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Symmetrized Random Permutations

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ABSTRACT. Selecting N random points in a unit square corresponds to selecting a random permutation. Placing symmetry restrictions on the points, we obtain special kinds of permutations: involutions, signed permutations and signed involutions. We are interested in the statistics of the length (in numbers of points) of the longest up/right path in each symmetry type as the number of points increases to infinity. The limiting distribution functions are expressed in terms of a Painlevé II equation. In addition to the Tracy–Widom distributions of random matrix theory, we also obtain two new classes of distribution functions interpolating between the GOE and GSE, and between the GUE and GOE^2 Tracy–Widom distribution functions. Applications to random vicious walks and site percolation are also discussed

1. Introduction

Suppose that we are selecting n points, $p_1, p_2, \dots, p_n$, at random in a rectangle, say $R = [0,1] \times [0,1]$ (see Figure 1). We denote by π the configuration of n random points. With probability 1, no two points have same x-coordinates nor y-coordinates. An up/right path of π is a collection of points $p_{i_1}, p_{i_2}, \dots, p_{i_k}$ such that $x(p_{i_1}) < x(p_{i_2}) < \dots < x(p_{i_k})$ and $y(p_{i_1}) < y(p_{i_2}) < \dots < y(p_{i_k})$. The length of such a path is defined by the number of the points in the path. Now we denote by $l_n(\pi)$ the length of the longest up/right path of a random points configuration π.

As one can see from Figure 1, a configuration of n points gives rise to a permutation. For the example at hand, the corresponding permutation is $\left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 3 & 2 & 4 \end{smallmatrix}\right)$. Therefore we can identity random points in R and random permutations, and we use the same notation π. In this identification, $l_n(\pi)$ is the length of the longest increasing subsequence of a random permutation.

The longest increasing subsequence has been of great interest for a long time (see [AD2], [OR], [BDJ1], for example). Especially as $n \to \infty$, it is known that $\mathrm{E}(l_n) \sim 2\sqrt{n}$ [LS], [VK1; VK2] (also [AD1; Se; Jo2]) and $\mathrm{Var}(l_n) \sim c_0 n^{1/3}$

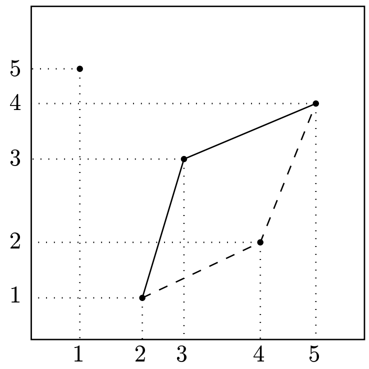

Figure 1. Random points in a rectangle.

[BDJ1] with some numerical constant $c_0 = 0.8132\cdots$. Moreover, the limiting distribution of l_n after proper scaling is obtained in [BDJ1] in terms of the solution to the Painlevé II equation (see Section 3 for precise statement). An interesting feature is that the limiting distribution function above is also the limiting distribution of the (scaled) largest eigenvalue of a random GUE matrix [TW1], the so-called "GUE Tracy–Widom distribution" F_2. In other words, properly centered and scaled, the length of the longest increasing subsequence of a random permutation behaves statistically for large n like the largest eigenvalue of a random GUE matrix. There have been many papers concerning the relations on combinatorics and random matrix theory: we refer the reader to [Re; Ge; Ke; Ra; Jo2; BDJ1; BDJ2; TW3; Bo; Jo1; Jo4; Ok; BOO; TW4; Jo3; BR1; BR2; ITW; St; PS2; PS1; Ba; BR3]. The purpose of this paper is to survey the analytic results of the recent papers [BR1; BR2] and discuss related topics.

In random matrix theory, three ensembles play important roles, GUE, GOE and GSE (see [Me], for example). Since random permutation is related to GUE, it would be interesting to ask which object in combinatorics is related to GOE and GSE. For this purpose, we consider symmetrized permutations. In terms of random points, 5 symmetry types of the rectangle R are considered, denoted by the symbols □, ⧄, ⧅, ⊡, and ⊠. Throughout this paper (and also in [BR1; BR2]), the symbol ⊛ is used to denote an arbitrary choice of the five possibilities above. Let $\delta = \{(t,t) : 0 \le t \le 1\}$, the diagonal line, and $\delta^t = \{(t, 1-t) : 0 \le t \le 1\}$, the anti-diagonal line. Consider the following random points selections:

- □ select n points in R at random.
- ⧄ select n points in $R \setminus \delta$ and m points in δ at random, and add their reflection images about δ.
- ⧅ select n points in $R \setminus \delta^t$ and m points in δ^t at random, and add their reflection images about δ^t.
- ⊡ select n points at random in R, and add their rotational images about the center $(1/2, 1/2)$.

$\boxtimes$ select n points in $R \setminus \delta$, m_+ points in δ and m_- points in δ^t at random, and add their reflection images about both δ and δ^t.

Define the map ι on S_n by $\iota(x) = n+1-x$. Let $\mathrm{fp}(\pi)$ denote the number of points satisfying $\pi(x) = x$, and $\mathrm{fpi}(\pi)$ denote the number of points satisfying $\pi(x) = \iota(x)$ (fpi represents negated points: see Remark 3 below). Each of the processes above corresponds to picking a random permutation from each of the following ensembles:

$$\begin{aligned}
S_n^{\square} &= S_n, \\
S_{n,m}^{\boxslash} &= \{\pi \in S_{2n+m} : \pi = \pi^{-1},\ \mathrm{fp}(\pi) = m\}, \\
S_{n,m}^{\boxbslash} &= \{\pi \in S_{2n+m} : \pi = \iota\pi^{-1}\iota,\ \mathrm{fpi}(\pi) = m\}, \\
S_n^{\boxdot} &= \{\pi \in S_{2n} : \pi = \iota\pi\iota\}, \\
S_{n,m_+,m_-}^{\boxtimes} &= \{\pi \in S_{4n+2m_++2m_-} : \pi = \pi^{-1},\ \pi = \iota\pi^{-1}\iota,\ \mathrm{fp}(\pi) = 2m_+,\ \mathrm{fpi}(\pi) = 2m_-\}.
\end{aligned}$$

We denote the length of the longest increasing subsequence (equivalently, the longest up/right path) of π in each of the ensemble respectively by

$$L_n^{\square}, \quad L_{n,m}^{\boxslash}, \quad L_{n,m}^{\boxbslash}, \quad L_n^{\boxdot}, \quad L_{n,m_+,m_-}^{\boxtimes}.$$

Remark 1. The map $\pi \mapsto \iota^{-1}\pi$ gives a bijection between $S_{n,m}^{\boxslash}$ and $S_{n,m}^{\boxbslash}$. Thus $L_{n,m}^{\boxbslash}$ has the same statistics with the length of the longest *decreasing* subsequence of a random involution with m fixed points taken from $S_{n,m}^{\boxslash}$. From the definition, $L_{n,m}^{\boxslash}$ is the random variable describing the length of the longest *increasing* subsequence of a random involution taken from the same ensemble.

Remark 2. We may identify S_{2n} with the set of bijections from

$$\{-n, \dots, -2, -1, 1, 2, \dots, n\}$$

onto itself. In this identification, $S_n^{\boxdot}$ becomes the set of signed permutations; $\pi(x) = -\pi(-x)$. The longest increasing subsequence problem of a random signed permutation is considered in [TW3] and [Bo].

Remark 3. Under the identification in Remark 2, $S_{n,m_+,m_-}^{\boxtimes}$ becomes the set of signed involutions with m_+ fixed points and m_- negated points (we call x a negated point if $\pi(x) = -x$.)

In this paper, we are interested in the statistics of $L^{\circledast}$ as $n \to \infty$. Especially for $\boxslash$, $\boxbslash$ and $\boxtimes$, we are interested in the cases when $m = [\sqrt{2n}\alpha]$ for $\boxslash$, $m = [\sqrt{2n}\beta]$ for $\boxbslash$, and $m_+ = [\sqrt{n}\alpha]$ and $m_- = [\sqrt{n}\beta]$ for $\boxtimes$ with fixed $\alpha, \beta \geq 0$ where $[k]$ denotes the largest integer less than or equal to k. Then for most cases, the expected values have the same asymptotics. Namely, if we set $N = n, 2n+m, 2n+m, 2n, 4n+2m_++2m_-$ for each of $\square, \boxslash, \boxbslash, \boxdot, \boxtimes$ case respectively, we have

$$\lim_{N \to \infty} \frac{\mathrm{E}(L^{\circledast})}{\sqrt{N}} = 2,$$

when $0 \leq \alpha \leq 1$ and $\beta \geq 0$ are fixed for ⧄, ⧅ and ⊠. When $\alpha > 1$, we have different expected value in the limit (see Section 3.)

On the other hand, the variance behaves asymptotically like $c_0 N^{1/3}$ but now with different constant c_0 depending on the symmetry type. It is because each symmetry type has different limiting distribution: $L_n^{\square}$ has GUE fluctuation, $L_{n,m}^{⧅}$ GOE fluctuation and $L_n^{\boxdot}$ GUE^2 fluctuation (see Section 3 below for precise statements). Here GUE^2 denotes the statistics of a superimposition of eigenvalues of two random GUE matrices. Similarly for GOE^2. The cases of ⧄ and ⊠ show more interesting features. For ⧄, the limiting distribution function changes depending on the value of $\alpha = m/\sqrt{2n}$. The fluctuation is GSE when $\alpha < 1$, GOE when $\alpha = 1$ and Gaussian when $\alpha > 1$. By taking suitable scaling limit $\alpha \to 1$, we can find a certain smooth transition between GSE and GOE. For ⊠, the value $\alpha = m_+/\sqrt{n}$ determines the limiting distribution ; the value m_- plays no role in the transition. The fluctuation is GUE when $\alpha < 1$, GOE^2 when $\alpha = 1$, and Gaussian when $\alpha > 1$.

In Section 2, we define the Tracy–Widom distributions for GUE, GOE and GSE as well as new classes of distribution functions describing the transition around $\alpha = 1$. Main results are stated in Section 3, and Section 4 includes some applications and the related problems. Most of the results in this article are taken from [BR1; BR2]. Theorems 3.1 and 3.5 for □ were first proved in [BDJ1], and Theorem 3.1 for ⊡ was first obtained in [TW4; Bo]. The only new result is Theorem 4.2.

2. Tracy–Widom Distribution Functions

Let $u(x)$ be the solution of the Painlevé II (PII) equation,

$$u_{xx} = 2u^3 + xu, \tag{2–1}$$

with the boundary condition

$$u(x) \sim -\operatorname{Ai}(x) \quad \text{as} \quad x \to +\infty, \tag{2–2}$$

where Ai is the Airy function. The proof of the (global) existence and the uniqueness of the solution was first established in [HM]: the asymptotics as $x \to -\infty$ are (see [HM; DZ2], for example)

$$u(x) = -\operatorname{Ai}(x) + O\left(\frac{e^{-(4/3)x^{3/2}}}{x^{1/4}}\right), \qquad \text{as } x \to +\infty, \tag{2–3}$$

$$u(x) = -\sqrt{\frac{-x}{2}}\left(1 + O\left(\frac{1}{x^2}\right)\right), \qquad \text{as } x \to -\infty. \tag{2–4}$$

Recall that $\mathrm{Ai}(x) \sim \dfrac{e^{-(2/3)x^{3/2}}}{2\sqrt{\pi}x^{1/4}}$ as $x \to +\infty$. Define

$$v(x) := \int_{\infty}^{x} u(s)^2\, ds, \tag{2–5}$$

so that $v'(x) = u(x)^2$.

We introduce the Tracy–Widom distributions. (Note that $q := -u$, which Tracy and Widom used in their papers, solves the same differential equation with the boundary condition $q(x) \sim +\mathrm{Ai}(x)$ as $x \to \infty$.)

Definition (Tracy–Widom distribution functions). Set

$$F(x) := \exp\left(\frac{1}{2}\int_x^\infty v(s)\, ds\right) = \exp\left(-\frac{1}{2}\int_x^\infty (s-x)u(s)^2\, ds\right),$$
$$E(x) := \exp\left(\frac{1}{2}\int_x^\infty u(s)\, ds\right),$$

and set

$$F_2(x) := F(x)^2 = \exp\left(-\int_x^\infty (s-x)u(s)^2\, ds\right),$$
$$F_1(x) := F(x)E(x) = F_2(x)^{1/2} e^{\frac{1}{2}\int_x^\infty u(s)\, ds},$$
$$F_4(x) := F(x)\big(E(x)^{-1} + E(x)\big)/2 = F_2(x)^{1/2}\, \frac{e^{-\frac{1}{2}\int_x^\infty u(s)\, ds} + e^{\frac{1}{2}\int_x^\infty u(s)\, ds}}{2}.$$

In [TW1; TW2], Tracy and Widom proved that under proper centering and scaling, the distribution of the largest eigenvalue of a random GUE/GOE/GSE matrix converges to $F_2(x)$ / $F_1(x)$ / $F_4(x)$ as the size of the matrix becomes large. We note that from the asymptotics (2–3) and (2–4), for some positive constant c,

$$F(x) = 1 + O\big(e^{-cx^{3/2}}\big) \qquad \text{as } x \to +\infty, \tag{2–6}$$
$$E(x) = 1 + O\big(e^{-cx^{3/2}}\big) \qquad \text{as } x \to +\infty, \tag{2–7}$$
$$F(x) = O\big(e^{-c|x|^3}\big) \qquad \text{as } x \to -\infty, \tag{2–8}$$
$$E(x) = O\big(e^{-c|x|^{3/2}}\big) \qquad \text{as } x \to -\infty. \tag{2–9}$$

Hence in particular, $\lim_{x\to+\infty} F_\beta(x) = 1$ and $\lim_{x\to-\infty} F_\beta(x) = 0$, $\beta = 1, 2, 4$. Monotonicity of $F_\beta(x)$ follows from the fact that $F_\beta(x)$ is the limit of a sequence of distribution functions. Therefore $F_\beta(x)$ is indeed a distribution function.

As indicated in Introduction, we need new classes of distribution functions to describe the phase transitions from GSE to GOE and from GUE to GOE^2. First we consider the Riemann–Hilbert problem (RHP) for the Painlevé II equation

[FN; JMU]. Let Γ be the real line $\mathbb{R}$, oriented from $+\infty$ to $-\infty$. Let $m(\cdot\,;x)$ be the solution of the following RHP:

$$\begin{cases} m(z;x) \qquad \text{is analytic in } z \in \mathbb{C}\setminus\Gamma, \\ m_+(z;x) = m_-(z;x)\begin{pmatrix} 1 & -e^{-2i(\frac{4}{3}z^3+xz)} \\ e^{2i(\frac{4}{3}z^3+xz)} & 0 \end{pmatrix} \quad \text{for } z\in\Gamma, \\ m(z;x) = I + O\big(\frac{1}{z}\big) \qquad \text{as } z\to\infty. \end{cases} \tag{2–10}$$

Here $m_+(z;x)$ and m_- are the limits of $m(z';x)$ as $z'\to z$ from the left and right of the contour Γ: $m_\pm(z;x) = \lim_{\varepsilon\downarrow 0} m(z\mp i\varepsilon;x)$. Relation (2–10) corresponds to the RHP for the PII equation with the special monodromy data $p=-q=1, r=0$ (see [FN; JMU], also [FZ; DZ2]). In particular if the solution is expanded at $z=\infty$,

$$m(z;x) = I + \frac{m_1(x)}{z} + O\Big(\frac{1}{z^2}\Big), \qquad \text{as } z\to\infty, \tag{2–11}$$

we have

$$2i(m_1(x))_{12} = -2i(m_1(x))_{21} = u(x),$$
$$2i(m_1(x))_{22} = -2i(m_1(x))_{11} = v(x),$$

where $u(x)$ and $v(x)$ are defined in Equations (2–1) to (2–5). Therefore the Tracy–Widom distributions above are expressed in terms of the residue at ∞ of the solution to the RHP (2–10). It is noteworthy that the new distributions below which interpolate the Tracy–Widom distributions require additional information of the solution of RHP.

Definition. Let $m(z;x)$ be the solution of RHP (2–10) and denote by $m_{jk}(z;x)$ the (jk)-entry of $m(z;x)$. For $w>0$, define

$$F^{\boxslash}(x;w) := F(x)$$
$$\times\Big(\big(m_{22}(-iw;x)-m_{12}(-iw;x)\big)E(x)^{-1}+\big(m_{22}(-iw;x)+m_{12}(-iw;x)\big)E(x)\Big)/2,$$

and for $w<0$, define

$$F^{\boxslash}(x;w) := e^{\frac{8}{3}w^3-2xw}F(x)$$
$$\times\Big(\big(-m_{21}(-iw;x)+m_{11}(-iw;x)\big)E(x)^{-1}-\big(m_{21}(-iw;x)+m_{11}(-iw;x)\big)E(x)\Big)/2.$$

Also define

$$F^{\boxtimes}(x;w) := m_{22}(-iw;x)F_2(x), \qquad w>0,$$
$$F^{\boxtimes}(x;w) := -e^{\frac{8}{3}w^3-2xw}m_{21}(-iw;x)F_2(x), \qquad w<0.$$

First $F^{\boxslash}(x;w)$ and $F^{\boxtimes}(x;w)$ are real from Lemma 2.1(i) below. Note that $F^{\boxslash}(x;w)$ and $F^{\boxtimes}(x;w)$ are continuous at $w=0$ since at $z=0$, the jump condition of the RHP (2–10) implies

$$(m_{12})_+(0;x) = -(m_{11})_-(0;x),$$
$$(m_{22})_+(0;x) = -(m_{21})_-(0;x).$$

In fact, $F^{\boxslash}(x;w)$ and $F^{\boxtimes}(x;w)$ are entire in $w\in\mathbb{C}$ from the RHP (2–10).

From (2–6)–(2–9) and Lemma 2.1(ii) below, we see that

$$\lim_{x\to+\infty} F^{\boxslash}(x;w), F^{\boxtimes}(x;w)=1, \qquad \lim_{x\to-\infty} F^{\boxslash}(x;w), F^{\boxtimes}(x;w)=0$$

for any fixed $w\in\mathbb{R}$. Also Theorem 3.3 below shows that $F^{\boxslash}(x;w)$ and $F^{\boxtimes}(x;w)$ are limits of distribution functions, implying that they are monotone in x. Therefore, $F^{\boxslash}(x;w)$ and $F^{\boxtimes}(x;w)$ are indeed distribution functions for each $w\in\mathbb{R}$.

We close this section summarizing some properties of $m(-iw;x)$ in the following lemma. In particular the lemma implies that $F^{\boxslash}(x;w)$ interpolates between $F_4(x)$ and $F_1(x)$, and $F^{\boxtimes}(x;w)$ interpolates between $F_2(x)$ and $F_1(x)^2$ (see Corollary 2.2).

LEMMA 2.1. *Let* $\sigma_3=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$, $\sigma_1=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, *and set* $[a,b]=ab-ba$.

(i) *For real* w, $m(-iw;x)$ *is real.*

(ii) *For fixed* $w\in\mathbb{R}$, *we have*

$$\begin{aligned}
m(-iw;x)&=\big(I+e^{-cx^{3/2}}\big)\begin{pmatrix}1 & -e^{\frac{8}{3}w^3-2xw}\\ 0 & 1\end{pmatrix}, && w>0,\ x\to+\infty,\\
m(-iw;x)&=\big(I+e^{-cx^{3/2}}\big)\begin{pmatrix}1 & 0\\ -e^{-\frac{8}{3}w^3+2xw} & 1\end{pmatrix}, && w<0,\ x\to+\infty,\\
m(-iw;x)&\sim\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}e^{(-\frac{4}{3}w^3+xw)\sigma_3}e^{(\frac{\sqrt{2}}{3}(-x)^{3/2}+\sqrt{2}w^2(-x)^{1/2})\sigma_3}, && \\
& && w>0,\ x\to-\infty,\\
m(-iw;x)&\sim\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}e^{(-\frac{4}{3}w^3+xw)\sigma_3}e^{(-\frac{\sqrt{2}}{3}(-x)^{3/2}-\sqrt{2}w^2(-x)^{1/2})\sigma_3}, && \\
& && w<0,\ x\to-\infty.
\end{aligned}$$

(iii) *For any* x, *we have*

$$\begin{aligned}
\lim_{w\to0^+} m(-iw;x)&=\lim_{w\to0^-}\sigma_1 m(-iw;x)\sigma_1\\
&=\begin{pmatrix}\frac12\big(E(x)^2+E(x)^{-2}\big) & -E(x)^2\\ \frac12\big(-E(x)^2+E(x)^{-2}\big) & E(x)^2\end{pmatrix}.
\end{aligned}\tag{2–12}$$

(iv) *For fixed* $w\in\mathbb{R}\setminus\{0\}$, $m(-iw;x)$ *solves the differential equation*

$$\frac{d}{dx}m=w[m,\sigma_3]+u(x)\sigma_1 m,$$

where $u(x)$ *is the solution of the PII equation* (2–1), (2–2).

COROLLARY 2.2. *We have*

$$\begin{aligned}
&F^{\boxslash}(x;0)=F_1(x)\ , && \lim_{w\to\infty}F^{\boxslash}(x;w)=F_4(x), && \lim_{w\to-\infty}F^{\boxslash}(x;w)=0,\\
&F^{\boxtimes}(x;0)=F_1(x)^2, && \lim_{w\to\infty}F^{\boxtimes}(x;w)=F_2(x), && \lim_{w\to-\infty}F^{\boxtimes}(x;w)=0.
\end{aligned}$$

PROOF. The values at $w=0$ follow from (2–12). For $w\to\pm\infty$, note that from the RHP (2–10), we have $\lim_{z\to\infty}m(z;x)=I$. ☐

3. Main Results

As in the Introduction, let N denote $n, 2n+m, 2n+m, 2n, 4n+2m_+ + 2m_-$ for each of $\square, \boxslash, \boxbslash, \boxdot, \boxtimes$ case respectively. We scale the random variables: for permutations and involutions,

$$\chi_n^{\square} = \frac{L_n^{\square} - 2\sqrt{N}}{N^{1/6}}, \qquad \chi_{n,m}^{\boxslash} = \frac{L_{n,m}^{\boxslash} - 2\sqrt{N}}{N^{1/6}}, \qquad \chi_{n,m}^{\boxbslash} = \frac{L_{n,m}^{\boxbslash} - 2\sqrt{N}}{N^{1/6}},$$

and for signed permutations and signed involutions,

$$\chi_n^{\boxdot} = \frac{L_n^{\boxdot} - 2\sqrt{N}}{2^{2/3}N^{1/6}}, \qquad \chi_{n,m_+,m_-}^{\boxtimes} = \frac{L_{n,m_+,m_-}^{\boxtimes} - 2\sqrt{N}}{2^{2/3}N^{1/6}}.$$

All the results in this section are taken from [BR2] which utilizes the algebraic work of [BR1].

First, we state the results for random permutations and random signed permutations. The result for random permutations was first obtained in [BDJ1], and the result for random signed permutations in [TW4; Bo].

Theorem 3.1. *For fixed $x \in \mathbb{R}$,*

$$\lim_{n\to\infty} \Pr\big(\chi_n^{\square} \le x\big) = F_2(x),$$
$$\lim_{n\to\infty} \Pr\big(\chi_n^{\boxdot} \le x\big) = F_2(x)^2.$$

For the involution cases, we have the following limits.

Theorem 3.2. *For each fixed α and β, and for fixed $x \in \mathbb{R}$, we have: for $\boxslash$,*

$$\lim_{n\to\infty} \Pr\big(\chi_{n,[\sqrt{2n}\alpha]}^{\boxslash} \le x\big) = F_4(x), \qquad 0 \le \alpha < 1,$$
$$\lim_{n\to\infty} \Pr\big(\chi_{n,[\sqrt{2n}]}^{\boxslash} \le x\big) = F_1(x),$$
$$\lim_{n\to\infty} \Pr\big(\chi_{n,[\sqrt{2n}\alpha]}^{\boxslash} \le x\big) = 0, \qquad \alpha > 1;$$

for $\boxbslash$,

$$\lim_{n\to\infty} \Pr\big(\chi_{n,[\sqrt{2n}\beta]}^{\boxbslash} \le x\big) = F_1(x), \qquad \beta \ge 0;$$

and for $\boxtimes$,

$$\lim_{n\to\infty} \Pr\big(\chi_{n,[\sqrt{n}\alpha],[\sqrt{n}\beta]}^{\boxtimes} \le x\big) = F_2(x), \qquad 0 \le \alpha < 1,\ \beta \ge 0,$$
$$\lim_{n\to\infty} \Pr\big(\chi_{n,[\sqrt{n}],[\sqrt{n}\beta]}^{\boxtimes} \le x\big) = F_1(x)^2, \qquad \beta \ge 0,$$
$$\lim_{n\to\infty} \Pr\big(\chi_{n,[\sqrt{n}\alpha],[\sqrt{n}\beta]}^{\boxtimes} \le x\big) = 0, \qquad \alpha > 1,\ \beta \ge 0.$$

This theorem shows that for $\boxslash$ and $\boxtimes$, the limiting distributions differ depending on the value of α. As indicated earlier in the Introduction, as $\alpha \to 1$ at a certain rate, we obtain smooth transitions. From Corollary 2.2, the following results are consistent with Theorem 3.2.

THEOREM 3.3. *For fixed $w \in \mathbb{R}$, $\beta \geq 0$ and $x \in \mathbb{R}$,*

$$\lim_{n\to\infty} \Pr\bigl(\chi^{\boxslash}_{n,m} \leq x\bigr) = F^{\boxslash}(x;w), \qquad m = [\sqrt{2n} - 2w(2n)^{1/3}],$$
$$\lim_{n\to\infty} \Pr\bigl(\chi^{\boxtimes}_{n,m_+,m_-} \leq x\bigr) = F^{\boxtimes}(x;w), \qquad m_+ = [\sqrt{n} - 2wn^{1/3}],\ m_- = [\sqrt{n}\beta].$$

When $\alpha > 1$, Theorem 3.2 shows that we have used inappropriate scaling. In a proper scaling, we obtain normal distribution $N(0,1)$.

THEOREM 3.4. *For fixed $\alpha > 1$ and $\beta \geq 0$, as $n \to \infty$,*

$$\frac{L^{\boxslash}_{n,[\sqrt{2n}\alpha]} - (\alpha + 1/\alpha)\sqrt{N}}{\sqrt{(1/\alpha - 1/\alpha^3)}N^{1/4}} \to N(0,1) \qquad \textit{in distribution},$$
$$\frac{L^{\boxtimes}_{n,[\sqrt{n}\alpha],[\sqrt{n}\beta]} - (\alpha + 1/\alpha)\sqrt{N}}{\sqrt{(1/\alpha - 1/\alpha^3)}N^{1/4}} \to N(0,1) \qquad \textit{in distribution}.$$

All the results above are on the convergence in distribution. We also have convergence of moments for all the cases. The case $\square$ was first obtained in [BDJ1].

THEOREM 3.5. *For each case of the preceding theorems, all the moments of the random variable converge to the moments of the corresponding limiting distribution.*

From this result, we can obtain the asymptotics of variances. Especially for $\square$, the variance is

$$\lim_{N\to\infty} \frac{\mathrm{Var}(l_n^{\square})}{N^{1/3}} = \int_{-\infty}^{\infty} x^2 dF_2(x) - \left(\int_{-\infty}^{\infty} x dF_2(x)\right)^2.$$

Evaluating the integrals numerically, we obtain $0.8132\ldots$ (see [TW1]).

The outline of the proofs is as follows. First we consider the Poisson generating function. It is to let the number of points be Poisson. For example, we define

$$P_l^{\square}(t) = e^{-t^2} \sum_{n=0}^{\infty} \frac{t^{2n}}{n!} \Pr\bigl(L_n^{\square} \leq l\bigr).$$

The de-Poissonization lemma [Jo2] tells us that in the limit $n \to \infty$, we have

$$\Pr\bigl(L_n^{\square} \leq l\bigr) \sim P_l^{\square}(n^2),$$

hence it is enough to obtain the asymptotics of the Poisson generating function. The crucial point is that there is a determinantal formula for each Poisson generating function. For the cases $\square$ and $\boxdot$, the determinant is of Toeplitz type [Ge; Ra], while for the rest, it is of Hankel type [BR1]. In fact, as in [Ge], there are general identities between sum of Schur functions and determinantal formulae (see [BR1] for details), which can be used to consider other type of Young tableaux problems (see Section 4.4 below). Now general theory connects Toeplitz/Hankel determinants and orthogonal polynomials. It turns out that to analyze all the cases above, only one set of orthogonal polynomials are needed, namely the orthogonal polynomials on the unit circle with respect to the weight $e^{2t\cos\theta}d\theta/(2\pi)$.

Now following Fokas, Its and Kitaev [FIK], there is a Riemann–Hilbert representation for orthogonal polynomials. Let Σ be the unit circle in the complex plane oriented counterclockwise. Let $Y(z)$ be a 2×2 matrix-valued function satisfying

$$\begin{aligned} &Y(z) \text{ is analytic in } \mathbb{C}\setminus\Sigma,\\ &Y_+(z) = Y_-(z)\begin{pmatrix} 1 & z^{-k}e^{t(z+z^{-1})} \\ 0 & 1\end{pmatrix} \quad \text{for } z\in\Sigma,\\ &Y(z)\begin{pmatrix} z^{-k} & 0 \\ 0 & z^k\end{pmatrix} = I + O(z^{-1}) \quad \text{as } z\to\infty, \end{aligned}$$

where $Y_+(z)$ and $Y_-(z)$ denote the limit of $Y(z')$ as $z'\to z$ satisfying $|z'|<1$ and $|z'|>1$, respectively. Then one finds that for example, the 11 entry of $Y(z)$ is the k-th monic orthogonal polynomial with respect to the weight $e^{2t\cos\theta}d\theta/(2\pi)$. Once we have a Riemann–Hilbert representation, we can employ the steepest-descent method (Deift–Zhou method) developed by Deift and Zhou [DZ1] to find asymptotics as parameters become large (or small). For our case, the parameters are t and k, and taking proper scaling, we obtain precise asymptotics which eventually yield the convergence in distribution and convergence of moments. We also mention that in our analysis, equilibrium measures play a crucial role as in the papers [DKMVZ; DKMVZ2; DKMVZ3]. The asymptotic analysis of $Y(z)$ was first obtained in [BDJ1], and further extended in [BR2].

4. Applications and Related Topics

4.1. Random Involutions and Random Signed Involutions. The ensemble $S^{\boxslash}_{n,m}$ is the set of involutions with n 2-cycles and m 1-cycles. In the previous section, we considered the limiting statistics when n and m are related by $m=[\sqrt{2n}\alpha]$ with α being finite ; either fixed or $\alpha\to 1$ with certain rate. It is of interest to consider the whole set of involutions without constraints on the number of fixed points. Similarly, the signed involutions without constraint on the number of fixed points and negated points is also of interest. We define the ensembles of involutions and signed involutions

$$\begin{aligned} \tilde S_n &= \{\pi\in S_n : \pi=\pi^{-1}\},\\ \tilde S^{\boxtimes}_n &= \{\pi\in S_{2n} : \pi=\pi^{-1},\ \pi=\iota\pi\iota\}, \end{aligned}$$

and denote by $\tilde L_n(\pi)$ and $\tilde L^{\boxtimes}_n(\pi)$ the length of the longest increasing subsequence of $\pi\in\tilde S_n$ and that of $\pi\in\tilde S^{\boxtimes}_n$, respectively.

Noting $\tilde S_n = \bigcup_{2k+m=n} S^{\boxslash}_{k,m}$, we have

$$\Pr\big(\tilde L_n\le l\big) = \frac{1}{|\tilde S_n|}\sum_{2k+m=n}\Pr\big(L^{\boxslash}_{k,m}\le l\big)|S_{k,m}|. \tag{4–1}$$

It is not difficult to check that (see [Kn, pp. 66-67]) as $n \to \infty$, the main contribution to the sum $|\tilde{S}_n| = \sum_{2k+m=n} |S_{k,m}|$ comes from $\sqrt{2k} - (2k)^{\varepsilon+1/4} \leq m \leq \sqrt{2k} + (2k)^{\varepsilon+1/4}$. Comparing with the scaling $m = [\sqrt{2k} - 2w(2k)^{1/3}]$ in Theorem 3.3, the main contribution to the sum (4–1) comes from when $w = 0$, or $\alpha = 1$. Thus we obtain GOE Tracy–Widom distribution function in the limit. Similarly, we can obtain the convergence of moments. The signed involution case is analogous.

THEOREM 4.1. *For fixed* $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \Pr\left(\frac{\tilde{L}_n - 2\sqrt{n}}{n^{1/6}} \leq x\right) = F_1(x),$$
$$\lim_{n\to\infty} \Pr\left(\frac{\tilde{L}_n^{\boxtimes} - 2\sqrt{n}}{2^{2/3}n^{1/6}} \leq x\right) = F_1(x)^2.$$

We also have convergence of all the moments.

This result should be compared with the results on random permutation and random signed permutation where the limiting distribution was $F_2(x)$ and $F_2(x)^2$ under the same scaling of the above (see [BDJ1], [TW3], [Bo] and Theorem 3.1).

REMARK. As in the permutation case, the length of the longest *increasing* subsequence and the length of the longest *decreasing* subsequence of random involutions have the same statistics. This can be seen by noting that there is a bijection (called the Robinson–Schensted correspondence; see [Kn], for example) between the set $\tilde{S}_n$ of involutions of n letters and the set of standard Young tableaux of size n, and the rows and the columns of standard Young tableaux have the same statistics under the push forward of the uniform probability distribution on $\tilde{S}_n$ under this bijection.

4.2. β-Plancherel Measure on the Set of Young Diagrams. Let Y_n be the set of Young diagrams, or equivalently partitions, of size n. Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$, let d_λ denote the number of standard Young tableaux of shape λ. We introduce the β-Plancherel measure M_n^β on Y_n defined by

$$M_n^\beta(\lambda) = \frac{d_\lambda^\beta}{\sum_{\mu\vdash n} d_\mu^\beta}, \qquad \lambda \in Y_n.$$

When $\beta = 2$, this is the Plancherel measure which arises in the representation theory. We are interest in the typical shape and the fluctuation of λ where λ is taken randomly from the probability space Y_n with M_n^β.

A motivation introducing the measure above is the result of Regev [Re], who proved that for fixed $\beta > 0$ and fixed l, as $n \to \infty$,

$$\sum_{\substack{\lambda\vdash n\\ \lambda_1\le l}} (d_\lambda)^\beta \sim \left(\frac{l^{l^2/2}l^n}{(\sqrt{2\pi})^{(l-1)/2}n^{(l-1)(l+2)/4}}\right)^\beta \frac{n^{(l-1)/2}}{l!}\int_{\mathbb{R}^l} e^{-\frac{1}{2}\beta l\sum_j x_j^2}\prod_{j<k}|x_j-x_k|^\beta d^l x.$$

The multiple integral on the right hand side is the Selberg integral which can be computed exactly for each β. When $\beta = 1, 2, 4$, this integral is the normalization constant of the probability density of eigenvalues in GOE, GUE and GSE, respectively (see [Me], for example). So the basic question is if the β in the definition of the β-Plancherel measure corresponds to the β in the random matrix theory.

The well-known Robinson–Schensted correspondence [Sc] establishes a bijection between S_n and the pairs of standard Young tableaux with the same shape of size n, RS : $\pi \mapsto (P(\pi), Q(\pi))$. Especially, we obtain $\sum_{\mu\vdash n} d_\mu^2 = |S_n| = n!$. Moreover, under RS, the length of the longest increasing subsequence of $\pi \in S_n$ is equal to the number of boxes in the first row of $P(\pi)$ (or equally of $Q(\pi)$). Therefore under RS, the Plancherel measure M_n^2 is simply the push forward of the uniform probability measure on S_n to Y_n, and the number of boxes in the first row of a random Young diagram and the length of the longest increasing subsequence of a random permutation have the same statistics: GUE ($\beta = 2$) fluctuation in the limit, by Theorem 3.1.

If $\mathrm{RS}(\pi) = (P, Q)$, then $\mathrm{RS}(\pi^{-1}) = (Q, P)$ (see [Kn], for example). Therefore the set of involutions $\tilde{S}_n$ is bijective to the set of (single) standard Young tableaux, and the number of boxes in the first row of a random Young diagram taken under the probability M_n^1 has the same statistics with the length of the longest increasing subsequence of a random involution: GOE ($\beta = 1$) fluctuation in the limit, by Theorem 4.1.

In fact, it is shown in [Ok; BOO; Jo1] that, for the case of $\beta = 2$, in the large n limit, the number of boxes in the k-th of a random Young diagram corresponds to the k-th largest eigenvalue of a random GUE matrix. Also the typical shape of λ is obtained in [LS] and [VK1] which is related to Wigner's semicircle law [Ke]. On the other hand, the second row of a random Young diagram for $\beta = 1$ is discussed in [BR2] implying that it corresponds to the second eigenvalue of a random GOE matrix. It would be interesting to obtain similar results for general row for $\beta = 1$ and also for general β.

4.3. Random Turn Vicious Walker Model. Random permutations and random involutions arise also in certain random walk process in a one dimensional integer lattice. We call a particle left-movable (resp. right-movable) if its left (resp. right) site is vacant. Initially, p particles are located at the points $1, 2, 3, \ldots, p$. At each time step t ($t = 1, 2, \ldots$), one particle among the left-movable particles is selected at random, and is moved to its left site (so at $t = 1$, the leftmost particle is moved.) Suppose this process is repeated for n time steps. It is found that [Fo] there is a bijection between all the possible configurations and the set of Young tableaux with at most p rows of size n: simply, in the k-th row, write the times when the particle is originally at the point k (see Figure 2).

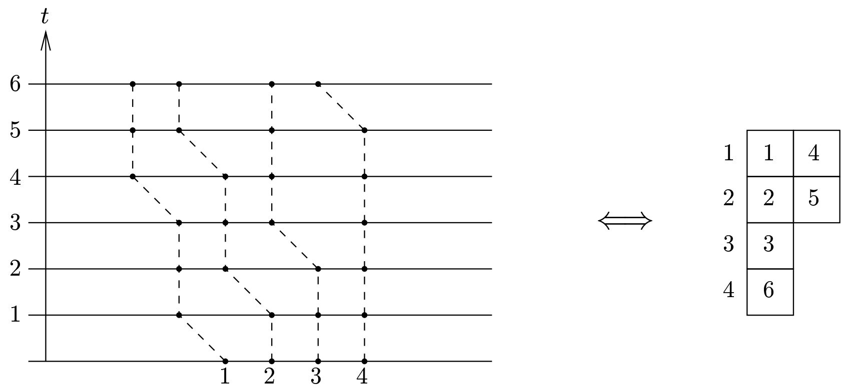

Figure 2. Random turn vicious walkers and Young tableaux.

In this correspondence the number of moves made by the k-th particle (counted from the left) is the number of boxes in the k-th row of the corresponding Young tableau. If we take the limit $p \to \infty$, we remove the constraint on the number of rows. In other words, we start with countably many particles located at $1, 2, 3, \ldots$ and at each time step, we move a randomly selected left-movable particle to its left site. Then the statistics of the number of moves made by the leftmost particle in n time steps is identical to that of the number of boxes in the first row of a random Young diagram under the probability M_n^1 (or the length of the longest increasing subsequence of a random involution). Hence, by Theorem 4.1, we obtain GOE fluctuation in the large n limit.

Now suppose after n steps of left moves, by taking n steps of right moves, we want the particles brought back to their original positions. The first n steps give a Young tableau and the next n steps give another Young tableau. So we are in the situation of pairs of Young tableaux, the case $\beta = 2$. Especially if the number of particles are infinite, the statistics of the moves made by the leftmost particle is identical to the length of the longest increasing subsequence of a random permutation: GUE fluctuation [Fo] in the large n limit.

4.4. Symmetrized Versions of Johansson's Model. In [Jo1], Johansson introduced a certain model which has several probabilistic interpretations, as a randomly growing Young diagram, a totally asymmetric one dimensional exclusion process, a certain zero-temperature directed polymer in a random environment or as a kind of first-passage site percolation model. Here we consider the symmetrized versions. The limiting distributions are parallel to the results of Section 3 depending on the symmetry type.

Each model is defined on $\mathbb{M}$, a subset of $\mathbb{Z}^2$, and at each site $(i,j) \in \mathbb{M}$, we define a random variable $w(i,j)$ of geometric distribution. We are interested in "the length of the longest up/right path". Let $g(q)$ denote the geometric distribution with parameter q, so that $\Pr\big(g(q) = k\big) = (1-q)q^k$ for $k = 0, 1, 2, \dots$. Let $g'(\beta, q)$ be the probability distribution defined by

$$\Pr(g'(\beta,q) = k) = \frac{1-q^2}{1+\beta q} \beta^{k \bmod 2} q^k.$$

Take $0 < q < 1$, and let $\alpha, \beta \geq 0$ such that $\alpha\sqrt{q}, \beta\sqrt{q} < 1$. Otherwise stated, the random variables $w(i,j)$ are independent each other. We denote by $(j,k) \nearrow (j',k')$ the set of up/right paths π from (j,k) to (j',k').

(i) Let $\mathbb{M} = \mathbb{Z}_+^2$. Let

$$w(i,j) \sim g(q).$$

Define

$$G^{\square}(N) = \max\left\{ \sum_{(i,j)\in\pi} w(i,j) : \pi \in (1,1) \nearrow (N,N) \right\}.$$

(ii) Let $\mathbb{M} = \mathbb{Z}^2$. Let $w(i,j) = 0$ if $i = 0$ or $j = 0$, and for $i, j \neq 0$, let

$$w(i,j) = w(-i,-j) \sim g(q).$$

Define

$$G^{\boxdot}(N) = \max\left\{ \sum_{(i,j)\in\pi} w(i,j) : \pi \in (-N,-N) \nearrow (N,N) \right\}.$$

(iii) Let $\mathbb{M} = \mathbb{Z}_+^2$. Let

$$w(i,j) = w(j,i) \sim g(q), \quad i \neq j,$$
$$w(i,i) \sim g(\alpha\sqrt{q}).$$

Define

$$G^{\boxslash}(N) = \max\left\{ \sum_{(i,j)\in\pi} w(i,j) : \pi \in (1,1) \nearrow (N,N) \right\}.$$

(iv) Let $\mathbb{M} = \mathbb{Z}_+ \times \mathbb{Z}_-$. Let

$$w(i,-j) = w(j,-i) \sim g(q), \quad i \neq -j,$$
$$w(i,-i) \sim g'(\beta\sqrt{q}).$$

Define

$$G^{\boxbslash}(N) = \max\left\{ \sum_{(i,j)\in\pi} w(i,j) : \pi \in (1,-N) \nearrow (N,-1) \right\}.$$

(v) Let $\mathbb{M} = \mathbb{Z}^2$. Let $w(i,j) = 0$ if $i = 0$ or $j = 0$. Otherwise

$$\begin{aligned} w(i,j) = w(-i,-j) &\sim g(q), \quad |i| \neq |j|, \\ w(i,i) = w(-i,-i) &\sim g(\alpha\sqrt{q}), \\ w(i,-i) = w(-i,i) &\sim g'(\beta\sqrt{q}). \end{aligned}$$

Define

$$G^{\boxtimes}(N) = \max\left\{ \sum_{(i,j)\in\pi} w(i,j) : \pi \in (-N,-N) \nearrow (N,N) \right\}.$$

We are interested in the limiting distributions of $G^{\circledast}(N)$.

THEOREM 4.2. *Set*

$$\eta(q) = \frac{2\sqrt{q}}{1-\sqrt{q}}, \qquad \rho(q) = \frac{q^{1/6}(1+\sqrt{q})^{1/3}}{1-\sqrt{q}}.$$

Then:

(i)

$$\lim_{N\to\infty} \Pr\left(\frac{G^{\square}(N;q) - \eta(q)N}{\rho(q)N^{1/3}} \leq x \right) = F_2(x).$$

(ii)

$$\lim_{N\to\infty} \Pr\left(\frac{G^{\boxdot}(N;q) - \eta(q)(2N)}{2^{2/3}\rho(q)(2N)^{1/3}} \leq x \right) = F_2(x)^2.$$

(iii)

$$\begin{aligned} &\lim_{N\to\infty} \Pr\left(\frac{G^{⧄}(N;q,\alpha) - \eta(q)N}{\rho(q)N^{1/3}} \leq x \right) = F_4(x), && 0 \leq \alpha < 1, \\ &\lim_{N\to\infty} \Pr\left(\frac{G^{⧄}(N;q,\alpha) - \eta(q)N}{\rho(q)N^{1/3}} \leq x \right) = F^{⧄}(x;w), && \alpha = 1 - \frac{2w}{\rho(q)N^{1/3}}, \\ &\lim_{N\to\infty} \Pr\left(\frac{G^{⧄}(N;q,\alpha) - \eta(q)N}{\rho(q)N^{1/3}} \leq x \right) = 0, && \alpha > 1. \end{aligned}$$

(iv)

$$\lim_{N\to\infty} \Pr\left(\frac{G^{⧅}(N;q,\beta) - \eta(q)N}{\rho(q)N^{1/3}} \leq x \right) = F_1(x), \qquad 0 \leq \beta.$$

(v)

$$\lim_{N\to\infty} \Pr\left(\frac{G^{\boxtimes}(N;q,\alpha,\beta)-\eta(q)(2N)}{2^{2/3}\rho(q)(2N)^{1/3}} \le x\right) = F_2(x), \qquad 0\le\alpha<1,\ \beta\ge 0,$$

$$\lim_{N\to\infty} \Pr\left(\frac{G^{\boxtimes}(N;q,\alpha,\beta)-\eta(q)(2N)}{2^{2/3}\rho(q)(2N)^{1/3}} \le x\right) = F^{\boxtimes}(x;w),$$

$$\alpha = 1-\frac{2w}{\rho(q)(2N)^{1/3}},\ \beta\ge 0,$$

$$\lim_{N\to\infty} \Pr\left(\frac{G^{\boxtimes}(N;q,\alpha,\beta)-\eta(q)(2N)}{2^{2/3}\rho(q)(2N)^{1/3}} \le x\right) = 0, \qquad \alpha>1,\ \beta\ge 0.$$

Remark. The result in part (i) is obtained in [Jo1] in the more general case of $(1,1)\nearrow(M,N)$, $M\neq N$.

The proof of the theorem is analogous to that of the result in Section 3. There again is a Toeplitz/Hankel determinantal formula for each case. In fact, the only change with the symmetrized permutation case is that we have $(1+q+2\sqrt{q}\cos\theta)^N d\theta/(2\pi)$ instead of $e^{2t\cos\theta}d\theta/(2\pi)$ for the weight. In analyzing the asymptotics of orthogonal polynomials, we need different scaling, but once we scale, the analysis is parallel. One important common property of the two weights above is that both of the supports of their equilibrium measures change from the full circle to a part of the circle with one gap depending on $2t/k<1$ and $2t/k>1$, and $N/k<\eta(q)^{-1}$ and $N/k>\eta(q)^{-1}$. And also in the one gap case, the equilibrium measures decay like a square root at the ends of the supports.

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