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Dual Isomonodromic Tau Functions and Determinants of Integrable Fredholm Operators

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ABSTRACT. The Hamiltonian approach to dual isomonodromic deformations in the setting of rational R-matrix structures on loop algebras is reviewed. The construction of a particular class of solutions to the deformation equations, for which the isomonodromic τ-functions are given by the Fredholm determinants of a special class of integrable integral operators, is shown to follow from the matrix Riemann–Hilbert approach of Its, Izergin, Korepin and Slavnov. This leads to an interpretation of the notion of duality in terms of the data defining the Riemann–Hilbert problem, and Laplace–Fourier transforms of the corresponding Fredholm integral operators.

1. Introduction

1a. Isomonodromic Deformation Equations. We consider rational covariant derivative operators on the punctured Riemann sphere, having the form

$$\mathcal{D}_\lambda = \frac{\partial}{\partial\lambda} - \mathcal{N}(\lambda),$$
$$\mathcal{N}(\lambda) := B + \sum_{i=1}^{n} \frac{N_i}{\lambda - \alpha_j}, \tag{1–1}$$

where

$$B = \mathrm{diag}(\beta_1, \dots, \beta_r), \quad N_j \in \mathfrak{gl}(r, \mathbb{C}).$$

They have regular singular points at $\{\lambda = \alpha_i\}_{i=1,\dots,n}$ and an irregular singularity at $\lambda = \infty$ with Poincare index 1. If the residue matrices $\{N_i\}_{i=1,\dots,n}$ are deformed differentiably with respect to the parameters $\{\alpha_i\}_{i=1,\dots,n}$ and $\{\beta_a\}_{a=1,\dots,r}$, the monodromy (including Stokes parameters and connection matrices) of the operator $\mathcal{D}_\lambda$ will be invariant under such deformations, as was shown in [Jimbo et al. 1980; Jimbo et al. 1981], provided the differential equations implied by the commutativity conditions

$$[\mathcal{D}_\lambda,\ \mathcal{D}_{\alpha_i}] = 0, \quad i = 1, \dots, n, \tag{1–2}$$

$$[\mathcal{D}_\lambda,\ \mathcal{D}_{\beta_a}] = 0, \quad a = 1, \dots, r, \tag{1–3}$$

are satisfied, where the differential operators $\mathcal{D}_{\alpha_j}$, $\mathcal{D}_{\beta_a}$ are defined by

$$\begin{aligned}
\mathcal{D}_{\alpha_i} &:= \frac{\partial}{\partial \alpha_i} - U_i, \\
\mathcal{D}_{\beta_a} &:= \frac{\partial}{\partial \beta_a} - V_a, \\
U_i &:= -\frac{N_i}{\lambda - \alpha_j}, \quad i = 1, \dots, n, \\
V_a &:= \lambda E_a + \sum_{\substack{b=1 \\ b \neq a}}^{r} \frac{E_a \left(\sum_{j=1}^n N_j\right) E_b + E_b \left(\sum_{j=1}^n N_j\right) E_a}{\beta_a - \beta_b}, \quad a = 1, \dots, r,
\end{aligned}$$

and E_a is the elementary $r \times r$ matrix with elements

$$(E_a)_{bc} := \delta_{ab}\delta_{ac}.$$

These also imply the commutativity conditions

$$[\mathcal{D}_{\alpha_i}, \mathcal{D}_{\alpha_j}] = [\mathcal{D}_{\alpha_i}, \mathcal{D}_{\beta_a}] = [\mathcal{D}_{\beta_a}, \mathcal{D}_{\beta_b}] = 0, \quad \text{for } i, j = 1, \dots, n \text{ and } a, b = 1, \dots, r. \tag{1–4}$$

Equations (1–2), (1–3), and (1–4) define a Frobenius integrable system of PDE's for the residue matrices $\{N_i\}_{i=1,\dots,n}$. They are "zero curvature" equations, implying the consistency of the overdetermined system

$$\mathcal{D}_\lambda \Psi = 0, \quad \mathcal{D}_{\alpha_i} \Psi = 0, \quad \mathcal{D}_{\beta_a} \Psi = 0. \tag{1–5}$$

They may also be interpreted as Hamiltonian equations [Harnad 1994] with respect to the Lie Poisson structure on the space $(\mathfrak{gl}(r))^{*n} = \{N_1, \dots, N_n\}$ of residue matrices in $\mathcal{N}(\lambda)$, defined by

$$\{(N_i)_{ab}, (N_j)_{cd}\} = \delta_{ij}[(N_i)_{ad}\delta_{bc} - (N_i)_{bc}\delta_{ad}] \tag{1–6}$$

(where we identify $\mathfrak{gl}(r)$ and its dual space $(\mathfrak{gl}(r))^*$ through the trace pairing $(X, Y) = \operatorname{tr}(XY)$). The Hamiltonians $\{H_i,\ K_a\}_{j=1,\dots,n,\ a=1,\dots,r}$ generating them are given by

$$\begin{aligned}
H_i &:= \tfrac{1}{2} \operatorname{res}_{\lambda=\alpha_i} \operatorname{tr}(\mathcal{N}(\lambda)^2) = \operatorname{tr}(BN_i) + \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{\operatorname{tr}(N_i N_j)}{\alpha_i - \alpha_j}, \\
K_a &:= \sum_{j=1}^{n} \alpha_j \, (N_j)_{aa} + \sum_{\substack{b=1 \\ b \neq a}}^{r} \frac{\left(\sum_{j=1}^n N_j\right)_{ab} \left(\sum_{k=1}^n N_k\right)_{ba}}{\beta_a - \beta_b}.
\end{aligned} \tag{1–7}$$

It follows from the Poisson bracket relations (1–6) that

$$\{\mathcal{N}(\lambda), H_i\} = [U_i, \mathcal{N}(\lambda)],$$
$$\{\mathcal{N}(\lambda), K_a\} = [V_a, \mathcal{N}(\lambda)],$$

which imply, together with the identities

$$\mathcal{N}(\lambda)_{\alpha_i} = \frac{\partial U_i}{\partial \lambda} = \frac{N_i}{(\lambda - \alpha_i)^2}, \tag{1–8}$$

$$\mathcal{N}(\lambda)_{\beta_a} = \frac{\partial V_a}{\partial \lambda} = E_a \tag{1–9}$$

(where the subscripts in $\mathcal{N}(\lambda)_{\alpha_i}$ and $\mathcal{N}(\lambda)_{\beta_a}$ denote derivation only with respect to the explicit dependence on the parameters appearing in the definition of $\mathcal{N}(\lambda)$), that the equations obtained by equating the residues at the poles $\{\lambda = \alpha_j\}$ in (1–2)–(1–3) and the leading terms at $\lambda = \infty$ are the nonautonomous Hamiltonian equations generated by the H_i's and K_a's when the α_i's and β_a's are identified with the respective "time" parameters.

The compatibility of these equations may be seen as a consequence of the fact that all the Hamiltonians Poisson commute:

$$\{H_i, H_j\} = 0, \quad \{H_i, K_a\} = 0, \quad \{K_a, K_b\} = 0,$$

for $i, j = 1, \dots, n$ and $a, b = 1, \dots, r$. This further implies [Jimbo et al. 1980; Jimbo et al. 1981] that the differential 1-form

$$\theta := \sum_{i=1}^{n} H_i \, d\alpha_i + \sum_{a=1}^{b} K_a \, d\beta_a, \tag{1–10}$$

on the parameter space, taken along any solution to this system of equations, is closed, and hence locally exact, implying the existence of the isomonodromic τ-function $\tau(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_r)$, defined up to a multiplicative constant by:

$$d \ln(\tau) = \theta. \tag{1–11}$$

The Hamiltonian structure of the above equations may be seen to follow from a more general setting, involving commuting Hamiltonians flows on loop algebras generated by spectral invariant functions, with respect to a rational R-matrix structure, adapted to the case of nonautonomous Hamiltonians [Harnad 1994]. For our purposes, it is sufficient to consider the Poisson space $\widetilde{\mathfrak{gl}}_{\mathrm{rat}}(r)$ consisting of $r \times r$ matrix-valued rational functions $X(\lambda)$ of the complex parameter λ, and to split $\widetilde{\mathfrak{gl}}_{\mathrm{rat}}(r)$ into the direct sum

$$\widetilde{\mathfrak{gl}}_{\mathrm{rat}}(r) := \mathfrak{g}_+ + \mathfrak{g}_-$$

of the subspaces $\mathfrak{g}_+$ consisting of polynomial $X(\lambda)$'s, and $\mathfrak{g}_-$ consisting of $X(\lambda)$'s satisfying $X(\infty) = 0$. Viewing each $X(\lambda)$ as an endomorphism of $\mathbb{C}^r$, we may concisely represent the Poisson bracket structure on the space $\widetilde{\mathfrak{gl}}_{\mathrm{rat}}(r)$ by simultaneously taking tensor products on the space of endomorphisms and giving an equation in the space $\mathrm{End}(\mathbb{C}^r \otimes \mathbb{C}^r)$ that determines the Poisson brackets of all the matrix elements of the $X(\lambda)$'s:

$$\{X(\lambda) \overset{\otimes}{,} X(\mu)\} = \big[r(\lambda-\mu),\, X(\lambda) \otimes \boldsymbol{I} + \boldsymbol{I} \otimes X(\mu)\big], \tag{1–12}$$

where

$$r(\lambda-\mu) := \frac{P_{12}}{\lambda-\mu} \in \mathrm{End}(\mathbb{C}^r \otimes \mathbb{C}^r), \qquad P_{12}(u \otimes v) := v \otimes u$$

is the rational classical R-matrix. We may view $\mathcal{N}(\lambda)$ as the image of a map

$$\begin{aligned} \mathcal{N}^A_B &: (\mathfrak{gl}(r))^{*n} \to \widetilde{\mathfrak{gl}}_{\mathrm{rat}}(r) \\ \mathcal{N}^A_B &: \{N_1, \dots, N_n\} \mapsto \mathcal{N}(\lambda) = B + \sum_{j=1}^{n} \frac{N_j}{\lambda - \alpha_j}. \end{aligned} \tag{1–13}$$

It is easily verified that this defines a Poisson embedding of $(\mathfrak{gl}(r))^{*n}$ as an affine subspace in $\widetilde{\mathfrak{gl}}_{\mathrm{rat}}(r)$. (If we take the union over all $r \times r$ matrices B, this becomes a linear Poisson subspace, but since the coefficients of the matrices B are in the centre of the Poisson algebra on this space, B may as well be chosen to have fixed constant values.)

Now let $\mathcal{I}$ denote the ring of polynomial functions of the coefficients of elements of $\widetilde{\mathfrak{gl}}_{\mathrm{rat}}(r)$ that are invariant under conjugation by λ-dependent invertible matrices, restricted to a finite dimensional Poisson submanifold such as, for example, the image of the map $\mathcal{N}^A_B$. This is just the ring of spectral invariants, generated by the coefficients of the characteristic polynomial

$$\det(X(\lambda) - z\boldsymbol{I}) := \mathcal{P}(\lambda, z).$$

The classical R-matrix theorem, adapted to the case of explicit time dependence in the Hamiltonians and in the elements $X(\lambda) \in \widetilde{\mathfrak{gl}}_{\mathrm{rat}}(r)$, then tells us that the elements of $\mathcal{I}$ Poisson commute, and the Hamiltonian equations generated by any $\phi \in \mathcal{I}$ may be expressed as

$$\frac{dX(\lambda)}{dt} = \pm[(d\phi)_\pm, X(\lambda)] + X(\lambda)_t, \tag{1–14}$$

where $X(\lambda)_t$ denotes the explicit time derivative, the differential $d\phi$ is identified as an element of the same space $\widetilde{\mathfrak{gl}}_{\mathrm{rat}}(r)$ through the dual pairing $\langle X, Y\rangle := \mathrm{res}_{\lambda=\infty} \mathrm{tr}((X(\lambda)Y\lambda))$ and $(d\phi)_\pm$ denotes projection to the subspaces $\mathfrak{g}_\pm$.

If it happens also that the term $X(\lambda)_t$ equals the λ derivative of either $d\phi_+$ or $-d\phi_-$

$$X(\lambda)_t = \pm\frac{\partial d(\phi_\pm)}{\partial\lambda}, \tag{1–15}$$

then equation (1–14) becomes a commutativity condition

$$\left[\frac{\partial}{\partial\lambda} - X(\lambda),\ \frac{\partial}{\partial t} \mp (d\phi)_\pm\right] = 0. \tag{1–16}$$

(More generally, we could replace $+d\phi_-$ and $-d\phi_+$ by any element along the line $(1+c)d\phi_+ + cd\phi_-$ through them.) In particular, this is the case if we choose ϕ as any of the Hamiltonians $\{H_i\}_{i=1,\dots,n}$

$$H_i = \frac{1}{2}\operatorname{res}_{\lambda=\alpha_i}\operatorname{tr}(X^2(\lambda)),$$

which clearly are elements of the spectral ring $\mathcal{I}$ which, when evaluated on $X(\lambda) = \mathcal{N}(\lambda)$ of the form (1–1), give

$$-(dH_i)_- = U_i = -\frac{N_i}{\lambda - \alpha_i}.$$

Condition (1–15) is satisfied on this subspace if we identify the time parameter as $t = \alpha_i$, since this reduces to the identity (1–8), while (1–16) gives the equations (1–2).

To obtain a similar interpretation of the equations (1–3), we note that the Hamiltonians $\{K_a\}_{a=1,\dots,r}$ can be expressed as follows:

$$K_a = \tfrac{1}{2}\operatorname{res}_{\lambda=\infty}\left(\operatorname{res}_{z=\beta_a}\lambda[(B - z\boldsymbol{I})^{-1}\mathcal{N}(\lambda)]^2 - 2\operatorname{tr}[(B - z\boldsymbol{I})^{-1}\mathcal{N}(\lambda)]\right),$$

which shows that they also belong to the spectral ring $\mathcal{I}$. Taking the $\mathfrak{g}_+$ projections of the differentials dK_a evaluated at $X(\lambda) = \mathcal{N}(\lambda)$ gives

$$(dK_a)_+ = V_a = \lambda E_a + \sum_{\substack{b=1\\ b\neq a}}^{r} \frac{E_a\left(\sum_{j=1}^{n} N_j\right)E_b + E_b\left(\sum_{j=1}^{n} N_j\right)E_a}{\beta_a - \beta_b}.$$

Identifying the time parameter as $t = \beta_a$, condition (1–15) reduces to (1–9), and (1–16) gives the equations (1–3).

1b. Symplectic Lift and Duality. The Hamiltonian structure of the isomonodromic deformation equations presented above involves a degenerate Poisson structure. (The center of the Poisson algebra consists of the elements of $\mathcal{I}$ obtained by localizing the spectral invariants at the points $\{\alpha_1, \dots, \alpha_n, \infty\}$.) It is possible, however, to view this space as a quotient of a symplectic space $\mathcal{M}$ under a suitable Hamiltonian group action. Moreover, doing so shows that the rôles of the deformation parameters $\{\alpha_1, \dots, \alpha_n\}$ and $\{\beta_1, \dots, \beta_r\}$ are in some sense interchangeable, and there exists another "dual" isomonodromic deformation system, obtained also as a Hamiltonian quotient of the system on $\mathcal{M}$, in which the parameters $\{\beta_1, \dots, \beta_r\}$ appear as the locations of the regular singular points, while $\{\alpha_1, \dots, \alpha_n\}$ become the eigenvalues at ∞ of the rational matrix defining this dual system.

To see this, suppose that the rank of the residue matrix N_i is k_i. We may express N_i in a factored form as the product of two maximal rank matrices of dimensions $r \times k_i$ and $k_i \times r$:

$$N_i = -G_i^T F_i, \qquad F_i, G_i \in \operatorname{Mat}^{k_i \times r}. \tag{1–17}$$

Of course, this factorization is arbitrary up to the following action of the group $\mathrm{GL}(k_i, \mathbb{C})$ on the space of such pairs (F_i, G_i):

$$g_i : (F_i, G_i) \mapsto (g_i F_i, (g_i^T)^{-1}), \quad g_i \in \mathrm{GL}(k_i, \mathbb{C}).$$

Making a similar factorization of all the residue matrices, we let

$$N := \sum_{i=1}^{n} k_i$$

and define the space $\mathcal{M}$ to consist of the set of pairs (F, G) of $N \times r$ matrices formed from n vertical blocks of $k_i \times r$ matrices of maximal rank:

$$F := \begin{pmatrix} F_1 \\ \cdot \\ F_i \\ \cdot \\ F_n \end{pmatrix}, \quad G := \begin{pmatrix} G_1 \\ \cdot \\ G_i \\ \cdot \\ G_n \end{pmatrix}. \tag{1–18}$$

Let $A \in \mathfrak{gl}(N, \mathbb{C})$ be the diagonal matrix with eigenvalues $(\alpha_1, \dots, \alpha_n)$ appearing with respective multiplicities $(k_1, \dots, k_n)$.

$$A = \mathrm{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n).$$

Using the resolvent matrix $(A - \lambda \boldsymbol{I})^{-1}$, we can express the rational matrix $\mathcal{N}(\lambda)$ as follows:

$$\mathcal{N}(\lambda) = B + G^T (A - \lambda \boldsymbol{I})^{-1} F,$$

where the different possible choices for the pairs (F, G) form an orbit under the block diagonal subgroup $G_A := \mathrm{GL}(k_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(k_n, \mathbb{C}) \subset \mathrm{GL}(N, \mathbb{C})$, under the action $G_A \times \mathcal{M} \to \mathcal{M}$ defined by

$$(g, (F, G)) \mapsto (gF, (g^T)^{-1} G), \ \text{for } g \in G_A. \tag{1–19}$$

This subgroup $G_A \subset \mathrm{GL}(N, \mathbb{C})$ is just the stabilizer of $A \in \mathfrak{gl}(N, \mathbb{C})$ under the adjoint (conjugation) action. Choosing the canonical symplectic structure

$$\omega := \mathrm{tr}(dF^T \wedge dG) \tag{1–20}$$

on $\mathcal{M}$, the G_A action is a free Hamiltonian group action generated by the equivariant moment map

$$\begin{aligned} J_k^N : (F, G) &\to (F_1 G_1^T, \dots, F_i G_i^T, \dots, F_n G_n^T) \\ &\in \mathfrak{gl}^*(k_1, \mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}^*(k_i, \mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}^*(k_n, \mathbb{C}), \end{aligned}$$

where the dual space $\mathfrak{gl}^*(k_i, \mathbb{C})$ is identified with the space $\mathfrak{gl}(k_i, \mathbb{C})$ through the trace pairing. The Poisson subspace of $(\mathfrak{gl}(r))^{*n}$ consisting of matrices $\{N_1, \dots, N_n\}$ having respective ranks $\{k_1, \dots, k_n\}$ may thus be viewed as a quotient $\mathcal{M}/G_A$ by the Hamiltonian group action (1–19). Composing the projection map $\pi_A^{\mathcal{M}} : \mathcal{M} \to \mathcal{M}/G_A$ with the Poisson embedding map $\mathcal{N}_B^A$ defined in (1–13), we obtain a Poisson map $J_B^A : \mathcal{M} \to \widetilde{\mathfrak{gl}}_{\mathrm{rat}}(r)$ given by

$$J_B^A : (F, G) \mapsto \mathcal{N}(\lambda) := B + G^T (A - \lambda \boldsymbol{I})^{-1} F, \tag{1–21}$$

whose fibres are the orbits under the free G_A-action (1–19), allowing us to identify the image J_B^A both as a quotient space $\mathcal{M}/G_A$ and a Poisson subspace of $\widetilde{\mathfrak{gl}}_{\mathrm{rat}}(r)$ (i.e., the space consisting of those $\mathcal{N}(\lambda)$'s for which the residue matrices $\{N_1, \ldots, N_n\}$ have ranks $\{k_1, \ldots, k_n\}$). Moreover, the ring $\mathcal{I}$ of spectral invariants, restricted to this subspace, may be pulled back to $\mathcal{M}$ to define a Poisson commuting ring

$$\mathcal{I}_B^A := J_B^{A*}(\mathcal{I})$$

of G_A-invariant functions on $\mathcal{M}$. The Hamiltonian vector fields generated by the elements of $\mathcal{I}_B^A$ project to the corresponding vector fields on the quotient, as do their integral curves. In particular, if we identify the parameters $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\beta_1, \ldots, \beta_r\}$ with multi-time variables associated to the pullbacks of the Hamiltonians $\{H_1, \ldots, H_n\}$ and $\{K_1, \ldots, K_r\}$, the corresponding nonautonomous Hamiltonian systems on $\mathcal{M}$ are given by

$$\begin{aligned} \frac{\partial F}{\partial \alpha_i} &= \{F_i,\ J_B^{A*} H_i\}, & \frac{\partial G}{\partial \alpha_i} &= \{G_i,\ J_B^{A*} H_i\}, \\ \frac{\partial G}{\partial \beta_a} &= \{G_i,\ J_B^{A*} K_a\}, & \frac{\partial G}{\partial \beta_a} &= \{G_i,\ J_B^{A*} K_a\}, \end{aligned} \tag{1–22}$$

where $\{\,\cdot\,,\cdot\,\}$ denotes the Poisson brackets on $\mathcal{M}$ determined by the symplectic form (1–20).

There is also a Hamiltonian action of the subgroup $G_B \subset \mathrm{GL}(r, \mathbb{C})$ stabilizing the matrix B under conjugation, which commutes with the G_A action (1–19), namely the action $G_B \times \mathcal{M} \to \mathcal{M}$ defined by

$$(g, (F, G)) \mapsto (F g^{-1}, G g^T), \qquad g \in G_B. \tag{1–23}$$

If the eigenvalues $\{\beta_1, \ldots, \beta_r\}$ are required to be distinct, the group G_B just consists of the invertible diagonal matrices in $\mathrm{GL}(r, \mathbb{C})$. (More generally, like $G_A \subset \mathrm{GL}(N, \mathbb{C})$, the group G_B is identified with the block diagonal subgroup $G_B := \mathrm{GL}(l_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(l_p, \mathbb{C}) \subset \mathrm{GL}(r, \mathbb{C})$, where $\{l_1, \ldots, l_p\}$ are the multiplicities of the eigenvalues of B.) We make this restriction henceforth, and also assume $N \geq r$. From its definition, the quotient map J_B^A intertwines the G_B action on $\mathcal{M}$ with the conjugation action of $G_B \subset \mathrm{GL}(r, \mathbb{C})$ on $\widetilde{\mathfrak{gl}}_{\mathrm{rat}}(r)$. Since the elements of the ring $\mathcal{I}$ are invariant under this action, they project to a Poisson commuting ring on the double quotient $G_B \backslash \mathcal{M}/G_A$.

It is natural to now ask what happens if we interchange the rôles of the matrices A and B and the corresponding groups G_A and G_B. We may consider the space $\widetilde{\mathfrak{gl}}_{\mathrm{rat}}(N)$ consisting of $N \times N$ matrices $Y(z)$ depending rationally on an auxiliary complex variable z, with the rational R-matrix structure (1–12) (with the replacements $r \to N$, $\lambda \to z$, $X \to Y$). Restricting analogously to the Poisson subspace consisting of elements of the form

$$\mathcal{M}(z) = A + \sum_{a=1}^{r} \frac{M_a}{z - \beta_a}, \qquad M_a \in \mathfrak{gl}(N, \mathbb{C}), \tag{1–24}$$

we must, consistently with our assumption that the matrix B has a simple spectrum, require the residue matrices $\{M_1, \dots, M_r\}$ to all have rank one. We may now repeat the entire Hamiltonian quotienting process as above by defining a Poisson map $J_A^B : \mathcal{M} \to \widetilde{\mathfrak{gl}}_{\mathrm{rat}}(N)$ as in (1–21):

$$J_A^B : (F, G) \mapsto \mathcal{N}(\lambda) = A + F(B - z\boldsymbol{I})^{-1} G^T. \tag{1–25}$$

The fibres of this map are the orbits of the free Hamiltonian G_B-action (1–23), so we may identify the quotient space (which we express as a left quotient) $G_B \backslash \mathcal{M}$ simultaneously as the image of the map J_A^B and as the Poisson subspace of $\widetilde{\mathfrak{gl}}_{\mathrm{rat}}(N)$ consisting of elements of the form (1–24) with rank one residue matrices at the poles $z = \beta_a$. Again, the Poisson map J_A^B intertwines the G_A action on $\mathcal{M}$ with the conjugation action of $G_A \subset \mathrm{GL}(N, \mathbb{C})$ on $\widetilde{\mathfrak{gl}}_{\mathrm{rat}}(N)$. We may define the ring $\tilde{\mathcal{I}}$ to consist of the spectral invariant polynomial functions formed from the $\mathcal{M}(z)$'s (i.e., generated by their characteristic polynomials), and obtain the Poisson commutative ring $\mathcal{I}_A^B := J_A^{B*}(\tilde{\mathcal{I}})$ by pulling back the elements of $\tilde{\mathcal{I}}$ under the map J_A^B. Defining the projections $\pi_A : \widetilde{\mathfrak{gl}}_{\mathrm{rat}}(r) \to \widetilde{\mathfrak{gl}}_{\mathrm{rat}}(r)/G_A$ and $\pi_B : \widetilde{\mathfrak{gl}}_{\mathrm{rat}}(N) \to G_B \backslash \widetilde{\mathfrak{gl}}_{\mathrm{rat}}(N)$ to the quotient space under the respective conjugation actions, we see that the composite maps $\pi_A \circ J_B^A$ and $\pi_B \circ J_A^B$ coincide, defining the projection from $\mathcal{M}$ to the double quotient $G_B \backslash \mathcal{M} / G_A$.

We can now consider the analog of the overdetermined system (1–5)

$$\widetilde{\mathcal{D}}_\lambda \widetilde{\Psi} = 0, \quad \widetilde{\mathcal{D}}_{\alpha_i} \widetilde{\Psi} = 0, \quad \widetilde{\mathcal{D}}_{\beta_a} \widetilde{\Psi} = 0 \tag{1–26}$$

with respect to the operators $\widetilde{\mathcal{D}}_\lambda$, $\widetilde{\mathcal{D}}_{\alpha_i}$, and $\widetilde{\mathcal{D}}_{\beta_a}$ defined by

$$\left.\begin{aligned}
\widetilde{\mathcal{D}}_z &:= \frac{\partial}{\partial z} - \mathcal{M}(z), \\
\widetilde{\mathcal{D}}_{\beta_a} &:= \frac{\partial}{\partial \alpha_i} - \widetilde{V}_a, \\
\widetilde{\mathcal{D}}_{\alpha_i} &:= \frac{\partial}{\partial \beta_a} - \widetilde{U}_i, \\
\widetilde{V}_a &:= -\frac{M_a}{z - \beta_a}, \quad a = 1, \dots, r, \\
\widetilde{U}_i &:= z E_i + \sum_{\substack{j=1 \\ j \neq i}}^{r} \frac{E_i \left(\sum_{b=1}^{r} M_b\right) E_j + E_j \left(\sum_{b=1}^{n} M_b\right) E_i}{\alpha_i - \alpha_j}, \\
&\qquad\qquad\qquad\qquad\qquad i = 1, \dots, n.
\end{aligned}\right\} \tag{1–27}$$

The Frobenius integrable set of compatibility conditions for this system are again given by the commutativity of the operators $\widetilde{\mathcal{D}}_\lambda$, $\widetilde{\mathcal{D}}_{\alpha_i}$, and $\widetilde{\mathcal{D}}_{\beta_a}$, and these may again be viewed as nonautonomous Hamiltonian systems either in the quotient space $G_B\backslash\mathcal{M}$ or lifted to $\mathcal{M}$. The Hamiltonians in the ring $\tilde{\mathcal{I}}$ corresponding to the α_i deformations and the β_a deformations are given respectively by

$$\begin{aligned}\widetilde{H}_i &= \tfrac{1}{2}\operatorname{res}_{z=\infty}\left(\operatorname{res}_{\lambda=\alpha_i} z[(A-\lambda\boldsymbol{I})^{-1}\mathcal{M}(z)]^2 - 2\operatorname{tr}[(A-\lambda\boldsymbol{I})^{-1}\mathcal{M}(z)]\right),\\ \widetilde{K}_a &= \tfrac{1}{2}\operatorname{res}_{z=\beta_a}\operatorname{tr}(\mathcal{M}^2(z)).\end{aligned} \tag{1–28}$$

The main result relating these systems to the ones introduced in the previous subsection is contained in the following theorem.

THEOREM 1.1 [Harnad 1994]. *The two Poisson commuting rings $\mathcal{I}_B^A$ and $\mathcal{I}_A^B$ coincide, and, in particular, we have the equalities*

$$J_B^{A*}(H_i) = J_A^{B*}(\widetilde{H}_i), \qquad J_B^{A*}(K_a) = J_A^{B*}(\widetilde{K}_a), \qquad i = 1,\dots,n,\ a = 1,\dots,r.$$

Therefore, the lifted systems in $\mathcal{M}$ coincide, as do the projected systems in $G_B\backslash\mathcal{M}/G_A$.

2. The Riemann–Hilbert Problem and Integrable Fredholm Operators

In this section, we show how a class of solutions to the isomonodromic deformation equations considered here result from the solution of a particular type of matrix Riemann–Hilbert problem, and how the corresponding τ-function may be identified as the Fredholm determinant of a special class of "integrable" integral operators. In this, we follow the general approach developed in [Its et al. 1990]. The results presented here are based on joint work with Alexander Its, and are presented in greater detail in [Harnad and Its 1997], where further developments may also be found.

2a. Riemann–Hilbert Problem. Let Γ be an oriented curve in the complex λ-plane passing sequentially through the points $\{\alpha_1,\dots,\alpha_n\}$. In the following, we take $n = 2m$ to be even, (although we may also let it be odd by considering ∞ as the last point). Denote by Γ_j be the segment of Γ from α_{2j-1} to α_{2j}. Now choose a set of constant pairs of maximal rank matrices $\{(f_j,g_j)\}_{j=1,\dots,m}$ of dimensions $\{r\times k_j\}_{j=1,\dots,m}$, where $k_j \le r$, satisfying the orthogonality conditions:

$$g_j^T f_j = 0, \quad j = 1,\dots,m. \tag{2–1}$$

Let $\theta_j(\lambda)$ denote the characteristic function for the segment G_j, viewed as a function defined along Γ, and define the piecewise constant functions

$$f_0(\lambda) := \sum_{j=1}^m f_j\theta_j(\lambda), \quad g_0(\lambda) := \sum_{j=1}^m g_j\theta_j(\lambda), \tag{2–2}$$

supported on $\cup_{j=1}^m \Gamma_j$. Now let $\Psi_0(\lambda)$ denote the exponential "vacuum" isomonodromic solution

$$\Psi_0(\lambda) := e^{\lambda B}$$

satisfying

$$\frac{\partial \Psi_0}{\partial \lambda} = B\Psi_0, \quad \frac{\partial \Psi_0}{\partial \alpha_i} = 0, \quad \frac{\partial \Psi_0}{\partial \beta_a} = \lambda E_a \Psi_0,$$

and let

$$f(\lambda) := \Psi_0(\lambda) f_0(\lambda), \qquad g(\lambda) := (\Psi_0^T(\lambda))^{-1} g_0(\lambda). \tag{2–3}$$

Define a $\mathrm{GL}(r,\mathbb{C})$-valued piecewise continuous exponential function along the curve Γ by

$$H(\lambda) := \Psi_0(\lambda) H_0(\lambda) \Psi_0^{-1}(\lambda) = \boldsymbol{I} + 2\pi f(\lambda) g^T(\lambda),$$

where $H_0(\lambda)$ is the piecewise constant $\mathrm{GL}(r,\mathbb{C})$-valued function

$$H_0(\lambda) := \boldsymbol{I} + 2\pi f_0(\lambda) g_0^T(\lambda).$$

In terms of these quantities, we pose the following Riemann–Hilbert problem: Find a $\mathrm{GL}(r,\mathbb{C})$-valued function $X(\lambda)$ that is holomorphic on the complement $\mathbb{C}\backslash\Gamma$ of the curve Γ and at $\lambda = \infty$, with the following asymptotic form near $\lambda = \infty$,

$$X(\lambda) = \boldsymbol{I} + O(\lambda^{-1}),$$

and such that the limits $X_\pm(\lambda)$ of the values of $X(\lambda)$ when approaching the curve Γ from the left $(+)$ and right $(-)$ are related by

$$X_+(\lambda) = X_-(\lambda) H(\lambda).$$

Moreover, we require that the local behaviour of the singularity in $X(\lambda)$ in a neighborhood of any of the points $\{\alpha_1, \dots, \alpha_n\}$ should be just logarithmic. (Its uniqueness follows from the analyticity conditions imposed.) Define

$$\Psi(\lambda) := X(\lambda)\Psi_0(\lambda). \tag{2–4}$$

Because of the orthogonality conditions (2–1), the limits

$$\begin{aligned} F_i &:= \lim_{\lambda\to\alpha_i} (X(\lambda) f(\lambda))^T, \\ G_i &:= (-1)^j \lim_{\lambda\to\alpha_i} (X(\lambda) g(\lambda))^T \end{aligned} \tag{2–5}$$

(taken within the curve segments Γ_j) exist. Now define the pair (F, G) entering in (1–21) as in (1–18), and the residue matrices in $\mathcal{N}(\lambda)$ by (1–17). Then:

THEOREM 2.1 [Harnad and Its 1997]. *The matrix-valued function* $\Psi(\lambda, a_1, \dots, n, \beta_1, \dots, \beta_r)$ *defined in* (2–4) *satisfies the linear equations* (1–5) *with the matrix* $\mathcal{N}(\lambda)$ *and its residues* N_i *determined through* (1–17) *in terms of the matrices* $\{F_i, G_i\}_{i=1,\dots,n}$ *defined in* (2–5). *The corresponding pairs* $(F, G) \in \mathcal{M}$ *satisfy*

Hamilton's equations (1–22). *The local behaviour of the function* $\Psi(\lambda)$ *in a neighborhood of the curve segment* Γ_j *is of the form*

$$\Psi(\lambda) \sim \Psi^j_{\mathrm{an}}(\lambda)\left(\frac{\lambda-\alpha_{2j-1}}{\lambda-\alpha_{2j}}\right)^{f_j g_j^T}, \tag{2–6}$$

where $\Psi^j_{\mathrm{an}}(\lambda)$ *is analytic in this neighborhood. Therefore the monodromy representation is generated by the following matrices* $\{M_i\}_{i=1,\dots,2m}$, *corresponding to simple positively oriented loops from an arbitrary base point* λ_0 *going once around the singular points* $\{a_i\}_{i=1,\dots,2m}$:

$$\begin{aligned} M_{2j-1} &= \exp(\ \ 2\pi i f_j g_j^T) = \boldsymbol{I} + 2\pi i f_j g_j^T, \\ M_{2j} &= \exp(-2\pi i f_j g_j^T) = \boldsymbol{I} - 2\pi i f_j g_j^T, \qquad j=1,\dots,m. \end{aligned}$$

(*There is no monodromy at* $\lambda=\infty$, *and the Stokes matrices are just the identity element.*)

The proof of this result is elementary; the local behaviour (2–6) follows from the conditions of the associated Riemann–Hilbert problem, and the differential equations (1–5) follow from explicit differentiation to obtain the local pole structure in $(\partial\Psi/\partial\lambda)\Psi^{-1}$, and application of Liouville's theorem. (Of course, the actual solution of the Riemann–Hilbert problem is highly nontrivial.)

2b. The Fredholm Determinant. We now choose all the ranks $\{k_i\}_{i=1,\dots,n}$ equal to k, and define a matrix Fredholm integral operator $\boldsymbol{K} : L^2(\Gamma,\mathbb{C}^k) \to L^2(\Gamma,\mathbb{C}^k)$ by

$$\boldsymbol{K}(\boldsymbol{v})(\lambda) = \int_\Gamma K(\lambda,\mu)\boldsymbol{v}(\mu)d\mu, \quad \boldsymbol{v} \in L^2(\Gamma,\mathbb{C}^k),$$

where the kernel is chosen to have the special form

$$K(\lambda,\mu) = \frac{f^T(\lambda)g(\mu)}{\lambda-\mu},$$

with $f(\lambda)$ and $g(\lambda)$ defined in (2–3). (For the case $r=2$, $k=1$, and $\beta_1=-\beta_2$, this is just the sine kernel occurring in the computation of spectral distributions for random matrices in the GUE [Tracy and Widom 1993; 1994]. Because of the orthogonality conditions (2–1), we have

$$f^T(\lambda)g(\lambda) = 0,$$

and hence the kernel is nonsingular, with diagonal values

$$K(\lambda,\lambda) = f'^T(\lambda)g(\lambda) = -f^T(\lambda)g'(\lambda).$$

The main result relating the Riemann–Hilbert problem discussed above with this Fredholm operator (see [Its et al. 1990]) is that its solution is equivalent to the determination of the resolvent operator

$$\boldsymbol{R} := (\boldsymbol{I}-\boldsymbol{K})^{-1}\boldsymbol{K}.$$

Specifically:

LEMMA 2.2. *The resolvent operator also has the special form*

$$\boldsymbol{R}(\boldsymbol{v})(\lambda) = \int_\Gamma R(\lambda,\mu)\boldsymbol{v}(\mu)d\mu$$

where the kernel is

$$R(\lambda,\mu) := \frac{F^T(\lambda)G(\mu)}{\lambda-\mu}.$$

with $F(\lambda)$, $G(\lambda)$ *given by*

$$F(\lambda) = X(\lambda)f(\lambda), \quad G(\lambda) = (X^T)^{-1}(\lambda)g(\lambda).$$

Conversely, the matrix $X(\lambda)$ *solving the above Riemann–Hilbert problem is given by the integral formula*

$$X(\lambda) = \boldsymbol{I}_r + \int_\Gamma \frac{F(\mu)g^T(\mu)}{\lambda-\mu}d\mu.$$

This result follows directly from the Cauchy integral representation for $X(\lambda)$, given its specified analytic properties [Its et al. 1990; Harnad and Its 1997]. Using it, we find a remarkable deformation formula for the Fredholm determinant $\det(\boldsymbol{I}-\boldsymbol{K})$:

THEOREM 2.3.

$$d\ln\det(\boldsymbol{I}-\boldsymbol{K}) = \sum_{k=1}^{n} H_k\, d\alpha_k + \sum_{a=1}^{r} K_a\, d\beta_a,$$

where the differential is understood as taken with respect to the deformation parameters $\{a_1,\dots,\alpha_n\}$ *and* $\{\beta_1,\dots,\beta_r\}$, *and the coefficients* $\{H_i\}_{i=1,\dots,n}$ *and* $\{K_a\}_{a=1,\dots,r}$ *are given by the formulae* (1–7) *defining the Hamiltonians generating the isomonodromic deformations, with the residue matrices* $\{N_i\}_{i=1,\dots,n}$ *given in terms of the matrices* $\{F_i,G_i\}_{i=1,\dots,n}$ *defined in* (2–5) *by* (1–17).

The proof of this result is given in [Harnad and Its 1997]. It implies that the Fredholm determinant $\det(\boldsymbol{I}-\boldsymbol{K})$ may be identified with the isomonodromic τ-function defined by formulae (1–10) and (1–11).

2c. The Dual Riemann–Hilbert Problem. The results of the previous subsection can of course be repeated with the rôles of the matrices A and B interchanged. For this, we assume that $r = 2s$ is even, the multiplicities k_i of the eigenvalues of A are all equal to 1, so $n = N$, and we choose the eigenvalues $\{\beta_a\}_{a=1,\dots,2s}$ of the matrix B to all have the same multiplicity $l \le n$ (the analog of k above), so the matrix B is now of dimension $2ls \times 2ls$. We also choose a set of $2s$ pairs $\{\tilde{f}_a,\widetilde{g_a}\}_{a=1\dots,s}$ of fixed maximal rank matrices of dimension $n \times l$ satisfying the orthogonality conditions

$$\tilde{f}_a^T\tilde{g}_b = 0, \quad a,b = 1,\dots,s.$$

As before, we choose an oriented simple curve $\widetilde{\Gamma}$ in the complex z-plane passing sequentially through the points $\{\beta_1, \dots, \beta_{2s}\}$, and denote the segments from β_{2a-1} to β_{2a} by $\widetilde{\Gamma}_a$. Now let

$$\tilde{f}(z) = e^{zA} \sum_{a=1}^{s} \widetilde{f}_a \theta_a(z), \quad \tilde{g}(z) = e^{-zA} \sum_{a=1}^{s} \widetilde{g}_a \theta_a(z), \tag{2–7}$$

where θ_a is now the characteristic function of the curve segment $\widetilde{\Gamma}_a$. As above, we associate a Riemann–Hilbert problem to this data, consisting of finding an $n \times n$ matrix valued function $\widetilde{X}(z)$ that is nonsingular and holomorphic on the complement of $\widetilde{\Gamma}$, with asymptotic form near $z = \infty$

$$\widetilde{X}(z) = \boldsymbol{I} + O(z^{-1}),$$

and discontinuities along $\widetilde{\Gamma}$ supported on the segments $\widetilde{\Gamma}_a$ defined by

$$\widetilde{X}_-(z) = \widetilde{X}_+(z) \widetilde{H}(z), \quad z \in \widetilde{\Gamma},$$

where

$$\widetilde{H}(z) = \boldsymbol{I} + 2\pi i \tilde{f}(z) \tilde{g}^T(z) = \exp 2\pi i \tilde{f}(z) \tilde{g}^T(z).$$

Once again, the solution of this Riemann–Hilbert problem allows us to define a matrix valued function

$$\widetilde{\Psi}(z) := \widetilde{X}(z) e^{zA}$$

that satisfies the linear system (1–26), where the operators (1–27) are determined in terms of the residue matrices M_a of the matrix $\mathcal{M}(z)$ at the points β_a by:

$$M_a := -\widetilde{F}_a \widetilde{G}_a^T,$$

with

$$\widetilde{F}_a := \lim_{z \to \beta_a} (\widetilde{X}(z) \tilde{f}(z)), \quad \widetilde{G}_a := (-1)^a \lim_{z \to \beta_a} (\widetilde{X}(z) \tilde{g}(z)).$$

As above, we may define a Fredholm integral operator along the curve acting on $\mathbb{C}^q$-valued functions $\tilde{\boldsymbol{v}}$ on $\widetilde{\Gamma}$ by

$$\widetilde{\boldsymbol{K}}(\tilde{\boldsymbol{v}})(z) = \int_{\widetilde{\Gamma}} \widetilde{K}(z, w) \tilde{\boldsymbol{v}}(w) \, dw,$$

where

$$\widetilde{\boldsymbol{K}}(z, w) = \frac{\tilde{f}^T(z) \tilde{g}(w)}{z - w}.$$

The corresponding resolvent operator

$$\widetilde{\boldsymbol{R}} = (\boldsymbol{I} - \widetilde{\boldsymbol{K}})^{-1} \widetilde{\boldsymbol{K}}$$

again has the form

$$\widetilde{\boldsymbol{R}}(\tilde{\boldsymbol{v}})(z) = \int_{\widetilde{\Gamma}} \widetilde{R}(z, w) \tilde{\boldsymbol{v}}(w),$$

where the kernel

$$\widetilde{R}(z, w) = \frac{\widetilde{F}^T(z) \widetilde{G}(w)}{z - w}$$

is given by

$$\widetilde{F}^T(z) = \widetilde{X}(z)\tilde{f}(z), \quad \widetilde{G}(z) = (\widetilde{X}^T(z))^{-1}\tilde{g}(z).$$

As before, the deformation formula for the Fredholm determinant gives

$$d \ln \det(1 - \widetilde{\boldsymbol{K}}) = d \ln \widetilde{\tau} = \sum_{j=1}^{n} \widetilde{H}_j \, d\alpha_j + \sum_{a=1}^{r} \widetilde{K}_a \, d\beta_a,$$

where the coefficients are given by the formulae (1–28).

Defining the pair of $n \times rl$ matrices $(\widetilde{F}, \widetilde{G})$ formed from the blocks $\{\widetilde{F}_a, \ \widetilde{G}_a\}$

$$\begin{aligned} \widetilde{F} &:= (\, \widetilde{F}_1 \quad \cdots \quad \widetilde{F}_a \quad \cdots \quad \widetilde{F}_r \,), \\ \widetilde{G} &:= (\, \widetilde{G}_1 \quad \cdots \quad \widetilde{G}_a \quad \cdots \quad \widetilde{G}_r \,), \end{aligned} \tag{2–8}$$

we may express the matrix $\mathcal{M}(z)$ as

$$\mathcal{M}(z) :== A + \widetilde{F}(B - z\boldsymbol{I})^{-1}\widetilde{G}^T.$$

Returning to the special case $l = 1$, we may ask whether there are choices of the matrices $\{(f_i, g_i)\}_{i=1,\ldots,m}$ and $\{(\tilde{f}_a, \tilde{g}_a)\}_{a=1,\ldots,s}$ defining the respective Riemann–Hilbert problems for which the $n \times r$ pairs of matrices (F, G) and $\widetilde{F}$, $\widetilde{G})$ defined by (1–18) and (2–8) coincide, defining the same solution to the respective dual Hamiltonian systems and dual isomonodromic deformation equations.

The following provides a particular answer to this question. Choosing $k = q = 1$, the (f_i, g_i)'s and $(\tilde{f}_a, \tilde{g}_a)$'s become pairs of r-component and n-component column vectors, respectively. We pick a fixed $m \times s$ matrix with elements $\{c_{ja}\}_{j=1,\ldots,m, \ a=1,\ldots,s}$ and choose the components of these vectors to be

$$\begin{aligned} (f_i)_{2a} &= (f_i)_{2a-1} = (\tilde{f}_a)_{2i} = (\tilde{f}_a)_{2i-1} = 1, \\ (\tilde{g}_i)_{2a} &= -(\tilde{g}_i)_{2a-1} = (\tilde{g}_a)_{2j} = -(\tilde{g}_a)_{2j-1} =: c_{ij}, \end{aligned}$$

for $i = 1, \ldots, m$ and $a = 1, \ldots, s$. Now define, on the product $\Gamma \times \widetilde{\Gamma}$, the locally constant function

$$\hat{K}(\lambda, z) := \sum_{j=1}^{m} \sum_{a=1}^{s} c_{ja} \theta_j(\lambda) \tilde{\theta}_a(z).$$

Taking the Fourier–Laplace transform with respect to the variables z and λ along the curves $\widetilde{\Gamma}$ and Γ, respectively, gives the two Fredholm kernels

$$\begin{aligned} K(\lambda, \mu) &= \int_{\widetilde{\Gamma}} \hat{K}(\mu, z) e^{z(\lambda - \mu)} dz = \frac{f^T(\lambda) g(\mu)}{\lambda - \mu}, \\ \widetilde{K}(w, z) &= \int_{\Gamma} \hat{K}(\mu, z) e^{\mu(w - z)} d\mu = \frac{\tilde{f}^T(w) \tilde{g}(z)}{w - z}, \end{aligned}$$

where $f(\lambda)$, $g(\lambda)$, $\tilde{f}(z)$, $\tilde{g}(z)$ define the Riemann–Hilbert data for these choices as in (2–2), (2–3), and (2–7). Then:

THEOREM 2.4. *The Fredholm determinants of these two operators are equal and so are the matrix pairs* (F, G) *and* $(\widetilde{F}, \widetilde{G})$ *constructed from the associated Riemann–Hilbert data.*

The proof is based on a straightforward application of the Neumann expansion for the resolvent and may be found in [Harnad and Its 1997]. In that work, further results are presented extending the above analysis to more general classes of isomonodromic deformation problems, corresponding to polynomial asymptotic terms in the matrix $\mathcal{N}(\lambda)$, as well as symplectic reductions by discrete symmetries. (Further cases corresponding to higher order poles in $\mathcal{N}(\lambda)$ and applications may be found in [Harnad and Routhier 1995; Harnad and Wisse 1996; Harnad et al. 1993].) The τ-functions associated with the special data discussed above in relation to the dual Fredholm operators $\boldsymbol{K}$ and $\widetilde{\boldsymbol{K}}$ are also shown in [Harnad and Its 1997], to be interpretable in a manner similar to multi-component KP τ-functions, as determinants of projection operators over suitably defined infinite dimensional Grassmannians. The interchange of data underlying the duality is then seen as an interchange of the rôles of the data determining the initial point W in the Grassmannian and the abelian group elements determining the flow.

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