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Universality and Scaling of Zeros on Symplectic Manifolds

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ABSTRACT. This article is concerned with random holomorphic polynomials and their generalizations to algebraic and symplectic geometry. A natural algebro-geometric generalization involves random holomorphic sections $H^0(M, L^N)$ of the powers of any positive line bundle $L \to M$ over any complex manifold. Our main interest is in the statistics of zeros of k independent sections (generalized polynomials) of degree N as $N \to \infty$. We fix a point P and focus on the ball of radius $1/\sqrt{N}$ about P. Magnifying the ball by the factor $\sqrt{N}$, we found in a prior work that the statistics of the configurations of simultaneous zeros of random k-tuples of sections tend to a universal limit independent of P, M, L. We review this result and generalize it further to the case of pre-quantum line bundles over almost-complex symplectic manifolds (M, J, ω). Following earlier work of Shiffman and Zelditch, we replace $H^0(M, L^N)$ in the complex case with the "asymptotically holomorphic" sections defined by Boutet de Monvel and Guillemin and (from another point of view) by Donaldson and Auroux. We then give a generalization to an m-dimensional setting of the Kac–Rice formula for zero correlations, which we use together with earlier results to prove that the scaling limits of the n-point correlation functions for zeros of random k-tuples of asymptotically holomorphic sections belong to the same universality class as in the complex case. In our prior work, we showed that the limit correlations are short range; here we show further that the limit "connected correlations" decay exponentially with respect to the square of the maximum distance between points.

1. Introduction

A well-known theme in random matrix theory (RMT), zeta functions, quantum chaos, and statistical mechanics, is the universality of scaling limits of correlation functions. In RMT, the relevant correlation functions are for eigenvalues of random matrices (see [De; TW; BZ; BK; So] and their references). In the case of zeta functions, the correlations are between the zeros [KS]. In

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quantum dynamics, they are between eigenvalues of 'typical' quantum maps whose underlying classical maps have a specified dynamics. In the 'chaotic case' it is conjectured that the correlations should belong to the universality class of RMT, while in integrable cases they should belong to that of Poisson processes. The latter has been confirmed for certain families of integrable quantum maps, scattering matrices and Hamiltonians (see [Ze2; RS; Sa; ZZ] and their references). In statistical mechanics, there is a large literature on universality of critical exponents [Car]; other rigorous results include analysis of universal scaling limits of Gibbs measures at critical points [Sin]. In this article we are concerned with a somewhat new arena for scaling and universality, namely that of RPT (random polynomial theory) and its algebro-geometric generalizations [Han; Hal; BBL; BD; BSZ1; BSZ2; SZ1; NV]. The focus of these articles is on the configurations and correlations of zeros of random polynomials and their generalizations, which we discuss below. Random polynomials can also be used to define random holomorphic maps to projective space, but we leave that for the future. Our purpose here is partly to review the results of [SZ1; BSZ1; BSZ2] on universality of scaling limits of correlations between zeros of random holomorphic sections on complex manifolds. More significantly, we give an improved version of our formula from [BSZ2] for determining zero correlations from joint probability distributions, and we apply this formula together with results in [SZ2] to extend our limit zero correlation formulas to the case of almost-complex symplectic manifolds.

Notions of universality depend on context. In RMT, one fixes a set of matrices (for example, a group $U(N)$ or a symmetric space such as $\mathrm{Sym}(N)$, the $N \times N$ real symmetric matrices) and endows it with certain kinds of probability measures μ_N. These measures bias towards certain types of matrices and away from others, and one may ask how the eigenvalue correlations depend on the $\{\mu_N\}$ in the large N limit. In RPT one could similarly endow spaces of polynomials of degree N with a variety of measures, and ask how correlations between zeros depend on them in the large N limit. However, the version of universality which concerns us in this article and in [BSZ1; BSZ2] lies in another direction. We are interested in very general notions of *polynomial* that arise in geometry, and in how the statistics of zeros depends on the geometric setting in which these polynomials live. We will always endow our generalized polynomials with Gaussian measures (or with essentially equivalent spherical measures).

The generalized polynomials studied in [SZ1; BSZ1; BSZ2] were holomorphic sections $H^0(M, L^N)$ of powers of a positive line bundle $L \to M$ over a compact Kähler manifold (M, ω) of a given dimension m. Such sections form the Hilbert space of quantum wave functions which quantize (M, ω) in the sense of geometric quantization [At; Wo]. Recall that geometric quantization begins with a symplectic manifold (M, ω) such that $\frac{1}{\pi}[\omega] \in H^2(M, \mathbb{Z})$. There then exists a complex Hermitian line bundle $(L, h) \to M$ and a Hermitian connection ∇ with curvature ω. To obtain a Hilbert space of sections, one needs additionally to fix a *polarization* of M, i.e. a Lagrangian sub-bundle $\mathcal{L}$ of TM, and we define *polarized sections* to be those satisfying $\nabla_v s = 0$ when v is tangent to $\mathcal{L}$. In the Kähler case, one takes $\mathcal{L} = T^{1,0}M$, the holomorphic sub-bundle. Thus, polarized sections are holomorphic sections. The power N plays the role of the inverse Planck constant, so that the high power $N \to \infty$ limit is the semiclassical limit.

When $M = \mathbb{CP}^m$ and $L = \mathcal{O}(1)$ (the hyperplane section line bundle), holomorphic sections of $L^N = \mathcal{O}(N)$ are just homogeneous holomorphic polynomials of degree N. (In general, one may embed $M \subset \mathbb{CP}^d$ for some d, and then holomorphic sections $s \in H^0(M, L^N)$ may be identified with restrictions of polynomials on $\mathbb{CP}^d$ to M, for $N \gg 0$ by the Kodaira vanishing theorem.) We equip L with a Hermitian metric h and endow M with the volume form dV induced by the curvature ω of L. The pair (h, dV) determine an $\mathcal{L}^2$ norm on $H^0(M, L^N)$ and hence a Gaussian probability measure μ_N. All probabilistic notions such as expectations or correlations are with reference to this measure. The basic theme of the results of [BSZ1; BSZ2] was that in a certain scaling limit, the correlations between zeros are universal in the sense of being independent of M, L, ω, and other details of the setting.

The geometric setting was extended even further by two of the authors in [SZ2] by allowing (M, ω) to be any compact symplectic manifold with integral symplectic form, i.e. $\frac{1}{\pi}[\omega] \in H^2(M, \mathbb{Z})$. Complex line bundles with $c_1(L) = \frac{1}{\pi}[\omega]$ are known in this context as 'pre-quantum line bundles' (cf. [Wo]). It has been known for some time [BG] that there are good analogues of holomorphic sections of powers of such line bundles in this context. Interest in symplectic analogues of holomorphic line bundles and their holomorphic sections has grown recently because of Donaldson's [Do1] use of asymptotically holomorphic sections of powers of pre-quantum line bundles over symplectic manifolds in constructing embedded symplectic submanifolds, Lefschetz pencils and other constructions of an algebro-geometric nature [Do1; Do2; Au1; Au2; AK; BU1; BU2; Sik]. Given an almost complex structure J on M which is compatible with ω, we follow Boutet de Monvel and Guillemin [BG] (see also [GU] and [BU1; BU2]) in defining spaces $H^0_J(M, L^N)$ of *almost holomorphic* sections of the pre-quantum line bundle $L \to M$ with curvature ω. A Hermitian metric h on L and ω determine an $\mathcal{L}^2$ norm and hence a Gaussian measure μ_N on $H^0_J(M, L^N)$.

Our main concern is with the zeros Z_s of k-tuples $s = (s_1, \dots, s_k)$ of holomorphic or almost-holomorphic sections. We let $|Z_s|$ denote Riemannian $(2m-2k)$-volume on Z_s, regarded as a measure on M:

$$(|Z_s|, \varphi) = \int_{Z_s} \varphi \, d\,\mathrm{Vol}_{2m-2k} \ .$$

As in [BSZ1; BSZ2], we introduce the punctured product

$$M_n = \{(z^1, \dots, z^n) \in \underbrace{M \times \cdots \times M}_{n} : z^p \neq z^q \quad \text{for } p \neq q\} \tag{1–1}$$

and consider the product measures on M_n,

$$|Z_s|^n := \big(\underbrace{|Z_s| \times \cdots \times |Z_s|}_{n}\big)\,.$$

The expectation $\mathbf{E}\,|Z_s|^n$ is called the *n-point zero correlation measure.* We write

$$\mathbf{E}\,|Z_s|^n = K^N_{nk}(z^1,\dots,z^n)\,dz\,,$$

where dz denotes the product volume form on M_n. The generalized function $K^N_{nk}(z^1,\dots,z^n)$ is called the *n-point zero correlation function*.

The main points are first to express these correlation measures in terms of the joint probability distribution

$$\widetilde{\mathbf{D}}^N_{(z^1,\dots,z^n)} = \widetilde{D}^N(x^1,\dots,x^n,\xi^1,\dots,\xi^n;z^1,\dots,z^n)\,dx\,d\xi$$

of the random variables $s(z^1),\dots,s(z^n),\nabla s(z^1),\dots,\nabla s(z^n)$, and secondly to prove that the latter has a universal scaling limit. Here dx denotes volume measure on $L^N_{z^1}\oplus\cdots\oplus L^N_{z^n}$, and $d\xi$ is volume measure on $(T^*_M\otimes L^N)_{z^1}\oplus\dots\oplus(T^*_M\otimes L^N)_{z^n}$. For more details and precise definitions, see § 4.1. As for the first point, we have the following formula for the correlation measures in terms of the joint probability distribution:

THEOREM 1.1. *The n-point zero correlation function for random almost holomorphic sections of $L^N\to M$ is given by*

$$K^N_{nk}(z) = \int d\xi\,\widetilde{D}^N_n(0,\xi,z)\prod_{p=1}^n\sqrt{\det(\xi^p\xi^{p*})}\,.$$

One of our main results is Theorem 4.3, which gives a general form of Theorem 1.1 with $H^0_J(M,L^N)$ replaced by a finite dimensional space of sections of an arbitrary vector bundle over a Riemannian manifold. Theorem 4.3 is a generalization of the Kac–Rice formula [Kac; Ri] (see also [BD; EK; Hal; SSm]) to higher dimensions. A special case of Theorem 4.3 was given by J. Neuheisel [Ne] in a parallel study of correlations of nodal sets (zero sets of eigenfunctions of the Laplacian) on spheres.

The correlations of course depend heavily on the geometry of the bundle. For instance, it was shown in [SZ1] that $Z_{s_N}\to\omega$ for almost every sequence $\{s_N\}$ of holomorphic sections of L^N. That is, zeros tend almost surely to congregate in highly (positively) curved regions. To find universal quantities, we scale around a point $z_0\in M$. The most vivid case is where $k=m$ so that almost surely the simultaneous zeros of the k-tuple of sections form a discrete set. The density of zeros in a unit ball $B_1(z_0)$ around z_0 then grows like N^m, so we rescale the zeros in the $1/\sqrt{N}$ ball $B_{1/\sqrt{N}}(z_0)$ by a factor of $\sqrt{N}$ to get configurations of zeros with a constant density as $N\to\infty$. Our problem is whether the statistics of these configurations tend to a limit and whether the limit is universal. In [BSZ2, Th. 3.4] (see also [BSZ1]), it was shown that when L is a positive holomorphic line bundle over a complex manifold M, the scaled n-point correlation functions $K^N_{nk}(\frac{z^1}{\sqrt{N}},\dots,\frac{z^1}{\sqrt{N}})$ converge in the high power limit to a universal correlation function $K^\infty_{nkm}(z^1,\dots,z^n)$ on the punctured product $(\mathbb{C}^m)_n$ depending only on the dimension m of the manifold and the codimension k of the zero set. Our main application of Theorem 1.1 is that this universality law for the scaling limits of the zero correlation functions extends to the general symplectic case:

THEOREM 1.2. *Let L be the pre-quantum line bundle over a $2m$-dimensional compact integral symplectic manifold (M,ω). Let $z_0 \in M$ and choose complex local coordinates $\{z_j\}$ centered at z_0 so that $\omega|_{z_0} = \frac{i}{2}\sum dz_j \wedge d\bar{z}_j$ and $(\partial/\partial z_j)|_{z_0} \in T^{1,0}M$ $(1 \le j \le m)$. Let $\mathcal{S} = H^0_J(M, L^N)^k$ $(k \ge 1)$, and give $\mathcal{S}$ the standard Gaussian measure μ. Then*

$$\frac{1}{N^{nk}} K^N_{nk}\left(\frac{z^1}{\sqrt{N}},\dots,\frac{z^n}{\sqrt{N}}\right) \to K^\infty_{nkm}(z^1,\dots,z^n)$$

(weakly in $\mathcal{D}'((\mathbb{C}^m)_n)$), where $K^\infty_{nkm}(z^1,\dots,z^n)$ is the universal scaling limit in the Kähler setting.

The proof of this result is similar to the holomorphic case [BSZ2]. Using Theorem 1.1, we reduce the scaling limit of $K_n(z)$ to that of the joint probability density $\widetilde{\mathbf{D}}^N_{(z^1,\dots,z^n)}$. It was shown in [SZ2, Theorem 5.4] that the latter has a universal scaling limit:

$$\widetilde{\mathbf{D}}^N_{(z^1/\sqrt{N},\dots,z^n/\sqrt{N})} \longrightarrow \mathbf{D}^\infty_{(z^1,\dots,n^n)}\,, \tag{1–2}$$

where $\mathbf{D}^\infty_{(z^1,\dots,z^n)}$ is a universal Gaussian measure supported on the holomorphic 1-jets, and $\{z_j\}$ are the complex local coordinates of Theorem 1.2.

Let us say a few words on the proof of (1–2). Recall that a Gaussian measure on $\mathbb{R}^p$ is a measure of the form

$$\gamma_\Delta = \frac{e^{-\frac{1}{2}\langle \Delta^{-1}x,x\rangle}}{(2\pi)^{p/2}\sqrt{\det\Delta}}dx_1\cdots dx_p\,, \tag{1–3}$$

where Δ is a positive definite symmetric $p\times p$ matrix. Since $\widetilde{\mathbf{D}}^N_{(z^1,\dots,z^n)}$ is the push-forward of a Gaussian measure, we have $\widetilde{\mathbf{D}}^N_{(z^1,\dots,z^n)} = \gamma_{\Delta^N}$ where Δ^N is the covariance matrix of the random variables $(s(z^p),\nabla s(z^p))$. The main step in the proof in [SZ2] was to show that the covariance matrices Δ^N underlying $\widetilde{\mathbf{D}}^N$ tend in the scaling limit to a semi-positive matrix Δ^∞. To deal with singular measures, we introduced a class of generalized Gaussians whose covariance matrices are only semi-positive definite. A generalized Gaussian is simply a Gaussian supported on the subspace corresponding to the positive eigenvalues of the covariance matrix. It followed that the scaled distributions $\widetilde{\mathbf{D}}^N$ tend to a generalized Gaussian γ_{Δ^∞} 'vanishing in the $\bar\partial$-directions.' To prove that $\Delta^N \to \Delta^\infty$, we expressed Δ^N in terms of the *Szegő kernel* $\Pi_N(x,y)$ and its derivatives. The Szegő kernel is essentially the orthogonal projection from $\mathcal{L}^2(M,L^N) \to H^0_J(M,L^N)$. Since it is more convenient to deal with scalar kernels than sections, we pass from

$L \to M$ to the associated principal S^1 bundle $X \to M$. Sections s of L^N are then canonically identified with equivariant functions $\hat{s}$ on X transforming by $e^{iN\theta}$ under the S^1 action. The space $H^0_J(M, L^N)$ then corresponds to a space $\mathcal{H}^2_N(X)$ of equivariant functions. In the holomorphic case, these functions are CR functions; i.e., they satisfy the tangential Cauchy–Riemann equations $\bar\partial_b \hat{s} = 0$. In the symplectic almost-complex case they are 'almost CR' functions in a sense defined by Boutet de Monvel and Guillemin. The scalar Szegő kernels are then the orthogonal projections $\Pi_N : \mathcal{L}^2(X) \to \mathcal{H}^2_N(X)$. The main ingredient in the proof of (1–2) was the *scaling asymptotics* of the Szegő kernels $\Pi_N(x, y)$. In 'preferred' local coordinates (z, θ) on X (see § 3.2), the scaling asymptotics read:

$$\Pi_N\Big(z_0 + \frac{u}{\sqrt{N}}, \frac{\theta}{N}, z_0 + \frac{v}{\sqrt{N}}, \frac{\varphi}{N}\Big) \sim e^{i(\theta-\varphi)} e^{u\cdot\bar{v} - \frac{1}{2}(|u|^2+|v|^2)} \Big\{1 + \frac{1}{\sqrt{N}} p_1(u, v; z_0) + \cdots\Big\}.$$

The universal limit correlation functions $K^\infty_{nkm}(z^1, \dots, z^n)$ are described in [BSZ2] (see also [BSZ1]). They are given in terms of the level 1 Szegő kernel for the (reduced) Heisenberg group (see § 2.3),

$$\Pi^{\mathbf{H}}_1(z, \theta; w, \varphi) = \frac{1}{\pi^m} e^{i(\theta-\varphi)+i\,\mathrm{Im}(z\cdot\bar{w}) - \frac{1}{2}|z-w|^2} = \frac{1}{\pi^m} e^{i(\theta-\varphi)+z\cdot\bar{w} - \frac{1}{2}(|z|^2+|w|^2)},$$

and its first and second derivatives at the points $(z, w) = (z^p, z^{p'})$. Indeed, the correlation functions are universal rational functions in $z^p_q, \bar{z}^p_q, e^{z^p\cdot\bar{z}^{p'}}$, and are smooth functions on $(\mathbb{C}^m)_n$. We let

$$\widetilde{K}_{nkm}(z^1, \dots, z^n) := (K^\infty_{1km})^{-n} K_{nkm}(z^1, \dots, z^n)$$

denote the "normalized" n-point limit correlation function, where

$$K^\infty_{1km} = \frac{m!}{\pi^k (m-k)!}$$

is the expected volume density of the zero set. For example [BSZ1; BSZ2],

$$\widetilde{K}^\infty_{21m}(z^1, z^2) = \frac{\left[\frac{1}{2}(m^2+m)\sinh^2 t + t^2\right]\cosh t - (m+1)t\sinh t}{m^2 \sinh^3 t} + \frac{(m-1)}{2m}, \quad \text{for } t = \frac{|z^1 - z^2|^2}{2}. \tag{1–4}$$

Formula (1–4) with $m = 1$ agrees with the scaling limit pair correlation function of Hannay [Han] (see also [BBL]) for zeros of polynomials in one complex variable, i.e. for $M = \mathbb{CP}^1$ and $L = \mathcal{O}(1)$.

The correlations are "short range" in the sense that $\widetilde{K}_{nkm}(z^1, \dots, z^n) = 1 + O(r^4 e^{-r^2})$, where r is the minimum distance between the points z^p [BSZ2]. We show in § 5.3 that in fact the "connected n-point correlations" are $o(e^{-R^2/n})$, where R is the maximum distance between the points.

2. Line Bundles on Complex Manifolds

We begin with some notation and basic properties of sections of holomorphic line bundles, their zero sets, and Szegő kernels. We also provide two examples that will serve as model cases for studying correlations of zeros of sections of line bundles in the high power limit.

2.1. Sections of Holomorphic Line Bundles. Let $L \to M$ be a holomorphic line bundle over a compact complex manifold. Thus, at each $z \in M$, $L_z \simeq \mathbb{C}$ is a complex line and locally, over a sufficiently small open set $U \subset M$, $L \simeq U \times \mathbb{C}$. For background on line bundles and other objects of complex geometry, we refer to [GH].

A key notion is that of *positive* line bundle. By definition, this means that there exists a smooth Hermitian metric h on L with positive curvature form

$$\Theta_h = -\partial\bar{\partial} \log \|e_L\|_h^2 ,$$

where e_L denotes a local holomorphic frame (= nonvanishing section) of L over an open set $U \subset M$, and $\|e_L\|_h = h(e_L, e_L)^{1/2}$ denotes the h-norm of e_L. A basic example is the hyperplane bundle $\mathcal{O}(1) \to \mathbb{CP}^m$, the dual of the tautological line bundle. When $m = 1$, its square is the holomorphic tangent bundle $T\mathbb{CP}^1$. Its positivity is equivalent to the positivity of the curvature of $\mathbb{CP}^1$ in the usual sense of differential geometry. Hyperbolic surfaces $\mathbf{H}^2/\Gamma$ have negatively curved tangent bundles, but their cotangent bundles $T^*(\mathbf{H}^2/\Gamma)$ are positively curved. In the case of complex tori $\mathbb{C}/\Lambda$ (where $\Lambda \subset \mathbb{C}$ is a lattice), both the tangent and cotangent bundles are flat. The positive 'pre-quantum' line bundle there is the bundle whose sections are theta functions.

Intuitively speaking, positive curvature at w creates a potential well which traps a particle near z. On the quantum level, this particle is a wave function (holomorphic section) $\Pi_N(z, w)$ which is concentrated at w. This wave function is known to mathematicians as the 'Szegő kernel', and to physicists as the 'coherent state' centered at w. The simplest (but non-compact) case is where $M = \mathbb{C}^m$ and where $\Theta_h = \sum_{j=1}^m dz_j \wedge d\bar{z}_j$ (cf. [Do1]). We note that $\Theta_h = dA$, where $A = \frac{1}{2}\sum_{j=1}^m z_j d\bar{z}_j - \bar{z}_j dz_j$ is a connection form on the trivial bundle $L = \mathbb{C}^m \times \mathbb{C} \to \mathbb{C}^m$. The associated covariant derivative on sections is given by $\bar{\partial}_A f = \bar{\partial} f + A^{0,1} f$, where $A^{0,1}$ is the $(0,1)$ component of A. Then $\bar{\partial}_A e^{-|z|^2/2} = 0$, i.e. there is a Gaussian holomorphic section concentrated at $w = 0$. As will be seen in § 2.3, it is essentially the Szegő kernel of the Heisenberg group.

According to the above intuition, positive line bundles should have a plentiful supply of global holomorphic sections. Indeed, the space $H^0(M, L^N)$ of holomorphic sections of $L^N = L \otimes \cdots \otimes L$ is a complex vector space of dimension $d_N = \frac{c_1(L)}{m!} N^m + \cdots$ given by the Hilbert polynomial ([Kl; Na]; see [SSo, Lemma 7.6]). It is in part because the dimension d_N increases so rapidly with N that probabilities and correlations simplify so much as $N \to \infty$.

To define the term 'Szegő kernel' we need to define a Hilbert space structure on $H^0(M, L^N)$: We give M the Hermitian metric corresponding to the Kähler form $\omega = \frac{\sqrt{-1}}{2}\Theta_h$ and the induced Riemannian volume form

$$dV_M = \frac{1}{m!}\omega^m \,. \tag{2–1}$$

Since $\frac{1}{\pi}\omega$ is a de Rham representative of the Chern class $c_1(L) \in H^2(M, \mathbb{R})$, it follows from (2–1) that $\mathrm{Vol}(M) = \frac{\pi^m}{m!}c_1(L)^m$.

The metric h induces Hermitian metrics h^N on L^N given by $\|s^{\otimes N}\|_{h^N} = \|s\|_h^N$. We give $H^0(M, L^N)$ the Hermitian inner product

$$\langle s_1, s_2 \rangle = \int_M h^N(s_1, s_2)\, dV_M \qquad (s_1, s_2 \in H^0(M, L^N))\,. \tag{2–2}$$

We first define the Szegő kernels as the orthogonal projections

$$\Pi_N : \mathcal{L}^2(M, L^N) \to H^0(M, L^N).$$

The projections Π_N can be given in terms of orthonormal bases $\{S_j^N\}$ of sections of $H^0(M, L^N)$ by

$$\Pi_N(z, w) = \sum_{j=1}^{d_N} S_j^N(z) \otimes \overline{S_j^N(w)}\,,$$

so that

$$(\Pi_N s)(w) = \int_M h_z^N\big(s(z), \Pi_N(z, w)\big)\, dV_M(z)\,, \quad s \in \mathcal{L}^2(M, L^N)\,.$$

Since we are studying the asymptotics of the Π_N as $N \to \infty$, we find it useful to instead view the Szegő kernels as projections on the same space of functions. We show how this is accomplished below.

2.2. Lifting the Szegő Kernel. As in [BG; Ze1; SZ1; BSZ2; SZ2] and elsewhere, we analyze the $N \to \infty$ limit by lifting the analysis of holomorphic sections over M to a certain S^1 bundle $X \to M$. We let L^* denote the dual line bundle to L, and we consider the circle bundle $X = \{\lambda \in L^* : \|\lambda\|_{h^*} = 1\}$, where h^* is the norm on L^* dual to h. Let $\pi : X \to M$ denote the bundle map; if $v \in L_z$, then $\|v\|_h = |(\lambda, v)|$, $\lambda \in X_z = \pi^{-1}(z)$. Note that X is the boundary of the disc bundle $D = \{\lambda \in L^* : \rho(\lambda) > 0\}$, where $\rho(\lambda) = 1 - \|\lambda\|_{h^*}^2$. The disc bundle D is strictly pseudoconvex in L^*, since Θ_h is positive, and hence X inherits the structure of a strictly pseudoconvex CR manifold. Associated to X is the contact form $\alpha = -i\partial\rho|_X = i\bar\partial\rho|_X$. We also give X the volume form

$$dV_X = \frac{1}{m!}\alpha \wedge (d\alpha)^m = \alpha \wedge \pi^* dV_M\,. \tag{2–3}$$

The setting for our analysis of the Szegő kernel is the Hardy space $\mathcal{H}^2(X) \subset \mathcal{L}^2(X)$ of square-integrable CR functions on X, i.e., functions that are annihilated by the Cauchy–Riemann operator $\bar\partial_b$ (see [St, pp. 592–594]) and are $\mathcal{L}^2$

with respect to the inner product

$$\langle F_1, F_2\rangle = \frac{1}{2\pi}\int_X F_1\overline{F_2}dV_X\,, \quad F_1, F_2 \in \mathcal{L}^2(X)\,. \tag{2–4}$$

Equivalently, $\mathcal{H}^2(X)$ is the space of boundary values of holomorphic functions on D that are in $\mathcal{L}^2(X)$. We let $r_\theta x = e^{i\theta}x$ $(x \in X)$ denote the S^1 action on X and denote its infinitesimal generator by $\frac{\partial}{\partial\theta}$. The S^1 action on X commutes with $\bar\partial_b$; hence $\mathcal{H}^2(X) = \bigoplus_{N=0}^\infty \mathcal{H}^2_N(X)$ where $\mathcal{H}^2_N(X) = \{F \in \mathcal{H}^2(X) : F(r_\theta x) = e^{iN\theta}F(x)\}$. A section s_N of L^N determines an equivariant function $\hat s_N$ on L^* by the rule

$$\hat s_N(\lambda) = \left(\lambda^{\otimes N}, s_N(z)\right)\,, \quad \lambda \in L^*_z\,,\ z \in M\,,$$

where $\lambda^{\otimes N} = \lambda\otimes\cdots\otimes\lambda$. We henceforth restrict $\hat s$ to X and then the equivariance property takes the form $\hat s_N(r_\theta x) = e^{iN\theta}\hat s_N(x)$. The map $s \mapsto \hat s$ is a unitary equivalence between $H^0(M, L^N)$ and $\mathcal{H}^2_N(X)$. (This follows from (2–3)–(2–4) and the fact that $\alpha = d\theta$ along the fibers of $\pi : X \to M$.)

We now define the (lifted) Szegő kernel to be the orthogonal projection $\Pi_N : \mathcal{L}^2(X) \to \mathcal{H}^2_N(X)$. It is defined by

$$\Pi_N F(x) = \int_X \Pi_N(x,y)F(y)\,dV_X(y)\,, \quad F \in \mathcal{L}^2(X)\,. \tag{2–5}$$

As above, it can be given as

$$\Pi_N(x,y) = \sum_{j=1}^{d_N} \widehat{S}_j^N(x)\overline{\widehat{S}_j^N(y)}\,, \tag{2–6}$$

where $S_1^N, \dots, S_{d_N}^N$ form an orthonormal basis of $H^0(M, L^N)$. Note that although the Szegő kernel Π_N is defined on X, its absolute value is well-defined on M. In particular, on the diagonal we have

$$\Pi_N(z,z) = \Pi_N(z,\theta; z,\theta) = \sum_{j=1}^{d_N} \|S_j^N(z)\|^2_{h^N}\,.$$

2.3. Model Examples. The Szegő kernels and their derivatives were worked out explicitly in [BSZ2] for two model cases, namely for the hyperplane section bundle over $\mathbb{CP}^m$ and for the Heisenberg bundle over $\mathbb{C}^m$, i.e. the trivial line bundle with curvature equal to the standard symplectic form on $\mathbb{C}^m$. These cases are important, since by universality, the scaling limits of correlation functions for all line bundles coincide with those of the model cases.

In fact, the two models are locally equivalent in the CR sense. In the case of $\mathbb{CP}^m$, the circle bundle X is the $2m+1$ sphere S^{2m+1}, which is the boundary of the unit ball $B^{2m+2} \subset \mathbb{C}^{m+1}$. In the case of $\mathbb{C}^m$, the circle bundle is the reduced Heisenberg group $\mathbf{H}^m_{\rm red}$, which is a discrete quotient of the simply connected Heisenberg group $\mathbb{C}^m \times \mathbb{R}$.

We summarize here the formulas for the Szegő kernels from [BSZ2] in these model cases; for further details see [BSZ2, § 1.3]. For the first example (see also [SZ1, § 4.2]), $M = \mathbb{CP}^m$ and L is the hyperplane section bundle $\mathcal{O}(1)$. Sections $s \in H^0(\mathbb{CP}^m, \mathcal{O}(1))$ are linear functions on $\mathbb{C}^{m+1}$, so that the zero divisors Z_s are projective hyperplanes. The line bundle $\mathcal{O}(1)$ carries a natural metric $h_{\rm FS}$ given by

$$\|s\|_{h_{\rm FS}}([w]) = \frac{|(s,w)|}{|w|}\,, \qquad w = (w_0,\dots,w_m) \in \mathbb{C}^{m+1}\,,$$

for $s \in \mathbb{C}^{m+1*} \equiv H^0(\mathbb{CP}^m, \mathcal{O}(1))$, where $|w|^2 = \sum_{j=0}^m |w_j|^2$ and $[w] \in \mathbb{CP}^m$ denotes the complex line through w. The Kähler form on $\mathbb{CP}^m$ is the Fubini–Study form

$$\omega_{\rm FS} = \frac{\sqrt{-1}}{2}\Theta_{h_{\rm FS}} = \frac{\sqrt{-1}}{2}\partial\bar\partial \log |w|^2\,.$$

The dual bundle $L^* = \mathcal{O}(-1)$ is the affine space $\mathbb{C}^{m+1}$ with the origin blown up, and $X = S^{2m+1} \subset \mathbb{C}^{m+1}$. The N-th tensor power of $\mathcal{O}(1)$ is denoted $\mathcal{O}(N)$. An orthonormal basis for the space $H^0(\mathbb{CP}^m, \mathcal{O}(N))$ of homogeneous polynomials on $\mathbb{C}^{m+1}$ of degree N is the set of monomials:

$$s_J^N = \left[\frac{(N+m)!}{\pi^m j_0!\cdots j_m!}\right]^{\frac12} z^J\,, \quad z^J = z_0^{j_0}\cdots z_m^{j_m}\,, \qquad J = (j_0,\dots,j_m),\ |J| = N$$

Hence the Szegő kernel for $\mathcal{O}(N)$ is given by

$$\Pi_N(x,y) = \sum_J \frac{(N+m)!}{\pi^m j_0!\cdots j_m!} x^J \bar y^J = \frac{(N+m)!}{\pi^m N!}\langle x,y\rangle^N\,.$$

Note that

$$\Pi(x,y) = \sum_{N=1}^\infty \Pi_N(x,y) = \frac{m!}{\pi^m}(1-\langle x,y\rangle)^{-(m+1)}\,,$$

which is the classical Szegő kernel for the $(m+1)$-ball.

The second example is the linear model $\mathbb{C}^m \times \mathbb{C} \to \mathbb{C}^m$ for positive line bundles $L \to M$ over Kähler manifolds and their associated Szegő kernels. Its associated principal S^1 bundle $\mathbb{C}^m \times S^1 \to \mathbb{C}^m$, which may be identified with the boundary of the disc bundle $D \subset L^*$ in the dual line bundle, is the *reduced Heisenberg group* $\mathbf{H}^m_{\rm red}$. Let us summarize its definition and properties. We start with the usual (simply connected) Heisenberg group $\mathbf{H}^m = \mathbb{C}^m \times \mathbb{R}$ with group law

$$(\zeta,t)\cdot(\eta,s) = (\zeta+\eta, t+s+\mathrm{Im}(\zeta\cdot\bar\eta)).$$

The identity element is $(0,0)$ and $(\zeta,t)^{-1} = (-\zeta,-t)$. The Lie algebra of $\mathbf{H}_m$ is spanned by elements $Z_1,\dots,Z_m,\bar Z_1,\dots,\bar Z_m,T$ satisfying the canonical commutation relations $[Z_j,\bar Z_k] = -i\delta_{jk}T$ (all other brackets are zero). Below we will select such a basis of left invariant vector fields.

We can regard $\mathbf{H}^m$ as a strictly convex CR manifold which may be embedded in $\mathbb{C}^{m+1}$ as the boundary of a strictly pseudoconvex domain, namely the upper half space $\mathcal{U}^m := \{z \in \mathbb{C}^{m+1} : \operatorname{Im} z_{m+1} > \frac{1}{2}\sum_{j=1}^m |z_j|^2\}$. $\mathbf{H}^m$ acts simply transitively on $\partial\mathcal{U}^m$ (cf. [St], XII), and we get an identification of $\mathbf{H}^m$ with $\partial\mathcal{U}^m$ by:

$$[\zeta, t] \to (\zeta, t + i|\zeta|^2) \in \partial\mathcal{U}^m .$$

The linear model for the principal S^1 bundle is the reduced Heisenberg group $\mathbf{H}^m_{\rm red} = \mathbf{H}^m/\{(0, 2\pi k) : k \in \mathbb{Z}\} = \mathbb{C}^m \times S^1$ with group law

$$(\zeta, e^{it}) \cdot (\eta, e^{is}) = (\zeta + \eta, e^{i[t+s+\operatorname{Im}(\zeta\cdot\bar\eta)]}).$$

It is the principal S^1 bundle over $\mathbb{C}^m$ associated to the line bundle $L_{\mathbf{H}} = \mathbb{C}^m \times \mathbb{C}$. The metric on $L_{\mathbf{H}}$ with curvature $\Theta = \sum dz_q \wedge d\bar z_q$ is given by setting $h_{\mathbf{H}}(z) = e^{-|z|^2}$; i.e., $|f|_{h_{\mathbf{H}}} = |f|e^{-|z|^2/2}$. The reduced group $\mathbf{H}^m_{\rm red}$ may be viewed as the boundary of the dual disc bundle $D \subset L^*_{\mathbf{H}}$ and hence is a strictly pseudoconvex CR manifold.

We then define the Hardy space $\mathcal{H}^2(\mathbf{H}^m_{\rm red})$ of CR holomorphic functions to be the functions in $\mathcal{L}^2(\mathbf{H}^m_{\rm red})$ satisfying the left-invariant Cauchy–Riemann equations $\bar Z^L_q f = 0$ $(1 \le q \le m)$ on $\mathbf{H}^m_{\rm red}$. Here, $\{\bar Z^L_q\}$ denotes a basis of the left-invariant anti-holomorphic vector fields on $\mathbf{H}^m_{\rm red}$. Let us recall their definition: we first equip $\mathbf{H}^m_{\rm red}$ with its left-invariant connection form $\alpha^L = \frac{1}{2}(\sum_q (u_q dv_q - v_q du_q) + d\theta)$ $(\zeta = u + iv)$, whose curvature equals the symplectic form $\omega = \sum_q du_q \wedge dv_q$. The left-invariant (CR-) holomorphic vector fields Z^L_q and anti-holomorphic vector fields $\bar Z^L_q$ are the horizontal lifts of the vector fields $\frac{\partial}{\partial z_q}$ and $\frac{\partial}{\partial \bar z_q}$, respectively, with respect to α^L. They span the left-invariant CR structure of $\mathbf{H}^m_{\rm red}$ and are given by

$$Z^L_q = \frac{\partial}{\partial z_q} + \frac{i}{2}\bar z_q \frac{\partial}{\partial\theta}, \quad \bar Z^L_q = \frac{\partial}{\partial \bar z_q} - \frac{i}{2} z_q \frac{\partial}{\partial\theta}.$$

The vector fields $\{\frac{\partial}{\partial\theta}, Z^L_q, \bar Z^L_q\}$ span the Lie algebra of $\mathbf{H}^m_{\rm red}$ and satisfy the canonical commutation relations above.

For $N = 1, 2, \ldots$, we define $\mathcal{H}^2_N \subset \mathcal{H}^2(\mathbf{H}^m_{\rm red})$ as the (infinite-dimensional) Hilbert space of square-integrable CR functions f such that $f \circ r_\theta = e^{iN\theta} f$ as before. The Szegő kernel $\Pi^{\mathbf{H}}_N(x, y)$ is the orthogonal projection to $\mathcal{H}^2_N$. It is given by

$$\Pi^{\mathbf{H}}_N(x, y) = \frac{1}{\pi^m} N^m e^{iN(t-s)} e^{N(\zeta\cdot\bar\eta - \frac{1}{2}|\zeta|^2 - \frac{1}{2}|\eta|^2)}, \qquad x = (\zeta, t)\,,\ y = (\eta, s)\,.$$

The Szegő kernels $\Pi^{\mathbf{H}}_N$ are Heisenberg dilates of the level 1 kernel $\Pi^{\mathbf{H}}_1$:

$$\Pi^{\mathbf{H}}_N(x, y) = N^m \Pi^{\mathbf{H}}_1(\delta_{\sqrt N}\, x, \delta_{\sqrt N}\, y)\,,$$

where the Heisenberg dilations (or scalings) δ_r are the automorphisms of $\mathbf{H}^m$

$$\delta_r(z, \theta) = (rz, r^2\theta)\,, \quad r \in \mathbb{R}^+\,.$$

(The dilation $\delta_{\sqrt{N}}$ descends to a homomorphism of $\mathbf{H}^m_{\rm red}$.)

Remark. The group $\mathbf{H}^m_{\rm red}$ acts by left translation on $\mathcal{H}^2_1$. The generators of this representation are the right-invariant vector fields $Z^R_q, \bar{Z}^R_q$ together with $\frac{\partial}{\partial\theta}$. They are horizontal with respect to the right-invariant contact form $\alpha^R = \frac{1}{2}(\sum_q (u_q dv_q - v_q du_q) - d\theta)$ and are given by:

$$Z^R_q = \frac{\partial}{\partial z_q} - \frac{i}{2}\bar{z}_q \frac{\partial}{\partial\theta}, \quad \bar{Z}^R_q = \frac{\partial}{\partial \bar{z}_q} + \frac{i}{2} z_q \frac{\partial}{\partial\theta}.$$

In physics terminology, Z^R_q is known as an annihilation operator and $\bar{Z}^R_q$ is a creation operator.

The representation $\mathcal{H}^2_1$ is irreducible and may be identified with the Bargmann–Fock space of entire holomorphic functions on $\mathbb{C}^n$ which are square integrable relative to $e^{-|z|^2}$. The identification goes as follows: the function $\varphi_o(z,\theta) := e^{i\theta} e^{-|z|^2/2}$ is CR-holomorphic and is also the ground state for the right invariant annihilation operator; i.e., it satisfies

$$\bar{Z}^L_q \varphi_o(z,\theta) = 0 = Z^R_q \varphi_o(z,\theta).$$

In the physics terminology, the level 1 Szegő kernel $\Pi^{\mathbf{H}}_1$, which is the left translate of φ_o by $(-w, -\varphi)$, is the coherent state associated to the phase space point w. Any element $F(z,\theta)$ of $\mathcal{H}^2_1$ may be written in the form $F(z,\theta) = f(z)\varphi_o$. Then $\bar{Z}^L_q F = (\frac{\partial}{\partial \bar{z}_q} f)\varphi_o$, so that F is CR if and only if f is holomorphic. Moreover, $F \in \mathcal{L}^2(\mathbf{H}^m_{\rm red})$ if and only if f is square integrable relative to $e^{-|z|^2}$.

3. Almost-Complex Symplectic Manifolds

In [SZ2], the study of the Szegő kernel was extended to almost-complex symplectic manifolds, and parametrices and resulting off-diagonal asymptotics for the Szegő kernel were obtained in this general setting. We now summarize the basic geometric and analytic constructions of [SZ2] for the almost-complex symplectic case.

We denote by (M,ω) a compact symplectic manifold such that $[\frac{1}{\pi}\omega]$ is an integral cohomology class. We also fix a compatible almost complex structure J satisfying $\omega(v, Jv) > 0$. We denote by $T^{1,0}M$, respectively $T^{0,1}M$, the holomorphic (respectively anti-holomorphic) sub-bundles of the complex tangent bundle, i.e. $J = i$ on $T^{1,0}$ and $J = -i$ on $T^{0,1}$. It is well known (see [Wo, Prop. 8.3.1]) that there exists a Hermitian line bundle $(L,h) \to M$ and a metric connection ∇ on L whose curvature Θ_L satisfies $\frac{i}{2}\Theta_L = \omega$. The 'quantization' of (M,ω) at Planck constant $1/N$ should be a Hilbert space of polarized sections of the N-th tensor power L^N of L ([GS, p. 266]). In the complex case, polarized sections are simply holomorphic sections. The notion of polarized sections is problematic in the non-complex symplectic setting, since the Lagrangian subspaces $T^{1,0}M$ defining the complex polarization are not integrable and there usually are no 'holomorphic' sections. A subtle but compelling replacement for the notion of polarized section has been proposed by Boutet de Monvel and Guillemin [BG], and it is this notion which was used in [SZ2].

To define these polarized sections, we work as above on the associated principal S^1 bundle $X \to M$ with $X = \{v \in L^* : |v|_h = 1\}$. We let α be the connection 1-form on X given by ∇; we then have $\frac{1}{\pi} d\alpha = \pi^*\omega$, and thus α is a contact form on X, i.e., $\alpha \wedge (d\alpha)^m$ is a volume form on X. In the complex case, X was a CR manifold. In the general almost-complex symplectic case it is an almost CR manifold. The almost CR structure is defined as follows: The kernel of α defines a horizontal hyperplane bundle $H \subset TX$. Using the projection $\pi : X \to M$, we may pull back J to an almost complex structure on H. We denote by $H^{1,0}$, respectively $H^{0,1}$ the eigenspaces of eigenvalue i, respectively $-i$, of J. The splitting $TX = H^{1,0} \oplus H^{0,1} \oplus \mathbb{C}\frac{\partial}{\partial\theta}$ defines the almost CR structure on TX. We also define local orthonormal frames $Z_1, \dots, Z_n$ of $H^{1,0}$, respectively $\bar{Z}_1, \dots, \bar{Z}_m$ of $H^{0,1}$, and dual orthonormal coframes $\vartheta_1, \dots, \vartheta_m$, respectively $\bar{\vartheta}_1, \dots, \bar{\vartheta}_m$. On the manifold X we have $d = \partial_b + \bar{\partial}_b + \frac{\partial}{\partial\theta} \otimes \alpha$, where $\partial_b = \sum_{j=1}^m \vartheta_j \otimes Z_j$ and $\bar{\partial}_b = \sum_{j=1}^m \bar{\vartheta}_j \otimes \bar{Z}_j$. Note that for an $\mathcal{L}^2$ section s^N of L^N, we have

$$(\nabla_{L^N} s^N)\widehat{\ } = d^h \hat{s}^N\,,$$

where $d^h = \partial_b + \bar{\partial}_b$ is the horizontal derivative on X.

3.1. The $\bar{D}$ Complex and Szegő Kernels. In the complex case, a holomorphic section s of L^N lifts to a function $\hat{s} \in \mathcal{L}^2_N(X)$ satisfying $\bar{\partial}_b \hat{s} = 0$. The operator $\bar{\partial}_b$ extends to a complex satisfying $\bar{\partial}_b^2 = 0$, which is a necessary and sufficient condition for having a maximal family of CR holomorphic coordinates. In the non-integrable case $\bar{\partial}_b^2 \neq 0$, and there may be no solutions of $\bar{\partial}_b f = 0$. To define polarized sections and their equivariant lifts, Boutet de Monvel and Guillemin [Bou; BG] defined a complex $\bar{D}_j$, which is a good replacement for $\bar{\partial}_b$ in the non-integrable case. Their main result is:

THEOREM 3.1 [BG, Lemma 14.11 and Theorem A 5.9]. *There exists an S^1-invariant complex of first order pseudodifferential operators $\bar{D}_j$ over X*

$$0 \to C^\infty(\Lambda_b^{0,0}) \xrightarrow{\bar{D}_0} C^\infty(\Lambda_b^{0,1}) \xrightarrow{\bar{D}_1} \cdots \xrightarrow{\bar{D}_{m-1}} C^\infty(\Lambda_b^{0,m}) \longrightarrow 0\,,$$

where $\Lambda_b^{0,j} = \Lambda^j(H^{0,1}X)^*$, such that:

(i) *$\sigma(\bar{D}_j) = \sigma(\bar{\partial}_b)$ to second order along $\Sigma := \{(x, r\alpha_x) : x \in X, r > 0\} \subset T^*X$;*
(ii) *The orthogonal projector $\Pi : \mathcal{L}^2(X) \to \mathcal{H}^2(X)$ onto the kernel of $\bar{D}_0$ is a complex Fourier integral operator which is microlocally equivalent to the Cauchy–Szegő projector of the holomorphic case;*
(iii) *$(\bar{D}_0, \frac{\partial}{\partial\theta})$ is jointly elliptic.*

We refer to the kernel $\mathcal{H}^2(X) = \ker \bar{D}_0 \cap \mathcal{L}^2(X)$ as the Hardy space of square-integrable 'almost CR functions' on X. The $\mathcal{L}^2$ norm is with respect to the inner product (2–4) as in the holomorphic case. Since the S^1 action on X commutes with $\bar{D}_0$, we have as before the decomposition $\mathcal{H}^2(X) = \bigoplus_{N=0}^{\infty} \mathcal{H}^2_N(X)$, where $\mathcal{H}^2_N(X)$ denotes the almost CR functions on X that transform by the factor $e^{iN\theta}$ under the action r_θ. By property (iii) above, they are smooth functions. We denote by $H^0_J(M, L^N)$ the space of sections corresponding to $\mathcal{H}^2_N(X)$ under the map $s \mapsto \hat{s}$. Elements of $H^0_J(M, L^N)$ are the 'almost holomorphic sections' of L^N. (Note that products of almost holomorphic sections are not necessarily almost holomorphic.) We henceforth identify $H^0_J(M, L^N)$ with $\mathcal{H}^2_N(X)$. By the Riemann–Roch formula of [BG, Lemma 14.14], the dimension of $H^0_J(M, L^N)$ (or $\mathcal{H}^2_N(X)$) is given by $d_N = \frac{c_1(L)}{m!} N^m + \cdots$ (for N sufficiently large), as before. (The estimate $d_N \sim \frac{c_1(L)}{m!} N^m$ also follows from [SZ2, § 4.2].)

As before, we let $\Pi_N : \mathcal{L}^2(X) \to \mathcal{H}^2_N(X)$ denote the orthogonal projection. The level N Szegő kernel $\Pi_N(x, y)$ is given as in the holomorphic case by (2–5) or (2–6), using an orthonormal basis $S^N_1, \dots, S^N_{d_N}$ of $H^0_J(N, L^N) \equiv \mathcal{H}^2_N(X)$.

3.2. Scaling Limit of the Szegő Kernel. Our analysis is based on the near-diagonal scaling asymptotics of the Szegő kernel from [SZ2]. These asymptotics are given in terms of the Heisenberg dilations $\delta_{\sqrt{N}}$, using local 'Heisenberg coordinates' at a point $x_0 \in X$. These coordinates are given in terms of *preferred coordinates* at $P_0 = \pi(x_0)$ and a *preferred frame* at P_0. A coordinate system $(z_1, \dots, z_m)$ on a neighborhood U of P_0 is said to be preferred if

$$(g - i\omega)|_{P_0} = \sum_{j=1}^{m} dz_j \otimes d\bar{z}_j\big|_0 .$$

Here g denotes the Riemannian metric $g(v, w) := \omega(v, Jw)$ induced by the symplectic form ω. Preferred coordinates satisfy the following three (redundant) conditions:

(i) $\partial/\partial z_j|_{P_0} \in T^{1,0}(M)$, for $1 \le j \le m$,
(ii) $\omega(P_0) = \omega_0$,
(iii) $g(P_0) = g_0$,

where ω_0 is the standard symplectic form and g_0 is the Euclidean metric:

$$\omega_0 = \frac{i}{2}\sum_{j=1}^{m} dz_j \wedge d\bar{z}_j = \sum_{j=1}^{m}(dx_j \otimes dy_j - dy_j \otimes dx_j)\,, \quad g_0 = \sum_{j=1}^{m}(dx_j \otimes dx_j + dy_j \otimes dy_j)\,.$$

A *preferred frame* for $L \to M$ at P_0 is a local frame (=nonvanishing section) e_L on U such that

(i) $\|e_L\|_{P_0} = 1$;
(ii) $\nabla e_L|_{P_0} = 0$;
(iii) $\nabla^2 e_L|_{P_0} = -(g + i\omega) \otimes e_L|_{P_0} \in T^*_M \otimes T^*_M \otimes L$.

A preferred frame can be constructed by multiplying an arbitrary frame by a function with specified 2-jet at P_0; any two such frames necessarily agree to third order at P_0.

DEFINITION. A *Heisenberg coordinate chart* at a point x_0 in the principal bundle X is a coordinate chart $\rho : U \approx V$ with $0 \in U \subset \mathbb{C}^m \times \mathbb{R}$ and $\rho(0) = x_0 \in V \subset X$ of the form

$$\rho(z_1, \dots, z_m, \theta) = e^{i\theta} h(z)^{1/2} e_L^*(z)\,,$$

where e_L is a preferred local frame for $L \to M$ at $P_0 = \pi(x_0)$, and $(z_1, \dots, z_m)$ are preferred coordinates centered at P_0. We require that P_0 have coordinates $(0, \dots, 0)$ and $e_L^*(P_0) = x_0$.

The following near-diagonal asymptotics of the Szegő kernel is the key analytical result on which our analysis of the scaling limit for correlations of zeros is based.

THEOREM 3.2 [SZ2, Theorem 2.3]. *Let $P_0 \in M$ and choose a Heisenberg coordinate chart about P_0. Then*

$$\begin{aligned} &N^{-m}\Pi_N\left(\frac{u}{\sqrt{N}}, \frac{\theta}{N}; \frac{v}{\sqrt{N}}, \frac{\varphi}{N}\right) \\ &\qquad = \Pi_1^{\mathbf{H}}(u, \theta; v, \varphi)\left[1 + \sum_{r=1}^{K} N^{-r/2} b_r(P_0, u, v) + N^{-(K+1)/2} R_K(P_0, u, v, N)\right], \end{aligned}$$

where $\|R_K(z_0, u, v, N)\|_{\mathcal{C}^j(\{|u| \le \rho,\ |v| \le \rho\}} \le C_{K,j,\rho}$ for $j \ge 0$, $\rho > 0$ and $C_{K,j,\rho}$ is independent of the point z_0 and choice of coordinates.

This asymptotic formula has several applications to symplectic geometry, in addition to our result on zero correlations. For example, Theorem 3.2 is used in [SZ2] to obtain symplectic versions of the following results in complex geometry:

(i) the asymptotic expansion theorem of [Ze1],
(ii) the Tian almost isometry theorem [Ti],
(iii) the Kodaira embedding theorem (see [GH] or [SSo]).

The symplectic forms of these theorems are based on the symplectic Kodaira maps $\Phi_N : M \to PH^0_J(M, L^N)^*$, which are defined as in the holomorphic case by $\Phi_N(z) = \{s^N : s^N(z) = 0\}$. Equivalently, we choose an orthonormal basis $S_1^N, \dots, S_{d_N}^N$ of $H^0_J(M, L^N)$ and write

$$\Phi_N : M \to \mathbb{CP}^{d_N - 1}\,, \qquad \Phi_N(z) = \left(S_1^N(z) : \dots : S_{d_N}^N(z)\right).$$

We now state the symplectic generalizations of the above three theorems:

THEOREM 3.3 [SZ2, Theorems 3.1–3.2]. *Let $L \to (M, \omega)$ be the pre-quantum line bundle over a $2m$-dimensional symplectic manifold, and let $\{\Phi_N\}$ be its Kodaira maps. Then*:

(i) *There exists a complete asymptotic expansion*:

$$\Pi_N(z,z) = a_0 N^m + a_1(z) N^{m-1} + a_2(z) N^{m-2} + \dots$$

for certain smooth coefficients $a_j(z)$ *with* $a_0 = \pi^{-m}$. *Hence*, *the maps* Φ_N *are well-defined for* $N \gg 0$.

(ii) *Let* ω_{FS} *denote the Fubini–Study form on* $\mathbb{CP}^{d_N-1}$. *Then*

$$\left\| \frac{1}{N} \Phi_N^*(\omega_{FS}) - \omega \right\|_{\mathcal{C}^k} = O\Big(\frac{1}{N}\Big)$$

for any k.

(iii) *For* N *sufficiently large*, Φ_N *is an embedding*.

For proofs we refer to [SZ2]. (See also [BU2] for a proof of a similar Kodaira embedding theorem.)

4. Correlations of Zeros

In § 5, we shall use Theorem 3.2 and the methods of [BSZ2] to extend the results of [BSZ1; BSZ2] on the universality of the scaling limit of the n-point zero correlations to the case of almost complex symplectic manifolds. The basis for our argument is Theorem 2.2 from [BSZ2], which generalizes a formula of Kac [Kac] and Rice [Ri] for zeros of functions on $\mathbb{R}^1$, and of [Hal] for zeros of (real) Gaussian vector fields (see also [BD; EK; Ne; SSm]). However, we shall need to consider the case where the joint probability distributions are singular, and hence we give below a complete proof of a more general result (Theorem 4.3) on the correlations of zeros of sections of $\mathcal{C}^\infty$ vector bundles.

4.1. General Formula for Zero Correlations. For our general setting, we let (V,h) be a $\mathcal{C}^\infty$ (real) vector bundle over an oriented $\mathcal{C}^\infty$ Riemannian manifold (M,g). (Here, h denotes a $\mathcal{C}^\infty$ metric on V.) Suppose that $\mathcal{S}$ is a finite dimensional subspace of the space $\mathcal{C}^\infty(M,V)$ of global $\mathcal{C}^\infty$ sections of V, and let $d\mu$ be a probability measure on $\mathcal{S}$ given by a semi-positive $\mathcal{C}^0$ 'rapidly decaying' volume form. We say that a $\mathcal{C}^0$ volume form $\psi dx_1 \wedge \cdots dx_d$ on $\mathbb{R}^d$ is *rapidly decaying* if $\psi(x) = o(\|x\|^{-N})$ for all $N \in \mathbb{Z}^+$. (In this paper, we are primarily interested in the case where $d\mu$ is a Gaussian measure.) The purpose of this section is to study the zero set Z_s of a random section $s \in \mathcal{S}$ and to obtain formulas for the expected value and n-point correlations of the volume measure $|Z_s|$. We shall later apply our results to the case where $V = L^N \oplus \cdots \oplus L^N$, for a complex line bundle L over a compact almost complex symplectic manifold M and where $\mathcal{S} = \mathcal{H}^2_N \oplus \cdots \oplus \mathcal{H}^2_N$. (Recall that $\mathcal{H}^2_N$ is the space of almost holomorphic sections of L^N.) Then the zero sets Z_s are the simultaneous zeros of (random) k-tuples of almost holomorphic sections.

Our formulation involving general vector bundles also allows us to reduce the study of n-point correlations to the case $n = 1$, i.e., to expected densities (or volumes) of zero sets. We first describe the formula (Theorem 4.2) for this expected zero density. This formula is given in terms of the 'joint probability density,' which is a measure on the space $J^1(M,V)$ of 1-jets of sections of V.

Recall that we have the exact sequence of vector bundles

$$0 \longrightarrow T^*_M \otimes V \xrightarrow{\iota} J^1(M,V) \xrightarrow{\pi_V} V \to 0\,. \tag{4–1}$$

We let

$$\mathcal{E} : M \times \mathcal{S} \to V\,, \quad \mathcal{E}(z,s) = s(z)$$

denote the evaluation map, and we say that $\mathcal{S}$ *spans* V if $\mathcal{E}$ is surjective, i.e., if $\{s(z) : s \in \mathcal{S}\}$ spans V_z for all $z \in M$. We are mainly interested in the jet map

$$\mathcal{J} : M \times \mathcal{S} \to J^1(M,V)\,, \quad \mathcal{J}(z,s) = J^1_z s = \text{ the 1-jet of } s \text{ at } z\,.$$

Note that $\mathcal{E} = \pi_V \circ \mathcal{J}$.

Note that a *measure* on an N-dimensional manifold Y is a current $\nu \in \mathcal{D}^0(Y)' = \mathcal{D}'^N(Y)$ of order 0. We can write $\nu = f d\,\mathrm{Vol}_Y$, where $f \in \mathcal{D}'^0(Y)$. (Recall that $\mathcal{D}^p(Y)$ denotes the space of compactly supported $\mathcal{C}^\infty$ p-forms on Y, and $\mathcal{D}'^p(Y) = \mathcal{D}^{N-p}(Y)'$.) Some authors refer to f as a measure, but to keep the distinction, we shall call elements of $\mathcal{D}'^0(Y)$ *generalized functions*.

To describe the induced volume forms on the total spaces of the bundles in (4–1), we write $g(z) = \sum g_{qq'}(z)\, du_q \otimes du_{q'}$, $h_{jj'} = h(e_j, e_{j'})$, where $\{u_1, \dots, u_m\}$ are local coordinates in M and $\{e_1, \dots, e_k\}$ is a local frame in V (here $m = \dim M$, $k = \operatorname{rank} V$). We let $G = \det(g_{qq'})$, $H = \det(h_{jj'})$. We further let $dz = \sqrt{G} du_1 \wedge \cdots \wedge du_m$ denote Riemannian volume in M, and we write

$$x = \sum_j x_j e_j(z) \in V_z, \quad dx = \sqrt{H(z)}\, dx_1 \wedge \cdots \wedge dx_k\,,$$

$$\xi = \sum_{j,q} \xi_{jq} du_q \otimes e_j|_z \in (T^*_M \otimes V)_z, \quad d\xi = G(z)^{-k/2} H(z)^{m/2} \prod_{j,q} d\xi_{jq}\,.$$

The induced volume measures on V and $T^*_M \otimes V$ are given by $dx\, dz$ and $d\xi dz$ respectively. We give V a connection that preserves h; its covariant derivative provides a splitting $\nabla : J^1(M,V) \to T^*_M \otimes V$ of (4–1), and hence $dx\, d\xi\, dz$ provides a volume form on $J^1(M,V)$.

Definition. The 1-*jet density* of μ is the measure

$$\mathbf{D} := \mathcal{J}_*(dz \times \mu)$$

on the space $J^1(M,V)$ of 1-jets. We write

$$\mathbf{D} = D(x,\xi,z)\, dx\, d\xi\, dz \quad D(x,\xi,z) \in \mathcal{D}'^0(J^1(M,V))\,.$$

We let ρ_ε denote a $\mathcal{C}^\infty$ 'approximate identity' on V of the form

$$\rho_\varepsilon(v) = \varepsilon^{-k}\rho(\varepsilon^{-1}v)\,,$$

with

$$\rho \in \mathcal{C}^\infty(V)\,,\ \int_{V_z} \rho(x,z)\,dx = 1\,, \quad \rho \geq 0\,, \quad \rho(v) = 0 \text{ for } \|v\| \geq 1\,.$$

We let $\tilde\rho_\varepsilon \in \mathcal{C}^\infty(J^1(M,V))$ be given by

$$\tilde\rho_\varepsilon(x,\xi,z) = \rho_\varepsilon(x,z)\,.$$

or formally, $\tilde\rho_\varepsilon = \rho_\varepsilon \circ \pi_V$.

Lemma 4.1. *Suppose that $\mathcal{E}$ spans V. Then there exists a unique positive measure $\mathbf{D}^0$ on $T^*_M \otimes V$ such that*

$$\iota_*\mathbf{D}^0 = \lim_{\varepsilon\to 0} \tilde\rho_\varepsilon \mathbf{D}\,.$$

Moreover, $\mathbf{D}^0$ *is independent of the choice of local frame* $\{e_j\}$, *connection* ∇, *and approximate identity* ρ_ε.

Proof. The surjectivity of $\mathcal{E} = \pi_V \circ \mathcal{J}$ guarantees that the normal bundle N_ι is disjoint from the wave front set of $D(x,\xi,z)$ and hence $\iota^* D(x,\xi,z)$ is well-defined (see [Hö, Th. 8.2.4]). Thus we can define

$$\mathbf{D}^0 := \iota^* D(x,\xi,z)\,d\xi\,dz\,. \tag{4–2}$$

To verify the equation of the lemma, it suffices by the continuity of ι^* to consider the case where $D(x,\xi,z) \in \mathcal{C}^\infty$. In this case, $\mathbf{D}^0 = D(0,\xi,z)\,d\xi\,dz$, and hence

$$\tilde\rho_\varepsilon\mathbf{D} \to \delta_0(x) D(0,\xi,z)\,dx\,d\xi\,dz = \iota_*\big(D(0,\xi,z)\,d\xi\,dz\big) = \iota_*\mathbf{D}^0\,.$$

Since dx and $d\xi$ are intrinsic volume forms, it follows that $\mathbf{D}^0$ is independent of the choice of local frame $\{e_j\}$ (and local coordinates). To show that $D(0,\xi,z)$ does not depend on the choice of connection on V, write $s = \sum x_j e_j$, $\nabla s = \sum \xi_{jq}\,dz_q \otimes e_j$, $\xi_{jq} = \frac{\partial x_j}{\partial z_q} + \sum_k x_k \theta^k_{jq}$. Then if we consider the flat connection $\nabla' s = \sum \xi'_{jq} dz_q \otimes e_j$, $\xi'_{jq} = \frac{\partial x_j}{\partial z_q}$, we have

$$\frac{\partial(\xi_{jq}, x_j)}{\partial(\xi'_{jq}, x_j)} = 1\,.$$

Hence, $dx\,d\xi' = dx\,d\xi$ so that $D'(0,\xi,z) = D(0,\xi,z)$. ☐

We note that

$$(J^1_{z_0})_*\mu = D(x,\xi,z_0)\,dx\,d\xi\,,$$

so that $D(x,\xi,z_0)\,dx\,d\xi$ is the joint probability distribution of the random variables $X^{z_0}_j$, $\Xi^{z_0}_{jq}$ on $\mathcal{S}$ given by

$$X^{z_0}_j(s) = x_j(z_0)\,, \quad \Xi^{z_0}_{jq}(s) = \xi_{jq}(z_0) \qquad (1 \leq j \leq k,\, 1 \leq q \leq m)\,.$$

This is a special case of the *n-point joint probability distribution* defined below.

For a vector-valued 1-form $\xi \in T^*_{M,z} \otimes V_z = \mathrm{Hom}(T_{M,z}, V_z)$, we let $\xi^* \in \mathrm{Hom}(V_z, T_{M,z})$ denote the adjoint to ξ (i.e., $\langle \xi^* v, t\rangle = \langle v, \xi t\rangle$). We consider the endomorphism $\xi\xi^* \in \mathrm{Hom}(V_z, V_z)$, and we write

$$|||\xi||| = \sqrt{\det(\xi\xi^*)}\,.$$

(Note that $|||\cdot|||$ is not a norm.) In terms of a local frame $\{e_j\}$,

$$|||\xi||| = \sqrt{H}\,\|\xi_1 \wedge \ldots \wedge \xi_k\|\,, \quad \xi = \sum_j \xi_j \otimes e_j\,. \tag{4–3}$$

To verify (4–3), write

$$\xi_j = \sum_{q=1}^m \xi_{jq} du_q\,;$$

then

$$\xi^* = \sum_{j,q} \xi^*_{jq} \frac{\partial}{\partial u_q} \otimes e^*_j\,, \qquad \xi^*_{jq} = \sum_{j',q'} h_{jj'} \gamma_{q'q} \xi_{j'q'}\,,$$

where $\big(\gamma_{qq'}\big) = \big(g_{qq'}\big)^{-1}$; hence we have

$$\xi\xi^* = \sum_{j,j',j'',q,q'} h_{j'j''} \xi_{jq} \gamma_{q'q} \xi_{j''q'}\, e_j \otimes e^*_{j'}\,.$$

Its determinant is given by

$$det(\xi\xi^*) = H \det\left(\sum_{q,q'} \xi_{jq}\gamma_{q'q}\xi_{j'q'} \right)_{1\le j,j'\le k} = H \det\langle \xi_j, \xi_{j'}\rangle = H\,\|\xi_1 \wedge \ldots \wedge \xi_k\|^2\,,$$

which gives (4–3).

Let us assume that $\mathcal{S}$ spans V. Then the incidence set $I := \{(z,s) \in M \times \mathcal{S} : s(z) = 0\}$ is a smooth submanifold and hence by Sard's theorem applied to the projection $I \to \mathcal{S}$, the zero set

$$Z_s = \{z \in M : z(s) = 0\}$$

is a smooth $(m-k)$-dimensional submanifold of M for almost all s. (In the holomorphic case, this is called Bertini's Theorem.) We let $|Z_s|$ denote Riemannian $(m-k)$-volume on Z_s, regarded as a measure on M:

$$(|Z_s|, \varphi) = \int_{Z_s} \varphi d\,\mathrm{Vol}_{m-k} \qquad \text{for a.a.} \quad s \in \mathcal{S}\,.$$

Its expected value is the positive measure $\mathbf{E}\,|Z_s|$ given by

$$(\mathbf{E}\,|Z_s|, \varphi) = \mathbf{E}\,(|Z_s|, \varphi) = \int_{\mathcal{S}} d\mu(s) \int_{Z_s} \varphi d\,\mathrm{Vol}_{m-k} \le +\infty\,,$$

for $\varphi \in \mathcal{C}^0(M)$ and $\varphi \geq 0$. (Recall that $\mathbf{E}$ denotes expectation.) In fact the following general density formula tells us that $(\mathbf{E}\,|Z_s|, \varphi) < +\infty$ if the test function φ has compact support.

THEOREM 4.2. *Let $M, V, \mathcal{S}, d\mu$ be as above, and suppose that $\mathcal{S}$ spans V. Then*

$$\mathbf{E}\,|Z_s| = \pi_*\big(\sqrt{\det(\xi\xi^*)}\,\mathbf{D}^0\big) \in \mathcal{D}'^m(M)\,, \tag{4–4}$$

where $\pi : T^*_M \otimes V \to M$ is the projection.

Note that although $\mathbf{D}^0$ depends on the metric h on V, the measure $\sqrt{\det(\xi\xi^*)}\mathbf{D}^0$ is independent of h. In the case where $D(x, \xi, z) \in \mathcal{C}^0$, (4–4) becomes

$$\mathbf{E}\,|Z_s| = K_1(z)\,dz\,, \quad K_1(z) = \int D(0, \xi, z)\sqrt{\det(\xi\xi^*)}\,d\xi\,. \tag{4–5}$$

Before proceeding further, we first give a heuristic explanation of (4–5). Suppose that $D \in \mathcal{C}^0$ and fix a point $z_0 \in M$. Let us consider the case where $\operatorname{rank} V = \dim M = m$ so that the zeros are discrete. Then the probability of finding a zero in a small ball $\mathbb{B}_r = \mathbb{B}_r(z_0)$ of radius r about z_0 is approximately $K_1(z_0)\operatorname{Vol}(\mathbb{B}_r)$. If the radius r is very small, we can suppose that the sections $s \in \mathcal{S}$ are approximately linear:

$$s(z) \approx X^{z_0} + \Xi^{z_0} \cdot (z - z_0)\,, \tag{4–6}$$

where we have written s in terms of a local frame for V and local coordinates in M. Here, $X^{z_0} = X^{z_0}(s) = \big(X^{z_0}_j(s)\big)$, respectively $\Xi^{z_0} = \Xi^{z_0}(s) = \big(\Xi^{z_0}_{jq}(s)\big)$, is a vector-valued, respectively matrix-valued, random variable on $\mathcal{S}$. Then the probability that the linearized section s given by (4–6) has a zero in $\mathbb{B}_r$ is given by

$$\begin{aligned}\mu\big\{s \in \mathcal{S} : X^{z_0} \in \Xi^{z_0}(\mathbb{B}_r)\big\} &= \int_{\mathbb{R}^{m^2}}\int_{\xi(\mathbb{B}_r)} D(x, \xi, z_0)\,dx\,d\xi \\ &\approx \int \operatorname{Vol}(\xi(\mathbb{B}_r))D(0, \xi, z_0)\,d\xi\,.\end{aligned}$$

Since $\operatorname{Vol}(\xi(\mathbb{B}_r)) = |||\xi|||\operatorname{Vol}(\mathbb{B}_r)$, we have

$$K_1(z_0) \approx \frac{\mu\big\{s \in \mathcal{S} : X^{z_0} \in \Xi^{z_0}(\mathbb{B}_r)\big\}}{\operatorname{Vol}(\mathbb{B}_r)} \approx \int D(0, \xi, z_0)|||\xi|||\,d\xi\,.$$

The linear approximation (4–6) leads to a similar explanation in the case where $\operatorname{rank} V < \dim M$; we leave this to the reader.

Before embarking on the proof of Theorem 4.2, we show how the theorem provides a generalization of Theorem 1.1 on the correlations between zeros. Let us first review the definition of these correlations.

DEFINITION. Let $M, V, \mathcal{S}, d\mu$ be as above, and suppose that $\mathcal{S}$ spans V. Let M_n denote the punctured product (1–1). The *n-point zero correlation measure* is the expectation $\mathbf{E}\,|Z_s|^n$, where

$$|Z_s|^n = \big(\underbrace{|Z_s| \times \cdots \times |Z_s|}_{n}\big)\,,$$

which is a well-defined measure on M_n for almost all $s \in \mathcal{S}$. We write

$$\mathbf{E}\,|Z_s|^n = K_n(z^1, \ldots, z^n)\,dz\,.$$

The generalized function $K_n(z^1, \ldots, z^n)$ is called the *n-point zero correlation function.*

We suppose $n \geq 2$ and write

$$\tilde{s}(z) = (s(z^1), \ldots, s(z^n))$$

for $(z^1, \ldots, z^n) \in M_n$, regarded as a section of the vector bundle

$$V_n := \bigoplus_{p=1}^{n} \pi_p^* V \longrightarrow M_n\,,$$

where $\pi_p : M_n \to M$ denotes the projection onto the p-th factor. We then have the evaluation map

$$\mathcal{E}_n : M_n \times \mathcal{S} \to V\,, \quad \mathcal{E}_n(z,s) = \tilde{s}(z)\,,$$

and the jet map

$$\mathcal{J}_n : M_n \times \mathcal{S} \to J^1(M_n, V_n)\,, \quad \mathcal{J}(z,s) = J^1_z\tilde{s} = (J^1_{z^1}s, \ldots, J^1_{z^n}s)\,.$$

We also write

$$\begin{aligned}
&x = (x^1, \ldots, x^n) \in V_n\,,\\
&\xi = (\xi^1, \ldots, \xi^n) \in (T^*_M \otimes V)_{z^1} \oplus \cdots \oplus (T^*_M \otimes V)_{z^n} \subset \left(T^*_{M_n} \otimes V_n\right)_z\,,\\
&dx = dx^1 \cdots dx^n\,, \quad d\xi = d\xi^1 \cdots d\xi^n\,, \quad dz = dz^1 \cdots dz^n\,.
\end{aligned}$$

DEFINITION. The *n-point density* at $(z^1, \ldots, z^n) \in M_n$ is the probability measure

$$\mathbf{D}_n := D_n(x, \xi, z)\,dx\,d\xi\,dz = \mathcal{J}_{n*}(dz \times \mu)$$

on the space $J^1(M_n, V_n)$. Note that this measure is supported on the sub-bundle

$$\pi_1^*(T^*_M \otimes V) \oplus \ldots \oplus \pi_n^*(T^*_M \otimes V) \subset T^*_{M_n} \otimes V_n\,.$$

The *(n-point) joint probability distribution* at $(z^1, \ldots, z^n)$ is the joint probability distribution $D_n(x, \xi, z)\,dx\,d\xi = (J^1_z)_*\mu$ of the (complex) random variables

$$X^z_{jp}(s) := x_j(z^p)\,, \quad \Xi^z_{jpq}(s) := \xi_{jq}(z^p) \qquad (1 \leq j \leq k,\ 1 \leq p \leq n,\ 1 \leq q \leq m)\,.$$

If the evaluation map $\mathcal{E}_n$ is surjective, we also write as before

$$\mathbf{D}_n^0 = \iota^* D(x,\xi,z)\, d\xi\, dz\,,$$

so that

$$\iota_* \mathbf{D}_n^0 = \lim_{\varepsilon\to 0} \tilde\rho_\varepsilon^n \mathbf{D}_n\,.$$

Thus, Theorem 4.2 applied to $V_n \to M_n$ yields our general formula for the n-point correlations of zeros:

THEOREM 4.3. *Let $V \to M$ be a $\mathcal{C}^\infty$ vector bundle over an oriented Riemannian manifold. Consider the ensemble $(\mathcal{S},\mu)$, where $\mathcal{S}$ is a finite-dimensional subspace of $\mathcal{C}^\infty(M,V)$ and μ is given by a $\mathcal{C}^0$ rapidly decaying volume form on $\mathcal{S}$. Suppose that $\mathcal{S}$ spans V_n, where n is a positive integer. Then*

$$\mathbf{E}\,|Z_s|^n = \pi_*\left(\sqrt{\textstyle\prod_{p=1}^n \det(\xi^p\xi^{p*})}\; \mathbf{D}^0\right)\,. \tag{4–7}$$

In the case where $D_n(x,\xi,z) \in \mathcal{C}^0$, (4–7) becomes

$$\mathbf{E}\,|Z_s|^n = K_n(z)\,dz\,,\quad K_n(z) = \int d\xi\, D_n(0,\xi,z) \prod_{p=1}^n \sqrt{\det(\xi^p\xi^{p*})}\,.$$

Our proof of Theorem 4.2 uses the following *coarea formula* of Federer:

LEMMA 4.4 [Fe, 3.2.12]. *Let $f : Y \to \mathbb{R}^k$ be a $\mathcal{C}^\infty$ map, where Y is an oriented m-dimensional Riemannian manifold. For $\gamma \in \mathcal{L}^1(Y)$, we have*

$$\int_{\mathbb{R}^k} dx_1\cdots dx_k \int_{f^{-1}(x)} \gamma\, d\,\mathrm{Vol}_{m-k} = \int_Y \gamma\, \|df_1\wedge\cdots\wedge df_k\|\, d\,\mathrm{Vol}_Y\ .$$

Recall that by Sard's theorem, $f^{-1}(x)$ is an $(m-k)$-dimensional submanifold for almost all $x \in \mathbb{R}^k$.

As a consequence of Lemma 4.4, for $\psi \in \mathcal{C}^0(\mathbb{R}^k)$ we have

$$\int_{\mathbb{R}^k} \psi(x)|f^{-1}(x)|\, dx_1\cdots dx_k = (\psi\circ f)\|df_1\wedge\cdots\wedge df_k\|\, d\,\mathrm{Vol}_Y \in \mathcal{D}'^m(Y)\,, \tag{4–8}$$

where $|f^{-1}(x)|$ denotes $(m-k)$-dimensional volume measure on $f^{-1}(x)$.

REMARK. Federer's coarea formula, which is actually valid for Lipschitz maps, can be regarded as in integrated form of the *Leray formula*

$$|f^{-1}(x)| = \|df_1\wedge\cdots\wedge df_k\| \left.\frac{d\,\mathrm{Vol}_Y}{df_1\wedge\cdots\wedge df_k}\right|_{f^{-1}(x)}\,.$$

PROOF OF THEOREM 4.2. We restrict to a neighborhood U of an arbitrary point $z_0 \in M$. Since $\mathcal{S}$ spans V, we can choose U so that there exist sections $e_1,\dots,e_k \in \mathcal{S}$ that form a local frame for V over U. Since $\mathbf{D}^0$ is independent of the connection, we can further assume that $\nabla|_U$ is the flat connection $\nabla s = \sum ds_j \otimes e_j$.

For a section $s \in \mathcal{S}$, we write $s(z) = \sum_{j=1}^{k} s_j(z) e_j(z)$ $(z \in U)$ and we let $\hat{s} = (s_1, \dots, s_k) : U \to \mathbb{R}^k$. Then

$$|||\nabla s||| = \sqrt{H}\, \|ds_1 \wedge \cdots \wedge ds_k\| .$$

Thus by (4–8),

$$\int_{\mathbb{R}^k} \rho_\varepsilon(x) |\hat{s}^{-1}(x)|\, dx = (\rho_\varepsilon \circ s)\, |||\nabla s|||\, dz \in \mathcal{D}'^m(U)\,, \tag{4–9}$$

where we write, as before, $dx = \sqrt{H(z)} dx_1 \cdots dx_k$.

Let π_U, π' denote the projections given in the commutative diagram:

$$\begin{tikzcd} U \times \mathcal{S} \arrow[r, "\mathcal{J}"] \arrow[dr, "\pi_U"'] & J^1(U, V) \arrow[d, "\pi'"] & T^*_U \otimes V \arrow[l, "\iota"'] \arrow[dl, "\pi"] \\ & U & \end{tikzcd}$$

Integrating (4–9) over $\mathcal{S}$, we obtain

$$\int_{\mathbb{R}^k} \rho_\varepsilon(x) \mathbf{E}\, |\hat{s}^{-1}(x)|\, dx = \pi_{U*}(\rho_\varepsilon \circ s\, |||\nabla s|||\, dz \times \mu) = \pi'_*\big(\rho_\varepsilon(x)\, |||\xi|||\, \mathbf{D}\big)$$
$$\to \pi'_*\big(|||\xi|||\, \iota_* \mathbf{D}^0\big) = \pi_*(|||\xi|||\, \mathbf{D}^0)\,. \tag{4–10}$$

To complete the proof of Theorem 4.2, it suffices to show that the map

$$\Psi : \mathbb{R}^k \to \mathcal{D}'^m(U)\,, \quad \Psi(x) = \mathbf{E}\, |\hat{s}^{-1}(x)|$$

is continuous; i.e., for all test functions $\varphi \in \mathcal{D}(U)$, the map $x \mapsto \mathbf{E}\, (|\hat{s}^{-1}(x)|, \varphi)$ is continuous. Indeed, if Ψ is continuous, then

$$\left(\int_{\mathbb{R}^k} \rho_\varepsilon(x) \mathbf{E}\, |\hat{s}^{-1}(x)|\, dx, \varphi \right) = \int \mathbf{E}\, (|\hat{s}^{-1}(x)|, \varphi) \rho_\varepsilon(x)\, dx$$
$$\to \mathbf{E}\, (|\hat{s}^{-1}(0)|, \varphi) = \mathbf{E}\, (|Z_s|, \varphi)\,,$$

and (4–4) follows from (4–10).

To verify the continuity of Ψ, we extend $\{e_1, \dots, e_k\}$ to a basis $\{e_1, \dots, e_k, \dots, e_d\}$ of $\mathcal{S}$, and we write

$$d\mu(s) = \psi(c_1, \dots, c_d)\, dc\,, \quad s = \sum_{i=1}^{d} c_i e_i\,.$$

We note that

$$\hat{s}^{-1}(x_1, \dots, x_k) = Z_{\left[s - \sum_1^k x_j e_j\right]}\,,$$

and therefore

$$\Psi(x_1, \dots, x_k) = \int |Z_s| \psi(c_1 + x_1, \dots, c_k + x_k, c_{k+1}, \dots, c_d)\, dc\,.$$

Write $c+x=(c_1+x_1,\dots,c_k+x_k,c_{k+1},\dots,c_d)$. We let $\tau: I\to\mathbb{R}^d$ denote the projection given by

$$\tau\Big(z,\sum c_ie_i\Big)=(c_1,\dots,c_d)\,.$$

For a test function $\varphi\in\mathcal{D}(U)$, we have

$$\begin{aligned}(\Psi(x),\varphi)&=\int_{\mathbb{R}^d}(|Z_s|,\varphi)\psi(c+x)\,dc=\int_{\mathbb{R}^d}\big(|\tau^{-1}(c)|,\varphi(z)\big)\,\psi(c+x)\,dc\\&=\int_I\varphi(z)\psi(c+x)\|dc_1\wedge\cdots\wedge dc_d\|_I\,d\operatorname{Vol}_I(z,c)\,,\end{aligned}\tag{4–11}$$

where the last equality is by the coarea formula (4–8) applied to τ.

Suppose that $x^\nu\to x^0\in\mathbb{R}^k$. In order to use (4–11) to show that $(\Psi(x^\nu),\varphi)\to(\Psi(x^0),\varphi)$, we note that $\|dc_1\wedge\cdots\wedge dc_d\|_I\le 1$ and hence

$$\varphi(z)\psi(c+x^\nu)\|dc_1\wedge\cdots\wedge dc_d\|_I\le\varphi(z)\gamma(\|c\|-R)\,,$$

where

$$\gamma(r)=\sup_{\|c\|\ge r}\psi(c)\,,\qquad R=\sup_\nu\|x^\nu\|\,.$$

We let $I(r)=\big\{(z,\sum c_ie_i)\in I:\|c\|=r\big\}$ denote the sphere bundle of radius r in the vector bundle $I\to M$. Then

$$\begin{aligned}\int_I\varphi(z)\gamma(\|c\|-R)\,d\operatorname{Vol}_I(z,c)&=\int_0^{+\infty}dr\;\gamma(r-R)\int_{I(r)}\varphi(z)\,d\operatorname{Vol}_{I(r)}\\&\le C\int_0^{+\infty}dr\;\gamma(r-R)r^{d-1}\,.\end{aligned}$$

Since by hypothesis $\gamma(r)=o(r^{-d-1})$, we conclude that the integral is finite and thus the Lebesgue dominated convergence theorem implies that $(\Psi(x^\nu),\varphi)\to(\Psi(x^0),\varphi)$. □

4.2. Zero Correlations on Complex Manifolds. We now describe the jet density $\mathbf{D}$ in the case where (V,h) is a complex Hermitian vector bundle. In this case, we choose a complex local frame $\{e_1,\dots,e_k\}$ and we let $H_{\mathbb{C}}=\det(h_{jj'})$, $h_{jj'}=h(e_j,e_{j'})$. We write

$$x=\sum_j x_je_j\,,\quad \xi=\sum_{j,q}\xi_{jq}du_q\otimes e_j=\sum_j\xi_j\otimes e_j\,,$$

where $\xi_{jq},\ x_j$ are complex. We then have

$$\mathbf{D}=D(x,\xi,z)\,dx\,d\xi\,dz\,,$$

where this time

$$dx=H_{\mathbb{C}}(z)\prod_j d\operatorname{Re}x_j\,d\operatorname{Im}x_j\,,\quad d\xi=G(z)^{-k/2}H_{\mathbb{C}}(z)^m\prod_{j,q}d\operatorname{Re}\xi_{jq}\,d\operatorname{Im}\xi_{jq}\,.$$

We also have

$$|||\xi|||=H_{\mathbb{C}}\|\xi_1\wedge\cdots\wedge\xi_k\wedge\bar\xi_1\wedge\cdots\wedge\bar\xi_k\|\,.$$

We can now specialize Theorems 4.2–4.3 to the case where V is a holomorphic line bundle over a complex manifold M and the sections in $\mathcal{S}$ are holomorphic. If we now let $\{z_q\}$ denote complex local coordinates, we can write

$$\xi = \xi' + \xi'' = \sum_{j,q} (\xi'_{jq}\, dz_q + \xi''_{jq} d\bar{z}_q) \otimes e_j \,.$$

Since $\bar{\partial} s = 0$ for all $s \in \mathcal{S}$, the support of the measure $\mathbf{D}$ is contained in $V \oplus (T^{*h}_M \otimes V)$, i.e., those (x, ξ) with $\xi''_{jq} \equiv 0$ (using a holomorphic frame $\{e_j\}$ and a connection ∇ of type (1,0)). Hence on the support of $\mathbf{D}$, we have $\xi_j \in T^{*h}_M$, and hence

$$|||\xi||| = H\, \|\xi_1 \wedge \cdots \wedge \xi_k\|^2 = \det(\xi\xi^*)_{\mathbb{C}} \,,$$

where $(\xi\xi^*)_{\mathbb{C}} \in \mathrm{Hom}_{\mathbb{C}}(V_z, V_z)$ denotes the complex endomorphism. Hence as a special case of Theorem 4.3, we obtain:

THEOREM 4.5. *Let $V \to M$ be a holomorphic line bundle over a complex manifold M and let $\mathcal{S}$ be a finite dimensional complex subspace of $H^0(M, V)$. We give $\mathcal{S}$ a semi-positive rapidly decaying volume form μ. If $\mathcal{S}$ spans V_n, then*

$$\mathbf{E}\, |Z_s|^n = \pi_* \left(\textstyle\prod_{p=1}^n \det(\xi\xi^*)_{\mathbb{C}}\, \mathbf{D}^0 \right) . \tag{4–12}$$

In the case where the image of $\mathcal{J}_n$ contains all the holomorphic 1-jets, we can write $\mathbf{D}_n = D_n(x, \xi', z)\, dx\, d\xi'\, dz$, $D_n(x, \xi, z) \in \mathcal{C}^0$. Then (4–12) yields the following result from [BSZ2, Th. 2.1]:

$$\mathbf{E}\, |Z_s|^n = K_n(z)\, dz\,, \quad K_n(z) = \int d\xi\, D_n(0, \xi, z) \prod_{p=1}^n \det(\xi^p \xi^{p*})_{\mathbb{C}} \,. \tag{4–13}$$

5. Universality of the Scaling Limit of the Correlations

We return to our complex Hermitian line bundle (L, h) on a compact almost complex $2m$-dimensional symplectic manifold M with symplectic form $\omega = \frac{i}{2}\Theta_L$, where Θ_L is the curvature of L with respect to a connection ∇. Theorem 1.1 follows from Theorem 4.3 applied to the vector bundle

$$V = \underbrace{L^N \oplus \cdots \oplus L^N}_{k}$$

and the (finite-dimensional) space of sections

$$\mathcal{S} = H^0_J(M, L^N)^k \subset \mathcal{C}^\infty(M, V)\,.$$

5.1. Gaussian Measures. Recalling (2–2), we consider the Hermitian inner product on $H^0_J(M, L^N)$:

$$\langle s_1, s_2\rangle = \int_M h^N(s_1, s_2)\frac{1}{m!}\omega^m \qquad (s_1, s_2 \in H^0_J(M, L^N))\,.$$

We give $\mathcal{S}$ the Gaussian probability measure $\mu_N = \nu_N \times \cdots \times \nu_N$, where ν_N is the 'normalized' complex Gaussian measure on $H^0_J(M, L^N)$:

$$\nu_N(s) = \left(\frac{d_N}{\pi}\right)^{d_N} e^{-d_N|c|^2} dc\,, \qquad s = \sum_{j=1}^{d_N} c_j S_j^N\,.$$

Here $\{S_j^N\}$ is an orthonormal basis for $H^0_J(M, L^N)$ (with respect to the Hermitian inner product (2–2)) and dc is $2d_N$-dimensional Lebesgue measure. The normalization is chosen so that $\mathbf{E}\,\langle s, s\rangle = 1$. This Gaussian is characterized by the property that the $2d_N$ real variables $\operatorname{Re} c_j, \operatorname{Im} c_j$ $(j = 1, \ldots, d_N)$ are independent identically distributed (i.i.d.) random variables with mean 0 and variance $\frac{1}{2d_N}$; i.e.,

$$\mathbf{E}\, c_j = 0, \quad \mathbf{E}\, c_j c_k = 0, \quad \mathbf{E}\, c_j \bar{c}_k = \frac{1}{d_N}\delta_{jk}\,.$$

Picking a random element of $\mathcal{S}$ means picking k sections of $H^0_J(M, L^N)$ independently and at random.

Remark. Since we are interested in the zero sets Z_s, which do not depend on constant factors, we could just as well suppose our sections lie in the unit sphere $SH^0_J(M, L^N)$ with respect to the Hermitian inner product (2–2), and pick random sections with respect to the spherical measure. This gives the same expectations for $|Z_s|^n$ as the Gaussian measure on $H^0_J(M, L^N)$.

We now review the concept of 'generalized Gaussian measures' from [SZ2], which is one of the ingredients in obtaining the (universal) scaling limit of the joint probability distribution, which in turn yields the universality of the scaling limit of the correlation of zeros on symplectic manifolds. (For further details and related results, see [SZ2, § 5.1].) To begin, a (non-singular) Gaussian measure γ_Δ on $\mathbb{R}^p$ given by (1–3) has second moments

$$\langle x_j x_k\rangle_{\gamma_\Delta} = \Delta_{jk}\,.$$

The measure γ_Δ is characterized by its Fourier transform

$$\widehat{\gamma_\Delta}(t_1, \ldots, t_p) = e^{-\frac{1}{2}\sum \Delta_{jk}t_j t_k}\,. \tag{5–1}$$

The push-forward of a Gaussian measure by a surjective linear map is also Gaussian. Since we need to push forward Gaussian measures (on the spaces $H^0_J(M, L^N)$) by linear maps that are sometimes not surjective, we shall consider the case where Δ is positive semi-definite. In this case, we can still use (5–1) to define a measure γ_Δ, which we call a *generalized Gaussian*. If Δ has null eigenvalues, then γ_Δ is a Gaussian measure on the subspace $\Lambda_+ \subset \mathbb{R}^p$ spanned by the positive eigenvectors. If γ is a generalized Gaussian on $\mathbb{R}^p$ and $L : \mathbb{R}^p \to \mathbb{R}^q$ is a (not necessarily surjective) linear map, then $L_*\gamma$ is a generalized Gaussian on $\mathbb{R}^q$. By studying the Fourier transform, it is easy to see that the map $\Delta \mapsto \gamma_\Delta$ is a continuous map from the positive semi-definite matrices to the space of positive measures on $\mathbb{R}^p$ (with the weak topology).

5.2. Densities and the Szegő Kernel. We now consider the n-point joint probability distribution of a (Gaussian) random almost holomorphic section $s \in H^0_J(M, L^N)$ having prescribed values $s(z^p) = x^p$ and prescribed derivatives $\nabla s(z^p) = \xi^p$ (for $1 \le p \le n$). We denote this density by $\widetilde{D}^N_n(x, \xi, z)\, dx\, d\xi$ as in [SZ2], where $z = (z^1, \dots, z^n)$. Having equipped $H^0_J(M, L^N)$ with the Gaussian measure ν_N, and recalling that the joint probability distribution

$$\widetilde{\mathbf{D}}^N_z := \widetilde{D}^N_n(x, \xi, z)\, dx\, d\xi = (J^1_z)_*\nu_N\,,$$

is the push-forward of ν_N by a linear map, we conclude that the joint probability distribution is a generalized Gaussian measure on the complex vector space of 1-jets of sections:

$$\widetilde{\mathbf{D}}^N_z = \gamma_{\Delta^N(z)}\,.$$

To be more precise, we consider the $n(2m+1)$ complex-valued random variables $X_p,\ \Xi_{pq}$ $(1 \le p \le n,\ 1 \le q \le 2m)$ on $\mathcal{H}^2_N(X) \equiv H^0_J(M, L^N)$ given by

$$X_p(s) = s(z^p, 0)\,, \quad \Xi_{pq}(s) = (\nabla_q)s(z^p, 0)\,,$$

where

$$\nabla_q = \frac{1}{\sqrt{N}}\frac{\partial^h}{\partial z_q}\,, \quad \nabla_{m+q} = \frac{1}{\sqrt{N}}\frac{\partial^h}{\partial \bar z_q} \qquad (1 \le q \le m)\,,$$

for $s \in \mathcal{H}^2_N(X)$. Here, $\partial^h/\partial z_q$ denotes the horizontal lift to X of the tangent vector $\partial/\partial z_q$ on M. The covariance matrix $\Delta^N(z)$ is given by the Szegő kernel and its covariant derivatives, as follows:

$$\Delta^N(z) = \begin{pmatrix} A^N & B^N \\ B^{N*} & C^N \end{pmatrix}\,,$$

$$\left(A^N\right)^p_{p'} = \mathbf{E}\left(X_p \bar X_{p'}\right) = \frac{1}{d_N}\Pi_N(z^p, 0; z^{p'}, 0)\,,$$

$$\left(B^N\right)^p_{p'q'} = \mathbf{E}\left(X_p \overline{\Xi}_{p'q'}\right) = \frac{1}{d_N}\overline{\nabla}^2_{q'}\Pi_N(z^p, 0; z^{p'}, 0)\,,$$

$$\left(C^N\right)^{pq}_{p'q'} = \mathbf{E}\left(\Xi_{pq} \overline{\Xi}_{p'q'}\right) = \frac{1}{d_N}\nabla^1_q\overline{\nabla}^2_{q'}\Pi_N(z^p, 0; z^{p'}, 0)\,,$$

for $p, p' = 1, \dots, n$ and $q, q' = 1, \dots, 2m$. Here, ∇^1_q and ∇^2_q denote the differential operator on $X \times X$ given by applying ∇_q to the first and second factors, respectively. (We note that $A^N,\ B^N,\ C^N$ are $n \times n,\ n \times 2mn,\ 2mn \times 2mn$ matrices, respectively; p, q index the rows, and p', q' index the columns.) In [BSZ2] we proved that the joint probability density has a universal scaling limit, and in [SZ1] this result was extended to the symplectic case:

THEOREM 5.1 [SZ2, Theorem 5.4]. *Let L be a pre-quantum line bundle over a $2m$-dimensional compact integral symplectic manifold (M,ω). Choose Heisenberg coordinates $\{z_j\}$ about a point $P_0 \in M$. Then*

$$\widetilde{\mathbf{D}}^N_{(z^1/\sqrt{N},\dots,z^n/\sqrt{N})} \longrightarrow \mathbf{D}^\infty_{(z^1,\dots,n^n)} = \gamma_{\Delta^\infty(z)}$$

where $\mathbf{D}^\infty_{(z^1,\dots,z^n)}$ is a universal Gaussian measure supported on the holomorphic 1-jets, and $\Delta^N(z/\sqrt{N}) \to \Delta^\infty(z)$.

Theorem 1.2 then follows immediately from Theorems 1.1 and 5.1. In fact, we have the error estimate

$$\left(\frac{1}{N^{nk}}K^N_{nk}\Big(\frac{z^1}{\sqrt{N}},\dots,\frac{z^n}{\sqrt{N}}\Big),\ \varphi\right) = \big(K^\infty_{nkm}(z^1,\dots,z^n),\ \varphi\big) + O\left(\frac{1}{\sqrt{N}}\right),$$

for all $\varphi \in \mathcal{D}^{mn}((\mathbb{C}^m)_n)$.

A technically interesting novelty in the proof of Theorem 5.1 is the role of the $\bar\partial$ operator. In the holomorphic case, $\widetilde{\mathbf{D}}^N_{(z^1,\dots,z^n)}$ is supported on the subspace of sections satisfying $\bar\partial s = 0$. In the almost complex case, sections do not satisfy this equation, so $\widetilde{\mathbf{D}}^N_{(z^1,\dots,z^n)}$ is a measure on a higher-dimensional space of jets. However, Theorem 5.1 says that the mass in the '$\bar\partial$-directions' shrinks to zero as $N \to \infty$.

An alternate statement of Theorem 5.1 involves equipping the unit spheres $H^0(M,L^N)$ with Haar probability measure, and letting $\mathbf{D}^N_{(z^1,\dots,z^n)}$ be the corresponding joint probability distribution on $SH^0_J(M,L^N)$. In [SZ1, Theorem 0.2], it was shown that these non-Gaussian measures $\mathbf{D}^N$ also have the same scaling limit $\mathbf{D}^\infty$.

The matrix Δ^∞ is given in terms of the Szegő kernel for the Heisenberg group:

$$\Delta^\infty(z) = \frac{m!}{c_1(L)^m}\begin{pmatrix} A^\infty(z) & B^\infty(z) \\ B^\infty(z)^* & C^\infty(z)\end{pmatrix}, \tag{5–2}$$

where

$$\begin{aligned}
A^\infty(z)^p_{p'} &= \Pi^{\mathbf{H}}_1(z^p,0;z^{p'},0),\\
B^\infty(z)^p_{p'q'} &= \begin{cases} (z^p_{q'} - z^{p'}_{q'})\Pi^{\mathbf{H}}_1(z^p,0;z^{p'},0) & \text{for } 1 \le q \le m,\\ 0 & \text{for } m+1 \le q \le 2m,\end{cases}\\
C^\infty(z)^{pq}_{p'q'} &= \begin{cases} \big(\delta_{qq'} + (\bar z^{p'}_q - \bar z^p_q)(z^p_{q'} - z^{p'}_{q'})\big)\Pi^{\mathbf{H}}_1(z^p,0;z^{p'},0) & \text{for } 1 \le q,q' \le m,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

For details, see [SZ2].

Equation (5–2) says that the variances in the anti-holomorphic directions vanish. If we remove the rows and columns of the matrices corresponding to $m+1 \le q \le 2m$, then we get the covariance matrix

$$\Delta^\infty_h(z) = \frac{m!}{c_1(L)^m}\begin{pmatrix} A^\infty(z) & B^\infty_h(z) \\ B^\infty_h(z)^* & C^\infty_h(z)\end{pmatrix} \tag{5–3}$$

for the joint probability distribution in the holomorphic case. (Here A^∞, B_h^∞, and C_h^∞ are $n\times n$, $n\times mn$, and $mn\times mn$ matrices, respectively.) In [BSZ2], we used (4–13) and (5–3) to obtain formulas for the scaling limit zero correlations K_{nkm}^∞. We briefly summarize here how it was done: Let us write

$$\mathbf{D}^\infty_{(z^1,\dots,z^n)} = \mathbf{D}^\infty_z = D^\infty(x,\xi,z)\,dx\,d\xi\,.$$

The function $D^\infty(0,\xi,z)$ is Gaussian in ξ, but is not normalized as a probability density. It is given by

$$D^\infty(0,\xi,z)\,d\xi = \frac{1}{\pi^n \det A^\infty(z)}\gamma_{\Lambda^\infty(z)}\,, \tag{5–4}$$

where

$$\Lambda^\infty(z) = C_h^\infty(z) - B_h^\infty(z)^* A^\infty(z)^{-1} B_h^\infty(z)\,. \tag{5–5}$$

We first consider the $k=1$ case of the limit correlation function for the zero divisor (complex hypersurface) of one random section. By (4–13), (5–4), and the identity $\det\Delta_h^\infty = \det\Lambda^\infty \det A^\infty$, we obtain

$$K_{n1m}^\infty(z^1,\dots,z^n) = \frac{1}{\pi^n \det A^\infty(z)} \int_{\mathbb{C}^{mn}} \prod_{p=1}^n \left(\sum_{q=1}^m |\xi_q^p|^2\right) d\gamma_{\Lambda^\infty(z)}(\xi)\,. \tag{5–6}$$

The integral in (5–6) is a sum of $(2n)$-th moments of the Gaussian measure $\gamma_{\Lambda^\infty(z)}$, and can be evaluated using the Wick formula. Indeed, in the pair correlation case $n=2$, (5–6) yields the explicit formula (1–4).

For the case of random k-tuples $s=(s^1,\dots,s^k)\in \mathcal{S} = H^0_J(M,L^N)^k$ (where the zero sets are of codimension k), the 1-jets $J^1_z\tilde s^1,\dots,J^1_z\tilde s^k$ are i.i.d. random vectors, and we have

$$\begin{aligned} &K_{nkm}^\infty(z^1,\dots,z^n)\\ &\quad= \frac{1}{[\pi^n \det A^\infty(z)]^k} \int_{\mathbb{C}^{kmn}} \prod_{p=1}^n \det_{1\le j,j'\le k}\left(\sum_{q=1}^m \xi^p_{jq}\bar\xi^p_{j'q}\right) d\gamma_{I_k\otimes\Lambda^\infty(z)}(\xi)\,, \end{aligned} \tag{5–7}$$

where I_k denotes the $k\times k$ identity matrix; i.e.,

$$\big(I_k\otimes\Lambda^\infty(z)\big)^{jpq}_{j'p'q'} = \delta^j_{j'}\Lambda^\infty(z)^{pq}_{p'q'}\,.$$

For further details and explicit formulas, see [BSZ2] for the case $k=n=2$, and see [BSZ3] for the point pair correlation case $n=2$, $k=m$. Indeed, we show in [BSZ3] that for small values of $r:=|z^1-z^2|$, we have

$$\widetilde K_{2mm}(z^1,z^2) = \frac{m+1}{4}r^{4-2m} + O(r^{8-2m})\,,\qquad m=1,2,3,\dots\,.$$

5.3. Decay of Correlations. Let us define the *normalized n-point scaling limit zero correlation function*

$$\widetilde{K}^{\infty}_{nkm}(z) = (K^{\infty}_{1km})^{-n} K^{\infty}_{nkm}(z) = \left(\frac{\pi^k (m-k)!}{m!}\right)^n K^{\infty}_{nkm}(z)\,. \tag{5–8}$$

In [BSZ2], we showed that the limit correlations are "short range" in the following sense:

THEOREM 5.2 [BSZ2, Theorem 4.1]. *The correlation functions satisfy the estimate*

$$\widetilde{K}^{\infty}_{nkm}(z^1,\dots,z^n) = 1 + O(r^4 e^{-r^2}) \quad \text{as } r\to\infty\,, \quad r = \min_{p\neq p'} |z^p - z^{p'}|\,.$$

PROOF. We review here the proof of this estimate. Writing

$$A = \pi^m A^\infty\,, \quad B = \pi^m B^\infty_h\,, \quad C = \pi^m C^\infty_h\,, \quad \Lambda = \pi^m \Lambda^\infty\,,$$

we have:

$$\begin{aligned} A^p_{p'} &= e^{i\,\mathrm{Im}(z^p\cdot \bar z^{p'})} e^{-\frac12 |z^p - z^{p'}|^2}\,, \\ B^p_{p'q'} &= (z^p_{q'} - z^{p'}_{q'}) A^p_{p'}\,, \\ C^{pq}_{p'q'} &= \left[\delta_{qq'} + (\bar z^{p'}_q - \bar z^p_q)(z^p_{q'} - z^{p'}_{q'})\right] A^p_{p'}\,. \end{aligned}$$

This implies that

$$\begin{aligned} A &= I + O(e^{-r^2/2})\,, \qquad A^p_p = 1\,, \\ B &= O(re^{-r^2/2})\,, \\ C &= I + O(r^2 e^{-r^2/2}) \quad \text{as } r\to\infty\,, \qquad C^{pq}_{pq} = 1\,. \end{aligned} \tag{5–9}$$

Recalling (5–5), we have

$$\Lambda = I + O(r^2 e^{-r^2/2})\,, \qquad \Lambda^{pq}_{pq} = 1 + O(r^2 e^{-r^2})\,, \qquad \text{as } r\to\infty\,. \tag{5–10}$$

We now use the Wick formula to evaluate the integral in (5–7). (Formula (5–7) is homogeneous of order 0 in the matrix entries, so is not affected when A^∞, Λ^∞ are multiplied by π^m.) Note that the Wick formula involves terms that are products of diagonal elements of Λ, and products that contain at least two off-diagonal elements of Λ. The former terms are of the form $1 + O(r^2 e^{-r^2})$, and the latter are $O(r^4 e^{-r^2})$. Similarly, $\det A = 1 + O(e^{-r^2})$, and the estimate follows. □

The theorem can be extended to estimates of the connected correlation functions (called also truncated correlation functions, cluster functions, or cumulants), as follows. The n-point connected correlation function is defined as (see, e.g., [GJ, p. 286])

$$\widetilde{T}^{\infty}_{nkm}(z^1,\dots,z^n) = \sum_G (-1)^{l+1}(l-1)! \prod_{j=1}^{l} \widetilde{K}^{\infty}_{n_j km}(z^{p_{j1}},\dots,z^{p_{jn_j}}), \tag{5–11}$$

where the sum is taken over all partitions $G = (G_1, \dots, G_l)$ of the set $(1, \dots, n)$ and $G_j = (p_{j1}, \dots, p_{jn_j})$. In particular, recalling that $\widetilde{K}^\infty_{1km} \equiv 1$,

$$\begin{aligned}
\widetilde{T}^\infty_{1km}(z^1) &= \widetilde{K}^\infty_{1km}(z^1) = 1\,,\\
\widetilde{T}^\infty_{2km}(z^1, z^2) &= \widetilde{K}^\infty_{2km}(z^1, z^2) - \widetilde{K}^\infty_{1km}(z^1)\widetilde{K}^\infty_{1km}(z^2) \;=\; \widetilde{K}^\infty_{2km}(z^1, z^2) - 1\,,\\
\widetilde{T}^\infty_{3km}(z^1, z^2, z^3) &= \widetilde{K}^\infty_{3km}(z^1, z^2, z^3) - \widetilde{K}^\infty_{2km}(z^1, z^2)\widetilde{K}^\infty_{1km}(z^3)\\
&\quad - \widetilde{K}^\infty_{2km}(z^1, z^3)\widetilde{K}^\infty_{1km}(z^2) - \widetilde{K}^\infty_{2km}(z^2, z^3)\widetilde{K}^\infty_{1km}(z^1)\\
&\quad + 2\widetilde{K}^\infty_{1km}(z^1)\widetilde{K}^\infty_{1km}(z^2)\widetilde{K}^\infty_{1km}(z^3)\\
&= \widetilde{K}^\infty_{3km}(z^1, z^2, z^3) - \widetilde{K}^\infty_{2km}(z^1, z^2)\\
&\quad - \widetilde{K}^\infty_{2km}(z^1, z^3) - \widetilde{K}^\infty_{2km}(z^2, z^3) + 2\,,
\end{aligned}$$

and so on. The inverse of (5–11) is

$$\widetilde{K}^\infty_{nkm}(z^1, \dots, z^n) = \sum_G \prod_{j=1}^{l} \widetilde{T}^\infty_{n_jkm}(z^{p_{j1}}, \dots, z^{p_{jn_j}}) \tag{5–12}$$

(Möbius theorem). The advantage of the connected correlation functions is that they go to zero if at least one of the distances $|z^i - z^j|$ goes to infinity (see Corollary 5.8 below). In our case the connected correlation functions can be estimated as follows. Define

$$d(z^1, \dots, z^n) = \max_{\mathcal{G}} \prod_{l \in L} |z^{i(l)} - z^{f(l)}|^2 e^{-|z^{i(l)} - z^{f(l)}|^2/2}. \tag{5–13}$$

where the maximum is taken over all oriented connected graphs $\mathcal{G} = (V, L)$ "with zero boundary" such that $V = (z^1, \dots, z^n)$. Here V denotes the set of vertices of $\mathcal{G}$, L the set of edges, and $i(l)$ and $f(l)$ stand for the initial and final vertices of the edge l, respectively. The graph $\mathcal{G}$ is said to be have zero boundary if $\sum\{l : l \in L\}$ is a 1-cycle; i.e., for each vertex $z^p \in V$, the number of edges beginning at z^p equals the number ending at z^p. (There must be at least one edge beginning at each vertex, since $\mathcal{G}$ is assumed to be connected. Graphs may have any number of edges connecting the same two vertices.) Observe that the maximum in (5–13) is achieved at some graph $\mathcal{G}$, because $te^{-t/2} \le 2/e < 1$ and therefore the product in (5–13) is at most $(2/e)^{|L|}$.

THEOREM 5.3. *The connected correlation functions satisfy the estimate*

$$\widetilde{T}^\infty_{nkm}(z^1, \dots, z^n) = O(d(z^1, \dots, z^n))\,,$$

provided that $\min_{p \ne q} |z^p - z^q| \ge c > 0$.

To prove the theorem, let us introduce the n-point functions

$$\begin{aligned}
\widehat{K}_n(z^1, \dots, z^n) &= \det A_{nkm}(z^1, \dots, z^n)\; \widetilde{K}^\infty_{nkm}(z^1, \dots, z^n) \qquad (5–14)\\
&= [(m-k)!/m!]^n \int \prod_{p=1}^{n} \det_{1 \le j, j' \le k} \left(\sum_{q=1}^{m} \xi^p_{jq} \bar{\xi}^p_{j'q} \right) d\gamma_{I_k \otimes \Lambda(z)}(\xi)
\end{aligned}$$

where $A_{nkm} = I_k \otimes A$, an $nk \times nk$ matrix. (Note that $\det A_{nkm} = (\det A)^k$. It was shown in [BSZ2, Lemma 3.3] that $\det A > 0$ at distinct points z^p.) We also consider the corresponding "connected functions"

$$\widehat{T}_n(z^1, \dots, z^n) = \sum_G (-1)^{l+1}(l-1)! \prod_{j=1}^{l} \widehat{K}_{n_j}(z^{p_{j1}}, \dots, z^{p_{jn_j}}) \,, \tag{5–15}$$

and we note that the Möbius inversion formula applies to $\widehat{K}_n$, $\widehat{T}_n$.

Observe that we can rewrite $\widehat{K}_n(z^1, \dots, z^n)$ as a sum over Feynman diagrams. Namely, each term in the Wick sum for the integral in (5–14) corresponds to a graph $\mathcal{F} = (V, L)$ (Feynman diagram) such that $V = (z^1, \dots, z^n)$ and the edges $l \in L$ connect the paired variables $\xi_{jq}^{i(l)}$, $\bar{\xi}_{jq'}^{f(l)}$ in the given Wick term. We have that

$$\widehat{K}_n(z^1, \dots, z^n) = [(m-k)!/m!]^n \sum_{\mathcal{F}} W_{\mathcal{F}}(z^1, \dots, z^n) \,, \tag{5–16}$$

where the function $W_{\mathcal{F}}(z^1, \dots, z^n)$ is the sum over all terms in the Wick sum corresponding to the Feynman diagram $\mathcal{F}$. (In other words, to get $W_{\mathcal{F}}(z^1, \dots, z^n)$ we fix the indices p, p' of the pairings $(\xi_{jq}^p, \bar{\xi}_{jq'}^{p'})$ prescribed by $\mathcal{F}$ and sum up in the Wick formula over all indices j, q at every z^p.) Note that each graph $\mathcal{F}$ in the sum (5–16), having arisen from a term in the Wick sum, has zero boundary.

A remarkable property of the "connected functions" is that they are represented by the sum over connected Feynman diagrams (see, e.g., [GJ]):

$$\widehat{T}_n(z^1, \dots, z^n) = [(m-k)!/m!]^n \sum_{\mathcal{F}}^{\text{conn}} W_{\mathcal{F}}(z^1, \dots, z^n) \,.$$

We conclude from (5–10) that for all connected Feynman diagrams $\mathcal{F}$,

$$W_{\mathcal{F}}(z^1, \dots, z^n) = O(d(z^1, \dots, z^n)) \,, \quad \text{provided that } \min_{p \neq q} |z^p - z^q| \geq c > 0 \,.$$

Summing over $\mathcal{F}$, we obtain the following estimate:

LEMMA 5.4. $\widehat{T}_n(z^1, \dots, z^n) = O(d(z^1, \dots, z^n))$, *provided that* $\min_{p \neq q} |z^p - z^q| \geq c > 0$.

It remains to relate $\widetilde{T}_{nkm}^\infty(z^1, \dots, z^n)$ to $\widehat{T}_n(z^1, \dots, z^n)$. To do this, we introduce the functions

$$Q_n(z^1, \dots, z^n) = \sum_G (-1)^{l+1}(l-1)! \prod_{j=1}^{l} \det\, A_{n_j km}(z^{p_{j1}}, \dots, z^{p_{jn_j}}),$$

which are the connected functions for $\det A_{nkm}(z^1, \dots, z^n)$, and

$$R_n(z^1, \dots, z^n) = \sum_G (-1)^{l+1}(l-1)! \prod_{j=1}^{l} \frac{1}{\det\, A_{n_j km}(z^{p_{j1}}, \dots, z^{p_{jn_j}})},$$

which are the connected functions for $\dfrac{1}{\det A_{nkm}(z^1,\dots,z^n)}$. Recall the Möbius inversion formula

$$\frac{1}{\det A_{nkm}(z^1,\dots,z^n)} = \sum_G \prod_{j=1}^{l} R_{n_j}(z^{p_{j1}},\dots,z^{p_{jn_j}}) \; . \tag{5–17}$$

We have the following relation between $\widetilde{T}^\infty_{nkm}(z^1,\dots,z^n)$ and $\widehat{T}_n(z^1,\dots,z^n)$.

Lemma 5.5.
$$\widetilde{T}^\infty_{nkm}(z^1,\dots,z^n) = \sum_{G,H}^{\text{conn}} \prod_{j=1}^{l} \widehat{T}_{n_j}(z^{p_{j1}},\dots,z^{p_{jn_j}}) \prod_{j=1}^{l'} R_{m_j}(z^{p'_{j1}},\dots,z^{p'_{jm_j}}), \tag{5–18}$$
where the sum is taken over all pairs $\{G=(G_1,\dots,G_l),\ H=(H_1,\dots,H_{l'})\}$ *of partitions of the set* $(1,\dots,n)$ *which are "mutually connected" in the sense that there is no proper subset* S *of the set* $(1,\dots,n)$ *such that* S *is a union of some subsets* G_j *and is also a union of some subsets* H_j. *In* (5–18), $G_j = (p_{j1},\dots,p_{jn_j})$ *and* $H_j = (p'_{j1},\dots,p'_{jm_j})$.

Proof. The proof is by induction on n. From (5–12),

$$\widetilde{T}^\infty_{nkm}(z^1,\dots,z^n) = \widetilde{K}^\infty_{nkm}(z^1,\dots,z^n) - {\sum_F}' \prod_{j=1}^{l} \widetilde{T}^\infty_{n_jkm}(z^{p_{j1}},\dots,z^{p_{jn_j}}) \tag{5–19}$$

where the summation goes over all partitions $F=(F_1,\dots,F_l)$ with at least two elements in the partition (i.e., $l\ge 2$). From (5–15) and (5–17), we have

$$\begin{aligned}\widetilde{K}^\infty_{nkm}(z^1,\dots,z^n) &= \widehat{K}_n(z^1,\dots,z^n)\frac{1}{\det A_{nkm}(z^1,\dots,z^n)} \\ &= \sum_{G,H} \prod_{j=1}^{l} \widehat{T}_{n_j}(z^{p_{j1}},\dots,z^{p_{jn_j}}) \prod_{j=1}^{l'} R_{m_j}(z^{p'_{j1}},\dots,z^{p'_{jm_j}}) \,,\end{aligned} \tag{5–20}$$

where the sum is taken over all pairs $\{G=(G_1,\dots,G_l),\ H=(H_1,\dots,H_{l'})\}$ of partitions of the set $(1,\dots,n)$. If we use the inductive assumption that (5–18) holds when the number of points is less than n and apply that assumption to $\widetilde{T}^\infty_{n_jkm}(z^{p_{j1}},\dots,z^{p_{jn_j}})$ in (5–19), we obtain that

$${\sum_F}' \prod_{j=1}^{l} \widetilde{T}^\infty_{n_jkm}(z^{p_{j1}},\dots,z^{p_{jn_j}}) = \sum_{G,H}^{\text{disconn}} \prod_{j=1}^{l} \widehat{T}_{n_j}(z^{p_{j1}},\dots,z^{p_{jn_j}}) \prod_{j=1}^{l'} R_{m_j}(z^{p'_{j1}},\dots,z^{p'_{jm_j}}) \,, \tag{5–21}$$

where the sum on the right is taken over all partitions $G=(G_1,\dots,G_l)$ of the set $(1,\dots,n)$ and all partitions $H=(H_1,\dots,H_p)$ of the set $(1,\dots,n)$ which are "mutually disconnected" in the sense that there is a proper subset S of the set

$(1,\dots,n)$ that is simultaneously a union of some subsets G_j and a union of some subsets H_j. When we substitute (5–20) and (5–21) into (5–19) and take the difference on the right of (5–19), disconnected pairs $\{G,H\}$ will be cancelled out and we will be left with mutually connected $\{G,H\}$. This proves the lemma. □

LEMMA 5.6. *The functions* $Q_n(z^1,\dots,z^n)$ *satisfy the estimate*

$$Q_n(z^1,\dots,z^n)=O(d(z^1,\dots,z^n))\,, \tag{5–22}$$

provided that $\min_{p\neq q}|z^p-z^q|\geq c>0$.

PROOF. By the determinant formula,

$$\det A_{nkm}(z^1,\dots,z^n)=(\det A)^k=\sum_\pi(-1)^{\sigma(\pi)}\prod_{j=1}^k\prod_{p=1}^n A_p^{\pi_j(p)}\,,$$

where the sum is over all k-tuples $\pi=(\pi_1,\dots,\pi_k)$ of permutations of $(1,\dots,n)$. We claim that

$$Q_n(z^1,\dots,z^n)=\sum_\pi^{\mathrm{conn}}(-1)^{\sigma(\pi)}\prod_{j=1}^k\prod_{p=1}^n A_p^{\pi_j(p)}\,, \tag{5–23}$$

where the summation on the right goes over the set of k-tuples $\pi=(\pi_1,\dots,\pi_k)$ such that no proper subset of $(1,\dots,n)$ is invariant under the group generated by the π_j. (Each such π corresponds to a connected graph consisting of edges beginning at p and ending at $\pi_j(p)$, for all p,j.) Indeed,

$$Q_n(z^1,\dots,z^n)=\det A_{nkm}(z^1,\dots,z^n)-\sum_F{}'\prod_{j=1}^l Q_{n_j}(z^{p_{j1}},\dots,z^{p_{jn_j}}),$$

where the summation on the right goes over all partitions $F=(F_1,\dots,F_l)$ with $l\geq 2$. Using this equation, we prove (5–23) by induction (cf. the proof of Lemma 5.5). The estimate (5–22) now follows from (5–23) and (5–9). □

LEMMA 5.7. *The functions* $R_n(z^1,\dots,z^n)$ *satisfy the estimate*

$$R_n(z^1,\dots,z^n)=O(d(z^1,\dots,z^n))\,, \tag{5–24}$$

provided that $\min_{p\neq q}|z^p-z^q|\geq c>0$.

PROOF. We have the identity

$$0=\sum_{G,H}^{\mathrm{conn}}\prod_{j=1}^l Q_{n_j}(z^{p_{j1}},\dots,z^{p_{jn_j}})\prod_{j=1}^{l'} R_{m_j}(z^{p'_{j1}},\dots,z^{p'_{jm_j}})\,,\quad n\geq 2\,. \tag{5–25}$$

The proof of this identity is the same as that of Lemma 5.5. Indeed, the connected functions of

$$\det A_{nkm}\,\frac{1}{\det A_{nkm}}=1$$

are equal to 0 (except that the 1-point connected function equals 1); hence (5–25) follows.

The identity (5–25) can be rewritten as

$$\det A_{nkm}(z^1,\dots,z^n)\, R_n(z^1,\dots,z^n) = -\sum_{G,H}^{\text{conn}'} \prod_{j=1}^{l} Q_{n_j}(z^{p_{j1}},\dots,z^{p_{jn_j}}) \prod_{j=1}^{l'} R_{m_j}(z^{p'_{j1}},\dots,z^{p'_{jm_j}}), \quad (5\text{–}26)$$

where the summation on the right goes over all mutually connected pairs of partitions $\{G,H\}$ with at least two elements in H (i.e., $l' \geq 2$). Now the estimate (5–24) follows by induction from Lemma 5.6 and identity (5–26). □

Theorem 5.3 follows from Lemmas 5.4, 5.5 and 5.7. The theorem yields the following more explicit estimate:

COROLLARY 5.8. *The connected correlation functions satisfy the estimate*

$$\widetilde{T}^{\infty}_{nkm}(z^1,\dots,z^n) = o\big(e^{-R^2/n}\big)\,, \qquad R = \max_{p,q} |z^p - z^q|\,,$$

provided that $\min_{p\neq q} |z^p - z^q| \geq c > 0$.

PROOF. We must show that

$$d(z^1,\dots,z^n) \leq o\big(e^{-R^2/n}\big)\,. \quad (5\text{–}27)$$

Assume without loss of generality that $|z^1 - z^n| = R$. Let $\mathcal{G} = (V,L)$ be an oriented connected graph with zero boundary as in the definition of $d(z^1,\dots,z^n)$. Since z^1 and z^n are connected by a chain of loops in $\mathcal{G}$, we can choose disjoint sets of edges $L', L'' \subset L$ such that L' forms a path starting at z^1 and ending at z^n, and L'' forms a path starting at z^n and ending at z^1. This means that there is a sequence $z^1 = z^{i_1}, z^{i_2}, \dots, z^{i_{n'}} = z^n$ such that $L' = \{l_1,\dots,l_{n'-1}\}$, where l_j begins at z^{i_j} and ends at $z^{i_{j+1}}$. By removing any loops in L', we can assume that the z^{i_j} are distinct and thus $n' \leq n$. A similar description holds for L''. Let $r_j = |z^{i_j} - z^{i_{j+1}}|$. We note that

$$R \leq \sum r_j \leq \Big((n'-1)\sum r_j^2\Big)^{1/2},$$

where the second inequality is by Cauchy–Schwarz. We then have

$$\begin{aligned}\prod_{l\in L'} |z^{i(l)} - z^{f(l)}|^2 e^{-|z^{i(l)}-z^{f(l)}|^2/2} &= \prod_{j=1}^{n'-1} r_j^2 e^{-r_j^2/2}\\ &\leq R^{2n'-2} e^{-\frac12 \sum r_j^2} \leq R^{2n'-2} e^{-R^2/(2n'-2)}\,,\end{aligned}$$

and hence

$$\prod_{l\in L'} |z^{i(l)} - z^{f(l)}|^2 e^{-|z^{i(l)}-z^{f(l)}|^2/2} \leq R^{2n-2} e^{-R^2/(2n-2)}\,, \quad R \geq 1\,.$$

The same inequality also holds for the product over the path L''. Since each term of the product in (5–13) is less than 1, we then have

$$\begin{aligned}\prod_{l\in L}|z^{i(l)}-z^{f(l)}|^2e^{-|z^{i(l)}-z^{f(l)}|^2/2} &\le \prod_{l\in L'\cup L''}|z^{i(l)}-z^{f(l)}|^2e^{-|z^{i(l)}-z^{f(l)}|^2/2}\\ &\le o\big(e^{-R^2/n}\big)\,.\end{aligned}$$

Taking the supremum over all graphs, we obtain (5–27). $\square$

Remark. The preceding proof gives the bound

$$d(z^1,\dots,z^n)\le R^{4n-4}e^{-R^2/(n-1)}\,,\quad R\ge 1\,.$$

Hence we actually have the estimate

$$\widetilde{T}^{\infty}_{nkm}(z^1,\dots,z^n)=O\big(R^{4n-4}e^{-R^2/(n-1)}\big)\,,\quad \text{provided that } \min_{p\neq q}|z^p-z^q|\ge c>0\,.$$

This implies Theorem 5.2 because of the inversion formula (5–12).

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