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# Overview

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Note: All bibliographic references are to articles in this volume.
See the individual articles for more references.

Even the most reserved among model theorists would no doubt agree that the subject has grown dramatically over the last thirty years. This period has produced a substantial and beautiful abstract theory as well as a range of remarkable applications that extend into several areas of mathematics and incorporate the most sophisticated theoretical developments in the field. During the last decade in particular, results obtained by model theorists have attracted the attention of researchers outside logic and have opened up broad avenues for interaction. The Model Theory of Fields program held at the Mathematical Sciences Research Institute from January to June 1998 sought to capitalize on the intense activity that has taken place in the discipline by bringing together for an extended period of time model theorists and mathematicians working in the areas of some of the most exciting applications.

Model theory's stock-in-trade is the analysis of the so-called *definable subsets* of a mathematical structure. The definable subsets of classical mathematical structures have long occupied a central position in algebraic and geometric investigations; the constructible sets in algebraic geometry and the semialgebraic sets in real geometry provide two notable examples. Although the model-theoretic viewpoint may have supplied these areas with some basic results, it did not offer enough until recently for practitioners to notice that the objects that they study with their own sophisticated methods could be illuminated by model theory.

For an area of mathematics to contribute significantly to another, the new point of view must add to the understanding of the objects in the area of application, and not, for example, merely provide convenient new terminology. In the case of the model theory of fields, the subject around which the articles in this volume are organized, the deep advances that have been made in abstract model

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theory over the last thirty or forty years have generated concepts and methods that have found meaning and application in classical mathematical contexts, ultimately leading to important new results in these areas.

This volume collects articles arising from lectures presented at the Introductory Workshop that inaugurated the MSRI model theory of fields program (January 1998). Section 1 of this overview introduces each article, providing a sense of our conception of the workshop minicourses. Sections 2 and 3 are intended to give a reader unfamiliar with Model Theory a gentle conceptual introduction to its main themes. We hope that these two modest aims shall whet the reader's appetite to explore the excellent contributions that follow, for which we herewith express our gratitude to the authors.

We also thank Silvio Levy for compiling the contributions into a book, and the Director and Deputy Director of MSRI for their encouragement of this project.

## 1. The Organization of This Volume

The goal of this volume, like the workshop on which it is based, is to serve as a guide to current developments in model theory and its many geometric applications in the model theory of fields. It attempts to provide the reader with a unified introductory account of contemporary pure model theory, the model theory of fields, and the different aspects of geometry in which the model theory has found its most significant recent applications.

The articles in this volume are organized around three themes which roughly speaking comprise the minicourses on which the workshop was based: the model theory of fields, dimension theory, and geometry. The expression "model theory of fields" connotes the analysis of various classes of fields, including identifying elementary classes, finding axioms for these classes, and determining the definable sets in fields belonging to each class (relative quantifier elimination). This tradition goes back to Alfred Tarski and Abraham Robinson. The more recent developments are in many instances informed by concepts from pure model theory such as forking and orthogonality. "Dimension theory" deals with the conceptual apparatus of pure model theory and the associated body of results. The term "dimension theory" is employed because this apparatus typically involves the assignment of dimensions to definable sets in suitable structures. "Geometry" refers (somewhat imprecisely) to those areas of mathematics in which most of the new applications of model-theoretic methods and analyses have been made.

The diversity of material within each of these topics suggested the somewhat unconventional approach that each minicourse be divided among several speakers. The organization of this volume reflects that of the workshop in that the articles forming each of the three minicourses are grouped together. Within each group the order of the papers is roughly parallel. Although by no means perfect, this parallelism is intended to emphasize how the several components of each minicourse correlate. It also should help a reader interested in a particular aspect of the subject to locate relevant material more easily. Bearing these organizational themes in mind, we now briefly review the contents of the articles.

David Marker's first article, *Introduction to model theory*, serves as a preamble to the three minicourses. It provides an excellent survey of elementary model theory that prepares a non-logician well for what is to come. It also covers some of the classical model theory of the first classes of fields to be studied and understood from the model-theoretic point of view: the algebraically closed and real closed fields.

The next three articles survey the model theory of fields. They examine fields with additional structure that have been treated successfully by model theory, reflecting the range of model-theoretic phenomena encountered in the abstract theory and also including several of the algebraic objects that arise in the applications. Lou van den Dries' paper, *Classical model theory of fields*, deals with the fields of real and $p$-adic numbers, and expansions of these fields obtained by adding analytic structure. Fields enriched by adjoining a derivation are discussed in Marker's *Model theory of differential fields*. Some of the most important recent applications of model theory have been to Diophantine questions, and Marker briefly indicates how differential fields enter into Hrushovski's model-theoretic proof of the Mordell–Lang conjecture for function fields of characteristic zero (see Section 5 of his article). In *A Survey of the model theory of difference fields*, Zoé Chatzidakis focuses on fields to which a distinguished automorphism has been adjoined. Hrushovski also has applied model-theoretic results concerning difference fields to Diophantine questions, and in Section 4 of her article Chatzidakis outlines his proof of the Manin–Mumford conjecture, including the explicit bounds his argument yields.

The next two articles deal with "dimension theory," that is, pure model theory. Dugald Macpherson's article, *Notes on o-minimality and variations*, concentrates on the body of pure model-theoretic results dating from the early 1980's that place the theory of semialgebraic and subanalytic sets from real geometry into an abstract context. The article by Bradd Hart, *Stability and its variants*, provides an introduction to the vast body of work on stable and simple theories. This aspect of pure model theory has its origins in Morley's proof of the Łos conjecture in the 1960's and took flight as a full-fledged theory with the deep work of Shelah in the 1970's around the classification of first order theories according to whether or not their class of models has a "structure theory". Even though o-minimality and stability/simplicity have developed separately, the two theoretical frameworks share strong conceptual similarities that emerge upon comparing the two articles. We shall comment further upon this in Section 3 below. Moreover, several of the technical notions that play a significant role in the abstract theory have clear meaning and significance in several of model theory's most striking applications in other areas of mathematics. This theme emerges in several articles in the volume.

The final three contributions to the volume, drawn from the introductory workshop's geometry minicourse and written by experts from outside model theory, accomplish several goals: they supply the mathematical background for many of the most notable applications of the model theory of fields, offer an introductory discussion of several of these developments, and raise provocative questions and/or suggest directions in which model theory might play a future role.

The article by Edward Bierstone and Pierre Milman, *Subanalytic geometry*, provides an introductory account of semialgebraic and subanalytic subsets of $\mathbb{R}^n$ (or $\mathbb{C}^n$) that indicates how this theory relates to o-minimality (see also van den Dries' article). It further suggests classes between the semialgebraic and subanalytic sets, based on local behavior of analytic mappings, that are "tame" from an algebraic or analytic point of view in much the same spirit as Grothendieck's vision of "tame topology". Jan Denef's article, *Arithmetic and geometric applications of quantifier elimination for valued fields*, surveys how the rich interaction between model theory and $p$-adic and rigid analytic geometry has yielded important number-theoretic and geometric applications. He also introduces some of the most recent exciting developments along the intersection between $p$-adic and rigid analytic geometry on the one hand, and model theory on the other. In Section 4 of his article, Denef discusses new invariants for algebraic varieties in which motivic measure and integration takes the place of $p$-adic integration. Barry Mazur's article, *Abelian varieties and the Mordell–Lang Conjecture*, the concluding paper in the volume, focuses on the mathematics surrounding the Mordell–Lang conjecture (see also Marker's article, *Introduction to the model theory of differential fields*). The article situates the Mordell–Lang conjecture in historical perspective as a counterpart in higher-dimensions to the Mordell conjecture proved by Faltings. Section 6 includes a proof of the Mordell–Lang conjecture over number fields in the rank one case, using Chabauty's method of embedding the situation in the $p$-adics and the theory of $p$-adic Lie groups. In Section 7 Mazur discusses the reduction of the Mordell–Lang conjecture to the number field case, and in Section 8 he mentions several questions regarding effectivity issues in the number of solutions of Diophantine equations that are very much in the spirit of model theory and are intended to suggest further interplay between the two fields.

## 2. Structures and Definable Sets

Our thematic overview of the volume begins with a brief primer of the objects studied in model theory. (Marker's *Introduction to model theory* offers a thorough treatment.) For a model theorist, a mathematical structure $\mathcal{M}$ is a set $M$ equipped with a set of operations on $M$, a set of relations on $M$, and a set of distinguished elements of $M$. One example is provided by the natural numbers $\mathbb{N}$ with the usual operations of addition and multiplication and the constant element 0: the structure $(\mathbb{N}, +, \cdot, 0)$. The ordered field of real numbers with addition, multiplication, the distinguished elements 0 and 1, and the binary relation $<$, that is, the structure $(\mathbb{R}, +, \cdot, 0, 1, <)$, provides another. At first glance, these examples might seem to be nothing more than mild variants on the kinds of structures dealt with by algebraists.

The model-theoretic point of view is distinguished by the unified perspective it provides for mathematical structures viewed with respect to the degree of generality described above. Thus on some level, the structures in the last paragraph are, to a model theorist, to be treated no differently than structures as diverse as the ordered field of real numbers augmented by all (partial) functions $f\colon [0,1]^n \to \mathbb{R}$ where $f$ is the restriction to $[0,1]^n$ of a function analytic in a neighborhood of $[0,1]^n$, or the difference field $(\mathbb{C}(t), +, \cdot, \sigma)$ where $\sigma$ is the "shift operator" defined by

$$\sigma \restriction \mathbb{C} = \text{identity} \quad \text{and} \quad \sigma(t) = t + 1.$$

Many further examples of fields enriched by additional structure figure in the articles in this volume and several assume central roles.

What is it then that provides this unified perspective? It is not just the scope and generality of structures in the above sense that makes the model-theoretic point of view distinctive. Rather it is the objects that model theory attaches to these structures, and the tools and methods that model theory employs to analyze and understand these objects that sets the subject apart. Just as a complex analyst, for example, concentrates on holomorphic functions on the field of complex numbers, model theory focuses on a particular class of relations and functions, the so-called definable ones.

For a set to be definable means simply that it can be "defined" by a formula in first-order logic in the language of the structure. But what is meant by this? We shall be informal, referring the reader to Section 1 of Marker's *Introduction to model theory* for the complete details. To each structure $\mathcal{M}$, there corresponds a formal language $\mathcal{L}$ which includes an $n$-place function symbol $\widehat{f}$ for each $n$-place function $f$ in $\mathcal{M}$, an $n$-place relation symbol $\widehat{R}$ for each $n$-place relation $R$ in $\mathcal{M}$, and a constant symbol $\widehat{c}$ for each distinguished element in $\mathcal{M}$. Each symbol in $\mathcal{L}$ is interpreted in $\mathcal{M}$ by the corresponding function, relation, or distinguished element. The $\widehat{\ }$ is usually deleted when writing the symbols in $\mathcal{L}$ where no confusion can arise. Formulas in the language $\mathcal{L}$ are the meaningful finite strings of symbols built from the symbols of $\mathcal{L}$, $=$, variables, the logical connectives $\neg$, $\wedge$, $\vee$, and the quantifiers $\exists$ and $\forall$. Here, "meaningful" refers to nothing more than a carefully executed version of the usual kind of symbolic expression that mathematicians write daily.

For those encountering formulas for the first time, two key restrictions should kept in mind: formulas are *finite* in length and quantification is limited to individual elements of the structure.[1] For example, the disjunction $\bigvee_{n\in\mathbb{N}} x^n = e$, indexed by the natural numbers, which captures the torsion elements of a group $(G, \cdot, e)$, is not a first-order formula because the disjunction is infinite, and hence the expression is infinite in length. As another example, quantification over all ideals of a ring $(R, +, \cdot, 0, 1)$ is not permitted since ideals are not *elements* of the ring.

The variables in an $\mathcal{L}$-formula $\varphi$ are either bound to a quantifier in $\varphi$ or not, in which case they are called free. Then, roughly speaking, an $\mathcal{L}$-formula $\varphi$ whose free variables are $x_1, \dots, x_n$ describes the definable set $X \subset M^n$ in a structure $\mathcal{M} = (M, \dots)$ consisting of all $(a_1, \dots, a_n) \in M^n$ which, when substituted for $x_1, \dots, x_n$, make the formula true when the symbols in $\mathcal{L}$ are interpreted by the corresponding objects in $\mathcal{M}$ and the set-theoretic operations corresponding to the logical symbols $\neg$ (complement), $\wedge$ (intersection), $\vee$ (union), and $\exists$ (projection) are performed as dictated by the formula.

Definable sets also can be given a simple and succinct set-theoretic characterization. We will be somewhat glib here; the precise characterization is given in Proposition 1.3 of Marker's article (page 18). Let $\mathcal{M} = (M, \dots)$ be a structure. For each $n \geq 1$ let $D^n$ be the smallest collection of subsets of $M^n$ that contains $M^n$, all $n$-ary relations in $\mathcal{M}$, and the graphs of all functions $f\colon M^{n-1} \to M$ in $\mathcal{M}$, and that is closed under taking generalized diagonals, complements, unions, intersections, and projections (from sets in $D^m$ for $m > n$). A set $X \subseteq M^n$, where $n \geq 1$, is definable in $\mathcal{M}$ if $X \in D^n$. Although correct and relatively easy to state, this set-theoretic version of definability reveals neither why definable sets are so natural and useful, nor, for that matter, why this class of sets is so named.

The power the point of view of definability confers comes from the fact that many of the sets that arise in mathematics can be described by formulas in exactly the way that mathematicians ordinarily do so. To illustrate, suppose that $\mathcal{R} = (\mathbb{R}, +, \cdot, -, <, 0, 1, f)$ where $f\colon \mathbb{R} \to \mathbb{R}$. Then the set of points at which $f$ is continuous or differentiable is definable in $\mathcal{R}$, since the the usual definitions of continuity and differentiability can be formalized in first-order logic in the language $\mathcal{L} = \{+, \cdot, -, <, 0, 1, f\}$. Likewise, most properties of elementary real analysis and topology of definable sets and functions are readily seen to be definable. Examples from a wide variety of contexts both of definable sets and, equally importantly, non-definable sets (such as the torsion elements in an arbitrary group $(G, \cdot, e)$) are provided in Marker's *Introduction to model theory* as well as throughout the volume.

Although first-order definability might appear to impose rather severe limitations on the objects that model theory studies, it does in fact supply rich and interesting classes of sets and functions. In fact, as mentioned in both Marker's

[1]Logicians have developed and studied logics that relax either or both of these restrictions with mixed success, but first-order logic appears to provide the best balance between expressibility and manageability.

introductory article and van den Dries' contribution, Gödel's Incompleteness Theorem implies that the definable sets in $(\mathbb{N}, +, \cdot, 0, 1)$ are complicated to the point of "wildness," and thus exhibit such poor model-theoretic behavior as to escape analysis. As will be evident from the diverse range of structures and applications discussed in this volume, manageable, or "tame", behavior does occur regularly enough. And even in contexts in which model-theoretic analysis might appear at first to be of limited value, it sometimes is possible (and desirable) to carry out the analysis in a setting with a richer collection of definable sets *and* with good model-theoretic properties. This point emerges in the striking applications of model theory to Diophantine problems as described in Marker's article on differential fields and Chatzidakis' article.

## 3. Analysis of Definable Sets and Applications

The preceding discussion of definable sets and structures is a necessary prelude to the most important aspect of the model-theoretic enterprise: the theoretical methods that have been developed for understanding definable sets and the applications that ensue.

As in any area of mathematics, model theorists analyze definable sets by devising measures of simplicity or tractability. Two main threads appear here. The first deals with the complexity of a definable set based on the "structural complexity" of a formula required to define the set. Just as the set-theoretic operation of projection can transport us outside of the class of Borel subsets of Euclidean space, projection, in the guise of existential quantification, typically adds complexity to definable sets. Indeed, mathematicians have often remarked that a proposition with more than three alternations of quantifiers strains the understanding. Thus, the "structural complexity" of a formula might be measured by whether or not it contains quantifiers or by counting the number of alternations of blocks of universal and existential quantifiers appearing in what is called prenex normal form of the formula. This kind of analysis typically shows that the definable sets of a structure satisfy a general hypothesis that subsequently permits the application of a general theoretical framework for the analysis of the definable sets, as we now describe.

The second approach to the analysis of definable sets that model theorists have developed classifies these sets by various abstract yet natural measures that assign to the sets a combinatorial, algebraic, or geometrically motivated notion of dimension. The two principal avenues of "dimension theory", as we have called this second approach to the analysis of definable sets, are stability and simplicity, the subject of Hart's article, and o-minimality and some of its variants as discussed in Macpherson's survey. In each case, central to the development of a dimension theory is a notion of independence, or freeness: for subsets $A, B$ and $C$ of a structure $\mathcal{M}$, the expression "$B$ is independent from $C$ over $A$" should mean "$C$ provides no more information about $B$ than $A$ does". For example, for algebraically or real closed fields, the dimension-theoretic definition of independence coincides with algebraic independence. The general theory has proven to be remarkably rich and has demonstrated its mettle in many applications.

This second mode of analysis can in principle ignore the syntactic shape of a defining formula for a set, and thus *prima facie* have little to do with the first approach. Yet some of the most important applications are often found where the two threads cross. As suggested earlier, if the definable sets of the structures in some class yield to the first kind of analysis, model theorists then may be able to show that the class of structures is amenable to the powerful model-theoretic tools and methods afforded by the second kind of analysis. We now take up in more detail these two approaches to understanding definable sets.

**Quantifier elimination and generalizations.** We begin with a simple illustration of the analysis of definable sets via the "structural complexity" of their defining formulas. Marker's introduction to model theory, his article on differential fields, and the contributions by van den Dries, Denef, and Bierstone and Milman include many more.

A set $X \subseteq M^n$ of a structure $\mathcal{M} = (M, \dots)$ is *quantifier-free definable* if there is a formula in the language of $\mathcal{M}$ not containing quantifiers that defines $X$. For a field $\mathcal{F} = (F, +, \cdot, 0, 1)$ the quantifier-free definable sets are exactly the constructible sets; boolean combinations of the zero sets of polynomials over $F$. Generally speaking, the class of definable sets in a field includes many more sets than the constructible sets (e.g., in the field of rational numbers; see Marker's introductory article). If $\mathcal{F}$ is algebraically closed, Chevalley's theorem that the projection of a constructible set is constructible implies — since existential quantification corresponds to projection — that the definable sets are exactly those that are quantifier-free definable. In this case the structure is said to have quantifier elimination. More is true. For a formula $\varphi(v_1, \dots, v_n)$ there is a quantifier-free formula $\psi(v_1, \dots, v_n)$ that defines the same set as $\varphi$ in *all* algebraically closed fields. This kind of uniformity plays a crucial role in both pure model theory and applications. Among the most well-known examples of theories — that is, consistent sets of first-order sentences — with quantifier elimination are the theory of real closed fields, the theory of differentially closed fields, and, for each $p$, the theory of $p$-adically closed fields. Others appear throughout the volume — in particular see van den Dries' and Denef's articles.

Quantifier elimination shows how the power of definability and the simplicity of definable sets can play off each other. The full strength of definability permits the definition of *a priori* complicated sets in the structure; that a set has a quantifier-free description implies on the other hand that it is in some sense simple. Several applications appear in Marker's introductory article, and Denef presents in Section 1 of his article his beautiful use of quantifier elimination for the $p$-adic numbers $\mathbb{Q}_p$, in the language with $+, \cdot, -, 0, 1$ augmented by predicates for $d$-th powers for all $d = 2, 3, 4, \dots$, to prove the rationality of several

Poincaré series. (See Section 2 of van den Dries' article for a detailed exposition of quantifier elimination for valued fields).

It should be noted that quantifier elimination is highly sensitive to the language of the class of structures under consideration. An artificial model-theoretic trick shows that if the language of a structure is enriched sufficiently then the structure can be made to have elimination of quantifiers (see Section 4 of Marker's introductory article). This artifice is of little use, though, since the quantifier elimination so obtained reveals nothing about the definable sets. Finding an appropriate language in which a class of structures has quantifier elimination is a difficult and subtle issue. For further discussion of this we refer again to Denef's and van den Dries' articles.

Sometimes quantifier elimination in a particular language fails but yet the definable sets in a structure still have a manageable and useful form. An instructive example of this is *model completeness*. One of the several equivalent definitions is that a theory is model complete if for every formula $\varphi$ there is an *existential formula* $\psi$, i.e., a formula consisting of a block of existential quantifiers followed by a quantifier-free formula, such that $\varphi$ and $\psi$ are equivalent, that is, they define the same sets in all structures satisfying the theory. Model completeness thus serves as the next best substitute for quantifier elimination.

An example may help bring this into sharper focus. The ordered real field $(\mathbb{R}, +, \cdot, 0, 1, <)$ has quantifier elimination, but an adaptation of an old argument of Osgood demonstrates that quantifier elimination fails in the real field augmented by any collection of total analytic functions. In particular, this is true in the real exponential field $(\mathbb{R}, +, \cdot, 0, 1, <, e^x)$, and so model completeness is the best that could be expected. A important theorem of Wilkie proved in 1991 shows that this structure is in fact model complete, and thus every definable set is the projection of a quantifier-free definable set. Results of Khovanskii from the 1970's provide a good understanding of the quantifier-free definable sets, and thus Wilkie's theorem yields a clear picture of all definable sets. The analysis afforded by model completeness in turn suffices to conclude that the real exponential field is o-minimal, and hence its definable sets enjoy the many good geometric properties that the theory of o-minimality provides (see Section 4.3 of van den Dries' articles). We shall say more about o-minimality in the "dimension theory" subsection below.

Yet another form of partial or relative quantifier elimination emerges in the first-order theory of modules. The language of modules over a ring $R$ includes a symbol $+$ for addition, 0, and, for each $r \in R$, a 1-place function symbol for multiplication by $r$. For a complete first-order theory $T$ of modules (see page 19 in Marker's article for the definition of completeness), every formula is equivalent to a boolean combination of "positive-primitive" formulas, that is, formulas in which a block of existential formulas precedes a conjunction of atomic formulas. This analysis implies that for a module $(M, +, \dots)$ every definable subset of $M^k$, where $k \in \mathbb{N}$, is a boolean combination of definable subgroups of $M^k$. In a more general context, the so-called 1-based groups (elaborated further in Section 3.3 of Hart's article and Section 4 of Chatzidakis' article), it also can be shown that all definable sets are boolean combinations of definable subgroups. This fact plays an crucial role in Hrushovski's applications of model theory to the Mordell–Lang and Manin–Mumford Conjectures (see Section 5 of both Marker's and Chatzidakis' articles).

**3.1. Dimension theory.** We turn now to the second major approach to the analysis of definable sets via what we call "dimension theory". This has constituted perhaps the central theme in the development of pure model theory for almost 40 years. There have been two main strands present here. The first, beginning with the seminal work of Morley in the 1960's and developed profoundly by Shelah in the following decade provides a combinatorial/algebraic account of dimension theory. Stable, and more generally simple theories are the subject of this analysis, which is elaborated in Hart's article. The second strand yields a more topological/algebraic version of dimension theory that comprises the main focus of Macpherson's article. Although these two dimension theories apply to disjoint classes of structures they share several common conceptual features. We here offer some introductory remarks that should highlight these themes.

The beginnings of the dimension-theoretic analysis of definable sets in structures that are stable or simple runs as follows. Let $\mathcal{M} = (M, \dots)$ be a structure. Since equality is included in every language, for every $n \in \mathbb{N}$ and $a_1, \dots, a_n \in M$, the sets $\{a_1, \dots, a_n\}$ and $M \setminus \{a_1, \dots, a_n\}$ must be definable subsets of $M$ in $\mathcal{M}$: they are defined by the formula

$$v = a_1 \vee \cdots \vee v = a_n$$

and its negation, respectively. Thus, the finite and cofinite subsets of the universe of a structure must always be definable. With a slight twist, the structures $\mathcal{M} = (M, \dots)$ that are the least complicated from the viewpoint of definability are those for which the definable subsets of $M$ are precisely the sets that must be definable, that is, the finite and cofinite sets. The twist is that reference must be made not to individual structures but rather to all structures for a particular language satisfying a set of axioms, that is, a theory: a theory $T$ is *strongly minimal* if for every structure $\mathcal{M} = (M, \dots)$ satisfying $T$ the definable subsets of $M$ are exactly the finite and cofinite sets. Elimination of quantifiers for the theory of algebraically closed fields shows that every definable subset of the universe of an algebraically closed field is defined by a formula which is a boolean combination of polynomial equalities in one variable. As the set given by such a formula is finite or cofinite, the theory of algebraically closed fields is strongly minimal.

As mentioned earlier, by adjoining new symbols to the language, a structure is endowed with a richer collection of definable sets. The balance to strike here is that the richer expressive power of the expanded structure should not yield an intractible definability theory. Thus, if a predicate for the natural numbers is adjoined to the complex numbers, the structure becomes as "wild" as $(\mathbb{N}, +, \cdot, 0, 1)$ itself and so subject to the Gödel phenomenon mentioned earlier. This is not always the case. The theory of differentially closed fields, discussed in depth in Marker's second article, provides an important example. A differential field is a field of characteristic zero equipped with a derivation. The language for this structure is the language of fields with a function symbol for the derivation adjoined. The theory of differentially closed fields is axiomatized by appropriate closure axioms asserting the existence of zeroes for differential polynomials. This theory too has quantifier elimination. It follows that the definable sets satisfy a hypothesis, $\omega$-stability, that ensures a highly manageable model-theoretic dimension theory in which there are definable sets having "transfinite dimension".

For the study of definable sets, strong minimality is relativized to definable sets in a structure. For a structure $\mathcal{M} = (M, \dots)$ and a definable set $X \subseteq M^n$, we say that $X$ is strongly minimal if every definable subset of $X$ in $\mathcal{M}$ is finite or cofinite. Strongly minimal sets can be thought of as "irreducible" sets of dimension one. They form the first layer of a dimensional analysis of definable sets based on what is called *Morley rank* (see Definition 1.16 in Hart's article). In the context of algebraically closed fields, the Morley rank of a definable (= constructible) set agrees with its algebro-geometric dimension. The $\omega$-stable theories can be shown to be exactly those theories for which every definable set has ordinal-valued Morley rank. The most general dimensional analysis, Shelah's theory of forking, applies to the class of stable, or more generally simple theories. The sweep of this theory is remarkably broad given the consequences that flow from it (see Sections 2 and 3 of Hart's article). Among the theories of fields discussed in this volume — see the articles by Marker and Chatzidakis — it embraces algebraically, differentially, and separably closed fields, pseudo-finite fields, and ACFA, the model companion of the theory of difference fields. The crucial point when it comes to Hrushovski's applications to diophantine geometry (see Marker's article on differential fields as well as those by Chatzidakis and by Mazur) is that objects of arithmetic type, such as the torsion points of an abelian variety or the points of an abelian variety over a function field can be embedded in definable groups in enriched structures to which the general theory applies.

Strongly minimal sets and more general versions of "1-dimensional" or "minimal" sets (see Hart's and Chatzidakis' articles) assume an important position in the model-theoretic analysis employed in these applications. Generally speaking, the structure of such sets also determines the structure of finite-dimensional sets, and so understanding the pregeometry given by model-theoretic dependence in "minimal" sets is imperative. For many years the only known examples of such a pregeometry were *trivial*, "module-like" (*locally modular*), or "field-like" (i.e., permit the interpretation in a precise model-theoretic sense of an infinite field) — see Section 4 of Marker's article and Example 1.26 in Hart's article. In the late 1970's, early on in the development of the theory, Zil'ber boldly conjectured that these are the only possible cases. This conjecture exercised a powerful and positive influence on model theory in the 1980's (see the statement of the Zil'ber principle right after Example 1.26 in Hart's article). Zil'ber's conjecture ultimately proved false: in the late 1980's Hrushovski found counterexamples. With the introduction of the notion of a *Zariski geometry*, however, Hrushovski and Zil'ber isolated an important class of strongly minimal sets for which the conjecture holds (see Section 4 of Marker's article on differential fields, as well as Hart's article). Furthermore, the "minimal" sets in the enriched structures that figure in Hrushovski's proof of the Mordell–Lang Conjecture can be shown to be Zariski geometries, and Hrushovski avails himself of this in his proof.

The field of real numbers presents a different situation. The ordering on $\mathbb{R}$ actually is definable in the field language:

$$x < y \iff \exists u\ (y = x + u^2 \ \wedge\ u \neq 0).$$

Hence, the order relation can be adjoined to the real field structure without altering the class of definable sets. It follows that the real field is not stable, or even simple — see Hart's article — and so cannot be analyzed by the machinery described above. Yet, as the ordered field of real numbers, actually the theory of real closed fields, has quantifier elimination, the definable sets are exactly those which are defined by boolean combinations of polynomial equalities and inequalities, that is, the semialgebraic sets. These have been studied with considerable success by real algebraic geometers (see the articles by Bierstone and Milman and van den Dries). Observe in particular that the definable subsets of $\mathbb{R}$ in the real field consist of finitely many open intervals and points. As these sets are those that must be definable in any linearly ordered structure, the definable subsets of $\mathbb{R}$ in the real field are uncomplicated if one adopts the right point of view.

A linearly ordered structure $\mathcal{M} = (M, <, \dots)$ is order-minimal, or *o-minimal* if every definable subset of $M$ is the union of finitely many points and open intervals, that is those that must be definable in the presence of a linear ordering. O-minimal structures — see Macpherson's article for a survey of this subject — permit a dimension-theoretic analysis of definable sets that accords with the geometry of the definable sets. In particular, many of the geometric and analytic properties of semialgebraic sets extend to o-minimal structures, particularly those whose underlying set is $\mathbb{R}$ (see van den Dries' and Macpherson's article). Furthermore, many analogues of theorems from the stable context can be proved under the hypothesis of o-minimality. For example, a version of Zil'ber's conjecture has been proved by Peterzil and Starchenko in the o-minimal setting. Notions of minimality relative to other basic predicates also are mentioned in Section 4 of Macpherson's contribution.

While *a priori* more limited in scope than stability and simplicity, what has made o-minimality successful is that it has been proved that o-minimality is preserved under adjoining many analytically important functions to the real field. Wilkie's theorem that the structure $(\mathbb{R}, +, \cdot, <, 0, 1, e^x)$ is (model complete and) o-minimal was the first dramatic result in this direction and Section 4 of van den Dries' article discusses many others. These theorems have in turn been applied to problems in real analytic and algebraic geometry, and recently have been invoked in work in the representation theory of Lie groups. To illustrate, the well-known semialgebraic fact that there are finitely many homeomorphism types in $\mathbb{R}^m$ of the zero sets of polynomials $p(x_1, \ldots, x_m) \in \mathbb{R}[x_1, \ldots, x_m]$ of some fixed degree $d$ can be extended via o-minimality to establish that the same holds true if "of some fixed degree $d$" is replaced by "with no more than $d$ monomials" (of arbitrary degree). The proof takes advantage of a form of uniformity in parameters that the model theory provides. Further afield, the o-minimality of expansions of the real field have seen applications in as seemingly distant subjects as neural nets and control theory.

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