{y})$ is assumed to be atomic, and the proof is based on induction on the complexity of a term $t(x)$ in the language. The main problems are posed by occurrences in a term of the binary function symbol D for division. One works in an elementary extension K, and, using a parametric Weierstrass Preparation Theorem, obtains a Preparation Theorem for a certain ring $K\{Y_1, \ldots, Y_n\}$ of definable functions on K^n. The idea is, given a term $t(x)$, to cover the valuation ring R of K with particularly nice sets known as 'connected affinoids'. Each connected affinoid F has a well-behaved associated ring $\mathcal{O}(F)$ of definable functions on it. The ring $\mathcal{O}(F)$ is a quotient of $K\{Y_1, \ldots, Y_n\}$, where $n-1$ is the number of 'holes' in F. It has nice divisibility properties — any non-zero element of $\mathcal{O}(F)$ has just finitely many zeros, and if it has no zeros then it is a unit of $\mathcal{O}(F)$. One uses this to show, by induction on the complexity, that any term is given piecewise by members of $\mathcal{O}(F)$ for various F. Theorem 4.2.5 follows immediately from this.

I conclude with an amalgam of results from [Macpherson et al. 1999] (Proposition 7.3), [Macpherson and Steinhorn 1996] (Proposition 3.4) and [Haskell and Macpherson 1997] (Proposition 7.1), which is suggested by Corollary 3.10 of [Pillay and Steinhorn 1986]. Recall that a theory T has the *independence property* [Shelah 1978] if there is $\mathcal{M} \models T$ and a formula $\phi(\bar{x}, \bar{y})$ with $l(\bar{x}) = m, l(\bar{y}) = n$, and $\bar{a}_i \in M^m$ (for all $i \in \omega$) such that: for any $S \subseteq \omega$, there is $\bar{b}_S \in M^n$ such that for all $i \in \omega$, $\mathcal{M} \models \phi(\bar{a}_i, \bar{b}_S)$ if and only if $i \in S$. No stable theory can have the independence property, but for example, pseudofinite fields do have it.

THEOREM 4.2.6. *No weakly o-minimal, C-minimal or P-minimal theory can have the independence property.*

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Dugald Macpherson
Department of Pure Mathematics
University of Leeds
Leeds LS2 9JT
England
pmthdm@amsta.leeds.ac.uk