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Abelian Varieties and the Mordell–Lang Conjecture

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ABSTRACT. This is an introductory exposition to background material useful to appreciate various formulations of the Mordell–Lang conjecture (now established by recent spectacular work due to Vojta, Faltings, Hrushovski, Buium, Voloch, and others). It gives an exposition of some of the elementary and standard constructions of algebro-geometric models (rather than model-theoretic ones) with applications (for example, via the method of Chabauty) relevant to Mordell–Lang. The article turns technical at one point (the step in the proof of the Mordell–Lang Conjecture in characteristic zero which passes from number fields to general fields). Two different procedures are sketched for doing this, with more details given than are readily found in the literature. There is also some discussion of issues of effectivity.

CONTENTS

The organizers of this MSRI workshop in Arithmetic and Model Theory gave me the agreeable task of lecturing on introductory background in the theory of abelian varieties, and especially those parts of the theory relevant to the Mordell–Lang Conjecture, which is the theme of some of the recent spectacular work (by Vojta, Faltings, Hrushovski, Buium, Voloch, and others). To keep this talk as "introductory" and as focussed as possible I will concentrate only on the version of the Mordell–Lang Conjecture that deals with abelian, rather than semi-abelian, varieties, and I will not discuss its elaborations that include "Manin–Mumford type" questions regarding torsion points. For this, see [Raynaud 1983b; 1983c; Coleman 1985b; Hindry 1988; McQuillan 1995], and the extensive bibliographies in these articles. For a general introduction to the model theory approach to Mordell–Lang, see [Bouscaren 1998], and particularly [Hindry 1998] therein. I have tried to make my expository article overlap as little as possible with Hindry's, given their nearly identical titles, and their similar missions.

We can look forward to further deep connections between model theory and arithmetic problems. For example, there is the model theory of difference fields which already has given rise to explicit bounds in certain arithmetic questions; see [Chatzidakis and Hrushovski 1999] and the bibliography there. Also there are important applications of model theory to aspects of the ABC Conjecture, in the work of Buium and that of Scanlon [1997a], which opens up a very promising avenue of research.

The "fundamentals" of the theory of abelian varieties are collected in Section 1. The main theorems are stated and discussed in Sections 2, 3, and 4. Motivated by the title of this conference, I also recorded a few of the much more modest construction of models, sometimes done explicitly, sometimes implicitly, in standard algebraic geometric arguments. These well known constructions are reviewed in Section 5. For fun, in Section 6 we put the "p-adic model" to work: we make use of the beautiful classical idea of Chabauty [1941] to give a proof of a (small) piece of Mordell–Lang. One can think of the strategy of Chabauty's method as being, first, to extend the groundfield to a rather large field (i.e., the p-adics) over which differential equations work well (i.e., giving us the Lie theory for p-adic analytic groups), and then reaping the benefits of being able to solve differential equations. Of course, I am saying it this way in order to force a kinship (albeit a distant one) between the Chabauty approach and the formidable methods (as in [Hrushovski 1996; Hrushovski and Zilber 1993; 1996]) that I am learning about in this conference, and therefore to justify including it in this article. Section 7 discusses the step in the proof of the Mordell–Lang Conjecture in characteristic 0 which passes from number fields to general fields; we give more details about this step than is found in the literature. Finally, Section 8 is devoted to some comments about effectivity.

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Introduction

The central result in the constellation of theorems to be discussed in this article is the classical 1922 conjecture of Mordell, proved some sixty years later by Faltings [1983] (which asserts that a curve of genus greater than 1 defined over a number field has only a finite number of points rational over that number field). As an example of an application of this theorem, choose your favorite polynomial $g(x)$ with rational coefficients, no multiple roots, and of degree ≥ 5, for example

$$g(x) = x(x-1)(x-2)(x-3)(x-4),$$

and let K be any number field (that is, any field of finite degree over $\mathbb{Q}$). Then Faltings' Theorem implies:

Corollary. *Let $g(x)$ be a polynomial in $K[x]$ with no multiple roots, and of degree ≥ 5. Then there are only finitely many elements $\alpha \in K$ for which $g(\alpha)$ is a square in K.*

We can use this corollary to illustrate some of the "degrees of effectivity" that are of interest in this problem, and in similar problems. We distinguish three grades:

1. **Number-effectivity.** Is there a "directly computable" function of the coefficients of $g(x)$ and of the number field K which provides an upper bound for the *number* of such elements $\alpha \in K$ (i.e., for the number of α's such that $g(\alpha)$ is a square in K)?
2. **Size-effectivity.** Is there a "directly computable" function of the coefficients of $g(x)$ and of the number field K which provides an upper bound for the *heights* of elements $\alpha \in K$ for which $g(\alpha)$ is a square in K?
3. **Uniform number-effectivity.** Is there a "directly computable" function of the *degree* of $g(x)$ and of the number field K which provides an upper bound for the *number* of elements $\alpha \in K$ for which $g(\alpha)$ is a square in K?

There is at least one somewhat ambiguous term in all of the questions raised above; namely, what does one accept as "directly computable"? But by any reasonable standard of "direct computability", the answer to Question 1 regarding number-effectivity is yes: in fact the proof of Faltings' Theorem readily provides a quite large, but effective, upper bound for the number of K-valued points on a curve of genus greater than 1 defined over K, in terms of K and the curve. Question 2 is open, and is the focus of a good deal of activity. Question 3 is also open (see [Caporaso et al. 1997] for the connection between Question 3 and certain conjectures of Lang). For further discussion of these issues, see Section 8 below.

Our main interest in the present article will be higher-dimensional analogues of Faltings' theorem, these analogues being expressed in terms of abelian varieties.

1. Complex Tori and Abelian Varieties

An excellent reference for the basics of this theory is [Mumford 1974]. Let V be a finite dimensional complex vector space, and call its dimension d. Let $\Lambda \subset V$ be a discrete additive subgroup of rank $2d$. It follows that the natural homomorphism of real vector spaces $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} \to V$ is an isomorphism, and that $2d$ is the maximal rank that a discrete additive subgroup of V can have. We refer to Λ as a *lattice* in V, and to the quotient (commutative, compact) complex Lie group $T := V/\Lambda$ as a *complex torus*. Given T we can reconstruct V up to unique isomorphism as the tangent space at the origin (i.e., the "Lie algebra") of the complex Lie group T. The elementary construction of passing to the quotient

$$(V, \Lambda) \mapsto T = V/\Lambda$$

is functorial from the category of lattices in finite-dimensional complex vector spaces to the category of complex tori. If $T = V/\Lambda$ we have the following diagram of endomorphism rings:

$$\begin{aligned} \mathrm{End}_{\text{cx tori}}(T) &= \mathrm{End}_{\text{ab gp}}(\Lambda) \cap \mathrm{End}_{\text{cx v. sp.}}(V) \\ &\subset \mathrm{End}_{\text{real v. sp.}}(\Lambda \otimes_{\mathbb{Z}} \mathbb{R}) = \mathrm{End}_{\text{real v. sp.}}(V). \end{aligned}$$

Since $\mathrm{End}_{\text{cx tori}}(T) \subset \mathrm{End}_{\text{ab gp}}(\Lambda)$ it follows that the underlying additive group of $\mathrm{End}_{\text{cx tori}}(T)$ is a finitely generated and free abelian group.

If the complex manifold $T \hookrightarrow \mathbb{P}^N$ admits a complex analytic imbedding into projective N-space, then by Chow's Theorem, T is the locus of zeroes of homogeneous polynomial equations in $\mathbb{P}^N$ and therefore carries the structure of projective algebraic variety; its group law is algebraic. By a *projective variety* in this article, we do not impose the condition that it be *irreducible*; we mean what is sometimes referred to as *algebraic set*, i.e., cut out by a system of homogenous forms as in [Hartshorne 1977, Chapter 1, Section 2]. If our varieties are irreducible, we will signal this explicitly by the phrase *irreducible variety*. By an *abelian variety* over a field F we mean a group object in the category of geometrically irreducible, proper (i.e., "complete") , algebraic varieties (over F). Thus, a complex analytic imbedding of a complex torus T in projective space endows T with the structure of abelian variety over the field $\mathbb{C}$. Any abelian variety over F is a projective variety, i.e., is isomorphic over F to a subvariety of projective space. Any abelian variety over $\mathbb{C}$, when viewed with its underlying complex Lie group structure is a complex torus. The theory of Weierstrass guarantees that any complex 1-*dimensional* torus admits a complex analytic imbedding (as a plane cubic) in $\mathbb{P}^2$, and therefore admits the structure of an *elliptic curve*, i.e., the structure of an abelian variety (of dimension 1) over $\mathbb{C}$. When $d > 1$ the "Riemann period relations" impose definite conditions on lattices $\Lambda \subset V$ in order that $T = V/\Lambda$ be imbeddable in projective space, or equivalently, that T have the structure of abelian variety over $\mathbb{C}$.

When T is an abelian variety, its complex analytic endomorphisms are all "algebraic", in the sense that they are endomorphisms of the abelian variety T, defined over $\mathbb{C}$. If A is an abelian variety over any field K, we have the Poincaré complete reducibility theorem [Mumford 1974, Section 19, Theorem 1], which says that given any abelian subvariety $Y \subset A$ there is a "complementary" abelian subvariety $Z \subset A$ (in the sense that $Y \cap Z$ is a finite group and Y and Z taken together span A). In standard terminology, we have that the natural homomorphism $Y \oplus Z \to A$ is an *isogeny* in the sense that it is a surjective homomorphism with finite kernel. It follows directly that, "up to isogeny" , any abelian variety over a field F is a direct sum of (a finite number of) *simple* abelian varieties over F. (An abelian variety X over F is called *simple* (over F) if any abelian subvariety of X defined over F is either $\{0\}$ or X.) It follows easily [Mumford 1974, Corollaries 1 and 2 of Section 19] that the algebra of endomorphisms of the abelian variety A, after being tensored with $\mathbb{Q}$,

$$\mathrm{End}_{\mathrm{ab\ var}\ /F}(A) \otimes_{\mathbb{Z}} \mathbb{Q},$$

is a (finite-dimensional) semi-simple algebra over $\mathbb{Q}$.

2. Configurations and Configuration-Closure

Let F be a field and let V be a finite dimensional vector space over F. By a *configuration* in V we will mean a subset $W \subset V$ that is the (possibly empty) finite union of translates of vector subspaces (over the same field F) of V. That is, W is a configuration in V if

$$W = \bigcup_{j=1}^{n} (v_j + V_j)$$

for elements $v_j \in V$ and vector subspaces $V_j \subset V$. The collection of configurations in a finite dimensional vector space V is closed under finite union and arbitrary intersection — that is, it forms the collection of closed sets for a topology of V; moreover, this collection satisfies the "noetherian" (descending chain) condition. That is, given any decreasing sequence $\cdots \subset W_{j+1} \subset W_j \subset \cdots$ of configurations in V, indexed by natural numbers j, the sequence eventually stabilizes. This allows us to define the notion of *configuration-closure:*

DEFINITION. Given an arbitrary subset $S \subset V$, the *configuration-closure*, $\mathrm{Conf}_V(S)$, of S in V is the smallest configuration in V containing S. Equivalently, $\mathrm{Conf}_V(S)$ is the intersection of all configurations containing S.

A linear transformation of vector spaces, $\phi : V \to W$, brings configurations in the domain V to configurations in W, and the full inverse image of a configuration in W (under ϕ) is a configuration in V. If $S \subset V$ is a subset,

$$\phi\big(\mathrm{Conf}_V(S)\big) = \mathrm{Conf}_W(\phi S).$$

We can carry the notion of "configuration" and "configuration-closure" to abelian varieties, or to complex tori. We do this explicitly for abelian varieties over an algebraically closed field K. Recall that if A is an abelian variety over K, and $S \subset A(K)$ is a set of K-rational points of A, we can consider, the *Zariski closure* of S in A (which we denote $\mathrm{Zar}_A(S) \subset A$) which is defined to be the smallest closed subvariety of A whose set of K-valued points contain S. Equivalently, if you imbed A in $\mathbb{P}^N$, you may think of $\mathrm{Zar}_A(S)$ as the projective subvariety of $\mathbb{P}^N$ cut out by the ideal of all homogenous polynomial forms which contain the set S in their zero-locus.

Now, following the pattern set in our discussion of vector spaces above, define a *configuration* in the abelian variety A to mean a subset $B \subset A$ which is (possibly empty) finite union of translates of abelian subvarieties of A. That is, B is a *configuration* in A if

$$B = \bigcup_{j=1}^{n} (a_j + A_j)$$

for elements $a_j \in A$ and abelian subvarieties $A_j \subset A$.

As with vector spaces, the collection of configurations in an abelian variety A is closed under arbitrary intersection and finite union, and satisfies the descending chain condition. Therefore, we can define the "configuration-closure" of a subset $S \subset A(K)$ in A:

Definition. Given an arbitrary subset $S \subset A(K)$, the *configuration-closure*, denoted $\mathrm{Conf}_A(S)$, of S in A is the smallest configuration C in A such that $S \subset C(K)$.

Clearly, $\mathrm{Conf}_A(S)$ is a closed subvariety of A containing S, and therefore

$$\mathrm{Zar}_A(S) \subset \mathrm{Conf}_A(S).$$

3. "Absolute" Mordell–Lang in Characteristic Zero: Theorems of Faltings and Vojta

Now we put these two notions of configuration (for abelian varieties and for vector spaces) together to give two equivalent formulations of the theorem of Faltings and Vojta. Let A be an abelian variety and $S \subset A(K)$ a subset of the group $A(K)$ of K-rational points of A.

THEOREM 3.1 ("ABSOLUTE" MORDELL–LANG IN CHARACTERISTIC 0, FIRST VERSION). *Let K be algebraically closed of characteristic 0. If $S \subset A(K)$ generates (or equivalently, is contained in) a finitely generated subgroup of $A(K)$, then the Zariski-closure of S in A is equal to its configuration-closure in A. In notation,*

$$\mathrm{Zar}_A(S) = \mathrm{Conf}_A(S).$$

This was first proved by Faltings [1994] (the proof in that reference is made explicit only for the case of $K = \bar{\mathbb{Q}}$; but see, for example, [McQuillan 1995] and Section 7 below). Earlier [1991] he had established the special case where $K = \bar{\mathbb{Q}}$ and $\mathrm{Zar}_A(S)$ contains no nontrivial translated abelian subvariety. The techniques of [Faltings 1991] have, as their starting point, Vojta's proof [1991] of the classical Mordell Conjecture. The original 1960 article of Lang, which formulated the special case of the Mordell–Lang conjecture proved in [Faltings 1991], is [Lang 1960], and this was followed by stronger versions of the conjecture [Lang 1965; 1974; 1986]. For further developments extending the proofs of the Mordell–Lang conjectures to include semi-abelian varieties see [Vojta 1996], and for the strongest statement of Mordell–Lang (on semi-abelian varieties combined with the Manin–Mumford conjecture) in characteristic 0, see [McQuillan 1995].

The "classical" application of Theorem 3.1. Let the abelian variety A be defined over a number field L and let K be an algebraically closed field containing L. Denote by $A(L)$ the group of L-rational points of A. Let $Z \subset A$ be a closed subvariety defined over L, and suppose that $S := Z(L)$, the set of L-rational points of Z, is *Zariski-dense* in Z. By the theorem of Mordell–Weil (see [Serre 1989, Section 4.3] or [Lang 1991, Chapter I, Theorem 4.1]), $A(L)$ is finitely generated, and so we have $Z = \mathrm{Zar}_A(S)$ and all the hypotheses of Theorem 3.1 above are satisfied ($S = Z(L) \subset A(L) \subset A(K)$, so S is contained in a finitely generated subgroup of $A(K)$). Therefore, Z is a configuration in A. To summarize:

COROLLARY. *Let A be an abelian variety defined over a number field L. Any closed subvariety of A defined over L which is the Zariski-closure of its set of L-rational points is a configuration in A; i.e., is a finite union of translates of abelian subvarieties of A.*

Bogomolov and Tschinkel [1999] introduced the term *potentially dense* to refer to algebraic varieties V defined over number fields which have the property that, over some (possibly larger) number field L, the variety V is the Zariski-closure of its set $V(L)$ of L-rational points. Another way of expressing the preceding corollary is to say that the closed subvarieties of an abelian variety that are *potentially dense* are precisely the configurations. One of the important projects in number theory these days is to understand in a more general context which algebraic varieties are potentially dense. There are many open issues here. It is even unknown, at present, whether there exist K3 surfaces which are not potentially dense! (But see [Bogomolov and Tschinkel 1999] for elliptic K3's!)

The "classical" Mordell Conjecture can be expressed as simply saying that a smooth projective irreducible algebraic curve defined over a number field is potentially dense (if and) only if its genus is 0 or 1. The "if" part of this statement is easy (and isn't usually packaged as part of the Mordell Conjecture). It is the "only if" assertion that is deep; it follows from the preceding corollary by noting that if C is a curve of positive genus defined over a number field, then by possible extension of the number field L we can suppose that C has an L-rational point $c \in C$. We then can view C as a subvariety of its jacobian, call it A, defined over L, by sending a point $x \in C$ to the linear equivalence class of the divisor $[x] - [c]$. The preceding corollary would then tell us that C is a *configuration* in A which can only be the case if its genus is 1.

The format of Theorem 3.1. Theorem 3.1 designates a class of algebraic varieties (i.e, abelian varieties) and says that if, within a variety in this class, S is a subset of points satisfying a certain "finiteness property" (i.e., is contained in a finitely generated subgroup of the Mordell–Weil group) then the Zariski-closure of S is a very restricted type of subvariety (i.e., is a configuration). Are there analogous results in other settings? For example, take, as class of algebraic varieties, smooth cubic hypersurfaces in $\mathbb{P}^N$. Define the "finiteness property" on a set S of points of a cubic hypersurface V to be: S is contained in a subset of V which is *finitely generated* in the sense of the chord-and-tangent process on V. I ask this question not because I have a sense that it is worthwhile to pursue (nor do I have any specific guess regarding the types of subvarieties of V that one can get as Zariski-closures of S's which are finitely generated in the above sense) but just in order to help us think, for moment, about the formal shape of Theorem 3.1. In the same vein, one can ask the analogous (Mordell–Lang) question about complex tori (which are not necessarily abelian varieties). I did so in an early draft of this article, and I am thankful to Anand Pillay for his affirmative answer to that question; see his proof [Pillay 1999] of the fact that if T is a complex torus and $S \subset T$ a subset which is contained in a finitely generated subgroup of T, then $\mathrm{Zar}_T(S) = \mathrm{Conf}_T(S)$ if we interpret $\mathrm{Zar}_T(S)$ as the smallest compact complex analytic subvariety of T containing S, and $\mathrm{Conf}_T(S)$ the smallest finite union of translates of complex subtori of T having the property that $S \subset \mathrm{Conf}_T(S)$.

Theorem 3.1 is equivalent to:

THEOREM 3.2 ("ABSOLUTE" MORDELL–LANG IN CHARACTERISTIC 0, SECOND VERSION). *Let K be algebraically closed of characteristic 0, A an abelian variety defined over K, and suppose that we are given a subset $S \subset \Gamma \subset A(K)$, where Γ is a finitely generated subgroup of $A(K)$. If Γ is Zariski-dense in A, and if the* *configuration-closure of the image of S in the real vector space $V = \Gamma \otimes \mathbb{R}$ is all of V, then S is Zariski-dense in A.*

PROOF OF EQUIVALENCE. Suppose the truth of Theorem 3.1 and the hypotheses of Theorem 3.2. We must prove that S is Zariski-dense in A. So

$$\operatorname{Zar}_A(S) = \operatorname{Conf}_A(S) = \bigcup_{j=1}^{n} (a_j + A_j) \subset A,$$

where the A_j's are abelian subvarieties of A and the $a_j \in A(K)$ are points. Put $\tilde{\Gamma}_j := (\Gamma \bigcap (a_j + A_j)$ so we have that $S \subset \bigcup_{j=1}^{n} \tilde{\Gamma}_j$. We have put the tilde on the $\tilde{\Gamma}_j$ to remind ourselves that $\tilde{\Gamma}_j$ isn't a subgroup of Γ but is rather a coset. Fix any element $\gamma_j \in \tilde{\Gamma}_j$ and we may write $\tilde{\Gamma}_j = \gamma_j + \Gamma_j$ for some subgroup $\Gamma_j \subset \Gamma$.

Therefore the image of S in $V = \Gamma \otimes \mathbb{R}$ is contained in the union of the translates by the image of γ_j of $\Gamma_j \otimes \mathbb{R}$ for $j = 1, \ldots, n$. Since S is configuration-dense in V we must have $V = \Gamma_j \otimes \mathbb{R}$ for some j, and for this j, $\Gamma_j \subset \Gamma$ must be a subgroup of Γ of finite index. But if Γ is Zariski-dense in A so is every (coset of every) subgroup of finite index in Γ. We deduce that $A_j = A$, and since a translate of A_j is contained in the Zariski-closure of S, it follows that S is Zariski-dense in A.

Now suppose the hypotheses of Theorem 3.1 and the truth of Theorem 3.2. We must prove that the conclusion of Theorem 3.1 holds. Consider the real vector space $V = \Gamma \otimes \mathbb{R}$. Let $\operatorname{Conf}_V(S) = \bigcup_{j=1}^{n}(v_j + V_j) \subset V$ be the configuration-closure of the image of S in $V = \Gamma \otimes \mathbb{R}$, where the V_j's are real vector subspaces of V and the $v_j \in V$ are points. Let $\tilde{S}_j \subset S$ be the inverse image of the affine subspace $v_j + V_j$, so that $S = \bigcup_{j=1}^{n} \tilde{S}_j$. Since Theorem 3.1 holds for S if it holds for each of the $\tilde{S}_j$'s, we need only prove Theorem 3.1 for each $\tilde{S}_j$. To do this we may, of course, take $\tilde{S}_j$ to be nonempty. Make a translation of $\tilde{S}_j$ by any element $\tilde{s}_j \in \tilde{S}_j$ to get a set which we denote $S_j := \tilde{S}_j - \tilde{s}_j$ and we have that $\operatorname{Conf}_V(S_j) = V_j$, i.e., is a vector subspace of V. If $\Gamma_j \subset \Gamma$ is the subgroup of all elements in Γ whose image in V lies in V_j, we have the inclusion $S_j \subset \Gamma_j$. By construction, $\operatorname{Conf}_{V_j}(S_j) = V_j$. Let $\tilde{A}_j \subset A$ be the Zariski-closure of the subgroup $\Gamma_j \subset A$.

If $\tilde{A}_j$ were connected, then it would be an abelian subvariety of A, and the triple $S_j, \Gamma_j, \tilde{A}_j$ would satisfy all the hypotheses required to apply Theorem 3.2, giving that

$$\operatorname{Zar}_{\tilde{A}_j}(S_j) = \tilde{A}_j,$$

thereby proving Theorem 3.1 for $\tilde{S}_j$. In general, let $A_j \subset \tilde{A}_j$ be the connected component containing the identity element, so that $\tilde{A}_j$ is a finite union of cosets of A_j. Breaking things up coset by coset, and applying the same argument as above, allows us to conclude the proof. □

About positive characteristic. In the setting of characteristic $p > 0$, the statement analogous to "absolute" Mordell–Lang (Theorem 3.1 or 3.2 above) is no longer true, as the following well-known counter-example will make clear. If C is any curve (of positive genus) defined over a finite field k of characteristic p, and if $K = k(C)$ is the function field of C, then let $C_{/K}$ be the curve over K obtained from C by extending the field of scalars from k to K. We can think of $C_{/K}$ as the generic fiber of the constant family $\pi : C \times C \to C$, where π is projection to the second factor. There is a natural K-valued point, call it c, on $C_{/K}$, which can be described geometrically as the restriction to the generic fiber of the diagonal section $C \to C \times C$. If $F = F_{/k} : C_{/k} \to C_{/k}$ is the Frobenius endomorphism of $C_{/k}$ (defined on local rings by the rule: $f \mapsto f^q$ where $q = \mathrm{card}(k)$), denote by the same letter $F = F_{/K} : C_{/K} \to C_{/K}$ the base change of the Frobenius endomorphism to K. For any natural number n, $F^n(c)$ is the restriction to the generic fiber of $C \to C \times C$ of the graph of the n-th iterate of $F_{/k}$. The set, $S = \{F^n(c) \mid n = 1, 2, \dots\} \subset C(K)$, of images of c under these iterations of $F_{/K}$ is an infinite subset whose Zariski closure is therefore the entire curve $C_{/K}$. The Frobenius endomorphism F acting on $C_{/K}$ induces an endomorphism Φ of the jacobian $A_{/K} = \mathrm{Pic}^o(C_{/K})$. Let $\mathcal{E} := \mathrm{End}_K(A)$ denote the endomorphism ring of the abelian variety A over K, so that $\Phi \in \mathcal{E}$. Since C is of positive genus, we have an imbedding $\iota : C_{/K} \hookrightarrow A_{/K}$ defined by sending $x \in C_{/K}$ to the linear equivalence class of $[x] - [c]$ in $A_{/K}$. For any $m \geq 0$ we have

$$\Phi^m(\iota(Fc)) = \Phi^m([Fc] - [c]) = [F^{m+1}c] - [F^m c] = \iota(F^{m+1}c) - \iota(F^m c),$$

and therefore

$$\iota(F^n c) = \sum_{j=0}^{n-1} \Phi^j(\iota(Fc))$$

for any $n \geq 1$, so that the image of S under ι is contained in $\mathcal{E} \cdot \iota(Fc) \subset A(K)$ which is a finitely generated subgroup of $A(K)$ (the endomorphism ring $\mathcal{E}$ of the abelian variety A being a finitely generated abelian group).

So $\mathrm{Zar}_A(\iota S) = \iota C \subset A$, and therefore if C is of genus > 1, $\mathrm{Zar}_A(\iota S)$ is *not* a configuration in A despite the fact that ιS is contained in a finitely generated subgroup of $A(K)$. Therefore one needs to appropriately modify the statement of "absolute Mordell–Lang" if one wishes to obtain a result which is valid in characteristic p. This is the subject of the next section.

4. "Relative" Mordell–Lang in All Characteristics: Theorems of Manin, Grauert, Buium, Voloch, Hrushovski, etc.

THEOREM 4.1 ("RELATIVE" MORDELL–LANG IN ALL CHARACTERISTICS). *Let $k \subset K$ be an inclusion of algebraically closed fields. Let A be an abelian variety over K having the property that no positive dimensional factor abelian variety* *of A comes by base extension from an abelian variety over k. Let $S \subset A(K)$ be a subset generating a finitely generated subgroup of $A(K)$. Then*

$$\mathrm{Zar}_A(S) = \mathrm{Conf}_A(S).$$

There is, to be sure, a large literature dealing with the "classical version" of this theorem by which I mean the special case where $X := \mathrm{Zar}_A(S)$ is a curve. This classical case was originally proved by Manin [1963] in characteristic zero. Another proof of it was given by Grauert [1965]. This latter proof was adapted by Samuel [1966] to make it work in characteristic p. In 1991, Voloch produced an extremely short, insightful, proof of this classical theorem under the auxiliary hypothesis that X is non-isotrivial, and its jacobian is ordinary [Voloch 1991].

To be sure, the statement of Theorem 4.1 in characteristic 0 is weaker than (and therefore follows immediately from) Theorem 3.1. So, it is only the characteristic p aspect of this theorem that is specifically new to our discussion. Progress towards the above theorem in characteristic p was made in [Abramovich and Voloch 1992], following on Voloch's approach. The full theorem is due to Hrushovski [1996], who writes that Buium's approach (to the characteristic 0 part of Theorem 4.1) inspired his own. Buium's ten-page paper [1992] is quite illuminating, and I would urge anyone who has not yet read it to do so! A certain universal jet space construction plays critical role in it. Briefly, given any affine smooth $\mathbb{C}$-scheme S with a derivation $\delta \in \mathrm{Der}(\mathcal{O}_S)$, and any S-scheme X, Buium constructs a "pro-X-scheme" (i.e., a projective system of X-schemes) $\mathrm{jet}(X/S, \delta)$ (call it J, for short) and he constructs a derivation δ_J on J lifting δ on S such that the pair (J, δ_J) satisfies the following universal property. For any pair $\eta = (Y, \delta_Y)$ where Y is an X-scheme and δ_Y is a derivation on Y lifting δ on S, there is a unique X-morphism $\mathrm{jet}(\eta) : Y \to J = \mathrm{jet}(X/S, \delta)$ which is *horizontal* in the sense that $\mathrm{jet}(\eta)$ intertwines the derivations δ_Y and δ_J. The method of Buium turns on the properties of $\mathrm{jet}(X/S, \delta)$ where specifically X is a group scheme over S; this method depends especially on the manner in which finitely generated groups of S-sections in group schemes X lift to $\mathrm{jet}(X/S, \delta)$. The fact that the theory of ordinary differential equations works so well in the complex analytic category is essential. In characteristic p, things aren't so smooth-going. People with some algebraic geometric background who wish to see a bridge between Buium's techniques and the model-theoretic techniques of Hrushovski might find it useful to read Chapter 2 of Scanlon's thesis [1997b], where he introduced the notion of $\mathcal{D}$-rings and $\mathcal{D}$-functors which serve as "jet-theoretic" technology suitable for use in characteristic p.

As with the Absolute Mordell–Lang Theorem, the "Relative" Mordell–Lang result we have just formulated has an alternate version:

THEOREM 4.2 ("RELATIVE" MORDELL–LANG IN ALL CHARACTERISTICS, SECOND VERSION). *Let $k \subset K$ be an inclusion of algebraically closed fields. Let A*

be an abelian variety over K having the property that no positive dimensional factor abelian variety of A comes by base extension from an abelian variety over k and suppose that we are given a subset $S \subset \Gamma \subset A(K)$ where Γ is a finitely generated subgroup of $A(K)$. If Γ is Zariski-dense in A and the configuration-closure of the image of S in the real vector space $V = \Gamma \otimes \mathbb{R}$ is all of V, then S is Zariski-dense in A.

The equivalence between Theorems 4.1 and 4.2 is proved in exactly the same way as the equivalence between Theorems 3.1 and 3.2 were proved.

5. Models in the Sense of Algebraic Geometry

If you start with the utterly general fields that appear in the statement of Theorems 3.1, 3.2, 4.1, and 4.2, here is the standard way of cutting down to the study of reasonably small fields. Consider, for example, the data of any of these theorems, say Theorem 4.2: that is, we are given (k, K, A, S) with $k \subset K$, an inclusion of algebraically closed fields, A an abelian variety over K having the property that no positive dimensional factor abelian variety of A comes by base extension from an abelian variety over k, and $S \subset \Gamma \subset A(K)$, a subset S generating Γ, a finitely generated subgroup of $A(K)$. Fix elements $\{\gamma_1, \ldots, \gamma_\nu\}$ that generate Γ.

If, contrary to the conclusion of Theorem 4.2, we were dealing with a counter-example to the assertion of that theorem, there would be some hypersurface $D \subset A$ which contains S but which does not contain some element $\gamma \in \Gamma - S$. Call such a pair (D, γ) a *witness to the fact that A, S, Γ is a counter-example* to Theorem 4.2, or just a *witness* for short. Similarly we can talk about the notion of "counter-example witness" to Theorem 3.2.

Let $k_0 \subset k$ denote the prime field (i.e., it is $\mathbb{Q}$ if we are in characteristic 0 and $\mathbb{F}_p$ if we are in characteristic p). Consider a specific set of equations defining the abelian variety A over K in projective space; for concreteness we can take A as given as an intersection of a finite collection of quadrics in some high-dimensional projective space, following Mumford [1966; 1967]). These equations have, all in all, only a finite number of coefficients and so do all the coordinates (in projective space) of the finitely many points $\gamma_1, \ldots, \gamma_\nu$. If we are presented with a *witness (D, γ) to the fact that A, S, Γ is a counter-example*, we add to this set of coefficients of these equations the coefficients of the equations defining the hypersurface D. Letting $\mathcal{C}$ denote the set of all the coefficents enumerated above, we see that $\mathcal{C} \subset K$ is finite. It follows that the subfield $K_0 := k_0(\mathcal{C}) \subset K$ is finitely generated over the prime field k_0, and moreover we have $S \subset \Gamma \subset A(K_0)$ (and in the case where we have prescribed witness (D, γ) we can get that D is defined over K_0 as well). We have then, a *finitely generated model* for our putative witnesses. For example, if we had a witnessed counter-example to Theorem 3.2 over any algebraically closed field K of characteristic 0 (that is, if we had A, $\Gamma \subset A(K)$, $S \subset \Gamma$, $S \subset D \subset A$, and $\gamma \in \Gamma$ with $\gamma \notin D$, satisfying the hypotheses required; i.e., that A is an abelian variety, Γ is a finitely generated subgroup of its group of K-rational points, S is configuration-dense in $\Gamma \otimes \mathbb{R}$, and D a hypersurface in A) the above discussion shows that we would also have such a witnessed counter-example all of whose ingredients are given over a field K_0 which is finitely generated over $\mathbb{Q}$.

Corollary 5.1. *To prove Theorems* 3.1, 3.2, 4.1 *and* 4.2 *it suffices to treat the case of fields K which are the algebraic closure of fields of finite transcendence degree over the prime fields*; *i.e., where K ranges through the algebraic closures of $k_0(x_1, \ldots, x_d)$ (for $d = 1, 2, \ldots$) where the k_0's are the prime fields* (*so that in the case of Theorems* 3.1 *and* 3.2, $k_0 = \mathbb{Q}$), *and where in the case of Theorem* 4.1 *and* 4.2, *k is an algebraic closure of k_0.*

Producing a "complex" or a "p-adic" model. In the case where K is of characteristic 0 it is sometimes useful to make use of complex analytic, or p-adic analytic methods. Imagine that we have given ourselves a witnessed counter-example to Theorem 3.2, which by Corollary 5.1 can be taken to be over a field K_0 which is of finite transcendence degree over $\mathbb{Q}$. Since any field of finite transcendence degree over $\mathbb{Q}$ is isomorphic to a subfield of $\mathbb{C}$, we may choose an imbedding $K_0 \hookrightarrow \mathbb{C}$ and make the base change (for our model) from K_0 to $\mathbb{C}$ gives us a model over the complex numbers. Similarly, let p be any prime number, and noting the fact that as an "abstract" field, $\bar{\mathbb{Q}}_p$ is the extension of $\mathbb{Q}$ given by adjoining an uncountable number of independent variables and then passing to the algebraic closure of the field so obtained, we see that there exists an imbedding $K_0 \subset \bar{\mathbb{Q}}_p$. Identify, then, K_0 with a subfield of $\bar{\mathbb{Q}}_p$ and form the compositum $E = \mathbb{Q}_p \cdot K_0 \subset \bar{\mathbb{Q}}_p$. Since K_0 is a finitely generated field extension of $\mathbb{Q}$, E is a finitely generated, algebraic, field extension of $\bar{\mathbb{Q}}_p$. It follows that $E/\bar{\mathbb{Q}}_p$ is of finite degree. As before, if we make the base change (for our model) from K_0 to E we get our sought-for model over E.

Corollary 5.2. *Let p be any fixed prime number. In proving Theorems* 3.1 *and* 3.2 *by reductio ad absurdum it suffices to assume that there is a counter-example defined over a finite field extension E of $\mathbb{Q}_p$, and then prove its nonexistence.*

Call such a putative counter-example a "p-adic model" of a counter-example. If we wish, we may make a further finite extension of our base field E so that the abelian variety A of our p-adic model over E has semi-stable reduction over the ring of integers of E (by a theorem of Grothendieck [1972]). As we shall see below, if we are willing to exclude a finite set of ("bad") primes p we may find finite field extensions $E/\mathbb{Q}_p$ and models over E for which the abelian variety A has good reduction over the ring of integers of E.

If our original putative counter-example is over a field of characteristic p, a similar argument as we have just given will allow us to produce a counter-example over the field $\mathbb{F}_q((t))$ of (finite-tailed) Laurent series with coefficients in a finite field $\mathbb{F}_q$ of cardinality $q =$ a power of p.

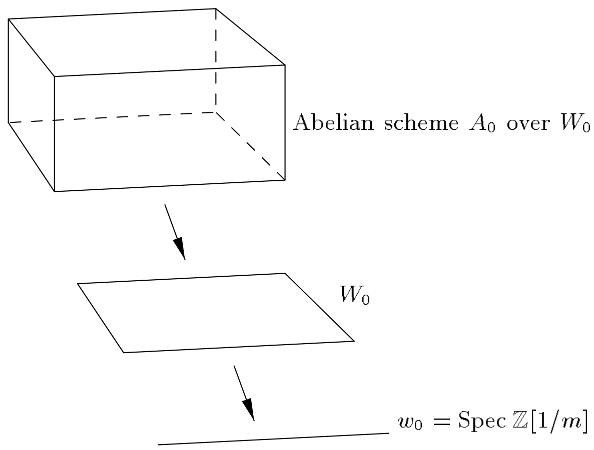
In our search for models, we needn't work only over fields: we can find subrings $r_0 \subset k_0$ and $r_0 \subset R_0 \subset K_0$ such that r_0 and R_0 are finitely generated rings over $\mathbb{Z}$, where $r_0 = \mathbb{F}_p$ if K_0 is of characteristic p, and $r_0 = \mathbb{Z}[1/m]$ for some positive integer m if K_0 is of characteristic zero, and such that

- the equations for A give us an abelian scheme, call it $\mathcal{A}_0$, such that
- the elements $\gamma_1, \ldots, \gamma_\nu$ are R_0-valued points of the R_0-scheme $\mathcal{A}_0$, and
- given a specific witness (D, γ), if it exists, the hypersurface D can be taken to be a relative Cartier divisor over R_0, and denoting $W_0 := \mathrm{Spec}\ R_0$ which is a scheme over $w_0 := \mathrm{Spec}\ r_0$ we may also arrange it so that (for fun)
- $W_0 \to w_0$ is a smooth surjective morphism of schemes.

We therefore have an abelian scheme

$$\mathcal{A}_0 \to W_0,$$

over the smooth w_0-scheme W_0 with the structures described above. The "picture", when $r_0 = \mathbb{Z}[1/m]$, looks as follows:



At this point, here are some things we can do:

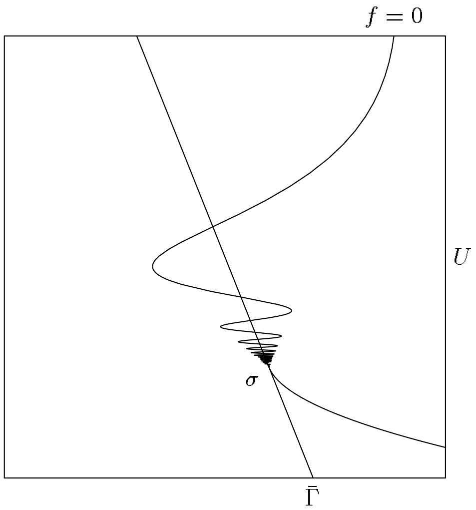
Producing a "p-adic model" with good reduction. If we are given a putative counter-example model over a field of characteristic 0, then $r_0 = \mathbb{Z}[1/m]$ for some nonzero integer m, and we have our geometric model $\mathcal{A}_0 \to W_0$, as above with witness (D, γ), where the hypersurface D is a relative Cartier divisor over R_0, and R_0 is as described above over $\mathbb{Z}[1/m]$, the mapping of schemes π : $\mathrm{Spec}(R_0) \to \mathbb{Z}[1/m]$ being smooth and surjective. After possibly "augmenting m", i.e., making the base change $\mathrm{Spec}(R_0) \mapsto \mathrm{Spec}(R_0 \otimes_{\mathbb{Z}[1/m]} \mathbb{Z}[1/m'])$ for m' a suitable nonzero multiple of m (we assume this done without changing our notation) we can prepare the ring R_0 via "Noether normalization" [Bourbaki 1972, Chapter V, Section 4, Corollary 1] so that R_0 contains a polynomial ring in a finite number of variables $\mathbb{Z}[1/m][x_1,\dots,x_\nu] \subset R_0$, and such that R_0 is an integral extension of $\mathbb{Z}[1/m][x_1,\dots,x_\nu]$. Let p be any prime number not dividing m. We choose any ring homomorphism $\xi : \mathbb{Z}[1/m][x_1,\dots,x_\nu] \to \mathbb{F}_p$ (and, of course, there are some). Now find any system of ν elements $\alpha_j \in \mathbb{Z}_p$ (for $j = 1,\dots,\nu$) which are transcendentally independent over $\mathbb{Z}$ taking care to choose them so that

$$\xi(x_j) = \alpha_j \mod p,$$

for $j = 1,\dots,\nu$ (and we can do this). We use this system of α_j's to imbed $\mathbb{Z}[1/m][x_1,\dots,x_\nu]$ in $\mathbb{Z}_p$. Call the imbedding $\alpha : \mathbb{Z}[1/m][x_1,\dots,x_\nu] \hookrightarrow \mathbb{Z}_p$. Since R_0 is an integral domain, and finitely generated and integral (hence finite) over $\mathbb{Z}[1/m][x_1,\dots,x_\nu]$, we may extend α to an imbedding of R_0 into a finite discrete valuation ring extension $\mathcal{O}$, of $\mathbb{Z}_p$. Making the base change from R_0 to $\mathcal{O}$, we transfer our model over R_0 to $\mathcal{O}$, and letting $E/\mathbb{Q}_p$ be the field of fractions of $\mathcal{O}$, we have constructed a "p-adic model" (over the discrete valued field E) for which the abelian variety $\mathcal{A}_0$ has good reduction over the residue field of E.

6. Proof of Absolute Mordell–Lang in Characteristic 0 for Curves of Rank 1

The beautiful classical argument of Chabauty may be adapted to give an elementary proof of Theorem 3.2 when the rank of Γ is equal to 1. Here is the briefest sketch of this argument; for a more general statement, see [Hindry 1988, Section 7]. If $\operatorname{rank}(\Gamma) = 1$ the condition that the configuration-closure of the image of S in the real vector space $V = \Gamma \otimes \mathbb{R}$ is *all* of V boils down to simply saying that S is infinite. Since Theorem 3.2 would be immediate if A were of dimension ≤ 1 we may assume that the dimension of A is 2 or more. Suppose that Theorem 3.2 were false. Then there would be a hypersurface D in A containing S but not containing Γ. Now there is a "p-adic model" for this. That is, there is a prime number p and a finite extension K of $\mathbb{Q}_p$ admitting the same situation, i.e., an abelian variety A over K and a hypersurface $D \subset A$ over K, and $S \subset \Gamma \subset A(K)$ all with the same properties as before (e.g., $S \subset D(K)$). The existence of such "p-adic models" is explained in Corollary 5.2 above. Choose such a p-adic model, and note that $A(K)$ has the structure of (compact) p-adic Lie group. Let $\bar\Gamma$ denote the topological closure of $\Gamma \subset A(K)$, and note that (by the basic theory of the p-adic logarithm, which identifies an open neighborhood of A(K) with an open "additive" subgroup in $K^{\dim A}$) we may identify $\bar\Gamma$ with a one-dimensional p-adic Lie subgroup of the p-adic Lie group $A(K)$. Since Γ is Zariski-dense in A, it follows that any open(-closed) subgroup of finite index in $\bar\Gamma$ is also Zariski-dense in A. Since S is infinite, it has at least one limit point in the p-adic Lie group $A(K)$ and all of the limit points of S lie in $D(K) \cap \bar\Gamma \subset A(K)$.



Fix a limit point σ of S. Consider a local equation f for D in a neighborhood $\mathfrak{U} \subset A(K)$ of σ. We view f as a p-adic analytic function on $\mathfrak{U} \subset A(K)$, and restricting it to the intersection of $\mathfrak{U}$ with the p-adic Lie subgroup $\bar{\Gamma}$, we see that f must *vanish* on some open neighborhood of σ in $\mathfrak{U} \cap \bar{\Gamma}$ because it vanishes on the infinite set of points of $\mathfrak{U} \cap S$ which has σ as accumulation point. It follows that $\bar{\Gamma}$ is contained in a finite union of translates of D, contradicting the Zariski-density of Γ.

About effectivity. Before we can discuss the status of *effectivity* of the above proof, we have to reformulate the theorem that it proves in a manner so that the result is *providing* something for us, and only then can we unambiguously ask whether or not our proof is providing that thing "effectively". One natural reformulation is to say that given the hypotheses of Theorem 3.2, in the case where V is of dimension one, and given any hypersurface $D \subset A$, there are only a finite number of points in the intersection $D \cap \Gamma$. Now even in this restricted setting– the above Chabauty-type argument seems to be giving us the same level of effectivity (and no more) that the general results of Vojta and Faltings provide. Namely, we can refine the above argument to give a *number-effective* statement, but not, it would seem, a *size-effective* one. But see [Coleman 1985a]!

7. The Reduction of the Proof of Mordell–Lang to Global Fields

It seems that there are (at least) two possible strategies for performing this reduction step. One approach is suggested in a few words in [Faltings 1994, Section 5]. We shall, up to a certain point, give a detailed account of how to follow this out. I say "up to a certain point" because, as the reader will see, we will be asserting that a demonstration given explicitly in the literature for number fields, also works for global fields. I am thankful to Paul Vojta for substantial help here. The alternative strategy requires, among other things, an excursus about the specialization properties of maximal abelian subvarieties in subvarieties in abelian varieties (sic). All the essential materials needed for this second strategy can be found in the literature. (See [Lang 1983, Chapter 9], especially Theorem 6.2 and Corollary 6.3, together with either [Hindry 1988, Appendice I] and, in particular, Lemma A there, or alternatively [Abramovich 1994; McQuillan 1994]. For further background I found [Raynaud 1983a, Section III] helpful.) We end this section with a sketch of how this second approach works, spelling out some (minor) aspects of the proof not found in the sources. I am thankful to Michael McQuillan and Dan Abramovich for substantial help with this.

First Approach. Recall that, as in Corollary 5.1 above, we know that to prove Theorems 3.1 and 3.2 it suffices to work over base fields K which are finitely generated over $\mathbb{Q}$. "Filtering" such a K by a sequence of subfields, each of transcendence degree 1 over the next, we shall see that it suffices to prove appropriately stated versions of Theorems 3.1 and 3.2:

(a) for number fields F, and
(b) for fields F of rational functions on curves C over base fields F_0, given that we have already proved (appropriately stated versions of) Theorems 3.1 and 3.2 for F_0.

In either case, our field F will therefore have the structure of a *global field* and therefore it may pay to recall this fundamental notion of *global field structure* (which was given center stage in Artin–Whaples' treatment of algebraic number theory and has kept that position in every subsequent treatment of the subject). We will be referring specifically to the discussion in [Lang 1983, Chapter 2, Section 1]. Recall that a valuation v on a field F of characteristic 0 is called *proper* if it is nontrivial, and its restriction to $\mathbb{Q}$ is either trivial, the negative logarithm of ordinary absolute value or a p-adic valuation for some prime number p. (If F is of positive characteristic, for the valuation v to be *proper*, it is also required to be "well-behaved", a condition automatically satisfied in characteristic 0; see [Lang 1983].) By a *global field structure* on a field F we mean a collection of "proper" valuations M_F of F, and of multiplicities $v \mapsto \lambda_v > 0$ for $v \in M_F$ such that for every $x \in F^*$, we have the *summation formula with the multiplicities* $v \mapsto \lambda_v$,

$$\sum_{v \in M_F} \lambda_v \cdot v(x) = 0.$$

Here we have written things additively, rather than multiplicatively as was done in [Lang 1983]. Our valuations $v : F^* \to \mathbb{R}$ are related to the absolute values $|\ \ |_v$ there by the standard formula $v(x) = -\log |x|_v$.

Recall that if F is endowed with a global field structure, and E/F is a finite field extension, there is a unique global field structure on E extending that on F (for details, again consult loc. cit.).

The examples of global field structures are precisely the fields that enter into the cases (a) and (b) above. More precisely, if F is a number field, we take M_F to be the standard collection of normalized absolute values corresponding to the archimedean and non-archimedean places of F and λ_v the collection of multiplicities as set out in [Lang 1983, Chapter 2, Section 1]. If F is expressed as a field of rational functions of a proper smooth curve C defined over a subfield $F_0 \subset F$, i.e., if we can write $F = F_0(C)$, we may impose a global field structure on K, again in the standard manner (see loc. cit.). Note that if F is of transcendence degree at least 2 over the prime field, there are *many* possible global field structures we can impose on F. Nevertheless (when no confusion can arise) a field F endowed with a specific global field structure will be called a "global field" and we will denote it (along with its global field structure) simply F.

Given a global field F, with $\bar{F}/F$ an algebraic closure, it is standard (see [Lang 1983, Chapters 3–5] for the details) to define the associated *height function* on $\bar{F}$-rational points of N-dimensional projective space (any N)

$$h_F : \mathbb{P}^N(\bar{F}) \to \mathbb{R}.$$

If $\iota : X \hookrightarrow \mathbb{P}^N$ is a projective variety defined over F, we can restrict the above height function to $\bar{F}$-rational points of X and divide by the degree of X to obtain a height function "on X" which we denote

$$h_{F,X,\iota} : X(\bar{F}) \to \mathbb{R};$$

that is, $h_{F,X,\iota}(P) = \frac{1}{d} \cdot h_F(\iota(P))$ where $d =$ the degree of the projective variety $\iota(F) \subset \mathbb{P}^N$. [In the special case where $\mathrm{Pic}(F) = \mathbb{Z}$ (e.g., if $F = C$ is a smooth projective curve), dividing by d has the effect of making the real-valued function $h_{F,X,\iota}$ on the set $F(\bar{K})$ independent (modulo bounded functions) of the projective imbedding ι; i.e., it depends only on the global field F and X (mod $O(1)$).]

Given a global field F, an abelian variety A defined over K, and a line bundle L over A defined over F, we also have the normalized Néron–Tate height function on $\bar{F}$-rational points of A [Lang 1983, Chapter 5, Section 3]

$$\hat{h}_{F,A,L} : A(\bar{K}) \to \mathbb{R}.$$

All of these height functions depend upon the global field structure of F.

The Ueno–Kawamata structures on subvarieties of abelian varieties. For a compendium of the results we are about to cite, and for complete references to the literature containing their proofs, see [Lang 1991, Chapter I, Section 6]. Recall *Ueno's Theorem* (loc. cit.), which says if $X \subset A$ is an irreducible subvariety of an abelian variety (over a field) and if $B \subset A$ is the connected component of the subgroup of translations of A that preserve X, then if $Y := X/B \subset A/B$

is the quotient variety of X under the action of the group B we have that Y is a variety of general type (or, in the terminology of [Lang 1991], is *pseudo-canonical*). Refer to the morphism $X \to Y$ as the *Ueno fibration* of X. Recall *Kawamata's Structure Theorem* that says if $X \subset A$ is an irreducible subvariety of general type (alias: pseudo-canonical) and the base field is of characteristic 0, then there is a finite number of proper subvarieties $Z_i \subset X$ $(i = 1, \ldots, \nu)$ whose Ueno fibrations $Z_i \to Y_i$ have positive fiber dimension, and such that any finite union of translates of nontrivial abelian subvarieties in X is actually contained in

$$Z := \bigcup_{i=1}^{\nu} Z_i \subset X.$$

Refer to $Z \subset X$ as the *Kawamata locus* of X.

THEOREM 7.1 (FALTINGS, EXTENDING THE METHOD OF VOJTA). *Let F be a global field of characteristic* 0, *A an abelian variety over F, and L any line bundle over A* (*defined over F*). *Let $X \subset A$ be an irreducible subvariety of general type* (*defined over F*) *and $Z \subset X$ its Kawamata locus. Then the set of F-rational points on X not lying in Z has bounded* (*Néron–Tate*) *height. That is, there is a bound B such that*

$$\hat{h}_{F,A,L}(x) \leq B$$

for all $x \in X(F) - Z(F)$.

For a proof of this the reader might consult [Faltings 1994]; in that reference Faltings only states the theorem for F a number field, but his proof goes through, word-for-word, for global fields of characteristic 0. Vojta [1993] has given another account of the proof of this same result (also stated only for number fields, but he assures me that his account of the proof works as well, with no change, in the context of global fields).

We shall now sketch the proof of why the preceding theorem implies Theorem 3.1 (and therefore also Theorem 3.2). Firstly, let A be an abelian variety over F and $S \subset A(F)$ as hypothesized in Theorem 3.1, where F is a field which is finitely generated over $\mathbb{Q}$. We shall prove Theorem 3.1, in effect, by induction on the transcendence degree of F, and the dimension of A. We view F with a given global field structure, coming either because F is a number field, or else by writing $F = F_0(C)$ where C is a curve. In the latter case, we may assume by induction that Theorem 3.1 holds for abelian varieties A_0 over the field F_0, and for subsets $S_0 \subset A_0(F_0)$ generating finitely generated subgroups. We may also assume that Theorem 3.1 holds for abelian varieties over F of strictly lower dimension than the dimension of A. Consider $\mathrm{Zar}_A(S)$, the Zariski-closure of the set S.

Step 1. Reduction to the case where $\mathrm{Zar}_A(S)$ is an irreducible variety of general type. Writing $\mathrm{Zar}_A(S) = \bigcup_j X_j$ as a finite union of irreducible varieties, we may cover the set S by a finite union of subsets $S_j = S \cap X_j(F)$ and note that it suffices to prove Theorem 3.1 for each of these S_j's separately. We may assume, therefore, that $X := \mathrm{Zar}_A(S)$ is irreducible, and then, using Ueno's Theorem, we pass to an appropriate quotient abelian variety of A in which X is of general type. Here, note that in this reduction step we have possibly reduced (but we have not increased) the dimension of A.

Step 2. Applying the theorem. We apply the above theorem for a choice of ample line bundle L over A to get that there is a bound B such that S breaks up into the union of two sets, the part of S contained in Z, $S_1 := S \cap Z(F)$ and the part of small height,

$$S_2 := \{x \in S \mid \hat{h}_{F,A,L}(x) \leq B\}.$$

Now we can apply Step 1 again to S_1, and note that here the application of Step 1 is guaranteed to *reduce* the dimension of the ambient abelian variety A; our inductive hypothesis therefore proves Theorem 3.1 for S_1. We may assume, then, that

$$S = S_2 = \{x \in S \mid \hat{h}_{F,A,L}(x) \leq B\}.$$

If F is a number field, it follows that S is finite, and so again Theorem 3.1 follows. We have reduced ourselves, therefore, to the case where $F = F_0(C)$ and S is of bounded height. After applying Step 1 again, we may assume that $X := \mathrm{Zar}_A(S)$ is irreducible (of general type). Let $B \subset A$ be the F/F_0-trace (that is, the "largest abelian subvariety of A defined over F_0"; [Lang 1991, Chapter I, Section 4]) and apply [Lang 1983, Chapter 6, Theorem 5.3] to guarantee that S is contained in a finite union of cosets of $B(F_0)$ (in $A(F)$). In particular, S is contained in a configuration in A consisting of a finite union of translates of B. We may assume that one of these translates covers X, and translating back to the origin, we may simply assume now that $S \subset B(F_0)$. Since, by our inductive hypothesis, Theorem 3.1 holds for the abelian variety B over the field F_0 and for $S \subset B(F_0)$, we are done.

Remarks regarding the second method. Here, as mentioned above, the relevant literature is [Lang 1983, Chapter 9] and either [Hindry 1988, Appendice I] or [Abramovich 1994; McQuillan 1994]. To be brief, we start by performing Step 1 of the first method, and therefore we have reduced ourselves to proving, simply, that if A is an abelian variety over $\mathcal{K}$ a field finitely generated over $\mathbb{Q}$, $\Gamma \subset A(\mathcal{K})$ is a finitely generated group, and $X \subset A$ is an irreducible subvariety of general type, then $X \cap \Gamma$ is not Zariski dense in X. For short, refer to the triple (A, Γ, X) as ξ. Also, by induction we assume that this has already been proved for all triples $\xi' = (A', \Gamma', X')$ satisfying the same properties (meaning that A' an abelian variety over $\mathcal{K}$, Γ' a finitely generated subgroup of the Mordell–Weil group of

A' ,and $X' \subset A'$ a subvariety of general type) where X' is of dimension strictly smaller than X and $\dim(A') \leq \dim(A)$, or A' is of dimension strictly smaller than A. Moreover, we assume Mordell–Lang for number fields!

Also, we will be availing ourselves of a *good* model of ξ over R, an integral domain and regular, finitely generated over $\mathbb{Q}$, whose field of fractions is $\mathcal{K}$. We will be explaining what we mean by *good* presently. But we begin with a model of the following form: an abelian scheme A over R, and $\Gamma \subset A(R)$ a finitely generated subgroup of the Mordell–Weil group over R, and $X \subset A$ a closed irreducible subvariety flat over R, such that the fiber of the triple $\xi_R = (A, \Gamma, X)$ over $\mathcal{K}$ is ξ.

Ueno–Kawamata collections. For any homomorphism $R \to K$, with K a field, let $\xi_{/K} = (A_{/K}, \Gamma_{/K}, X_{/K})$ denote the "fiber" of our triple over K. We may apply the Ueno–Kawamata theory to the subvariety $X_{/K} \subset A_{/K}$ obtaining the Kawamata locus $Z(X_{/K}, A_{/K}) \subset X_{/K}$ and after possible finite field extension of K we may write $Z(X_{/K}, A_{/K}) = \bigcup_{j=1}^{\nu} Z_j(X_{/K}, A_{/K})$ with Z_j geometrically irreducible, and having Ueno fibrations denoted $\eta_j : Z_j \to X_j$ where the mapping η_j is induced from the natural projection $A \to A/A_j$ for A_j some abelian variety of positive dimension, and such that $X_j \subset A/A_j$ is of general type. We define inductively, a (finite) set $\mathcal{C}(A_{/K}, X_{/K})$ of pairs (B, Y), such a pair consisting of an abelian subvariety of positive dimension $B \subset A$ and an irreducible subvariety of general type $Y \subset A/B$. For this purpose we note that single points are deemed to be of general type (and, of course, the Kawamata locus of a point is empty). An inductive definition of the *Ueno–Kawamata Collection* can be culled from the following axioms:

0. If $A_{/K} = 0$ then $\mathcal{C}(A_{/K}, X_{/K})$ is empty.
1. If the Kawamata locus of $X_{/K}$ is empty, and $A_{/K}$ is of positive dimension, then $\mathcal{C}(A_{/K}, X_{/K})$ consists of the single element $(A_{/K}, X_{/K})$.
2. Keep the notation as in the preceding paragraph, but drop the subscript $/K$ in some of the terms to make it easier to read. For each index $1 \leq j \leq \nu$ and each pair $(B', Y') \in \mathcal{C}(A/A_j, X_j)$ form the pair (B, Y) where $B \subset A$ is the abelian subvariety which is the inverse image in A of $B' \subset A/A_j$ under the homomorphism $A \to A/A_j$, and where $Y \subset A/B$ is the subvariety $Y' \subset (A/A_j)/B' = A/B$. The elements of the set $\mathcal{C}(A_{/K}, X_{/K})$ are precise all these pairs (B, Y) together with the pair $(A_{/K}, X_{/K})$.

We have a partial ordering on Ueno–Kawamata collections where, by definition, $(B', Y') \leq (B, Y)$ if $B \subset B'$ and the inverse image of Y' in A/B is contained in Y.

"Good" Models. Consider $\xi_R = (A, \Gamma, X)$, as described above, and the Ueno–Kawamata collections $\mathcal{C}(A_{/K}, X_{/K})$ for each homomorphism $R \to K$ where K is a field. We say that our model is *good* if there is a collection which we can denote $\mathcal{C}(A_{/R}, X_{/R})$ of pairs $(B_{/R}, Y_{/R})$ where $B_{/R} \subset A_{/R}$ is an abelian subscheme over

R, and $Y_{/R} \subset (A/B)_{/R}$ is a closed subscheme, flat over R, with the following property:

"The Ueno–Kawamata collection specializes well": *For each $R \to K$, with K a field, the "fiber" of the collection of pairs $\mathcal{C}(A_{/R}, X_{/R})$ over K (i.e., the set of pairs $(B_{/K}, Y_{/K})$ obtained by restriction to the fiber over K of the pairs $(B_{/R}, Y_{/R})$ in $\mathcal{C}(A_{/R}, X_{/R})$) is the Ueno–Kawamata collection $\mathcal{C}(A_{/K}, X_{/K})$.*

Here one has a choice. One may use the results in [Hindry 1988, Appendice 1] (in particular, Lemme A), which guarantee that the generic Kawamata locus specializes to the special Kawamata locus outside a proper closed subscheme in $\mathrm{Spec}(R)$. An alternative argument for this using a method of Abramovich [1994] is found in [McQuillan 1994], where (loc. cit., Theorem 1.2) it is shown that there is a closed subscheme of $X_{/R}$ whose fibers (over R) are the Kawamata loci of the fibers of $X_{/R}$.

An inductive application of these results to the successive tiers in the Ueno–Kawamata collection allows us to conclude that we have such a "good model".

Specialization of Γ. Fix a "good model" as described above.

Lemma 1. *There exists a closed point u of $\mathrm{Spec}(R)$ such that for every pair $(B_{/R}, Y_{/R}) \in \mathcal{C}(A_{/R}, X_{/R})$ the specialization mappings of $\Gamma/\Gamma \cap B \subset (A/B)(R)$ to $(A/B)(k(u))$ is injective; i.e., the specialization mapping allows us to identify $\Gamma/\Gamma \cap B$ with a subgroup $\Gamma/\Gamma \cap B \hookrightarrow (A/B)(k(u))$. Here $k(u)$ is the residue field of the point $u \in \mathrm{Spec}(R)$.*

Proof. Apply [Lang 1974, Chapter 9, Theorem 6.2] to the finitely generated subgroup $\prod \Gamma/\Gamma \cap B$ of the Mordell–Weil group of the abelian scheme $\prod A/B$, where the products range over all abelian subschemes B occurring in pairs in the Ueno–Kawamata collection $\mathcal{C}(A_{/R}, X_{/R})$. $\square$

Specializing points in the complement of the Kawamata locus. Fix such a closed point $u \in \mathrm{Spec}(R)$ with the properties guaranteed by Lemma 1. Let S denote the set of points in $\Gamma_{/\mathcal{K}} \cap X_{/\mathcal{K}}$ which lie outside the Kawamata locus $Z(X_{/\mathcal{K}}, A_{/\mathcal{K}})$. (We are eventually aiming to prove that S is finite.) For a pair $(B, Y) \in \mathcal{C}(A_{/R}, X_{/R})$ consider the subsets $S(B, Y) \subset S$ consisting of those elements of S, which when projected to A/B and specialized to $k(u)$ land in $Y(k(u))$. Clearly,

$$S = \bigcup S(B, Y),$$

where the union is taken over all pairs in the Ueno–Kawamata collection. Given a point $y \in Y(k(u))$ let $S(B, Y, y) \subset S$ consist of the elements of $S(B, Y)$, which when projected to A/B and specialized to $k(u)$ map to y.

Lemma 2. *The sets $S(B, Y, y)$ are finite.*

Proof. By the definition of $S(B, Y, y)$ and the injectivity guaranteed by Lemma 1, we have that the projection of the subset $S(B, Y, y) \subset \Gamma$ to $\Gamma/\Gamma \cap B$ consists of a single point, and therefore $S(B,Y,y)$ lies in a coset of the abelian subvariety B. Since B is of strictly smaller dimension than A, our inductive hypothesis guarantees that the Zariski-closure of $S(B,Y,y)$ is a union of translates of abelian subvarieties of B, and since $S(B,Y,y)$ is external to the Kawamata locus of X this union must only contain abelian varieties of dimension zero, i.e., $S(B,Y,y)$ is finite. □

Conclusion of the proof by downwards induction on the dimension of the Y's.

PROPOSITION. *The set S is finite.*

PROOF. Consider the following statement:

$\mathbf{P}(N)$**:** The union of the subsets $S(B,Y)$ where (B,Y) range through all pairs in the Ueno–Kawamata locus such that $\dim(Y) \leq N$ has *finite* complement in S.

By Lemma 2, we see that $\mathbf{P}(0)$ implies that S is finite. Clearly $\mathbf{P}(d)$ is true for $d = \dim(X)$. For $N > 0$, suppose $\mathbf{P}(N)$ and we shall show $\mathbf{P}(N-1)$. There are only finitely many pairs (B,Y) in the Ueno–Kawamata locus with $\dim(Y) = N$. Fix one such pair (B,Y). Now invoke Mordell–Lang for $Y_{/k(u)}$: there are only a finite number of $k(u)$-rational points of $Y_{/k(u)}$ which lie outside its Kawamata locus. By Lemma 2, only a finite subset of S which specializes to this finite set of points. Excluding those points, the remainder specialize to the Kawamata locus of Y, and therefore the complement of a finite subset of S lies in the union of the subsets $S(B',Y')$ where (B',Y') range through all pairs in the Ueno–Kawamata locus such that $\dim(Y') \leq N-1$, giving our proposition. □

8. Number-Effectivity Revisited

In the introduction we asked briefly about possible levels of effectivity we might hope for in connection with the finiteness of numbers of solutions of a certain class of Diophantine equations (we chose *hyper-elliptic curves* as our illustrative class). In this section I would like to discuss number-effectivity questions in the general context of curves of genus at least 2 but with some attention paid to uniformity over families of curves and the base fields. As Teresa de Diego [1997] has explained, the methods of Vojta, Faltings, and Bombieri produce number-effective upper bounds of a very precise nature. Let X be a smooth projective curve of genus > 0 defined over $\bar{\mathbb{Q}}$, and consider the "canonical" mapping of X to its jacobian (abelian) variety A,

$$j : X \hookrightarrow A$$

by the rule $x \mapsto$ the linear equivalence class of the divisor $(2g-2)[x] - \kappa$, where κ is (a choice of) canonical divisor for the curve X. Now consider the Néron–Tate height function $\hat{h} : A(\bar{\mathbb{Q}}) \to \mathbb{R}$, where $\hat{h} := \hat{h}_{\bar{\mathbb{Q}},A,\mathcal{L}}$ in the notation that we introduced earlier, where $\mathcal{L}$ is the "Poincaré" line bundle on the jacobian

$A = \mathrm{jac}(X)$. Composing the $\hat{h} : A(\bar{\mathbb{Q}}) \to \mathbb{R}$ with $j : X(\bar{\mathbb{Q}}) \to A(\bar{\mathbb{Q}})$, gives us the height function on algebraic points of X, which we will continue to refer to as "Néron–Tate height":

$$\hat{h}(X, P) := \hat{h}(j(P)) \in \mathbb{R}.$$

Our height function $\hat{h}(X, P)$ is "canonical", in the sense that it depends upon nothing more than X and P. For our purposes below we need to make a further *normalization* of this height function, scaling it relative to the "height" of the curve X itself. At present, we must do this in a slightly ad hoc way. Eventually we might want a thoroughly *canonical* normalization (e.g., perhaps the correct thing to do is to replace $\hat{h}(X, P)$ by the ratio

$$\frac{\hat{h}(X, P)}{\sqrt{\omega_X^2}},$$

where ω_X is the canonical divisor attached to a semi-stable model of the curve X over the ring of integers in an appropriate number field, as given in Arakelov's Theory; see [Lang 1988]). But a perfectly serviceable, and more down-to-earth if less canonical, normalization, which works uniformly for families of curves parametrized by specific quasi-projective varieties, is as follows. Let $\iota : T \hookrightarrow \mathbb{P}^N$ be an irreducible quasi-projective variety which is a parameter space for a family of curves of genus > 1. That is, we give ourselves a smooth proper family $\mathcal{X} \to T$ with fibers $\mathcal{X}_t$ equal to (smooth proper) curves of genus > 1 as t ranges through the $\bar{\mathbb{Q}}$-rational points of T. Now, for an algebraic point $P \in \mathcal{X}_t(\bar{\mathbb{Q}})$, define

$$h(\mathcal{X}_t, P) := \frac{\hat{h}(\mathcal{X}_t, P)}{1 + \sqrt{h_{\bar{\mathbb{Q}}, T, \iota}}}.$$

For example, we can take the hyper-elliptic family $y^2 = g(x)$ discussed in the introduction, where $g(x) = \sum_{j=0}^{d} g_j x^j$ is a polynomial of degree d (at least 5) with no multiple roots, and the variety $T = T_d$ is the open variety in affine $(d+1)$-space over $\mathbb{Q}$ (parametrized by $g_0, \ldots, g_d$) which is the complement of $g_d = 0$ and the discriminant locus.

But fix any quasi-projective variety $\iota : T \hookrightarrow \mathbb{P}^N$ over a number field $k \subset \bar{\mathbb{Q}}$ which parametrizes a family, $\mathcal{X} \to T$, of (smooth projective) curves of genus ≥ 2. For C a positive number, define a *C-small point of* $\mathcal{X}_t$ to be an algebraic point $P \in \mathcal{X}_t(\bar{\mathbb{Q}})$ such that

$$h(\mathcal{X}_t, P) \leq C,$$

and a *C-large point of* X to be one for which the reverse inequality holds, i.e.,

$$h(\mathcal{X}_t, P) > C.$$

For $\mathcal{X}_t$ be a curve in our family, and an intermediate field $k \subset L \subset \bar{\mathbb{Q}}$, partition the set, $\mathcal{X}_t(L)$, of L-rational points on $\mathcal{X}_t$ into the set of C-small and C-large points,

$$\mathcal{X}_t(L) = \mathcal{X}_t(L)_{\text{small}} \coprod \mathcal{X}_t(L)_{\text{large}}.$$

We visibly have a *size-estimate* on $\mathcal{X}_t(L)_{\text{small}}$ and, for appropriate choice of C, we seek a *number-estimate* on $\mathcal{X}_t(L)_{\text{large}}$.

Problem. Given our parametrized family $\mathcal{X} \to T \hookrightarrow \mathbb{P}^N$ over the number field k, as above, find three constants C, D and E for which an estimate of the following form holds. For *any* number field L containing k, and $t \in T(L)$,

$$\left|\mathcal{X}_t(L)_{C-\text{large}}\right| \leq D \cdot E^{r(\mathcal{X}_t, L)},$$

where $r(\mathcal{X}_t, L)$ is the rank of the Mordell–Weil group of the jacobian of $\mathcal{X}_t$ over L.

As mentioned, de Diego [1997], who establishes uniform estimates, developing upon methods as given, for example, in [Bombieri 1990], shows that there exist constants C, D, and E that solve this problem. In fact, de Diego shows that (for any family as above) there is a choice of C for which:

$$|\mathcal{X}_t(L)_{C-\text{large}}| \leq \tfrac{55}{2} \cdot 7^{r(\mathcal{X}_t, L)},$$

for all number fields L containing k and points $t \in T(L)$. Of course, what is missing here is an explicit evaluation of a constant $C = C(\mathcal{X}, T, \iota)$ that does the above job.

In the recent work of A. Pacheco, there are similar estimates for curves over finite fields replacing number fields (but with further complications due to inseparability phenomena).

In view of a solution to the preceding problem, we might hope for an upper bound with better uniformity in the sense that it would involve only the *genus* of the curve and the quantity $r(X, L)$. We might ask for the following stronger assertion than the above (this being, at the same time, weaker than the question about uniform number-effectivity posed on page 201 (Question 3).

Question. If X is a (smooth projective) curve over a field L, let $g(L)$ denote its genus; let $X(L)$ denote the set of its L-rational points, and $r(X, L)$ the rank of the group of L-rational points of the jacobian of X. Is there a function $N(g, L, r)$ with the property that

$$|X(L)| \leq N\left(g(X),\ L,\ r(X, L)\right),$$

where L ranges through all number fields, and X ranges through all (smooth projective) curves over L of genus ≥ 2?

In the direction of achieving effective results in characteristic p, Buium and Voloch [1996] establish an affirmative answer to the question analogous to the question above in the case where X is a curve defined over a field K (of characteristic p) and the jacobian of X is *ordinary* and such that no non-trivial factor of the jacobian is definable over K^p. Specifically, they show that

$$|X(L)| \leq p^{r(X,K)} \cdot (3p)^g \cdot (8g-2)g!,$$

where $g = g(X) \geq 2$.

Returning to characteristic 0, Buium [1993; 1994] has shown:

PROPOSITION (BUIUM). *Let X be a closed (possibly singular) curve in an abelian variety A (over the complex numbers) and let Γ be a subgroup of $A(\mathbb{C})$ of rank r. Assume that X is of geometric genus $g \geq 2$ and its normalisation is not defined over the field of algebraic numbers. Set $N = \max\{g, r, 4\}$. Then the cardinality of $X(\mathbb{C}) \cap \Gamma \subset A(\mathbb{C})$ is at most $N(!)^{6N+6}$, where $(!)^m$ means factorial iterated m times.*

In e-mail correspondence, Buium noted to me the curious fact that the bound just quoted for characteristic zero is much worse than the one proved in [Buium and Voloch 1993] for characteristic p. In the higher-dimensional situation (but over number fields) one has the very interesting preprint [Hrushovski and Pillay 1998], which gives a related uniform upper bound. This work makes use of the powerful methods explained in this conference; their context is as follows. One is given a subvariety $X \subset A$ in a (semi-) abelian variety both defined over a number field contained in $\mathbb{C}$ and one assumes that X contains no subvarieties of the form $X_1 + X_2 \subset A$, for X_1, X_2 positive dimensional varieties. One considers finitely generated subgroups $\Gamma \subset A(\mathbb{C})$ and seeks an upper bound for the number of *nonalgebraic* points in $X \cap \Gamma$, i.e., for $|X \cap \Gamma - X(\bar{\mathbb{Q}})|$. Letting $r := \mathrm{rank}(\Gamma)$, the upper bound given in [Hrushovski and Pillay 1998] for $|X \cap \Gamma - X(\bar{\mathbb{Q}})|$ is doubly exponential in r. More specificially, the upper bound is of the form

$$a^{(br)^{cr}},$$

where a, b, c are explicit, and quite computable, functions of X and A.

The double exponential upper bound in this result of Hrushovski and Pillay raises the hope (as Buium mentioned to me in e-mail correspondence) that the iterated factorial upper bound obtained in the non-isotrival context (i.e., in the proposition of Buium quoted above) will eventually be significantly improved.

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