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z-Measures on Partitions, Robinson–Schensted–Knuth Correspondence, and $\beta = 2$ Random Matrix Ensembles

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Abstract. We suggest a hierarchy of all the results known so far about the connection of the asymptotics of combinatorial or representation theoretic problems with "$\beta = 2$ ensembles" arising in the random matrix theory. We show that all such results are, essentially, degenerations of one general situation arising from so-called generalized regular representations of the infinite symmetric group.

Introduction

In last few years there appeared a number of papers indicating a strong connection of certain asymptotic problems of enumerative combinatorics and representation theory of symmetric groups with the random matrix theory; see [Baik et al. 1999a; 1999b; Baik and Rains 1999a; 1999b; Borodin 1998a; 1998b; 1999; $\geq$ 2001; Borodin and Olshanski 1998a; 1998b; 2000a; Borodin et al. 2000; Johansson 2000; 1999; Okounkov 1999b; 1999a; Olshanski 1998a; 1998b; Tracy and Widom 1998; 1999], for a partial list. Such a connection was also anticipated in earlier works [Regev 1981; Kerov 1993; 1994]. For other interesting connections see also [Borodin 2000b; Borodin and Okounkov 2000; Okounkov 2001].

In this paper we suggest a hierarchy of all the results known so far about the connection of the asymptotics of combinatorial or representation theoretic problems with so-called "$\beta = 2$ ensembles" arising in random matrix theory. (These ensembles are characterized by the property that their correlation functions have determinantal form with a scalar kernel; see below.) We show that all such results are, essentially, degenerations of one general situation arising from so-called generalized regular representations of the infinite symmetric group; see [Kerov et al. 1993] and Section 3 below.

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It is worth noting that though the hierarchy provides a clear understanding why this or that problem should have this or that asymptotics, the technical aspects of the proof are sometimes nontrivial and should not be underestimated.

Many claims cited below were recently proved by Kurt Johansson, we would like to thank him for keeping us informed about his work.

1. z-Measures

For $n = 1, 2, \dots$, let $\mathbb{Y}_n$ denote the set of partitions of n, which will be identified with Young diagrams with n boxes. We agree that $\mathbb{Y}_0$ consists of a single element — the zero partition or the empty diagram $\varnothing$.

Given $\lambda \in \mathbb{Y}_n$, we write $|\lambda| = n$ and denote by $d = d(\lambda)$ the number of diagonal boxes in λ. We shall use the Frobenius notation [Macdonald 1995]

$$\lambda = (p_1, \dots, p_d \mid q_1, \dots, q_d).$$

Here $p_i = \lambda_i - i$ is the number of boxes in the i-th row of λ on the right of the i-th diagonal box; likewise, $q_i = \lambda'_i - i$ is the number of boxes in the i-th column of λ below the i-th diagonal box (λ' stands for the transposed diagram).

Note that

$$p_1 > \cdots > p_d \geq 0, \qquad q_1 > \cdots > q_d \geq 0, \qquad \sum_{i=1}^{d} (p_i + q_i + 1) = |\lambda|.$$

The numbers p_i, q_i are called the *Frobenius coordinates* of the diagram λ.

Let $b = (i, j)$ be a box of λ; here i, j are the row number and the column number of b. Recall the definition of the *content* and the *hook length* of b:

$$c(b) = j - i, \quad h(b) = (\lambda_i - j) + (\lambda'_j - i) + 1.$$

We will consider two complex parameters z, z' such that the numbers $(z)_k (z')_k$ and $(-z)_k (-z')_k$ are real and strictly positive for any $k = 1, 2, \dots$. Here and below

$$(a)_k = a(a+1) \dots (a+k-1), \qquad (a)_0 = 1,$$

denotes the Pochhammer symbol.

The above assumption on z, z' means that one of the following two conditions holds:

- either $z' = \bar{z}$ and $z \in \mathbb{C} \setminus \mathbb{Z}$
- or $z, z' \in \mathbb{R}$ and there exists $m \in \mathbb{Z}$ such that $m < z, z' < m + 1$.

We set

$$t = zz'$$

and note that $t > 0$.

For a Young diagram λ let $\dim \lambda$ denote the number of the standard Young tableaux of shape λ. Equivalently, $\dim \lambda$ is the dimension of the irreducible representation (of the symmetric group of degree $|\lambda|$) corresponding to λ; see

[Macdonald 1995]. The well-known *hook formula* for $\dim\lambda$ has the following form (see [Macdonald 1995], for example):

$$\dim\lambda = \frac{|\lambda|!}{\prod_{b\in\lambda} h(b)}.$$

In the Frobenius coordinates this formula takes the form

$$\dim\lambda = \frac{|\lambda|!}{\prod_{i=1}^{d} p_i!q_i!}\det\left[\frac{1}{p_i+q_j+1}\right].$$

We introduce a function on the Young diagrams depending on the parameters z, z':

$$\begin{aligned} M_{z,z'}(\lambda) &= \frac{\prod_{b\in\lambda}(c(b)+z)(c(b)+z')}{(t)_{|\lambda|}}\cdot\frac{\dim^2\lambda}{|\lambda|!} \\ &= \frac{|\lambda|!}{(t)_{|\lambda|}}\cdot\prod_{b\in\lambda}\frac{(c(b)+z)(c(b)+z')}{h^2(b)}. \end{aligned} \tag{1–1}$$

We agree that $M_{z,z'}(\varnothing)=1$. Thanks to our assumption on the parameters, $M_{z,z'}(\lambda)>0$ for all λ.

Proposition 1.1. *For any* $n=0,1,2,\dots,$

$$\sum_{\lambda\in\mathbb{Y}_n} M_{z,z'}(\lambda)=1,$$

so that the restriction of $M_{z,z'}$ *to* $\mathbb{Y}_n$ *is a probability distribution on* $\mathbb{Y}_n$.

We shall denote this distribution by $M^{(n)}_{z,z'}$ and call it the *n-th level z-measure.*

Proposition 1.1 is an easy corollary of Proposition 3.1 below.

Let $\mathbb{Y}=\mathbb{Y}_0\sqcup\mathbb{Y}_1\sqcup\dots$ denote the set of all Young diagrams. Consider the negative binomial distribution on the nonnegative integers, which depends on t and an additional parameter ξ, $0<\xi<1$:

$$\pi_{t,\xi}(n) = (1-\xi)^t\,\frac{(t)_n}{n!}\,\xi^n, \qquad n=0,1,\dots .$$

For $\lambda\in\mathbb{Y}$ we set

$$M_{z,z',\xi}(\lambda) = M_{z,z'}(\lambda)\,\pi_{t,\xi}(|\lambda|).$$

By the construction, $M_{z,z',\xi}(\cdot)$ is a probability distribution on $\mathbb{Y}$, which can be viewed as a mixture of the finite distributions $M^{(n)}_{z,z'}$. From the formulas for $M_{z,z'}$ and $\pi_{t,\xi}$ we get an explicit expression for $M_{z,z',\xi}$:

$$\begin{aligned} M_{z,z',\xi}(\lambda) &= (1-\xi)^t\xi^{|\lambda|}\prod_{b\in\lambda}\frac{(c(b)+z)(c(b)+z')}{h^2(b)} \\ &= (1-\xi)^t\xi^{\sum_{i=1}^{d}(p_i+q_i+1)}t^d\prod_{i=1}^{d}\frac{(z+1)_{p_i}(z'+1)_{p_i}(-z+1)_{q_i}(-z'+1)_{q_i}}{p_i!\,p_i!\,q_i!\,q_i!}\det{}^2\left[\frac{1}{p_i+q_j+1}\right]. \end{aligned}$$

We shall call $M_{z,z',\xi}$ the *mixed z-measure.* Following a certain analogy with models of statistical physics (compare [Vershik 1996]) one may call $(\mathbb{Y}, M_{z,z',\xi})$ the *grand canonical ensemble.*

Let $\mathbb{Z}'$ denote the set of half-integers,

$$\mathbb{Z}' = \mathbb{Z} + \tfrac{1}{2} = \{\dots, -\tfrac{3}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, \dots\},$$

and let $\mathbb{Z}'_+$ and $\mathbb{Z}'_-$ be the subsets of positive and negative half-integers, respectively. It will be sometimes convenient to identify both $\mathbb{Z}'_+$ and $\mathbb{Z}'_-$ with $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ by making use of the correspondence $\pm(k + \frac{1}{2}) \leftrightarrow k$, where $k \in \mathbb{Z}_+$.

Denote by $\operatorname{Conf}(\mathbb{Z}')$ the space of all finite subsets of $\mathbb{Z}'$ which will be called *configurations*. We define an embedding $\lambda \mapsto X$ of the set $\mathbb{Y}$ of Young diagrams into the set $\operatorname{Conf}(\mathbb{Z}')$ of configurations in $\mathbb{Z}'$ as follows:

$$\lambda = (p_1, \dots, p_d \,|\, q_1, \dots, q_d) \;\mapsto\; X = \{p_1 + \tfrac{1}{2}, \dots, p_d + \tfrac{1}{2}, -q_1 - \tfrac{1}{2}, \dots, -q_d - \tfrac{1}{2}\}. \tag{1–2}$$

Under the identification $\mathbb{Z}' \simeq \mathbb{Z}_+ \sqcup \mathbb{Z}_+$, the map $\lambda \mapsto X$ is simply associating to λ the collection of its Frobenius coordinates. The image of the map consists exactly of the configurations X with the property $|X \cap \mathbb{Z}'_+| = |X \cap \mathbb{Z}'_-|$. We call such configurations *balanced.*

Under the embedding $\lambda \mapsto X$ the probability measure $M_{z,z',\xi}$ on $\mathbb{Y}$ turns into a probability measure on the balanced configurations in $\mathbb{Z}'$. According to the conventional terminology [Daley and Vere-Jones 1988], we get a point process on $\mathbb{Z}'$; let us denote it as $\mathcal{P}_{z,z',\xi}$.

The *n-th correlation function* $\rho_n^{(z,z',\xi)}(x_1, \dots, x_n)$ of $\mathcal{P}_{z,z',\xi}$ is the probability that the random point configuration contains the points $x_1, \dots, x_n$.

In [Borodin and Olshanski 2000a] we have computed all the correlation functions of $\mathcal{P}_{z,z',\xi}$. To state the result we need some notation.

Consider the following functions in u depending on z, z', ξ as parameters (compare [Borodin and Olshanski 2000a]):

$$\begin{aligned}
\psi_\pm(u) &= t^{1/2}\, \xi^{u+1/2}\, (1-\xi)^{\pm(z+z')}\, \frac{\Gamma(u+1\pm z)\Gamma(u+1\pm z')}{\Gamma(1\pm z)\Gamma(1\pm z')\Gamma(u+1)\Gamma(u+1)}, \\
P_\pm(u) &= (\psi_\pm(u))^{1/2}\, F\big(\mp z, \mp z'; u+1; \tfrac{\xi}{\xi-1}\big), \\
Q_\pm(u) &= \frac{t^{1/2}\xi^{1/2}\,(\psi_\pm(u))^{1/2}}{1-\xi}\; \frac{F\big(1\mp z, 1\mp z'; u+2; \tfrac{\xi}{\xi-1}\big)}{u+1}.
\end{aligned}$$

Here $F(a,b;c;w)$ is the Gauss hypergeometric function.

THEOREM 1.2 [Borodin and Olshanski 2000a]. *The correlation functions of $\mathcal{P}_{z,z',\xi}$ have the form*

$$\rho_n^{(z,z',\xi)}(x_1, \dots, x_n) = \det[K(x_i, x_j)]_{i,j=1}^n, \qquad x_1, \dots, x_n \in \mathbb{Z}',$$

where

$$K(x,y)=\frac{F_1(x)G_1(y)+F_2(x)G_2(y)}{x-y}, \tag{1–3}$$

with

$$\begin{aligned}
F_1(x)&=\begin{cases}-Q_+\left(x-\frac12\right) & \textit{for } x>0,\\ P_-\left(-x-\frac12\right) & \textit{for } x<0;\end{cases}\\
F_2(x)&=\begin{cases}P_+\left(x-\frac12\right) & \textit{for } x>0,\\ Q_-\left(-x-\frac12\right) & \textit{for } x<0;\end{cases}\\
G_1(x)&=\begin{cases}P_+\left(x-\frac12\right) & \textit{for } x>0,\\ -Q_-\left(-x-\frac12\right) & \textit{for } x<0;\end{cases}\\
G_2(x)&=\begin{cases}Q_+\left(x-\frac12\right) & \textit{for } x>0,\\ P_-\left(-x-\frac12\right) & \textit{for } x<0.\end{cases}
\end{aligned} \tag{1–4}$$

We call $K(x,y)$ the *hypergeometric kernel.*

Remarks 1.3. 1. The hypergeometric kernel has no singularity on the diagonal: the numerator of (1.3) vanishes if $x=y$.

2. The hypergeometric kernel satisfies the relation

$$K(x,y)=\operatorname{sgn}(x)\operatorname{sgn}(y)K(y,x). \tag{1–5}$$

This shows that the kernel is Hermitian with respect to the indefinite inner product in $\ell^2(\mathbb{Z}')=\ell^2(\mathbb{Z}'_+)\oplus\ell^2(\mathbb{Z}'_-)$ given by the operator $\mathrm{id}\oplus(-\mathrm{id})$.

3. The restriction of the hypergeometric kernel to $\mathbb{Z}'_+$ has the form

$$\frac{P_+(x-\frac12)Q_+(y-\frac12)-P_+(y-\frac12)Q_+(x-\frac12)}{x-y}.$$

Note that this kernel is symmetric. We will call it the *positive part* of the hypergeometric kernel.

4. Kernels with the symmetry (1.5) appeared before in works of mathematical physicists on solvable models of systems with positive and negative charged particles; see [Alastuey and Forrester 1995; Cornu and Jancovici 1987; 1989; Gaudin 1985; Forrester 1986; 1988; 1989] and references therein. The mixed z-measure can also be interpreted as a model for positive and negative particles on $\mathbb{Z}'$: positive particles may occupy locations in $\mathbb{Z}'_+$, negative — in $\mathbb{Z}'_-$. The square of the Cauchy determinant

$$\det{}^2\left[\frac{1}{p_i+q_j+1}\right]=\frac{\prod_{i<j}[(p_i-p_j)(q_i-q_j)]^2}{\prod_{i,j}(p_i+q_j+1)^2}$$

in the formula for $M_{z,z',\xi}$ above encodes the logarithmic interaction of the charged particles.

5. The papers [Okounkov 1999a; 2001] contain another derivation of Theorem 1.2 and a generalization of the mixed z-measures $M_{z,z',\xi}$.

2. Three Versions of the Robinson–Schensted–Knuth Correspondence

A description of the RSK algorithm can be found in [Fulton 1997; Sagan 1991].

We start with the "widest" version of the RSK correspondence due to Knuth [1970].

Denote by $B^n_{k,l}$ the set of 'bijections' between two sets of size n, the first set consists of (possibly repeated) numbers from 1 to k and the second set consists of (possibly repeated) numbers from 1 to l. Such bijections are in one-to-one correspondence with matrices of size $k\times l$ with nonnegative integral entries, total sum of entries equal to n: the (i,j)–entry shows how many times the element $i\in\{1,\dots,k\}$ is associated with the element $j\in\{1,\dots,l\}$. Clearly,

$$|B^n_{k,l}| = \binom{kl+n-1}{n} = \frac{kl(kl+1)\cdots(kl+n-1)}{n!}.$$

The RSK algorithm establishes a bijection of $B^n_{k,l}$ and the set of ordered pairs of semi-standard Young tableaux of the same shape with n boxes, the first tableau has entries from the set $\{1,\dots,k\}$, while the second — from the set $\{1,\dots,l\}$. (Recall that a semi-standard Young tableau is a tableau whose entries are weakly increasing along the rows and strictly increasing along the columns. In a standard tableau we have strictly increasing entries in both directions.)

As is well-known, the number of semi-standard Young tableaux of shape λ with entries from $\{1,\dots,k\}$ is equal to the value of the Schur symmetric function $s_\lambda(1,1,\dots,1,0,0,\dots)$ where the number of 1's equals k. This value can be written in the following form (see [Macdonald 1995, I.3, Ex. 4], for example),

$$s_\lambda(\underbrace{1,1,\dots,1}_{k},0,0,\dots) = \prod_{b\in\lambda}\frac{c(b)+k}{h(b)}.$$

Recall also that the number of standard Young tableaux of shape λ is $\dim\lambda$.

Hence, if we consider the uniform probability distribution on $B^n_{k,l}$, then, with respect to its image on the set of Young diagrams with n boxes, the probability of a Young diagram $\lambda\in\mathbb{Y}_n$ equals

$$\frac{n!}{kl(kl+1)\cdots(kl+n-1)}\prod_{b\in\lambda}\frac{(c(b)+k)(c(b)+l)}{h^2(b)}.$$

Comparing this with (1.1) we conclude that this distribution coincides with $M^{(n)}_{z,z'}$ for $z=k$, $z'=l$.

Note that these values of z,z' do not satisfy our conditions on the parameters imposed in Section 1. The reason is that for such z,z' the values of $M^{(n)}_{z,z'}$ can be zero, for example $M^{(n)}_{z,z'}(\lambda)=0$ for all λ with length (number of nonzero parts) greater than $\min\{k,l\}$. However, all values of $M^{(n)}_{z,z'}$ remain nonnegative. We consider such situation as a specific degeneration of the regular picture (when the values $M^{(n)}_{z,z'}$ are strictly positive).

Two other (earlier) versions of the RSK correspondence are due to Robinson [1938] and Schensted [1961].

Denote by $B^n_{k,\infty}$ the set of words of length n built from the alphabet $\{1,\dots,k\}$ (our notation will become clear soon). It is a subset of $B^n_{k,n}$ characterized by the property that the numbers in the second set are all distinct (they encode the order of letters $\{1,\dots,k\}$ in the word). It means that in the corresponding matrices of size $k\times n$ every column has exactly one nonzero element which is equal to 1. Obviously, $|B^n_{k,\infty}|=k^n$.

In this case the RSK algorithm establishes a bijection of $B^n_{k,\infty}$ and the set of ordered pairs of Young tableaux of the same shape with n boxes; the first tableau is semi-standard and it is filled with numbers from 1 to k, and the second tableau is standard. This means that the probability of a Young diagram $\lambda\in\mathbb{Y}_n$ with respect to the image of the uniform distribution on $B^n_{k,\infty}$ equals

$$k^{-n}\prod_{b\in\lambda}\frac{c(b)+k}{h(b)}\cdot\dim\lambda.$$

It is easy to see from (1.1) that this is the limit of $M^{(n)}_{z,z'}$ for $z=k$ and $z'\to\infty$.

Finally, if we forbid for both sets in the definition of $B^n_{k,l}$ to have repetitions, then we get the symmetric group S_n. It would be logical to denote the symmetric group by $B^n_{\infty,\infty}$; see below. In the language of matrices, it means that we consider $n\times n$ matrices with 0's and 1's such that in each row and each column there is exactly one nonzero element. Clearly, $|S_n|=n!$.

The RSK algorithm provides a bijection of the set of permutations of n symbols and the set of ordered pairs of standard Young tableaux of the same shape with n boxes. Hence, the probability of a Young diagram $\lambda\in\mathbb{Y}_n$ with respect to the distribution coming from the uniform distribution on S_n equals $\dim^2\lambda/n!$. This distribution on the Young diagrams is called the *Plancherel distribution*. The relation (1.1) easily implies that the Plancherel distribution is the limit of $M^{(n)}_{z,z'}$ as $z,z'\to\infty$.

For bijections from $B^n_{k,l}$, $B^n_{k,\infty}$, $B^n_{\infty,\infty}$ we define a weakly increasing subsequence to be a sequence of pairs of associated elements, first element is from the first set, second element is from the second set, which weakly increase in each element. Under the RSK correspondence the length of the longest weakly increasing subsequence of a bijection coincides with the length of the first row of the corresponding Young diagram in all three cases described above; see [Knuth 1970; Schensted 1961].

3. Harmonic Analysis on the Infinite Symmetric Group

For more detailed discussion of the material of this section see [Kerov et al. 1993; Vershik and Kerov 1981a; Olshanski 1998a].

We define the infinite symmetric group $S(\infty)$ as the inductive limit of the finite symmetric groups S_n with respect to natural embeddings $S_n \to S_{n+1}$. Equivalently, $S(\infty)$ is the group of *finite* permutations of the set $\{1, 2, \ldots\}$.

By a *character* of $S(\infty)$ (in the sense of von Neumann) we mean any central, positive definite function χ on $S(\infty)$, normalized by the condition $\chi(e) = 1$. We assign to χ a function $M(\lambda)$ on the set $\mathbb{Y} = \sqcup \mathbb{Y}_n$ of Young diagrams as follows: for any $n = 1, 2, \ldots,$

$$\chi \mid_{S_n} = \sum_{\lambda \in \mathbb{Y}_n} M(\lambda) \frac{\chi^\lambda}{\dim \lambda},$$

where χ^λ denotes the irreducible character of S_n (in the conventional sense), indexed by $\lambda \in \mathbb{Y}_n$, and $\dim \lambda = \chi^\lambda(e)$ is its dimension. Let $M^{(n)}$ stand for the restriction of the function M to $\mathbb{Y}_n$; this is a probability distribution on $\mathbb{Y}_n$. Conversely, let $M = \{M^{(n)}\}$ be a function on $\mathbb{Y}$ such that each $M^{(n)}$ is a probability distribution; then M corresponds to a character χ if (and only if) the distributions $M^{(n)}$ obey a natural coherence relation, which comes from the classical Young branching rule for the irreducible characters of the finite symmetric groups; see [Vershik and Kerov 1981a; Olshanski 1998a]. (Equivalently, the function $\varphi(\lambda) = M(\lambda)/\dim \lambda$ must be a *harmonic function on the Young graph* $\mathbb{Y}$ in the sense of Vershik and Kerov; see [Vershik and Kerov 1981a; Olshanski 1998a].)

Proposition 3.1. *The z-measures $M^{(n)}_{z,z'}$ introduced in Section* 1 *satisfy the coherence relation mentioned above and*, *consequently*, *define a character $\chi_{z,z'}$ of $S(\infty)$.*

Several direct proofs of the proposition are known. A simple proof is given in [Olshanski 1998a, § 7]. About generalizations, see [Kerov 2000; Borodin and Olshanski 2000b].

Note that the degenerations $M^{(n)}_{k,l}$, $M^{(n)}_{k,\infty}$, and $M^{(n)}_{\infty,\infty}$ of the z-measures also correspond to certain characters, which will be denoted as $\chi_{k,l}$, $\chi_{k,\infty}$, and $\chi_{\infty,\infty}$, respectively. The character $\chi_{\infty,\infty}$ is easily described: it takes value 1 at $e \in S(\infty)$ and vanishes at all other elements of the group.

By the very definition of the characters of $S(\infty)$, they form a convex set. The extreme points of that set are called the *indecomposable* characters, and the other points are called *decomposable* characters.

According to a remarkable theorem due to Thoma [1964] (see also [Vershik and Kerov 1981a; Wassermann 1981; Kerov et al. 1998]), the indecomposable characters of $S(\infty)$ are parametrized by the points of the infinite dimensional simplex

$$\Omega = \left\{ \alpha_1 \geq \alpha_2 \geq \ldots \geq 0,\ \beta_1 \geq \beta_2 \geq \ldots \geq 0 \;\middle|\; \sum_{i=1}^{\infty} (\alpha_i + \beta_i) \leq 1 \right\},$$

which is called the *Thoma simplex.* Given a point $\omega = (\alpha, \beta) \in \Omega$, we denote by $\chi^{(\omega)}$ the corresponding indecomposable character.

The characters $\chi_{k,\infty}$ and $\chi_{\infty,\infty}$ are indecomposable: the former corresponds to the point ω with $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 1/k$ (all other coordinates are zero), and the latter corresponds to the point $\omega = (0,0)$ (all coordinates are zero). The characters $\chi_{z,z'}$ (with z, z' satisfying the conditions of Section 1) and $\chi_{k,l}$ are decomposable.

Every character can be uniquely represented as a convex combination of the indecomposable ones,

$$\chi = \int_\Omega \chi^{(\omega)} P(d\omega).$$

Here P is a probability measure on Ω, which is called the *spectral measure* of the character χ. Moreover, any probability measure on Ω is a spectral measure of a character, so that the set of characters of $S(\infty)$ is isomorphic, as a convex set, to the set of probability measures on the Thoma simplex. Under this isomorphism, indecomposable characters correspond to delta measures on Ω.

Given a concrete decomposable character χ, a natural problem is to describe explicitly its spectral measure P. This will be referred to as the *problem of harmonic analysis.*

This problem is readily solved for the degenerate characters $\chi_{k,l}$:

Proposition 3.2. *Let $\chi = \chi_{k,l}$ with $k \le l$. Set $a = l - k$. Then the spectral measure is concentrated on the $(k-1)$-dimensional subsimplex*

$$\alpha_1 + \cdots + \alpha_k = 1, \qquad \alpha_{k+1} = \alpha_{k+2} = \cdots = \beta_1 = \beta_2 \ldots = 0$$

of Ω and has density

$$\text{const} \cdot \prod_{1 \le i < j \le k} (\alpha_i - \alpha_j)^2 \cdot \prod_{i=1}^k \alpha_i^a$$

with respect to the Lebesgue measure.

For the characters $\chi_{z,z'}$ with nonintegral parameters the problem of harmonic analysis is highly nontrivial and will be briefly discussed at the end of Section 8. One of the first results in this direction is as follows (recall that two measures are called disjoint if there exist disjoint Borel sets supporting them).

Proposition 3.3. *Let $P_{z,z'}$ denote the spectral measure of $\chi_{z,z'}$. Except the obvious equality $P_{z,z'} = P_{z',z}$, the measures $P_{z,z'}$ are pairwise disjoint.*

Notice the following general result which relates the spectral measure P of a character χ to the finite probability distributions $M^{(n)}$. Let us embed $\mathbb{Y}_n$ into Ω by:

$$\lambda = (p_1, \ldots, p_d \,|\, q_1, \ldots, q_d) \in \mathbb{Y}_n$$
$$\mapsto \left\{ \frac{p_1 + \frac12}{n}, \ldots \frac{p_d + \frac12}{n}, 0, 0, \ldots; \frac{q_1 + \frac12}{n}, \ldots, \frac{q_d + \frac12}{n}, 0, 0, \ldots \right\} \in \Omega. \quad (3\text{–}1)$$

PROPOSITION 3.4. *As $n \to \infty$, the push-forwards of the measures $M^{(n)}$ under these embeddings weakly converge to P.*

This is a special case of a more general result proved in [Kerov et al. 1998].

The characters of $S(\infty)$ can be related to representations in two ways.

The first way is rather evident. Each character χ is a positive definite function on $S(\infty)$, so that it determines a unitary representation of $S(\infty)$, which will be denoted as $\Pi(\chi)$. When χ is indecomposable, $\Pi(\chi)$ is a *factor* representation of finite type in the sense of von Neumann; see [Thoma 1984].

The second way is a bit more involved. Set $G = S(\infty) \times S(\infty)$ and let K denote the diagonal subgroup in G, which is isomorphic to $S(\infty)$. We interpret χ as a function on the first copy of $S(\infty)$, which is a subgroup of G, and then extend it to the whole group G by the formula

$$\psi(g_1, g_2) = \chi(g_1 g_2^{-1}), \qquad g_1, g_2 \in S(\infty).$$

Note that ψ is the only extension of χ that is a K-biinvariant function on G. The function ψ is also positive definite, so that one can assign to it a unitary representation in the canonical way. This representation of the group G will be denoted by $T(\chi)$. By the very construction, it possesses a distinguished K-invariant vector.

Note that $\Pi(\chi)$ coincides with the restriction of $T(\chi)$ to the first copy of $S(\infty)$. If χ is indecomposable, $\chi = \chi^{(\omega)}$, then $T(\chi) = T(\chi^{(\omega)})$ is irreducible. The representations of the form $T(\chi^{(\omega)})$ are exactly the irreducible unitary representations of the group G possessing a K-invariant vector (such a vector is unique, within a scalar factor). Thus, the Thoma simplex can be identified with the *spherical dual* to (G, K). It is worth noting that the irreducible representations of the form $T(\chi^{(\omega)})$ (except two trivial cases) are *not* tensor products of irreducible representations of the factors $S(\infty)$. For more details about the representations $T(\chi^{(\omega)})$, see [Vershik and Kerov 1981b; Wassermann 1981; Olshanski 1989].

The representation $T(\chi_{\infty,\infty})$ is readily described: it coincides with the natural representation of the group G realized in the Hilbert space $\ell^2(G/K)$. Note that G/K is identified with the group $S(\infty)$ on which G acts by left and right shifts, so that $T(\chi_{\infty,\infty})$ may be called the *regular* representation of G. As for $\Pi(\chi_{\infty,\infty})$, it provides a classical realization of the hyperfinite von Neumann factor of type II_1.

The representations $T(\chi_{z,z'})$ are called the *generalized regular* representations of G. The term is motivated by the fact that each $T(\chi_{z,z'})$ can be realized as the inductive limit of a chain of the form

$$\cdots \to (\mathrm{Reg}_n, v_n) \to (\mathrm{Reg}_{n+1}, v_{n+1}) \to \cdots,$$

where Reg_n stands for the (bi)regular representation of the group $S_n \times S_n$ in the space of functions on S_n and v_n is a certain vector in that space, depending on the parameters z, z' (v_n is given by a certain central function on S_n). When

$z' = \bar{z}$, the generalized regular representations admit a very nice realization in certain L^2 spaces of functions defined on a compactification of the group $S(\infty)$. We refer to [Kerov et al. 1993] for the exposition of this construction.

Finally, note that for any decomposable character χ, the decomposition of $T(\chi)$ into irreducible representations is governed by the spectral measure P:

$$T(\chi) = \int_\Omega T(\chi^{(\omega)}) P(d\omega).$$

4. Mixing

Theorem 1.2 computes the correlation functions of a point process obtained from the distributions $M^{(n)}_{z,z'}$ mixed together by the negative binomial distribution with parameters (t, ξ); see Section 1. In this section we consider the degenerations of the mixing procedure in the cases when z and z' are integers, when z is an integer and $z' \to \infty$, and when z and z' both tend to infinity while $\xi \to 1$.

If $z = k$ and $z' = l$ are positive integers then nothing interesting happens— we have to mix the corresponding measures on $\mathbb{Y}_n$'s by the negative binomial distribution with parameters (kl, ξ).

If $z = k$ is a positive integer and $z' \to \infty$, or $z \to \infty$ and $z' \to \infty$ then $t = zz'$ goes to ∞. If we keep $\theta = t\xi$ fixed (hence, $\xi \to 0$) then the negative binomial distribution degenerates to the Poisson distribution with parameter θ. The mixing procedure with Poisson distribution is called *poissonization.*

The degeneration $\xi \to 1$ is a bit more delicate. Let us embed $\mathbb{Z}'$ into the punctured line $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and then rescale the lattice by multiplying the coordinates of its points by $(1-\xi)$. Then the coordinates of the point configuration in $\mathbb{R}^*$ that corresponds to $\lambda \in \mathbb{Y}_n$ (as defined in (1.2)) after rescaling differ from the coordinates of the image of λ in Ω (as defined in (3.1)) by the scaling factor $(1-\xi)n$.

The discrete distribution on the positive semiaxis with

$$\operatorname{Prob}\{(1-\xi)n\} = (1-\xi)^t \, \frac{(t)_n}{n!} \, \xi^n, \quad n = 0, 1, 2, \dots,$$

which depends on the parameter $\xi \in (0,1)$, converges, as $\xi \to 1$, to the gamma distribution with parameter t

$$\gamma(ds) = \frac{s^{t-1}}{\Gamma(t)} \, e^{-s} ds.$$

This brings us to the following construction. Consider the space $\widetilde{\Omega} = \Omega \times \mathbb{R}_+$ with the probability measure

$$\widetilde{P}_{z,z'} = P_{z,z'} \otimes \frac{s^{t-1}}{\Gamma(t)} \, e^{-s} ds.$$

Let us embed $\mathbb{Y} = \mathbb{Y}_0 \sqcup \mathbb{Y}_1 \sqcup \mathbb{Y}_2 \sqcup \dots$ into $\widetilde{\Omega} = \Omega \times \mathbb{R}_+$ by sending a Young diagram $\lambda \in \mathbb{Y}_n$ to the pair consisting of its image in Ω and the number $(1-\xi)n$.

PROPOSITION 4.1. *The push-forwards of $M_{z,z',\xi}$ under the embeddings described above converge*, *as* $\xi \to 1$, *to* $\widetilde{P}_{z,z'}$.

Exact claims with a detailed description of this convergence will appear in [Borodin and Olshanski $\geq$ 2001a].

5. Ensembles

We introduce several terms that will be used below.

By the word *ensemble* throughout this paper we will mean a stochastic point process (i.e, a probability measure on the space of point configurations) whose correlation functions $\rho_n(x_1, \dots, x_n)$ are given by determinantal formulas of the form

$$\rho_n(x_1, \dots, x_n) = \det[K(x_i, x_j)]_{i,j=1}^n\,,$$

where $K(x, y)$ is a certain kernel. We will call $K(x, y)$ the *correlation kernel.*

The process $\mathcal{P}_{z,z',\xi}$ is an example; the ensemble lives on $\mathbb{Z}'$ and the correlation kernel is the hypergeometric kernel; see Theorem 1.2. We will call it the *discrete z-ensemble.*

In all our examples the points of the ensembles will vary in discrete or continuous subsets of the real line. Such a subset will be called the *phase space* of the corresponding ensemble. For example, $\mathbb{Z}'$ is the phase space of $\mathcal{P}_{z,z',\xi}$.

There is a class of *orthogonal polynomial ensembles* characterized by the condition of having a fixed finite number of points, say k, the joint probability distribution of which has the density

$$\text{const}\cdot \prod_{1\leq i,j\leq k} (x_i - x_j)^2 \prod_{i=1}^k w(x_i)$$

with respect to either the Lebesgue measure, if the phase space is continuous, or counting measure, if the phase space is discrete. A standard argument due to Dyson [1962] (see also [Mehta 1991]) shows that the correlation kernel is the Christoffel–Darboux kernel of order k for orthogonal polynomials on the phase space with respect to the weight function $w(x)$. If a_n denotes the top degree coefficient of p_n,

$$p_n(x) = a_n x^n + \{\text{lower degree terms}\},$$

and $h_n = \|p_n\|^2$ then the kernel has the form

$$K(x, y) = \frac{a_{k-1}}{a_k h_{k-1}} \frac{p_k(x)p_{k-1}(y) - p_{k-1}(x)p_k(y)}{x - y} \sqrt{w(x)w(y)}.$$

Below we will consider the following orthogonal polynomial ensembles:

- *Laguerre ensemble*: phase space $\mathbb{R}_+$, weight function $w(x) = x^a e^{-x}$, $a > -1$;
- *Hermite ensemble*: phase space $\mathbb{R}$, weight function $w(x) = e^{-x^2}$;
- *Charlier ensemble*: phase space $\mathbb{Z}_+$, weight function $w(x) = \theta^x/x!$, $\theta > 0$;

- *Meixner ensemble*: phase space $\mathbb{Z}_+$, weight function $w(x) = (a+1)_x \xi^x / x!$, $a > -1$, $\xi \in (0,1)$.

Corresponding normalizing constants for the orthogonal polynomials can be found in [Erdélyi 1953b; Koekoek and Swarttouw 1998; Nikiforov et al. 1991]. The Christoffel–Darboux kernels for these ensembles will be called *Laguerre, Hermite, Charlier*, and *Meixner kernels*, respectively.

We will also deal with the *Airy ensemble* (see [Forrester 1993; Tracy and Widom 1994a]): the phase space is $\mathbb{R}$, the correlation kernel is

$$\frac{A(x)A'(y) - A'(x)A(y)}{x-y}$$

where $A(x)$ is the Airy function.

Two other ensembles that we will need are the ensemble arising from poissonized Plancherel distributions for symmetric groups (see [Borodin et al. 2000] and Section 6 below) with the phase space $\mathbb{Z}'$ and the kernel of the form (1.3), (1.4) where

$$P_\pm(x) = \theta^{\frac{1}{4}} J_x(2\sqrt{\theta}), \quad Q_\pm(x) = \theta^{\frac{1}{4}} J_{x+1}(2\sqrt{\theta}), \tag{5–1}$$

$\theta > 0$ is a parameter, $J_\nu(x)$ is the Bessel function; and the ensemble arising from the problem of harmonic analysis on $S(\infty)$ described in Section 3 (see [Borodin and Olshanski 1998a] and Section 6 below) with the phase space $\mathbb{R}^*$ and the kernel of the form (1.3) where

$$F_1(x) = \begin{cases} -\mathcal{Q}_+(x) & \text{for } x > 0, \\ \mathcal{P}_-(-x) & \text{for } x < 0; \end{cases}$$

$$F_2(x) = \begin{cases} \mathcal{P}_+(x) & \text{for } x > 0, \\ \mathcal{Q}_-(-x) & \text{for } x < 0; \end{cases}$$

$$G_1(x) = \begin{cases} \mathcal{P}_+(x) & \text{for } x > 0, \\ -\mathcal{Q}_-(-x) & \text{for } x < 0; \end{cases}$$

$$G_2(x) = \begin{cases} \mathcal{Q}_+(x) & \text{for } x > 0, \\ \mathcal{P}_-(-x) & \text{for } x < 0, \end{cases}$$

where we have defined

$$\mathcal{P}_\pm(x) = \frac{(zz')^{1/4}}{(\Gamma(1\pm z)\Gamma(1\pm z')\,x)^{1/2}} W_{\frac{\pm(z+z')+1}{2}, \frac{z-z'}{2}}(x),$$

$$\mathcal{Q}_\pm(x) = \frac{(zz')^{3/4}}{(\Gamma(1\pm z)\Gamma(1\pm z')\,x)^{1/2}} W_{\frac{\pm(z+z')-1}{2}, \frac{z-z'}{2}}(x),$$

z, z' satisfy the assumptions stated in Section 1, and $W_{\kappa,\mu}(x)$ is the Whittaker function; see [Erdélyi 1953a]. We will call these ensembles the *Plancherel ensemble* and the *continuous z-ensemble*, respectively. The kernel for the first one will be called the *Plancherel kernel*, for the second one — the *Whittaker kernel*.

When an ensemble lives on $\mathbb{R}^*$ or $\mathbb{Z}'$, one may single out its *positive part* — the restriction to $\mathbb{R}_+ \subset \mathbb{R}^*$ or $\mathbb{Z}'_+ \subset \mathbb{Z}'$, respectively. The correlation kernel of the positive part is the corresponding restriction of the correlation kernel of the initial ensemble. We will use the term "positive part of the kernel" for such restrictions.

The positive part of the Plancherel kernel has been independently found in [Johansson 1999] where it was called the *discrete Bessel kernel*; see Section 9.

The random point configurations of the discrete z-ensemble and of the Plancherel ensemble are finite with probability 1. The random point configuration of the Airy ensemble is, with probability 1, infinite, bounded from above, unbounded from below, and has no finite accumulation points. The random point configuration of the continuous z-ensemble is, with probability 1, infinite, bounded from above and below, and has zero (which is *not* in the phase space) as its only accumulation point.

6. Correlations After Mixing

In accordance with the notation of Section 2, it is natural to denote the measures on Young diagrams with n boxes coming from $B^n_{k,l}$, $B^n_{k,\infty}$, $B^n_{\infty,\infty}$ as $M^{(n)}_{k,l}$, $M^{(n)}_{k,\infty}$, $M^{(n)}_{\infty,\infty}$, respectively, and the corresponding mixtures (i.e., measures on the set of all Young diagrams) as $M_{k,l,\xi}$, $M_{k,\infty,\theta}$, $M_{\infty,\infty,\theta}$. We want to see how the hypergeometric kernel will behave in these degenerate cases.

We start with the case when z, z' are positive integers, say, $z = k$, $z' = l$, $k \le l$. Set $a = l - k$.

Proposition 6.1 [Borodin and Olshanski 2000a], [Johansson 2000]. *Let $\lambda = (p_1, \dots, p_d \,|\, q_1, \dots, q_d) \in \mathbb{Y}$ be distributed according to $M_{k,l,\xi}$. Then the distribution of points $\{k + p_1, \dots, k + p_d\}$ coincides with the restriction of the k-point Meixner ensemble with parameters (a, ξ) to the set $\{k, k + 1, \dots\}$.*

This claim corresponds to the fact that the hypergeometric functions entering the hypergeometric kernel become Meixner polynomials if z or z' is integral; see [Borodin and Olshanski 2000a]. Furthermore, the positive part of the hypergeometric kernel becomes the Christoffel–Darboux kernel for Meixner polynomials (shifted by k).

Now we pass to $M_{k,\infty,\theta}$.

Proposition 6.2 [Johansson 1999]. *Let $\lambda = (p_1, \dots, p_d \,|\, q_1, \dots, q_d) \in \mathbb{Y}$ be distributed according to $M_{k,\infty,\theta}$. Then the distribution of points $\{k + p_1, \dots, k + p_d\}$ coincides with the restriction of the k-point Charlier ensemble with parameter θ to the set $\{k, k + 1, \dots\}$.*

The easiest way to see this is to examine the degeneration of Meixner polynomials with parameters (a, ξ) to Charlier polynomials with parameter θ when $a \to \infty$, $\theta = k(k + a)\xi$ is fixed.

Next, consider the situation when z and z' both go to ∞.

PROPOSITION 6.3 [Borodin et al. 2000]. *Let* $\lambda = (p_1, \dots, p_d \,|\, q_1, \dots, q_d) \in \mathbb{Y}$ *be distributed according to* $M_{\infty,\infty,\theta}$. *Then the random point configuration*

$$\{p_1 + \tfrac{1}{2}, \dots, p_d + \tfrac{1}{2}, -q_1 - \tfrac{1}{2}, \dots, -q_d - \tfrac{1}{2}\}$$

forms the Plancherel ensemble with parameter θ.

This claim corresponds to the degeneration of the hypergeometric function to the Bessel J-function when first two parameters go to infinity and the argument goes to zero so that the product of these three numbers is fixed (and equals θ).

As for the representation theoretic picture, we have the following claim.

PROPOSITION 6.4 [Borodin and Olshanski 1998a; Borodin $\geq$ 2001]. *Let*

$$((\alpha, \beta), s) \in \widetilde{\Omega} = \Omega \times \mathbb{R}_+$$

be distributed according to $\widetilde{P}_{z,z'}$. *Then the random point configuration*

$$(s\alpha_1, s\alpha_2, \dots, -s\beta_1, -s\beta_2, \dots)$$

forms the continuous z*-ensemble.*

REMARK 6.5. When one of the parameters z, z' becomes integral, say, $z = k \in \{1, 2, \dots\}$, and $z' = z + a$, $a > -1$, the Whittaker kernel degenerates to the Laguerre kernel of order k with parameter a. Then Proposition 6.4 implies that the measure $\widetilde{P}_{z,z'}$ gets concentrated on the finite-dimensional subset of $\widetilde{\Omega} = \Omega \times \mathbb{R}_+$ where $\alpha_{k+1} = \alpha_{k+2} = \dots = \beta_1 = \beta_2 = \dots = 0$, and on this subset, in the new coordinates $x_i = s\alpha_i$ (s is the coordinate on $\mathbb{R}_+$), it equals

$$\text{const} \cdot \prod_{1 \leq i < j \leq k} (x_i - x_j)^2 \prod_{i=1}^{k} x_i^a e^{-x_i} dx_i.$$

See [Borodin and Olshanski 1998b, Remark 2.4]. This agrees with Proposition 3.2.

7. Asymptotics When Mixing Parameters Tend to a Limit

We start with $M_{k,l,\xi}$. Assume that $a = l - k \geq 0$. Then Proposition 3.4, the degeneration of the hypergeometric kernel to the Whittaker kernel (Proposition 6.4) and the coincidence of the Whittaker kernel with the Laguerre kernel when at least one parameter is integral (Remark 6.5) imply the following claim.

PROPOSITION 7.1. *Let* $\lambda \in \mathbb{Y}$ *be distributed according to* $M_{k,l,\xi}$. *Then the random point configuration* $\{(1-\xi)\lambda_1, \dots, (1-\xi)\lambda_k\}$ *converges, as* $\xi \to 1$, *to the Laguerre ensemble.*

Now we pass to $M_{k,\infty,\theta}$. The fact that the character of $S(\infty)$ corresponding to $M_{k,\infty}^{(n)}$ is indecomposable and corresponds to the point $\alpha_1 = \dots = \alpha_k = 1/k$ in Ω (see Section 3) leads to the following statement.

Consider the embedding of the set of Young diagrams with length at most k into $\mathbb{R}_+^k$ defined by dividing the lengths of rows of a Young diagram by θ.

Proposition 7.2. *Under the embeddings described above $M_{k,\infty,\theta}$ weakly converges to the delta measure at the point $(1/k,\dots,1/k)$ as $\theta\to\infty$.*

One can also ask about fluctuations of $M_{k,\infty,\theta}$ around the limit delta measure. Johansson proved the following statement.

Proposition 7.3 [Johansson 1999]. *Let $\lambda\in\mathbb{Y}$ be distributed according to $M_{k,\infty,\theta}$. Then the random point configuration*

$$\left\{\frac{\lambda_1-\theta/k}{\sqrt{2\theta/k}},\dots,\frac{\lambda_k-\theta/k}{\sqrt{2\theta/k}}\right\}$$

converges, as $\theta\to\infty$, to the k-point Hermite ensemble.

The convergence of distribution of the first point of the random configuration from Proposition 7.3 was proved by Tracy and Widom, [Tracy and Widom 1999].

Propositions 7.2 and 7.3 correspond to a certain degeneration of Charlier polynomials to Hermite polynomials which follows from a more general degeneration of Laguerre polynomials with large argument and parameter to Hermite polynomials; see [Temme 1990].

The most interesting case is $M_{\infty,\infty,\theta}$. The reason is simple: the number of points (rows of Young diagrams) is unbounded in this case. One can look at at least two different regimes when $\theta\to\infty$: "in the bulk of spectrum" or "at the edge of spectrum".

Proposition 7.4 [Borodin et al. 2000], [Johansson 1999]. *Let $\lambda\in\mathbb{Y}$ be distributed according to $M_{\infty,\infty,\theta}$. Then the random point configuration*

$$\left\{\frac{\lambda_1-2\sqrt{\theta}}{\theta^{\frac16}},\frac{\lambda_2-2\sqrt{\theta}}{\theta^{\frac16}},\dots\right\}$$

converges, as $\theta\to+\infty$, to the Airy ensemble.

The convergence of distributions of the first and the second points of the random configuration from Proposition 7.4 was proved earlier in [Baik et al. 1999a; 1999b].

Proposition 7.4 is the result of degeneration of the Bessel functions (5.1) to the Airy function and its derivative.

For the results on the asymptotics "in the bulk of spectrum", see [Borodin et al. 2000]. These results correspond to the degeneration of the Plancherel kernel to the *discrete sine kernel*

$$\frac{\sin(a(x-y))}{\pi(x-y)},\qquad x,y\in\mathbb{Z},\quad 0<a\le\frac{\pi}{2}\,.$$

It was also mentioned in [Johansson 1999] that under a certain limit procedure the Plancherel kernel degenerates to the conventional sine kernel $\sin(\pi(x-y))/(\pi(x-y))$ on $\mathbb{R}$.

One can also consider "double limits" of $M_{k,l,\xi}$ and $M_{k,\infty,\theta}$ (or, equivalently, Meixner and Charlier ensembles) when at least two parameters tend to critical values. Then the scaling procedure must involve at least two large parameters. For $M_{k,\infty,\theta}$ the asymptotics looks as follows.

PROPOSITION 7.5 [Johansson 1999]. *Let $\lambda \in \mathbb{Y}$ be distributed according to $M_{k,\infty,\theta}$. Then the random point configuration*

$$\left\{ \frac{\lambda_1 - \theta/k - 2\sqrt{\theta}}{(1+\sqrt{\theta}/k)^{\frac{2}{3}}\theta^{\frac{1}{6}}}, \frac{\lambda_2 - \theta/k - 2\sqrt{\theta}}{(1+\sqrt{\theta}/k)^{\frac{2}{3}}\theta^{\frac{1}{6}}}, \dots \right\}$$

converges, as $k \to \infty$ and $\theta \to \infty$, to the Airy ensemble.

The result corresponds to a degeneration of Charlier polynomials to the Airy function [Johansson 1999].

For $M_{k,l,\xi}$ a similar result was proved for $k, l \to +\infty$ in [Johansson 2000].

8. Asymptotics of Nonmixed Measures for Large n

As we have seen above, after mixing the study of our measures is not very difficult — we just need to look at the corresponding degenerations of the hypergeometric kernel. The picture before mixing is more subtle.

For $M_{k,l}^{(n)}$ and $M_{k,\infty}^{(n)}$ the asymptotics before and after mixing are different. In comparison to the mixed cases, there appear restrictions on the supports of the limit measures. These restrictions come from the trivial condition that the sum of lengths of rows of a Young diagram with n boxes is equal to n.

Consider the embedding of the set of Young diagrams with n boxes and length at most k into $\mathbb{R}_+^k$ defined by normalizing the lengths of rows of a Young diagram by n.

Proposition 3.2 and Proposition 3.4 lead to the following result.

PROPOSITION 8.1. *As $n \to \infty$, the images of the measures $M_{k,l}^{(n)}$ under the embeddings defined above converge to a measure concentrated on the set*

$$\{(x_1, \dots, x_k) \in \mathbb{R}_+^k \mid x_1 \ge x_2 \ge \dots \ge x_k,\ \sum_{i=1}^k x_i = 1\}.$$

The density of the limit measure with respect to the Lebesgue measure equals

$$\text{const} \cdot \prod_{1 \le i < j \le k} (x_i - x_j)^2 \prod_{i=1}^k x_i^a$$

(*recall that $a = l - k \ge 0$*).

As in Proposition 7.2, we have:

PROPOSITION 8.2. *As $n \to \infty$, the images of the measures $M_{k,\infty}^{(n)}$ under the embeddings defined above converge to the delta measure at the point* $(1/k, \dots, 1/k)$.

Again, the fluctuations around the limit delta measure were determined by Johansson [1999].

Define an embedding of the set of Young diagrams with n boxes and length $\leq k$ into $\mathbb{R}^k$ setting the i-th coordinate of the image of $\lambda \in \mathbb{Y}_n$ equal to

$$\frac{\lambda_i - n/k}{\sqrt{2n/k}}$$

(compare Proposition 7.3).

PROPOSITION 8.3 [Johansson 1999]. *As $n \to \infty$, the images of the measures $M_{k,\infty}^{(n)}$ under the embeddings defined above converge to a measure concentrated on the set*

$$\left\{(x_1, \dots, x_k) \in \mathbb{R}^k \mid x_1 \geq x_2 \geq \dots \geq x_k,\ \sum_{i=1}^{k} x_i = 0\right\}.$$

The density of the limit measure with respect to the Lebesgue measure equals

$$\text{const} \cdot \prod_{1 \leq i < j \leq k} (x_i - x_j)^2 \cdot e^{-x_1^2 - \dots - x_k^2}.$$

For the values of $M_{k,\infty}^{(n)}$ on functions depending only on λ_1 the claim was proved by Tracy and Widom [1999].

In a sense, $M_{\infty,\infty}^{(n)}$ is the most pleasant measure. In this case the asymptotics of $M_{\infty,\infty}^{(n)}$ in the bulk of spectrum and at the edge of spectrum as $n \to \infty$ is exactly the same as the asymptotics of $M_{\infty,\infty,\theta}$ as $\theta \to \infty$. We can say that the asymptotics admits *depoissonization*; see [Borodin et al. 2000; Johansson 1999]. Let us explicitly state the analog of Proposition 7.4.

PROPOSITION 8.4 [Borodin et al. 2000; Johansson 1999]. *Let $\lambda \in \mathbb{Y}_n$ be distributed according to $M_{\infty,\infty}^{(n)}$. Then the random point configuration*

$$\left\{\frac{\lambda_1 - 2\sqrt{n}}{n^{\frac{1}{6}}}, \frac{\lambda_2 - 2\sqrt{n}}{n^{\frac{1}{6}}}, \dots\right\}$$

converges, as $n \to +\infty$, to the Airy ensemble.

Again, the convergence of distributions of first two points was proved in [Baik et al. 1999a; 1999b].

Depoissonization of the result in the bulk of spectrum requires different ideas from those used in the proof of Proposition 8.4. For the discussion of this case see [Borodin et al. 2000].

Proposition 7.5 also admits depoissonization.

PROPOSITION 8.5 [Johansson 1999]. *Let $\lambda \in \mathbb{Y}_n$ be distributed according to $M^{(n)}_{k,\infty}$. Then the random point configuration*

$$\left\{ \frac{\lambda_1 - n/k - 2\sqrt{n}}{(1+\sqrt{n}/k)^{\frac{2}{3}} n^{\frac{1}{6}}}, \frac{\lambda_2 - n/k - 2\sqrt{n}}{(1+\sqrt{n}/k)^{\frac{2}{3}} n^{\frac{1}{6}}}, \dots \right\}$$

converges, as $k \to \infty$, $n \to \infty$ so that $(\ln n)^{\frac{1}{6}}/k \to 0$, to the Airy ensemble.

The structure of spectral z-measures $P_{z,z'}$ defined in Section 3 for general z and z' is fairly complicated. Note that $P_{z,z'}$ is the limit of the n-th level z-measures $M^{(n)}_{z,z'}$; see Proposition 3.4.

Every probability measure on Ω (definition in Section 3) can be viewed as a point process on $\mathbb{R}^*$, if we associate to every point $(\alpha, \beta) \in \Omega$ the point configuration $(\alpha_1, \alpha_2, \dots, -\beta_1, -\beta_2, \dots)$ (compare Proposition 6.4). The correlation functions of the process corresponding to $P_{z,z'}$ were all explicitly computed in [Borodin 1998a]. They do not have determinantal form and can be expressed through multivariate hypergeometric functions.

The situation after mixing is substantially simpler: the process associated to $\widetilde{P}_{z,z'}$ is the Whittaker ensemble (Proposition 6.4).

We refer to [Borodin 1998a; 1998b; 2000a; $\geq$ 2001; Borodin and Olshanski 1998a; 1998b; Olshanski 1998a; 1998b] for a detailed discussion of measures $P_{z,z'}$, $\widetilde{P}_{z,z'}$ and associated point processes.

9. Limit Transitions

The fact that numerous kernels and ensembles described above originated from the same hypergeometric kernel suggests a number of different limit transitions between them.

On the top of the hierarchy we have the hypergeometric kernel which degenerates to all ensembles described above. This corresponds to the fact that the hypergeometric function is on the top of the hierarchy of classical special functions in one variable. The kernel depends on three parameters z, z', ξ, and lives on the lattice $\mathbb{Z}'$.

The Meixner kernel is the specialization of the positive part of the hypergeometric kernel when one of the parameters z, z' is integral. To be concrete, we will assume below that $z \in \{1, 2, \dots\}$.

The Charlier kernel and the Whittaker kernel are one step below — they both depend on two parameters, (z, θ) and (z, z'), respectively. The Charlier kernel is obtained from the Meixner kernel by taking the limit $z' - z \to +\infty$ with $\theta = zz'\xi$ fixed, the Whittaker kernel is obtained from the hypergeometric kernel via a scaling limit when $\xi \to 1$. The Charlier kernel lives on $\mathbb{Z}_+$, the Whittaker kernel lives on $\mathbb{R}^*$.

The Laguerre kernel is a particular case of the positive part of the Whittaker kernel when one of parameters (z, z') is integral. It can be also obtained from the Meixner kernel by taking the limit $\xi \to 1$ (Proposition 7.1). The Laguerre kernel depends on two parameters $(z, a = z' - z)$ and lives on $\mathbb{R}_+$.

The Plancherel kernel is one more step below — it lives on $\mathbb{Z}'$, depends on one parameter θ and can be obtained from the hypergeometric kernel via the limit $z, z' \to \infty$, $\xi \to 0$, $\theta = zz'\xi$ fixed. Its positive part can be obtained either from the Meixner kernel by letting $z, z' \to \infty$ with $\theta = zz'\xi$ fixed, or from the Charlier kernel by taking the limit $z \to \infty$. These two transitions are thoroughly discussed in [Johansson 1999].

The Hermite kernel also depending on one integral parameter z can be obtained from the Charlier kernel via the limit $\theta \to \infty$ (Proposition 7.3).

The Airy kernel is at the bottom — it has no parameters. It can be obtained in a number of different ways. For example, one can obtain the Airy kernel in the limit $\theta \to +\infty$ of the Plancherel kernel at the edge of spectrum (Proposition 7.4), or as the limit at the edge of spectrum of the Hermite kernel and the Laguerre kernel with parameter a fixed when the order z of these polynomial ensembles goes to infinity [Forrester 1993; Tracy and Widom 1994a]. It can also be obtained as a double limit of Charlier or Meixner kernels; see the end of Section 7, as well as [Johansson 2000; 1999].

Of course, this is not the end of the story. The discrete sine kernel and the conventional sine kernel can be obtained from the Plancherel kernel as $\theta \to \infty$; see Section 7. The so-called Bessel kernel can be extracted from the Laguerre kernel "at the hard edge of spectrum" [Forrester 1993; Nagao and Wadati 1993; Tracy and Widom 1994b]. The sine kernel can be obtained from the Laguerre and Hermite kernels in the bulk of spectrum; see [Nagao and Wadati 1991], for example. A number of new kernels can be obtained from the Whittaker kernel; see [Olshanski 1998b]. Presumably, all these kernels can also be obtained as double or triple limits of the hypergeometric kernel.

Thus, a variety of kernels known so far can be obtained from the hypergeometric kernel, often in several different ways. As we tried to demonstrate above, sometimes such degenerations also carry information about the asymptotic behavior of certain combinatorial objects.

Addendum

Recent developments in the harmonic analysis on the infinite *unitary* group led to a substantial expansion of the hierarchy of correlation kernels described above. See [Borodin and Olshanski $\geq$ 2001b].

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