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Solvable Matrix Models

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ABSTRACT. We review some old and new methods of reduction of the number of degrees of freedom from $\sim N^2$ to $\sim N$ in the multimatrix integrals.

1. Introduction

Multimatrix integrals of various types appear in many mathematical and physical applications, such as combinatorics of graphs, topology, integrable systems, string theory, theory of mesoscopic systems or statistical mechanics on random surfaces.

A general Q-matrix integral of the form

$$Z = \int \prod_{q=1}^{Q} d^{N^2} M_q \exp S(M_1, \ldots, M_Q)$$

usually goes over the $N \times N$ hermitian, real symmetric or symplectic matrices M_q with the action S and the measure symmetric under the simultaneous group rotation: $M_q \to \Omega^+ M_q \Omega$. Some other multimatrix integrals, such as these with complex matrices or with general real matrices, can be reduces to those three basic cases.

We will consider here only the case of hermitian matrices for which Ω belongs to the $U(N)$-group.

In many applications "to solve" the corresponding matrix model usually means to reduce the number of variables by explicit integrations over most of the variables in such a way that instead of QN^2 original integrations (matrix elements) one would be left in the large N limit only with $\sim N$ integration variables. In this case the integration over the rest of the variables can be performed, at least in the widely used large N limit, by means of the saddle point approximation. A more sophisticated double scaling limit [1] is also possible (if possible at all) only after such a reduction. The key of success is in the fact that after reduction the effective action at the saddle point is still of the order $\sim N^2$ whereas the corrections given by the logarithm of determinant of the second variation of the action cannot be bigger than $\sim N$ (the "entropy" of the remaining variables). The problem is thus reduced to the solution of the "classical" saddle point equations, instead of the "quantum" problem of functional (in the large N limit) integration over the original matrix variables.

Such an explicit reduction of the number of "degrees of freedom" is in general possible only for a few rather restricted, though physically and mathematically interesting, classes of multimatrix integrals. The purpose of our present notes is to review the basic old and new methods of such a reduction. Before going to the particular cases let us stress the importance of the search for new methods of such a reduction: any nontrivial finding on this way leads immediately to numerous fruitful applications.

2. Some Old Examples

The best known example of such a reduction of the number of degrees of freedom is the one-matrix integral

$$Z = \int d^{N^2}M \exp N \operatorname{Tr} S(M),$$

where $S(M)$ is an arbitrary function of one variable. We use the decomposition:

$$M = \Omega^{+} x \Omega,$$

where $x = diag(x_1, \dots, x_N)$ is a diagonal matrix of the eigenvalues and Ω is the $U(N)$ group variable. The corresponding (Dyson) measure can be written as

$$d^{N^2}M = d[\Omega]_{U(N)} \Delta^2(x) \prod_{k=1}^{N} dx_k \tag{2–1}$$

where $\Delta(x) = \prod_{i>j}(x_i - x_j)$ is the Vandermonde determinant. The integrand as an invariant function does not depend at all on Ω (the integration over it produces just a group volume factor which we will always omit). The remaining integral over the eigenvalues reads

$$Z = \int \prod_{k=1}^{N} dx_k \exp\bigl(NS(x_k)\bigr) \Delta^2(x).$$

In the large N limit the corresponding saddle point equation takes the form

$$\frac{1}{N}\frac{\partial S}{\partial x_k} = S'(x_k) + \frac{1}{N}\sum_{j \neq k} \frac{1}{x_k - x_j} = 0.$$

These arguments were successfully used for an interesting combinatorial problem: enumeration of graphs of fixed two dimensional topologies [14; 11]. There exist powerful methods to analyze this equation but it is not our present goal to review them here.

The next fruitful example is the so called two matrix model:

$$Z = \int d^{N^2}A \; d^{N^2}B \exp N \operatorname{Tr}\big(-A^2 - B^2 + cAB + U(A) + V(B)\big), \qquad (2\text{–}2)$$

where U and V are some arbitrary functions of one variable. After the decomposition $A = \Omega_1^+ x \Omega_2$, $B = \Omega_2^+ y \Omega_2$ we are left, due to the term $\operatorname{Tr}(AB)$ in the action, with one nontrivial unitary integral over the variable $\Omega = \Omega_1 \Omega_2^+$. Fortunately, this integral was explicitly calculated by Harish-Chandra [10] and by Itzykson and Zuber [11]:

$$\int d[\Omega]_{U(N)} \exp \operatorname{Tr}(\Omega^+ x \Omega y) = \prod_{k=1}^{N-1} k! \, \frac{\det_{ij} e^{x_i y_j}}{\Delta(x)\Delta(y)}. \qquad (2\text{–}3)$$

Substituting (2–3) and the Dyson measure (2–1) into (2–2) we are left again with only $2N$ variables x_k and y_k and we can write again the saddle point equations in the large N limit. They are more complicated than in the one matrix integral but can be nevertheless solved quite explicitly. The first solution of that kind was found in [16] in an indirect way, using the method of orthogonal polynomials, but the direct solution is also possible; see [18].

This model was used in [17] to solve exactly the first example of new statistical mechanical models of interacting spins on random planar graphs: in this case it was a model of Ising spins on random planar graphs.

An obvious generalization of the two matrix model is the matrix chain model:

$$Z = \int \prod_{q=1}^{Q} d^{N^2} M_q \; \exp \operatorname{Tr}\bigg(\sum_{p=1}^{Q} V_q(M_q) + \sum_{p=1}^{Q-1} M_{p-1} M_p \bigg). \qquad (2\text{–}4)$$

One easily notices that the same unitary decomposition $M_q = \Omega_q^+ x_q \Omega_q$ leads to $Q-1$ independent integrals over the variables $U_q = \Omega_{q-1}^+ \Omega_q$ of the type (2–3). We are left again with only QN eigenvalues instead of QN^2 matrix elements and are ready to apply the saddle point approximation to this integral. This model was first analyzed by the method of orthogonal polynomials by [15]. It was shown in [19] that by special choices of the potential V the model can be described by the KP integrable flow with respect to the coupling constant of the potential.

Note that if we imposed the periodicity condition $M_1 = M_q$ on this matrix chain and add the term $M_1 M_q$ to the action the problem would become much more complicated (and actually not solved so far), since this would give an extra condition $\prod_q U_q = I$ making the variables U_q not independent.

Another solvable matrix chain describing the statistical RSOS RSOS models on random planar graphs was proposed and solved in [20]. Similar models were considered in [32].

Some multimatrix models can be reduced to the solvable ones by means of simple matrix integral transformations. The first example of such transformation was described in [21] for the matrix integral describing the Q-state Potts model on random dynamical planar graphs. Its partition function is

$$Z = \int \prod_{q=1}^{Q} d^{N^2} M_q \exp \mathrm{Tr} \biggl(\sum_{q=1}^{Q} V_q(M_q) + \sum_{p,q=1}^{Q} M_p M_q \biggr).$$

One can represent the last factor under the integral as

$$\int d^{N^2} X \exp \mathrm{Tr} \biggl(-\tfrac{1}{2} X^2 + X \sum_{q=1}^{Q} M_q \biggr).$$

We consider the case $V_1 = \cdots = V_Q = V$. Then the whole integral can be expressed as

$$Z = \int d^{N^2} X \exp(-\tfrac{1}{2} \,\mathrm{Tr}\, X^2) \Bigl(\int d^{N^2} M \exp \mathrm{Tr} \left(XM + V(M) \right) \Bigr)^Q .$$

The integrals in this expression can be reduced to the eigenvalues: in the integral under the power the only nontrivial “angular” integration over the relative $U(N)$-"angle” can be done by means of the formula (2–3) and the external one will also depend only on the eigenvalues of X. The solution of the corresponding saddle point equations was found in [22] and analyzed in [23] and [24].

Combining these methods in the obvious ways one can generalize the large N solvability on a certain larger class of multimatrix models.

3. Matrix Quantum Mechanics

In the limit when $Q \to \infty$ and with the special scaling of coupling constants the matrix chain (2–4) becomes matrix quantum mechanics. It is defined by the Hamiltonian

$$\hat{H}_M = -\Delta_M + \mathrm{Tr}\, V(M), \tag{3–1}$$

where Δ_M is the usual $U(N)$ invariant Laplacian on the homogeneous space of hermitian matrices and the potential $V(M)$ can actually explicitly depend on time t.

The Schrödinger equation can be written in the form of a minimization principle:

$$min_\Psi \int d^{N^2} M \,\mathrm{Tr} \left(\tfrac{1}{2} |\partial_M \Psi(M)|^2 + V(M) |\Psi(M)|^2 \right) \tag{3–2}$$

To reduce this problem to the eigenvalues we use the $U(N)$ symmetry of our model and look for a wave function $\Psi(M)$ transforming according to a certain irreducible representation R of $U(N)$:

$$\Psi_R^I(\Omega^+ M \Omega) = \sum_J \Omega_R^{IJ} \Psi_R^J(M),$$

where Ω_R is a group element Ω in representation R and I, J are the indices of the representation. Such a function may be decomposed as

$$\Psi_R^I(M) = \sum_J \Omega_R^{IJ} \psi_R^J(x). \tag{3–3}$$

Here $\psi_R^I(x_1, \ldots, x_N)$ is a vector in the representation R.

Near the unity element on the group space $\Omega \simeq I + \omega$ we have $\Omega_R \simeq P_R + \sum_{ij} \omega_{ij} T_{ij}^R$ where P_R is a projector (unity element) in the R space, ω is a small deviation from it and T_{ij}^R are the $u(N)$ algebra generators. This gives

$$\frac{\partial}{\partial M_{ij}} = \delta_{kj} \frac{\partial}{\partial x_k} + \sum_{m=1}^{N} \frac{1}{x_k - x_m} \frac{\partial}{\partial \omega_{mj}},$$

and we finally obtain from (3–2) the variational principle

$$\min_{\psi_R} \int \prod_k dx_k \Delta^2(x) \\ \times \operatorname{Tr}_R \left(\frac{1}{2} \sum_j \left| \frac{\partial}{\partial x_j} \psi_R(x) \right|^2 + \frac{1}{2} \sum_{i \neq j} \left| T_{ij}^R \psi_R \right|^2 + \sum_m V(x_m) \left| \psi_R \right|^2 \right),$$

where all the quantities and operators with the subscript R are subjected to the corresponding matrix operations in the matrix space of representation.

The Schrödinger equation now reads:

$$-\sum_k \Delta^{-2}(x) \frac{\partial}{\partial x_k} \Delta^2(x) \frac{\partial}{\partial x_k} \psi_R(x) - \sum_{i \neq j} T_{ij}^R T_{ji}^R \psi_R(x) = \left(E - \sum_k V(x_k) \right) \psi_R(x). \tag{3–4}$$

It is useful to introduce a new function $\phi_R(x) = \dfrac{1}{\Delta(x)} \psi_R(x)$ obeying the equation

$$-\sum_k \left(\frac{\partial}{\partial x_k} \right)^2 \phi_R(x) - \sum_{i \neq j} \frac{T_{ij}^R T_{ji}^R}{(x_i - x_j)^2} \phi_R(x) = \left(E - \sum_i V(x_i) \right) \phi_R(x) \tag{3–5}$$

Note that any translation $\omega_{ij} \to \omega_{ij} + \delta_{ij} \varepsilon$ does not change the wave function Ψ_R. That means that we are looking only for the states on which the condition

$$T_{kk}^R \psi_R = 0, \quad \text{for } k = 1, \ldots, N$$

is imposed.

At first sight, we fulfilled our main task for the matrix quantum mechanics: we reduced it to an eigenvalue problem and are now dealing with only N variables. But the Schrödinger equation (3–4) contains the Hamiltonian which is a matrix in the representation space acting on the wave function which is a vector in this space. For small representations whose Young tableaux contain $\ll N^2$ boxes the problem is still solvable in the large N limit (as we will demonstrate below). For a very interesting case of big representations ($\sim N^2$ boxes in the Young tableaux) the problem remains a serious challenge.

In the simplest case of singlet representation (solved long ago in [14]) the wave function is a scalar and the last term in the right-hand side of the Schrödinger equation (3–5) drops out. The problem appears to be equivalent to the quantum mechanical system of N non-interacting fermions (due to the antisymmetry of $\phi(x)$) in a potential $V(x)$. It was used in many applications, including the solution of the non-critical string theory in 1+1 dimensions [9].

The next smallest representation is adjoint. The adjoint wave function satisfying the relation (3–3) should be a function of the type

$$\Psi(M;x) = \sum_{a=0}^{N-1} C_a(x) M^a$$

where the coefficients C_A possibly depend on the invariants (eigenvalues). If we denote $\phi_{adj}(x_i;x) \equiv \phi_i(x)$ (depending of course on all N x_i) we can write the Schrödinger equation for the adjoint wave function in the following form [13]:

$$\sum_i \Big(-\left(\frac{\partial}{\partial x_i}\right)^2 + V(x_i)\Big)\phi_k(x) - \frac{1}{N^2} \sum_{i(\neq k)} \frac{\phi_i(x) - \phi_k(x)}{(x_i - x_k)^2} = E\phi_k(x).$$

One can see that the last term in the left-hand side of this equation is $\sim N^2$ smaller than the other terms and can be regarded as a small perturbation on the background of the free fermion solution of the singlet sector.

For one of physically most interesting applications, the 1+1 dimensional string theory, we need to solve the model in the inverted oscillatory potential $V(M) = -M^2$. The model is unstable and one needs to specify the boundary conditions for big M's. Usually one considers the boundary conditions when the absolute value of any of the eigenvalues of M cannot exceed some maximum value Λ (a cut-off wall). In the case of the large N limit one takes $\Lambda \sim N$ and it happens that the spectrum density of the model depends in a very universal (logarithmic) way on Λ. In the singlet state the spectrum is that of N independent fermions (eigenvalues) in the same potential and the eigenfunctions are the Slater determinants of the parabolic cylinder functions [25]. In the non-singlet sectors the eigenvalues start interacting and obey a more complicated statistics corresponding to the symmetry of the Young tableau of representation (see the review [30] for details). Although the problem is clearly integrable the spectrum of the non-singlet sectors of the inverted matrix quantum oscillator is still unknown. For the large N estimates of the mass gap of adjoint representation see [13; 12; 8].

It was conjectured in [12] and shown in [8] that the adjoint representation describe the vortex anti-vortex sector in the 1+1 dimensional string theory with one compact dimension. Higher representation describe higher numbers of vortex anti-vortex pairs (corresponding to the number of boxes in the Young tableau of the representation).

4. Character Expansion and New Solvable (Multi) Matrix Models

The group character expansion has shown its power in the lattice gauge theory long time ago, starting from the work of A. Migdal [26].

The character expansion method proposed in the papers [2]–[4] and inspired by the result of paper [27] is the most general approach for the reduction of the number of degrees of freedom from $\sim N^2$ to $\sim N$ in a new big class of (multi) matrix integrals. The matrix integral considered in these papers looks as follows:

$$Z = \int d^{N^2}M \exp\bigl(-\operatorname{Tr} M^2 + \operatorname{Tr} V(AM)\bigr), \tag{4–1}$$

where $V(y) = \sum_{k>2} t_k y^k$ is an arbitrary potential and A is an arbitrary hermitian matrix (which can be taken diagonal without a loss of generality). We again diagonalize the matrix M as

$$M = \Omega^+ X \Omega. \tag{4–2}$$

The integral over the $U(N)$ variable Ω looks difficult to do directly since the Itzykson-Zuber formula (2–3) seems to be of little use here. Instead, we expand $\exp\bigl(\operatorname{Tr} V(AM)\bigr)$ as an invariant function of the variable AM in terms of the characters $\chi_R(AM)$ of irreducible representations R of the $\mathrm{GL}(N)$ group:

$$\exp\bigl(\operatorname{Tr} V(AM)\bigr) = \sum_R f_R \chi_R(AM), \tag{4–3}$$

where the coefficients f_R are the functions of N highest weight components of a representation

$$R = \{0 \le m_N \le m_{N-1} \le \dots \le m_1 < \infty\}.$$

The sum $\sum_R$ is nothing but the sum over N ordered integers. They can be calculated due to the orthogonality of characters as the following unitary integrals:

$$f_R = \int d[\Omega]_{U(N)} \exp\bigl(\operatorname{Tr} V(\Omega)\bigr)\chi_R(\Omega^+). \tag{4–4}$$

This integral can be represented as an explicit integrals only over the Cartan subgroup $\Omega = \{e^{i\omega_1}, \dots, e^{i\omega_N}\}$ and thus contains only N integration variables. We have $\operatorname{Tr} V(\Omega) = \sum_k V(e^{i\omega_k})$ and $d[\Omega]_{U(N)} \to \prod_k d\theta_k \prod_{i>j} \sin^2 \frac{1}{2}(\theta_i - \theta_j)$. Now if we plug (4–3) into (4–1) we realize that the decomposition (4–2) is actually useful and we can integrate over Ω using the following orthogonality relation between matrix elements of representation R:

$$\int d[\Omega]_{U(N)} \chi_R(A\Omega^+ X \Omega) = \frac{1}{\dim_R} \chi_R(A)\chi_R(X), \tag{4–5}$$

where $\dim_R$ is the dimension of a representation R. We see that we achieved our main goal: due to the formulas (4–3), (4–4) and (4–5) we reduced the original matrix integral (4–1) to an integral over only N eigenvalues $x_1, \dots, x_N$ of the matrix M and the sum over N highest weight components $m_1, \dots, m_N$. In the large N limit, if we scale appropriately the constants in the potential $V(M)$, the sums over m's can be replaced by integrals and we can again apply the saddle point approximation in all $2N$ integration variables. To get explicitly the right large N scaling of the couplings one usually changes $e^V \to e^{NV}$. Then the effective action at the saddle point is always of the order $1/N^2$ and the new couplings of the potential V can be kept finite in this limit.

As was shown in [27] (see also [2]), the integral over $x_1, \dots, x_N$ can be calculated exactly and the remaining sum over strictly ordered nonnegative integers $h_i = -m_i + N - i$ (shifted highest weights) reads

$$Z = \sum_{h_1<h_2<\cdots<h_N} \frac{\prod(h^e-1)!!\, h^o!!}{\prod(h^e - h^o)} \chi_R(A)\chi_R(t), \tag{4–6}$$

where $\{h^e\}$ and $\{h^o\}$ are the collections of even and odd integers h_k (their number is equal). Only the representations with equal amounts of even and odd h's contribute to (4–6). The products in the numerator go over all even and odd h's and the product in the denominator goes over all couples h^e, h^o. $\chi_R(t)$ is a character of the coupling constants t_k written in the Schur form

$$\chi_R = \det_{ij} P_{h_i - j}(t),$$

and the Schur polynomials $P_k(t)$ are defined as usually: $\sum_n P_n(t) z^n = e^{\sum_k t_k z^k}$.

So in the large N limit we have to do the saddle point calculation only with respect to N summation variables $h_1, \dots, h_N$.

The details of these formulas can be found in [27], [2]–[4]. One can also find in these papers the geometrical interpretation of the integral (4–1) in terms of the so called dually weighted planar graphs. It gives the generating function of planar graphs where both vertices and faces are weighted by the generating parameters depending on their orders. In [2]–[4] one can find the solutions of some combinatorial problems related to the enumeration of planar graphs which were possible only due to the power of the character expansion method. The particular solutions of the saddle point equations could be very tricky but it is already a "classical" problem of solution of various integral equations rather than a "quantum" problem of functional integration over infinite matrices. In that sense this model is solvable.

It is obvious that there exist many ways to generalize the model (4–1) to other matrix integrals. An immediate generalization is to substitute the $\operatorname{Tr} M^2$ term in (4–1) by an arbitrary function $W(M)$. In that case we cannot calculate explicitly the coefficients f_R (except when W is a monomial: $W(M) = M^k$) but we still get an explicit integral over $3N$ variables x_i, ω_i and m_i. So the model is again solvable.

Another solvable matrix model of this kind involving general complex matrices was proposed and investigated in [33; 28]. Its free energy gives a generating functional counting branched coverings of two dimensional surfaces.

The most general solvable two matrix model reads as

$$Z = \int d^{N^2}A\, d^{N^2}B \exp N \operatorname{Tr}\big(U(AB) + V(A) + W(B)\big), \tag{4–7}$$

where U,V and W are arbitrary functions. The way to reduce it to $\sim N$ degrees of freedom is again to expand in characters

$$\exp \operatorname{Tr} U(AB) = \sum_R u_R \chi_R(AB),$$

diagonalize the matrices A and B and integrate over the $U(N)$ variable between them by means of (4–5). In the particular case

$$Z = \int d^{N^2}A\ d^{N^2}B \exp N \operatorname{Tr}\big(\tfrac{1}{2}(A^2+B^2) - \tfrac{1}{4}\alpha(A^4+B^4) - \tfrac{1}{4}\beta(AB)^2\big),$$

the model describes a special trajectory of the 8-vertex model on random graphs. It was completely solved in [29]. Again it was possible, using character orthogonality relations, to integrate over the relative angle between A and B; this leads to separation into one-matrix integrals:

$$\mathrm{Z}(\alpha,\beta) \sim \sum_{\{h\}} (N\beta/2)^{\#h/2} c_{\{h\}} (P_{\{h\}}(\alpha))^2, \tag{4–8}$$

where

$$c_{\{h\}} = \frac{1}{\prod_i \lfloor h_i/2 \rfloor! \prod_{i,j}(h_i^e - h_j^o)}$$

is a coefficient and the one-matrix integral

$$S_{\{h\}}(\alpha) = \int d^{N^2}M\ \chi_{\{h\}}(M) \exp N\big(-\tfrac{1}{2}\operatorname{Tr} M^2 + \tfrac{1}{4}\alpha \operatorname{Tr} M^4\big)$$

appears squared in (4–8) because the contributions from the two matrices A and B are identical.

Now we can reduce the calculation of the one-matrix integral $S_{\{h\}}$ to eigenvalue integrations:

$$S_{\{h\}}(\alpha) = \int \prod_k d\lambda_k\, \Delta(\lambda) \det\left(\lambda_k^{h_j}\right) \exp N\left(-\frac{1}{2}\sum_k \lambda_k^2 + \frac{\alpha}{4}\sum_k \lambda_k^4\right),$$

where $\Delta(\lambda) = \det\left(\lambda_k^{N-j}\right) = \prod_{j<k}(\lambda_j - \lambda_k)$.

We are left only with N degrees of freedom and the action of the order N^2, so the integration is reduced to the saddle point calculation with respect to the eigenvalues λ_k (see [29] for the details).

We can immediately propose some solvable generalization of the general two matrix model (4–7) to a multimatrix chain

$$Z = \int \prod_{q=1}^{Q} d^{N^2}M_q \exp \operatorname{Tr}\left(\sum_{q=1}^{Q} V_q(M_q) + \sum_{p=1}^{Q-1} W(M_{p-1}M_p)\right),$$

where V and W are arbitrary functions.

Another interesting model solvable by the character expansion method can be written in the general form

$$Z=\int\prod_{q=1}^{Q}d^{N^2}M_q\exp\left(\operatorname{Tr}V_q(M_q)+V\left(\prod_{p=1}^{Q}M_p\right)\right).$$

It can be solved by character expansion with respect to the last factor by means of the formulas (4–3), (4–4) and the multiple application of the formula (4–5) by induction.

The results in terms of the sum over N highest weight components of the representations R reads

$$Z=\sum_R\frac{f_R}{\dim_R^{Q-1}}\prod_{q=1}^{Q}[S_{\{h\}}^{(q)}],$$

where

$$S_{\{h\}}^{(q)}=\int d^{N^2}M\,\chi_{\{h\}}(M)\exp\operatorname{Tr}V_q(M).$$

The last integral can be immediately reduced to the integrations of a type (4) over eigenvalues of the matrix M.

5. Comments and Unsolved Problems

1. The non-singlet sectors in the matrix quantum mechanics (3–1) can be effectively studied for the oscillatory potential $V(M)=M^2$. In this case the Hamilton-Ian is a collection of N^2 independent oscillators represented by the matrix elements of M. The spectrum of Hamilton-Ian of this model in a given irreducible representation of $U(N)$, is encoded into the partition functions $Z_R(q)$ for finite inverse temperature β (where $q=e^\beta$) in a given representation R. The effective way to study $Z_R(q)$ can be found in [8] or [5].
2. The character expansion is nothing but the Fourier expansion on a group manifold. As trivial as it looks for us now the Fourier transform was always a powerful method of solving problems using their symmetries. Many of the matrix models presented in the previous section and solved by this method seemed hopeless just a few years ago.
3. One of the interesting and not well studied questions is how to classify all the matrix integrals which can be reduced from $\sim N^2$ to $\sim N$ integrals or sums by use of the character expansion.
4. In many physically interesting cases we don't need a general form of potentials $V(M)$ or $W(M)$ mentioned through this paper. For example, as it was mentioned, to study the universal behavior of the large N matrix quantum mechanics near the instability point we need to know only the solution in the vicinity of a quadratic top of the potential $V(M)\simeq-\operatorname{Tr}(M-M_0)^2$. The rest

of the potential $V(M)$ has small influence on the behavior of the eigenvalues and serves only as a $U(N)$ invariant cutoff wall. It simplifies greatly the problem. For instance, all the applications in string theory, two-dimensional quantum gravity and most of statistical-mechanical applications need only the analyses of the vicinity of such critical points. The lesson to draw from it is that for some physically most interesting regimes the seemingly hopeless matrix integrals become not so hopeless and look "almost Gaussian". May be a general method of the investigation of these instability points can be worked out.

5. Another question is related to the integrability properties of sums and integrals after such reduction. The partition functions of some of them (such as the old one matrix and two matrix models) are known to be τ-functions of some integrable hierarchies of classical differential or difference equations, like Toda hierarchy [31; 5] or KP hierarchy [6] (see also [34; 35]). But many others, like the model (4–1), cannot be represented by free fermions. On the other hand, the Itzykson–Di Francesco formula (4–6) suggests that it might exist some interacting fermion representation of the partition function of the model of dually weighted graphs (4–1).

The method of character expansion as well as all other methods of calculation of the large N matrix integrals presented here represent just another refining and generalization of the usual method of reduction to the matrix eigenvalues invented long ago by Dyson. Its range of applicability is quite limited although it includes quite a few important matrix integrals known from physics and mathematics. Many more interesting matrix integrals look not hopeless for the investigation in the large N limit. The search for new tricks of integration over matrices is a fascinating and potentially extremely rewarding research direction.

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