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Random Permutations and the Discrete Bessel Kernel

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ABSTRACT. Let l_N denote the length of a longest increasing subsequence of a random permutation from S_N. If we write $l_N = 2\sqrt{N} + N^{1/6}\chi_N$, then χ_N converges in distribution to a random variable χ with the Tracy-Widom distribution of random matrix theory. We give an outline of the basic steps in a proof of this result which does not use the asymptotics of Toeplitz determinants, and which, in a sense, explain why the largest eigenvalue distribution occurs.

1. Introduction

Consider the length $l_N(\sigma)$ of a longest increasing subsequence in a permutation $\sigma \in S_N$; if $\sigma = i_1 i_2 \dots i_N$ and $i_{k_1} < \dots < i_{k_r}$, then $i_{k_1}, \dots, i_{k_r}$ is an increasing subsequence of length r. If we give S_N the uniform probability distribution, $l_N(\sigma)$ becomes a random variable and we want to investigate its distribution. This problem was first addressed by Ulam [1961], who made Monte Carlo simulations and concluded that the expectation $E[l_N]$ seems to be of order $\sqrt{N}$. The first rigorous result was obtained by Hammersley [1972], who considered the following variant of the problem. Consider a Poisson process in the square $[0,1]\times[0,1]$ with intensity α, so that the number M of points in the square is Poisson distributed with mean α. Let $x_1 < x_2 < \dots < x_M$ and $y_1 < y_2 < \dots < y_N$ be the x- and y-coordinates of the points $(x_j, y_{\sigma(j)})$, $1 \le j \le M$, in the square. This associates a permutation $\sigma \in S_M$ with each point configuration, and if we condition M to be fixed, equal to N say, we get the uniform distribution on S_N. We see that $l_M(\sigma)$ equals the number of points, $L(\alpha)$, in an up/right path from $(0,0)$ to $(1,1)$ through the points, and containing as many points as possible. The random variable $L(\alpha)$ is the Poissonization of l_N,

$$P[L(\alpha) \le n] = e^{-\alpha} \sum_{N=0}^{\infty} \frac{\alpha^N}{N!} P[l_N(\sigma) \le n]. \tag{1.1}$$

Using subadditivity Hammersley showed that $E[L(\alpha)]/\sqrt{\alpha} \to c$ as $\alpha \to \infty$ with a positive constant c. Numerical simulations [Baer and Brock 1968] indicated that $c = 2$, and this was proved by Vershik and Kerov [1977]. That $c = 2$ has also been proved using Hammersley's picture in [Aldous and Diaconis 1995; Seppäläinen 1996]. Large deviation results have been obtained in [Deuschel and Zeitouni 1999; Seppäläinen 1998; Johansson 1998]. For more background on the problem see [Aldous and Diaconis 1999]. Hammersley's Poisson model also has an interesting interpretation as a certain 1+1-dimensional random growth model called polynuclear growth; see [Prähofer and Spohn 2000]. The fluctuations around the mean has been an open problem for a long time. Numerical simulations by Odlyzko and Rains [1998], indicate that the standard deviation for l_N is like a constant times $N^{1/6}$, and this can also be seen heuristically from the large deviation formulas. A precise result for the fluctuations was proved by Baik, Deift and Johansson in [Baik et al. 1999]. To state the result we need some definitions. Let u be the unique solution to the Painlevé II equation

$$u'' = 2u^3 + xu,$$

which satisfies $u(x) \sim \mathrm{Ai}(x)$, as $x \to \infty$, where $\mathrm{Ai}(x)$ is the Airy function. Such a solution exits [Hastings and McLeod 1980; Deift and Zhou 1995]. Put

$$F(t) = \exp\bigl(-\int_t^\infty (x-t)u(x)^2\,dx\bigr). \tag{1.2}$$

The result in [Baik et al. 1999] is:

THEOREM. *The random variable* $(l_N - 2\sqrt{N})/N^{1/6}$ *converges in distribution to a random variable with distribution function* (1.2),

$$\lim_{N\to\infty} P[l_N \le 2\sqrt{N} + tN^{1/6}] = F(t). \tag{1.3}$$

Also, we have convergence of all moments.

The proof of (1.3) in [Baik et al. 1999] is based on the following formula due to Gessel [1990], which expresses the probability in (1.1) as a Toeplitz determinant,

$$\begin{aligned} P[L(\alpha) \le n] &= e^{-\alpha} D_n(e^{2\sqrt{\alpha}\cos\theta}) \\ &= \det\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{2\sqrt{\alpha}\cos\theta - i(j-k)\theta}d\theta\right)_{j,k=1}^{n}. \end{aligned} \tag{1.4}$$

The Toeplitz determinant can be expressed in terms of the leading coefficients of the normalized orthogonal polynomials on the unit circle, $\mathbb{T}$, with respect to the weight $e^{2\sqrt{\alpha}\cos\theta}$. These orthogonal polynomials can be obtained as the solution of a certain matrix Riemann–Hilbert problem on $\mathbb{T}$, [Fokas et al. 1991], and using the powerful steepest descent argument for Riemann–Hilbert problems developed by Deift and Zhou [1993], it is possible to show that

$$\lim_{\alpha\to\infty} P[L(\alpha) \le 2\sqrt{\alpha} + t\alpha^{1/6}] = F(t) \tag{1.5}$$

for each $\in \mathbb{R}$. This is closely related to the so called double scaling limit in unitary random matrix models [Periwal and Shevitz 1990]. From (1.5) the result (1.3) can be deduced by a de-Poissonization argument [Johansson 1998].

The distribution function $F(t)$ also has a different expression. Let

$$A(x,y)=\int_0^\infty \mathrm{Ai}(x+t)\,\mathrm{Ai}(y+t)dt,$$

be the *Airy kernel*. Then,

$$F(t)=\det(I-A)_{L^2[t,\infty)}, \tag{1.6}$$

where the right hand side is the Fredholm determinant

$$\det(I-A)_{L^2[t,\infty)}=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\int_{[t,\infty)^m}\det(A(x_i,x_j))_{i,j=1}^m d^m x. \tag{1.7}$$

The fact that (1.2) and (1.7) are equal was proved by Tracy and Widom [1994], and $F(t)$ is often referred to as the *Tracy–Widom distribution*. The interesting thing about (1.6) is that the right hand side of (1.6) is the asymptotic distribution for the appropriately scaled largest eigenvalue of a random matrix from the Gaussian Unitary Ensemble (GUE) [Mehta 1991], as the size of the matrix goes to infinity [Tracy and Widom 1994]. Thus $l_N(\sigma)$ behaves, for large N, as the largest eigenvalue of a large random hermitian matrix!

In this paper we will outline a proof of (1.5) that does not use Gessel's formula (1.4) and which makes it clearer why the largest eigenvalue distribution appears. The presentation is based on [Johansson 2000; 2001], to which we refer for more details. Below we will not give all the technical details, in particular those concerning the precise justification of different limits. A closely related proof appears in [Borodin et al. 2000], see the remark at the end of section 3.

In the next section we will outline the main ingredients that go into the proof. The actual argument for (1.5) will be given in the last section.

2. Main Ingredients

2a. The Robinson–Schensted–Knuth Correspondence. General references for this subsection and the next are [Fulton 1997; Sagan 1991; Stanley 1999]. A *partition* of K is a sequence $\lambda=(\lambda_1,\dots,\lambda_N)$, $\lambda_1\ge\cdots\ge\lambda_N\ge 0$, of integers, such that $\lambda_1+\cdots+\lambda_N=K$, and can be illustrated by a *Young diagram* with λ_j left justified boxes in row j. If we write numbers from $\{1,\dots,N\}$ in the boxes in such away that the numbers in each row are weakly increasing and the numbers in each cloumn are strictly increasing, we get a *semistandard Young tableaux* T of shape λ, $\mathrm{sh}(T)=\lambda$, with entries in $\{1,\dots,N\}$. Let $m_i(T)$ denote the number of i's in T.

An integer $N \times N$ matrix $A = (w_{ij})$, $w_{ij} \in \mathbb{N}$, can be described by a *generalized permutation*

$$\sigma = \begin{pmatrix} i_1 & \dots & i_K \\ j_1 & \dots & j_K \end{pmatrix}, \qquad i_r, j_r \in \{1, \dots, N\},$$

where $i_r \le i_{r+1}$ and if $i_r = i_{r+1}$, then $j_r \le j_{r+1}$. The matrix A is mapped bijectively to the σ in which a pair $\binom{i}{j}$ occurs w_{ij} times.

The *Robinson–Schensted–Knuth correspondence* [Knuth 1970], or RSK correspondence, maps σ bijectively to a pair of semistandard tableaux (P, Q) of the same shape λ with entries in $\{1, \dots, N\}$. The numbers $j_1, \dots, j_K$ go into P and $i_1, \dots, i_K$ go into Q, so λ is a partition of $K = \sum_{i,j} w_{ij}$. Note that $m_i(Q) = \sum_j w_{ij}$ and $m_j(P) = \sum_i w_{ij}$. This correspondence has the property that λ_1, the length of the first row, equals the length $l(\sigma)$ of the longest weakly increasing subsequence in $j_1, \dots, j_K$. In terms of the matrix A this has the following interpretation. Let $\pi = \{(m_r, n_r)\}_{r=1}^{2N-1}$ be an *up/right path* from $(1,1)$ to (N, N), i.e. $(m_{r+1}, n_{r+1}) - (m_r, n_r) = (1, 0)$ or $(0, 1)$, $(m_1, n_1) = (1, 1)$ and $(m_{2N-1}, n_{2N-1}) = (N, N)$. Set

$$L_N(A) = \max_{\pi} \sum_{(i,j) \in \pi} w_{ij}. \tag{2.1}$$

It is not difficult to see that $L_N(A)$ is equal to $l(\sigma)$ and hence $L_N(A) = \lambda_1$.

2b. The Schur Polynomial. Let $\lambda = (\lambda_1, \dots, \lambda_N)$ be a partition of K as above. Then the *Schur polynomial* $s_\lambda(x_1, \dots, x_N)$ is a certain symmetric, homogeneous polynomial of degree K. Let $\Delta_N(x) = \prod_{1 \le i < j \le N}(x_i - x_j)$ denote the Vandermonde determinant. We have two different expressions for $s_\lambda(x)$,

$$s_\lambda(x) = \sum_{T \,:\, \mathrm{sh}(T) = \lambda} \prod_{i=1}^{N} x_i^{m_i(T)} = \frac{1}{\Delta_N(x)} \det(x_j^{\lambda_k + N - k})_{j,k=1}^{N}, \tag{2.2}$$

where the sum is over all semistandard tableaux with shape λ and entries in $\{1, \dots, N\}$. The second formula is known as the Jacobi–Trudi identity.

2c. Orthogonal Polynomial Ensembles. Consider the probability measure

$$\frac{1}{Z_N} \Delta_N(x)^2 \prod_{j=1}^{N} w(x_j) d\mu(x_j) \tag{2.3}$$

on $\mathbb{R}^N$, where $w(x)$ is a nonnegative weight function and the measure μ typically is the Lebesge measure on $\mathbb{R}$, some interval in $\mathbb{R}$ or the counting measure on $\mathbb{N}$. These type of measures occur as eigenvalue measures in invariant ensembles of hermitian matrices. GUE for example has the eigenvalue measure (2.3) with $w(x) = e^{-x^2}$ and μ the Lebesgue measure on $\mathbb{R}$. An important property of measures of the form (2.3) is that all marginal distributions (correlation functions) are given by certain determinants. The k-th marginal probability is

$$\frac{(N-k)!}{N!}\det(K_N(x_i,x_j))_{i,j=1}^k d\mu(x_1)\dots d\mu(x_k), \tag{2.4}$$

where

$$K_N(x,y)=\sum_{n=0}^{N-1} p_k(x)p_k(y)(w(x)w(y))^{1/2},$$

and $p_k(x)$, $k=0,1,\dots$, are the normalized orthogonal polynomials with respect to the measure $w(x)d\mu(x)$,

$$\int_{\mathbb{R}} p_j(x)p_k(y)w(x)d\mu(x)=\delta_{jk},$$

see [Mehta 1991] or [Tracy and Widom 1998]. For GUE the relevant orthogonal polynomials are the Hermite polynomials. Using the fact that the marginal probabilities are given by (2.4) we see that

$$\begin{aligned}
&P[\max_{1\le j\le N} x_j\le t]\\
&\qquad=\frac{1}{Z_N}\int_{\mathbb{R}^N}\prod_{j=1}^N(1-\chi_{(t,\infty)}(x_j))\Delta_N(x)^2\prod_{j=1}^N w(x_j)d\mu(x_j)\\
&\qquad=\sum_{m=0}^N\frac{(-1)^m}{m!}\int_{\mathbb{R}^m}\det(K_N(x_i,x_j)\chi_{(t,\infty)}(x_j))_{i,j=1}^m d\mu(x_1)\dots d\mu(x_m)\\
&\qquad=\det(I-K_N\chi_{(t,\infty)})_{L^2(\mathbb{R},d\mu)}.
\end{aligned} \tag{2.5}$$

The last expression is the Fredholm determinant with kernel $K_N(x,y)\chi_{(t,\infty)}(y)$ on $L^2(\mathbb{R},d\mu)$. By using asymptotic formulas for Hermite polynomials, (2.5) can be used to show that the appropriately scaled largest eigenvalue of a GUE-matrix has a limiting distribution given by (1.6).

In our proof of (1.5) we will get (2.3) with $d\mu(x)$ counting measure on $\mathbb{N}$ and $w(x)=q^x$, for $0<q<1$ a fixed parameter. The orthogonal polynomials are then the Meixner polynomials, and we turn to them next.

2d. Asymptotics for Meixner Polynomials and the Discrete Bessel Kernel. The polynomials $M_n^q(x)$ (which are multiples of the standard Meixner polynomials $m_n^{1,q}(x)$; see [Chihara 1978; Nikiforov et al. 1991]), satisfy

$$\sum_{x=0}^\infty M_n^q(x)M_m^q(x)q^x=\delta_{nm},$$

and have the integral representation [Chihara 1978]

$$M_n^q(x)=\frac{q^{n/2}\sqrt{1-q}}{2\pi i}\int_{\Gamma_{\sqrt{q},q}}\frac{(1+w/q)^x}{(1+w)^{x+1}w^{n+1}}dw, \tag{2.6}$$

where $\Gamma_{r,q}$ is the circle $re^{i\theta}$, $-\pi < \theta < \pi$, together with the line segments from $-r + i0$ to $-rq + i0$ and from $-rq - i0$ to $-r - i0$. The integral formula is a consequence of the generating function for the Meixner polynomials. The kernel we need in (2.4) is the *Meixner kernel*

$$K_N^q(x, y) = \sum_{n=0}^{N-1} M_n^q(x) M_n^q(y) q^{(x+y)/2}. \tag{2.7}$$

If we make the change of variables $w = z\sqrt{q}$ in (2.6) we see that

$$\begin{aligned}
&\lim_{N\to\infty} M_{N-n}^{\alpha/N^2}(x + N)\Big(\frac{\alpha}{N^2}\Big)^{(x+N)/2} \\
&\quad = \lim_{N\to\infty} \frac{\sqrt{1 - \alpha/N^2}}{2\pi i} \int_{\Gamma_{1,\alpha/N^2}} \frac{\big(1 + \sqrt{\alpha}/(zN)\big)^N}{\big(1 + z\sqrt{\alpha}/N\big)^N} z^{x+n} \frac{\big(1 + \sqrt{\alpha}/zN\big)^x}{\big(1 + z\sqrt{\alpha}/N\big)^{x+1}} \frac{dz}{z} \\
&\quad = \frac{1}{2\pi i} \int_{\Gamma_{1,0}} e^{\sqrt{\alpha}(1/z - z)} z^{x+n} \frac{dz}{z} = J_{x+n}(2\sqrt{\alpha}),
\end{aligned} \tag{2.8}$$

where $J_n(t)$ is the standard Bessel function. Using (2.7) and (2.8) we see that

$$\begin{aligned}
&\lim_{N\to\infty} K_N^{\alpha/N^2}(x+N, y+N) \\
&\quad = \lim_{N\to\infty} \sum_{n=1}^{N} M_{N-n}^{\alpha/N^2}(x + N)\Big(\frac{\alpha}{N^2}\Big)^{(x+N)/2} M_{N-n}^{\alpha/N^2}(y + N)\Big(\frac{\alpha}{N^2}\Big)^{(y+N)/2} \\
&\quad = \sum_{k=1}^{\infty} J_{x+k}(2\sqrt{\alpha}) J_{y+k}(2\sqrt{\alpha}) =: B^\alpha(x, y).
\end{aligned} \tag{2.9}$$

We call $B^\alpha(x, y)$ the *discrete Bessel kernel*. The Bessel function $J_n(t)$ has the following asymptotics for n and t of the same order,

$$\lim_{\alpha\to\infty} \alpha^{1/6} J_{2\sqrt{\alpha}+\xi\alpha^{1/6}}(2\sqrt{\alpha}) = \mathrm{Ai}(\xi). \tag{2.10}$$

Note the similarity with (1.5)! Combining the definition (2.9) of $B^\alpha(x, y)$ and (2.10) we see that we should have

$$\begin{aligned}
&\lim_{\alpha\to\infty} \alpha^{1/6} B^\alpha(2\sqrt{\alpha} + \xi\alpha^{1/6}, 2\sqrt{\alpha} + \eta\alpha^{1/6}) \\
&\quad = \lim_{\alpha\to\infty} \sum_{k=1}^{\infty} \alpha^{1/6} J_{2\sqrt{\alpha}+\xi\alpha^{1/6}+k}(2\sqrt{\alpha}) \alpha^{1/6} J_{2\sqrt{\alpha}+\eta\alpha^{1/6}+k}(2\sqrt{\alpha}) \alpha^{-1/6} \\
&\quad = \lim_{\alpha\to\infty} \sum_{k=1}^{\infty} \mathrm{Ai}(\xi + k\alpha^{-1/6})\, \mathrm{Ai}(\eta + k\alpha^{-1/6}) \alpha^{-1/6} \\
&\quad = \int_0^\infty \mathrm{Ai}(\xi + t)\, \mathrm{Ai}(\eta + t) dt = A(\xi, \eta).
\end{aligned} \tag{2.11}$$

3. The Proof

Let w_{ij}, $1 \le i, j \le N$, be independent, geometrically distributed random variables, $P[w_{ij} = x] = (1-q)q^x$, $x \in \mathbb{N}$, and let $A = (w_{ij})_{i,j=1}^N$. We let $L_N(A)$ denote the maximal sum along an up/right path as in (2.1). Let P_N^q denote the measure we get on the set of integer matrices. If q is small then most of the w_{ij}'s will be zero and a few will be equal to one. In particular, if we take $q = \alpha/N^2$, then with probability going to 1 as $N \to \infty$, there will be at most one w_{ij} equal to 1 in each row and each column, and no w_{ij} will be ≥ 2. Put a particle in the square $[(i-1)/N, i/N] \times [(j-1)/N, j/N]$ if $w_{ij} = 1$ and no particle if $w_{ij} = 0$. As $N \to \infty$ this will converge to a Poisson process in $[0,1] \times [0,1]$ with intensity α, so the number of particles in the unit square will be Poisson distributed with mean α. Some thought shows that $L_N(A)$ will converge to the maximal number of points in an up/right path from $(0,0)$ to $(1,1)$ through the Poisson points, that is to the random variable $L(\alpha)$ that we defined in the introduction. Thus we have,

$$P[L(\alpha) \le n] = \lim_{N\to\infty} P_N^{\alpha/N^2}[L_N(A) \le n]. \tag{3.1}$$

Next, we will see that we can use the RSK correspondence and the Schur polynomial to derive an expression for the probability in the right hand side of (3.1). If A is mapped to (P, Q) by the RSK correspondence, then

$$\begin{aligned} P_N^q[A = (a_{ij})] &= (1-q)^{N^2} q^{\sum_{i,j} a_{ij}} \\ &= (1-q)^{N^2} \prod_{i=1}^N (\sqrt{q})^{\sum_j a_{ij}} \prod_{j=1}^N (\sqrt{q})^{\sum_i a_{ij}} \\ &= (1-q)^{N^2} \prod_{i=1}^N (\sqrt{q})^{m_i(Q)} \prod_{j=1}^N (\sqrt{q})^{m_j(P)}. \end{aligned}$$

Consequently, by the RSK correspondence and the first equality in (2.2),

$$\begin{aligned} &P_N^q[L_N(A) \le n] \\ &\quad= \sum_{A\,:\,L_N(A)\le n} P_N^q[A] \\ &\quad= \sum_{\lambda:\lambda_1\le n} \sum_{P:\mathrm{sh}(P)=\lambda} \sum_{Q\,:\,\mathrm{sh}(Q)=\lambda} (1-q)^{N^2} \prod_{i=1}^N (\sqrt{q})^{m_i(Q)} \prod_{j=1}^N (\sqrt{q})^{m_j(P)} \\ &\quad= (1-q)^{N^2} \sum_{\lambda\,:\,\lambda_1\le n} s_\lambda(\sqrt{q}, \dots, \sqrt{q}) s_\lambda(\sqrt{q}, \dots, \sqrt{q}) \\ &\quad= (1-q)^{N^2} \sum_{\lambda\,:\,\lambda_1\le n} s_\lambda(1, \dots, 1)^2 q^{\sum_j \lambda_j}. \end{aligned} \tag{3.2}$$

Since semistandard Young tableaux are strictly increasing in columns, λ can have at most N nonzero parts, $\lambda = (\lambda_1, \dots, \lambda_N)$. We can now use the second equality in (2.2) to get

$$s_\lambda(1,\dots,1)=\lim_{r\to 1}s_\lambda(1,r,\dots,r^{N-1})=\prod_{1\le i<j\le N}\frac{\lambda_i-\lambda_j+j-i}{j-i}. \tag{3.3}$$

If we introduce the new variables $h_j=\lambda_j+N-j$, $h_1>h_2>\dots>h_N\ge 0$, we see that (3.2) and (3.3) give

$$P_N^q[L_N(A)\le n]=\frac{1}{Z_N}\sum_{h\in\{0,\dots,n+N-1\}^N}\Delta_N(h)^2\prod_{j=1}^N q^{h_j}, \tag{3.4}$$

where $Z_N=q^{N(N-1)/2}(1-q)^{-N^2}\prod_{j=1}^{N-1}j!^2$. Note that (3.4) has exactly the form of an orthogonal polynomial ensemble. The relevant orthogonal polynomials are the Meixner polynomials (2.6). The computation (2.5) gives

$$P_N^q[L_N(A)\le n]=\det(I-\mathcal{K}_N^q)_{\ell^2(\{n,n+1,\dots\})}, \tag{3.5}$$

where we have the kernel

$$\mathcal{K}_N^q(x,y)=K_N^q(x+N,x+N),$$

and K_N^q is the Meixner kernel (2.7).

We can now combine (3.1) and (3.5) and use (2.9) to see that

$$\begin{aligned}P[L(\alpha)\le n]&=\lim_{N\to\infty}\det(I-\mathcal{K}_N^{\alpha/N^2})_{\ell^2(\{n,n+1,\dots\})}\\&=\det(I-B^\alpha)_{\ell^2(\{n,n+1,\dots\})}.\end{aligned} \tag{3.6}$$

Note that combining (1.4) and (3.6) gives an interesting equality between a Toeplitz and a Fredholm determinant, which has been generalized by Borodin and Okounkov [2000]; see also [Basor and Widom 2000].

When we have the formula (3.6) we are not far from a proof of (1.5). Write the Fredholm expansion of the right hand side in (3.6)

$$\begin{aligned}P[L(\alpha)\le 2\sqrt{\alpha}+t\alpha^{1/6}]=&\sum_{m=0}^\infty\frac{(-1)^m}{m!}\\&\times\sum_{h\in\mathbb{N}^m}\det(\alpha^{1/6}B^\alpha(2\sqrt{\alpha}+t\alpha^{1/6}+h_i,2\sqrt{\alpha}+t\alpha^{1/6}+h_j))_{i,j=1}^m(\alpha^{-1/6})^m.\end{aligned}$$

By (2.11) this is approximately

$$\sum_{m=0}^\infty\frac{(-1)^m}{m!}\sum_{h\in\mathbb{N}^m}\det(A(t+h_i\alpha^{-1/6},t+h_j\alpha^{-1/6}))_{i,j=1}^m(\alpha^{-1/6})^m \tag{3.7}$$

for large α, and in the limit $\alpha \to \infty$, the Riemann sums in (3.7) converge to integrals and we get

$$P[L(\alpha) \leq 2\sqrt{\alpha} + t\alpha^{1/6}] = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_{[t,\infty)^m} \det(A(x_i, x_j))_{i,j=1}^m \, d^m x$$
$$= \det(I - A)_{L^2[t,\infty)} = F(t),$$

by (1.6). This establishes (1.5).

The RSK correspondence actually transforms the Poissonization of the uniform measure on S_N (that is, we regard N as a Poisson random variable) to a measure on partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$, called the Poissonized Plancherel measure. Above we have studied the asymptotic distribution of λ_1, but the analysis can be extended to the lengths of the other rows as well [Johansson 2001]. The Poissonized Plancherel measure has also been investigated by Borodin, Okounkov and Olshanski [Borodin et al. 2000], coming from measures on partitions motivated by representation theory. They also obtain the discrete Bessel kernel and are able to give a closely related proof of (1.5).

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