Random Matrices and Their Applications
MSRI Publications
Volume **40**, 2001

Integration over Angular Variables for Two Coupled Matrices

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Abstract. An integral over the angular variables for two coupled $n \times n$ real symmetric, complex hermitian or quaternion self-dual matrices is expressed in terms of the eigenvalues and eigenfunctions of a hamiltonian closely related to the Calogero hamiltonian. This generalizes the known result for the complex hermitian matrices. The integral can thus be evaluated for $n = 2$ and reduced to a single sum for $n = 3$.

1. Introduction

The remarkable and useful formula

$$\int dU \,\exp\Big(-\frac{1}{2t}\,\mathrm{tr}(A - UA'U^{-1})^2\Big) = t^{n(n-1)/2}\Big(\prod_{j=0}^{n-1} j!\Big) \times \big(\Delta(\boldsymbol{x})\Delta(\boldsymbol{x}')\big)^{-1} \det\Big[\exp\Big(-\frac{1}{2t}(x_j - x'_k)^2\Big)\Big]_{j,k=1,\ldots,n} \tag{1–1}$$

has been known for the last two decades; see [Itzykson and Zuber 1980; Mehta 1981; Mehta 1991, Appendix A.5]. Here A and A' are $n \times n$ complex hermitian matrices having eigenvalues $\boldsymbol{x} := \{x_1, \ldots, x_n\}$ and $\boldsymbol{x}' := \{x'_1, \ldots, x'_n\}$ respectively, integration is over the $n \times n$ complex unitary matrices U with the invariant Haar measure dU normalized such that $\int dU = 1$. The function $\Delta(\boldsymbol{x})$ is the product of differences of the x_j:

$$\Delta(\boldsymbol{x}) := \begin{cases} 1 & \text{if } n = 1, \\ \prod_{1\le j<k\le n}(x_k - x_j) & \text{if } n \ge 2. \end{cases}$$

We would like to have a similar formula when A and A' are $n \times n$ real symmetric or quaternion self-dual matrices and the integration is over $n \times n$ real orthogonal or quaternion symplectic matrices U, a formula not presently known. These three cases are usually denoted by a parameter β taking values 1, 2 and 4 corresponding

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respectively to the integration over $n \times n$ real orthogonal, complex unitary and quaternion symplectic matrices U. We will show that the integral in (1–1) with a measure dU invariant under the appropriate group can be expressed in terms of the eigenvalues and eigenfunctions of a particular hamiltonian. This hamiltonian is closely related to the Calogero model [1969a; 1969b; 1971] where one considers the quantum n-body problem with the hamiltonian

$$H := -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \frac{1}{2}\beta(\beta-2) \sum_{1 \le j < k \le n} (x_j - x_k)^{-2}. \tag{1–2}$$

For $n = 2$ and $n = 3$ the complete set of the relevant eigenfunctions and eigenvalues are known. The three integrals (namely, for $\beta = 1,\ 2,\ 4$) can thus be explicitly computed for $n = 2$ and reduced to a single infinite sum for $n = 3$ and $\beta = 1$ or 4. For $n > 3$ and $\beta = 1$ or $\beta = 4$, the question remains open.

Integral (1–1) over the orthogonal group has also been of interest. Muirhead [1982, Chapters 7 and 9; 1975] gives this integral in terms of a hypergeometric function of two matrix variables, expressed itself as a series of matrix zonal polynomials. But no general formula for these zonal polynomials is known.

2. The Diffusion Equation

Recall that, D_j being constants, the partial differential (or diffusion) equation

$$\frac{\partial \xi}{\partial t} = \frac{1}{2} \sum_{j=1}^{n} D_j \frac{\partial^2 \xi}{\partial x_j^2} \tag{2–1}$$

with the initial condition

$$\xi(\boldsymbol{x}; 0) := \eta(\boldsymbol{x})$$

has, for $t \ge 0$, the unique solution

$$\xi(\boldsymbol{x}; t) = \int d\boldsymbol{x}'\ K(\boldsymbol{x}, \boldsymbol{x}'; t)\eta(\boldsymbol{x}'), \quad \text{with } d\boldsymbol{x}' := dx_1' \dots dx_n',$$
$$K(\boldsymbol{x}, \boldsymbol{x}'; t) := \prod_{j=1}^{n} \left((2\pi D_j t)^{-1/2} \exp\Big(-\frac{1}{2D_j t}(x_j - x_j')^2\Big)\right).$$

Indeed, for $t > 0$, $K(\boldsymbol{x}, \boldsymbol{x}'; t)$ satisfies (2–1) and it reduces to $\prod_{j=1}^{n} \delta(x_j - x_j')$ as $t \to 0$.

Now, in these formulae, we use as variables x_j the $n+\beta n(n-1)/2$ real variables that determine the matrix A, namely, the n real diagonal elements A_{jj} and the β components $A_{jk(r)}$ (with $r = 1, \dots, \beta$) of each of the $n(n-1)/2$ nondiagonal elements A_{jk}. For real symmetric matrices, only one component A_{jk} is present; for complex hermitian matrices, the two components are the real and imaginary parts of A_{jk}; for the quaternion self-dual matrices, A_{jk} has four components, one scalar and three vectorial parts. Similarly, the x'_j are the $n + \beta n(n-1)/2$ real variables that determine the matrix A'. The kernel $K(\boldsymbol{x}, \boldsymbol{x}'; t)$ then becomes

$$\begin{aligned} K(A, A'; t) := \prod_{j=1}^{n} \Bigl((2\pi t)^{-1/2} \exp\Bigl(-\frac{1}{2t}(A_{jj} - A'_{jj})^2\Bigr)\Bigr) \\ \times \prod_{1 \le j < k \le n} \prod_{r=1}^{\beta} \Bigl((\pi t)^{-1/2} \exp\Bigl(-\frac{1}{t}(A_{jk(r)} - A'_{jk(r)})^2\Bigr)\Bigr) \\ = (2\pi t)^{-n/2} (\pi t)^{-\beta n(n-1)/4} \exp\Bigl(-\frac{1}{2t} \operatorname{tr}(A - A')^2\Bigr). \end{aligned} \tag{2–2}$$

The integral

$$\xi(A; t) = \int dA' \; K(A, A'; t) \eta(A'), \tag{2–3}$$

with the measure

$$dA' := \Bigl(\prod_{j=1}^{n} dA'_{jj} \Bigr) \prod_{1 \le j < k \le n} \prod_{r=1}^{\beta} dA'_{jk(r)} \tag{2–4}$$

satisfies the diffusion equation

$$\frac{\partial \xi}{\partial t} = \frac{1}{2} \nabla_A^2 \xi, \quad \nabla_A^2 := \sum_{j=1}^{n} \frac{\partial^2}{\partial A_{jj}^2} + \frac{1}{2} \sum_{1 \le j < k \le n} \sum_{r=1}^{\beta} \frac{\partial^2}{\partial A_{jk(r)}^2}, \tag{2–5}$$

and the initial condition

$$\xi(A; 0) = \eta(A).$$

The dA' in (2–4) is a measure invariant under the automorphism $A' \to UA'U^{-1}$ for any U in the group $\mathcal{G}_\beta$ of $n \times n$ real orthogonal, complex unitary or quaternion symplectic matrices, respectively for $\beta = 1$, 2 or 4. Let us assume from now on that $\eta(A')$ is invariant under the same transformation; that is, $\eta(A') = \eta(UA'U^{-1})$ for any U in $\mathcal{G}_\beta$. From the invariance of the measure dA' in (2–3), (2–4) and the cyclic invariance of the trace in (2–2), it follows that $\xi(A; t)$ is also invariant under the same transformation; that is, $\xi(A; t) = \xi(UAU^{-1}; t)$. We can choose a matrix U_A in $\mathcal{G}_\beta$ to diagonalize A, [1]

$$A = U_A \boldsymbol{X} U_A^{-1}, \quad \boldsymbol{X} := [x_i \delta_{ij}], \tag{2–6}$$

and similarly for A'. The invariance of η and ξ implies that $\eta(A')$ and $\xi(A; t)$ are symmetric functions of the eigenvalues of A' and A respectively; we denote them as

$$\eta(\boldsymbol{x}') := \eta(A') \qquad \xi(\boldsymbol{x}; t) := \xi(A; t).$$

The hyperplanes $x_j = x_k$ for $1 \le j < k \le n$ divide the n-dimensional space into $n!$ sectors (sometimes called Weyl chambers). Taking advantage of the symmetry

[1] Diagonalization of quaternion matrices is not as well known as that of real or complex matrices. See [Mehta 1989, Chapters 4 and 8], for example.

property of $\xi(\boldsymbol{x};t)$, from now on we will restrict our attention to one of the sectors where $\Delta(\boldsymbol{x}) \geq 0$, namely the sector $\mathcal{S}$ defined by the conditions

$$\mathcal{S} := \{\boldsymbol{x}; x_1 \leq x_2 \leq \cdots \leq x_n\}. \tag{2–7}$$

Changing the variables from matrix element components to the n (real) eigenvalues and the $\beta n(n-1)/2$ real "angle" variables on which U_A and U'_A depend, we have as usual [Mehta 1991, Chapter 3][2]

$$dA' = \pi^{\beta n(n-1)/4}\left(\prod_{j=1}^{n}\frac{\Gamma(1+\beta/2)}{\Gamma(1+\beta j/2)}\right)\left|\Delta(\boldsymbol{x}')\right|^{\beta} d\boldsymbol{x}'\, dU'_A, \tag{2–8}$$

where dU'_A is the invariant measure over the group $\mathcal{G}_\beta$ normalized such that $\int dU'_A = 1$. Hence equation (2–3) reads, for any $\boldsymbol{x}$ in $\mathcal{S}$,

$$\xi(\boldsymbol{x};t) = \int_{R^n} d\boldsymbol{x}' \left|\Delta(\boldsymbol{x}')\right|^{\beta} \mathcal{K}(\boldsymbol{x},\boldsymbol{x}';t)\,\eta(\boldsymbol{x}'), \tag{2–9}$$

where

$$\mathcal{K}(\boldsymbol{x},\boldsymbol{x}';t) = (2\pi t)^{-n/2} t^{-\beta n(n-1)/4}\left(\prod_{j=1}^{n}\frac{\Gamma(1+\beta/2)}{\Gamma(1+\beta j/2)}\right) \times \int dU\, \exp\Big(-\frac{1}{2t}\operatorname{tr}(\boldsymbol{X} - U\boldsymbol{X}'U^{-1})^2\Big). \tag{2–10}$$

It follows from the integration over the group $\mathcal{G}_\beta$ with the invariant measure dU, and from the cyclic invariance of the trace, that $\mathcal{K}(\boldsymbol{x},\boldsymbol{x}';t)$ is a function symmetric in the x_j and symmetric in the x'_j. Since the measure $d\boldsymbol{x}' \left|\Delta(\boldsymbol{x}')\right|^{\beta}$ and $\eta(\boldsymbol{x}')$ are symmetric in the x'_j, the integral in (2–9) can be restricted to the sector $\mathcal{S}$ and multiplying it by a factor $n!$; thus

$$\xi(\boldsymbol{x};t) = n!\int_{\mathcal{S}} d\boldsymbol{x}' \left(\Delta(\boldsymbol{x}')\right)^{\beta}\mathcal{K}(\boldsymbol{x},\boldsymbol{x}';t)\eta(\boldsymbol{x}'). \tag{2–11}$$

We recognize in the right-hand side of (2–10) precisely the quantity we are interested in. Namely, apart from explicitly known constant factors, this is the left-hand side of (1–1), including now the cases where A, A' are $n \times n$ real symmetric or quaternion self-dual matrices and the integration is over $n \times n$ real orthogonal or quaternion symplectic matrices. Our problem thus amounts to constructing the kernel $\mathcal{K}(\boldsymbol{x},\boldsymbol{x}';t)$ of the evolution operator of the diffusion equation (2–5).

Separating the laplacian ∇_A^2 into parts depending on $\boldsymbol{x}$ and on U_A, we get (see Appendix A for proof)

$$\nabla_A^2 = \left(\Delta(\boldsymbol{x})\right)^{-\beta}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left(\Delta(\boldsymbol{x})\right)^{\beta}\frac{\partial}{\partial x_j} + \mathcal{D}^2_{U_A}, \tag{2–12}$$

[2] There the constant factor in (2–8) is not evaluated. The evaluation can be done by computing $\int dA e^{-\operatorname{tr} A^2}$ both directly and using (3.3.10) of the reference.

where $\mathcal{D}^2_{U_A}$ involves derivatives with respect to the angle variables entering U_A, one sees that $\xi(\boldsymbol{x};t)$ satisfies the diffusion equation, for all $\boldsymbol{x}$ in $\mathcal{S}$:

$$\frac{\partial\xi}{\partial t} = -\frac{1}{2}\mathcal{H}\xi, \tag{2–13}$$

$$\begin{aligned}\mathcal{H} &:= -\big(\Delta(\boldsymbol{x})\big)^{-\beta}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\big(\Delta(\boldsymbol{x})\big)^{\beta}\frac{\partial}{\partial x_j}\\ &= -\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2} - \beta\sum_{1\le j<k\le n}\frac{1}{x_j-x_k}\left(\frac{\partial}{\partial x_j}-\frac{\partial}{\partial x_k}\right),\end{aligned} \tag{2–14}$$

with the initial value $\xi(\boldsymbol{x};0)=\eta(\boldsymbol{x})$. Equation (2–13) is similar to a Schrödinger equation with the hamiltonian $\mathcal{H}$ and a purely imaginary time $-i\hbar t/2$.

We now have to specify the space of functions ξ and η, in which we solve the preceding equations.

3. The Evolution Operator

To construct the evolution operator of equation (2–13), we build a Hilbert space $\mathcal{L}^2(\mathcal{S},\mu)$ of functions $f(\boldsymbol{x})$. These functions are supposed to be the restrictions to the sector $\mathcal{S}$, defined in (2–7), of symmetric functions of the variables x_j in R^n. The scalar product using Dirac's notation is defined as

$$\langle f|g\rangle := \int_{\mathcal{S}} d\boldsymbol{x}\,\mu(\boldsymbol{x})\,f^*(\boldsymbol{x})g(\boldsymbol{x})$$

and we choose the weight $\mu(\boldsymbol{x})$ in such a way that $\mathcal{H}$ be hermitian. From (2–14) one deduces for twice differentiable functions f and g the identity

$$\mu\,(f\mathcal{H}g-g\mathcal{H}f) = -\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\big(\mu\,W_j(f,g)\big)+\sum_{j=1}^{n}\left(\frac{\partial\mu}{\partial x_j}-\beta\mu\sum_{k\ne j}\frac{1}{x_j-x_k}\right)W_j(f,g), \tag{3–1}$$

where W_j is the wronskian $W_j(f,g) := f\,\partial g/\partial x_j - g\,\partial f/\partial x_j$. By integrating both sides over the sector $\mathcal{S}$, the first term in the right-hand side of (3–1), which is a divergence, gives an integral over the boundary $\partial\mathcal{S}$ of $\mathcal{S}$. The points at finite distance of $\partial\mathcal{S}$ do not contribute: because of the symmetry of f and g extended to R^n, the vector with components $W_j(f,g)$ has no component normal to $\partial\mathcal{S}$. Also the points at infinity of $\partial\mathcal{S}$ do not contribute if f and g vanish fast enough at infinity; that is, if they belong to the domain of $\mathcal{H}$. As for the second term, it vanishes identically if and only if μ satisfies n linear first order partial differential equations, with the solution unique up to a constant c,

$$\mu(\boldsymbol{x}) = c\,\big(\Delta(\boldsymbol{x})\big)^{\beta}. \tag{3–2}$$

With such a weight μ, $\mathcal{H}$ defines a hermitian operator in $\mathcal{L}^2(\mathcal{S},\mu)$.

Looking at factorized solutions of the form $\varphi_{\boldsymbol{\alpha}}(\boldsymbol{x})\exp(-\mathcal{E}_{\boldsymbol{\alpha}}t/2)$ of (2–13), $\varphi_{\boldsymbol{\alpha}}(\boldsymbol{x})$ is the eigenfunction with the eigenvalue $\mathcal{E}_\alpha$ of the t-independent hamiltonian $\mathcal{H}$. In Dirac's bracket notation $\varphi_{\boldsymbol{\alpha}}(\boldsymbol{x}) := \langle \boldsymbol{x}|\varphi_{\boldsymbol{\alpha}}\rangle$ satisfies the "Schrödinger equation"

$$\mathcal{H}|\varphi_{\boldsymbol{\alpha}}\rangle = \mathcal{E}_{\boldsymbol{\alpha}}|\varphi_{\boldsymbol{\alpha}}\rangle. \tag{3–3}$$

Here, $\boldsymbol{\alpha}$ denotes a convenient set of indices. The orthogonality and closure relations for the basis $\{|\boldsymbol{x}\rangle\}$ are

$$\langle \boldsymbol{x}|\boldsymbol{x}'\rangle = \big(\mu(\boldsymbol{x})\big)^{-1}\delta(\boldsymbol{x}-\boldsymbol{x}'), \qquad \int_{\mathcal{S}} d\boldsymbol{x}\,\mu(\boldsymbol{x})|\boldsymbol{x}\rangle\langle\boldsymbol{x}| = \mathbf{1} \tag{3–4}$$

and those for the basis $\{|\varphi_{\boldsymbol{\alpha}}\rangle\}$ are

$$\langle \varphi_{\boldsymbol{\alpha}}|\varphi_{\boldsymbol{\alpha}'}\rangle = \big(\rho(\boldsymbol{\alpha})\big)^{-1}\delta(\boldsymbol{\alpha}-\boldsymbol{\alpha}'), \qquad \int d\boldsymbol{\alpha}\ \rho(\boldsymbol{\alpha})|\varphi_{\boldsymbol{\alpha}}\rangle\langle\varphi_{\boldsymbol{\alpha}}| = \mathbf{1}, \tag{3–5}$$

where the positive weight $\rho(\boldsymbol{\alpha})$ can be chosen at will. With these notations, it follows from (2–11) and (3–2)–(3–5) that the kernel $\mathcal{K}$ equals

$$\mathcal{K}(\boldsymbol{x},\boldsymbol{x}';t) = \frac{c}{n!}\langle\boldsymbol{x}|e^{-\mathcal{H}t/2}|\boldsymbol{x}'\rangle = \frac{c}{n!}\int d\boldsymbol{\alpha}\ \rho(\boldsymbol{\alpha})\varphi_{\boldsymbol{\alpha}}(\boldsymbol{x})e^{-\mathcal{E}_{\boldsymbol{\alpha}}t/2}\varphi^*_{\boldsymbol{\alpha}}(\boldsymbol{x}'). \tag{3–6}$$

Using (2–10), this last equation allows us to extend (1–1) to the cases $\beta = 1$ and $\beta = 4$:

$$\int dU\ \exp\Big(-\frac{1}{2t}\operatorname{tr}(\boldsymbol{X}-U\boldsymbol{X}'U^{-1})^2\Big) = \frac{c}{n!}(2\pi t)^{n/2}t^{\beta n(n-1)/4}\bigg(\prod_{j=1}^{n}\frac{\Gamma(1+\beta j/2)}{\Gamma(1+\beta/2)}\bigg)$$
$$\times\int d\boldsymbol{\alpha}\,\rho(\boldsymbol{\alpha})\varphi_{\boldsymbol{\alpha}}(\boldsymbol{x})e^{-\mathcal{E}_{\boldsymbol{\alpha}}t/2}\varphi^*_{\boldsymbol{\alpha}}(\boldsymbol{x}'). \tag{3–7}$$

4. Connection with the Calogero Model

If we wanted to eliminate the linear derivative terms in (2–13) and (2–14), we would change the unknown function for all $\boldsymbol{x}$ in the sector $\mathcal{S}$ defined in (2–7), as follows

$$\psi(\boldsymbol{x};t) := \big(\Delta(\boldsymbol{x})\big)^{\beta/2}\xi(\boldsymbol{x};t). \tag{4–1}$$

A straightforward calculation shows that this $\psi(\boldsymbol{x};t)$ satisfies the partial differential equation

$$\frac{\partial\psi}{\partial t} = -\tfrac{1}{2}H\psi \tag{4–2}$$

where H is precisely the Calogero hamiltonian (1–2). Looking at factorized solutions of the form $\phi_{\boldsymbol{\alpha}}(\boldsymbol{x})\exp(-E_{\boldsymbol{\alpha}}t/2)$ of (4–2), $\phi_{\boldsymbol{\alpha}}(\boldsymbol{x})$ is the eigenfunction with the eigenvalue E_α of the t-independent hamiltonian H. In Dirac's bracket notation $\phi_{\boldsymbol{\alpha}}(\boldsymbol{x}) := \langle\boldsymbol{x}|\phi_{\boldsymbol{\alpha}}\rangle$ satisfies the "Schrödinger equation"

$$H|\phi_{\boldsymbol{\alpha}}\rangle = E_{\boldsymbol{\alpha}}|\phi_{\boldsymbol{\alpha}}\rangle. \tag{4–3}$$

The discussion of the hermitian character of $\mathcal{H}$, (3–1) and what follows, can be applied to H. For the function ψ of the form (4–1), with $\xi(\boldsymbol{x};t)$ analytic in $\boldsymbol{x}$, one can check that H is hermitian in $\mathcal{S}$ with $\mu(\boldsymbol{x}) = 1$. Hence finding a solution ϕ of (4–3) in $\mathcal{L}^2(\mathcal{S}, 1)$ is equivalent to calculating the integral (1–1). Indeed the considerations leading from (2–13) to (3–6) when applied to (4–2) yields for all $\boldsymbol{x}$ in $\mathcal{S}$

$$\psi(\boldsymbol{x};t) = \int_{\mathcal{S}} d\boldsymbol{x}' \langle \boldsymbol{x}| e^{-\frac{1}{2}Ht} |\boldsymbol{x}'\rangle \big(\Delta(\boldsymbol{x}')\big)^{\beta/2} \eta(\boldsymbol{x}') \tag{4–4}$$

with

$$\langle \boldsymbol{x}| e^{-\frac{1}{2}Ht} |\boldsymbol{x}'\rangle = \int d\boldsymbol{\alpha} \phi_{\boldsymbol{\alpha}}(\boldsymbol{x}) e^{-\frac{1}{2}E_{\boldsymbol{\alpha}} t} \phi^*_{\boldsymbol{\alpha}}(\boldsymbol{x}'). \tag{4–5}$$

Here the orthogonality and closure relations for both bases $\{|\boldsymbol{x}\rangle\}$ and $\{|\phi_{\boldsymbol{\alpha}}\rangle\}$ are

$$\langle \boldsymbol{x}|\boldsymbol{x}'\rangle = \delta(\boldsymbol{x}-\boldsymbol{x}'), \qquad \int_{\mathcal{S}} d\boldsymbol{x}\, |\boldsymbol{x}\rangle\langle \boldsymbol{x}| = \mathbf{1},$$

$$\langle \phi_{\boldsymbol{\alpha}}|\phi_{\boldsymbol{\alpha}'}\rangle = \delta(\boldsymbol{\alpha}-\boldsymbol{\alpha}'), \qquad \int d\boldsymbol{\alpha}\, |\phi_{\boldsymbol{\alpha}}\rangle\langle \phi_{\boldsymbol{\alpha}}| = \mathbf{1}.$$

The $\xi(\boldsymbol{x};t)$ defined in $\mathcal{S}$ from (4–1) and (4–4) is then extended to R^n by the requirement that it is a symmetric function of the x_j.

In spite of a slight difference of the point of view, our problem is similar to that of Calogero. In sections 2 and 3 we deal with either one particle in n dimensions and completely symmetric wave functions in $(x_1, \ldots, x_n)$ or with n bosons in one dimension. Calogero considers n particles in one dimension which can be bosons, fermions or boltzmannions. Calogero's particles, all on the real line, cannot cross each other due to the singular potential (see Appendix B). The phase space is thus naturally divided in $n!$ sectors, each sector corresponding to a certain order of the n particles. For Calogero it is sufficient to find the eigenfunctions and eigenvalues in any one sector, say when $x_1 \le x_2 \le \cdots \le x_n$, the solutions in other sectors being obtained by the proper symmetry according as the particles satisfy Boltzmann, Bose or Fermi statistics [Calogero 1969a]. In our case, the "Schrödinger equation" (4–3) for a single particle in n dimensions is well defined only in $\mathcal{S}$ and the singular potential requires a special treatment as detailed in Appendix B for the case $n = 2$.

When $\beta = 2$, the hamiltonian (1–2) reduces to the "kinetic energy"

$$-\sum_{j=1}^{n} \partial^2/\partial x_j^2,$$

each variable x_j in the "Schrödinger equation" (4–3) separates, and the normalized solutions with the corresponding eigenvalues are

$$\phi_{\boldsymbol{\alpha}}(\boldsymbol{x}) = \prod_{j=1}^{n} (2\pi)^{-1/2} \exp(ik_j x_j),$$

$$E_{\boldsymbol{\alpha}} = \sum_{j=1}^{n} k_j^2, \qquad \boldsymbol{\alpha} = \{k_1, \ldots, k_n\},$$

where k_j are real varying from $-\infty$ to ∞.

Gaussian integrals in (4–5) over the k_j can be performed and on extending it by symmetry to the whole of R^n [Itzykson and Zuber 1980] one gets back (1–1), as one should.

When $\beta = 1$ or $\beta = 4$, the solutions of (3–3) or of (4–3) are not completely known for general n.

5. The Case $n = 2$

The Hilbert space considered in section 3 is $\mathcal{L}^2\big(x_1 \le x_2, 2^{-\beta/2}(x_2 - x_1)^\beta\big)$ and the hermitian hamiltonian reads

$$\mathcal{H} := -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\beta}{(x_1 - x_2)}\Big(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\Big).$$

Following Calogero [1969a] with a slight modification, we change the variables to

$$X := \frac{x_1 + x_2}{\sqrt{2}}, \quad x := \frac{x_2 - x_1}{\sqrt{2}} = \frac{\Delta(\boldsymbol{x})}{\sqrt{2}},$$

where X varies from $-\infty$ to ∞ and x from 0 to ∞. This is an orthogonal change of variables, and (3–3) becomes

$$\mathcal{H}\varphi = -\Big(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial x^2} + \frac{\beta}{x}\frac{\partial}{\partial x}\Big)\varphi = \mathcal{E}\varphi.$$

We look for solutions of the form

$$\varphi(X, x) = f(X)g(x).$$

The variables can be separated. Letting primes denote differentiation, we get

$$f''(X) = -K^2 f(X), \tag{5–1}$$

$$g''(x) + \frac{\beta}{x}g'(x) + k^2 g(x) = 0, \tag{5–2}$$

and

$$\mathcal{E} \equiv \mathcal{E}_{K,k} := K^2 + k^2,$$

where $\mathcal{E}_{K,k}$ is real. Since $-d^2/dX^2$ is the square of the hermitian operator $-id/dX$, K^2 is real nonnegative. As a consequence k^2 is also real. Then (5–1) has only a continuous spectrum labelled by K real and a complete set of normalized solutions

$$f_K(X) = \frac{1}{\sqrt{2\pi}}\exp(iKX) \qquad K \text{ real.} \tag{5–3}$$

Setting $g(x) := x^{-\nu}J(x)$ and $\nu := (\beta - 1)/2$, equation (5–2) changes to Bessel's differential equation

$$x^2 J''(x) + xJ'(x) + \big(k^2x^2 - \nu^2\big)\, J(x) = 0,$$

having two linearly independent solutions. They are: for $k^2 \ge 0$ the Bessel functions $J_\nu(kx)$ and $Y_\nu(kx)$ with $k \ge 0$, and for $k^2 = -\kappa^2 < 0$ the modified

Bessel functions $I_\nu(\kappa x)$ and $K_\nu(\kappa x)$ with $\kappa > 0$. Only the solution J_ν provides a function

$$g_k(x) = (kx)^{-\nu} J_\nu(kx), \quad \nu = (\beta-1)/2 \tag{5–4}$$

which is square-integrable for $x \geq 0$ with the measure x^β. Then (5–2) has only a continuous spectrum labelled by k real nonnegative. The orthogonality and closure relations (3–5) for the $g_k(x)$ with $k \geq 0$ read

$$\int_0^\infty dx\; x^\beta g_k(x) g_{k'}(x) = k^{-\beta}\delta(k-k')$$
$$\int_0^\infty dk\; k^\beta g_k(x) g_k(x') = x^{-\beta}\delta(x-x').$$

They can be verified by taking the limit $\rho \to 0$ in [Gradshteyn and Ryzhik 1991, page 718, formula 6.633.2]:

$$\int_0^\infty dk\, k\, e^{-\rho^2 k^2} J_\nu(kx) J_\nu(kx') = \frac{1}{2\rho^2} e^{-(x^2+x'^2)/(4\rho^2)} I_\nu\Big(\frac{xx'}{2\rho^2}\Big). \tag{5–5}$$

Thus

$$\langle \boldsymbol{x} | e^{-\mathcal{H}t/2} | \boldsymbol{x}' \rangle = \int_{-\infty}^\infty dK \int_0^\infty dk\; k^\beta e^{-(K^2+k^2)t/2} f_K(X) f_K^*(X') g_k(x) g_k(x').$$

The integration over K is a gaussian integral while for the integration over k one can use (5–5). One finally gets from (3–7)

$$\begin{aligned}\int dU\, \exp\Big(-\frac{1}{2t}\operatorname{tr}(A-UA'U^{-1})^2\Big) &= \frac{\sqrt{\pi}}{2}\, t^{(\beta-1)/2} \frac{\Gamma(1+\beta)}{\Gamma(1+\beta/2)} \\ &\quad\times \big((x_2-x_1)(x_2'-x_1')\big)^{(1-\beta)/2} I_{(\beta-1)/2}\Big(\frac{(x_2-x_1)(x_2'-x_1')}{2t}\Big) \\ &\quad\times \exp\Big(-\frac{1}{2t}\big(x_1^2+x_2^2+{x_1'}^2+{x_2'}^2-(x_1+x_2)(x_1'+x_2')\big)\Big). \end{aligned}\tag{5–6}$$

This result can be directly verified for $\beta = 1$. For $\beta = 2$, we have

$$I_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\, \sinh z,$$

and (5–6) gives back the known result (1–1), as it should. More effort is needed to verify directly the result for $\beta = 4$.

6. The Case $n = 3$

The Hilbert space considered in section 3 is $\mathcal{L}^2\big(x_1 \leq x_2 \leq x_3,\, 2^{\beta/2}(\Delta(\boldsymbol{x}))^\beta\big)$, and the hermitian hamiltonian reads

$$\begin{aligned}\mathcal{H} := -\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\Big) - \beta\Big(&\frac{1}{(x_2-x_3)}\Big(\frac{\partial}{\partial x_2}-\frac{\partial}{\partial x_3}\Big) \\ &+\frac{1}{(x_3-x_1)}\Big(\frac{\partial}{\partial x_3}-\frac{\partial}{\partial x_1}\Big)+\frac{1}{(x_1-x_2)}\Big(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}\Big)\Big).\end{aligned}$$

Following Calogero [1969a] with some slight modifications, we change the variables to

$$X := \frac{1}{\sqrt{3}}(x_1 + x_2 + x_3), \tag{6–1}$$
$$x := \frac{1}{\sqrt{2}}(-x_1 + x_2) =: r\cos\theta,$$
$$y := \frac{1}{\sqrt{6}}(-x_1 - x_2 + 2x_3) := r\sin\theta.$$

It is convenient to make this change of variables in two steps: from x_1, x_2, x_3 to X, x, y and then to X, r, θ. Thereby one gets

$$\Delta(\boldsymbol{x}) = -\frac{1}{\sqrt{2}}r^3\cos(3\theta).$$

Therefore, in the sector $x_1 \le x_2 \le x_3$, X varies from $-\infty$ to ∞; x, y and r vary from 0 to ∞; and θ varies from $\pi/6$ to $\pi/2$. Using the identity

$$\tan\theta + \tan\Big(\theta + \frac{\pi}{3}\Big) + \tan\Big(\theta + \frac{2\pi}{3}\Big) = 3\tan 3\theta,$$

one has

$$\mathcal{H} = -\Big(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\Big) - 3\beta\Big(\frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\tan 3\theta\,\frac{\partial}{\partial\theta}\Big).$$

Now we look for factorized solutions of the form

$$\xi(X, r, \theta) = f(X)g(r)h(\theta).$$

The variables can again be separated. The equation $(\mathcal{H} - \mathcal{E})\xi = 0$ splits into three equations

$$f''(X) = -K^2 f(X), \tag{6–2}$$
$$r^2 g''(r) + (1 + 3\beta)rg'(r) + (k^2r^2 - L^2)g(r) = 0, \tag{6–3}$$
$$h''(\theta) - 3\beta\tan(3\theta)h'(\theta) + L^2 h(\theta) = 0, \tag{6–4}$$

where we have introduced three constants K^2, k^2 and L^2, with

$$\mathcal{E} \equiv \mathcal{E}_{K,k} := K^2 + k^2.$$

Like (5–1), equation (6–2) has a continuous spectrum labelled by K real, a complete set of normalized solutions given by (5–3) and k^2 is real.

Setting $g(r) := r^{-3\beta/2}J(r)$ in (6–3) one gets Bessel's differential equation for J:

$$z^2J''(z) + zJ'(z) + (z^2 - \nu^2)J(z) = 0, \quad z = kr, \quad \nu^2 = L^2 + (3\beta/2)^2.$$

As in (5–4), only the Bessel function $J_\nu(z)$ with k real nonnegative and $\nu = l + 3\beta/2$ with l an integer, that is,

$$L^2 = \nu^2 - (3\beta/2)^2 = l(l + 3\beta) \tag{6–5}$$

gives for (6–3) a square integrable solution $g(r)$ over $x_1 \le x_2 \le x_3$ with the measure $r^{3\beta+1}$.

The singularities of (6–4) are at the points θ with $\cos 3\theta = 0$, that is, the end points of the interval $[\pi/6, \pi/2]$. It is convenient to shift from the variable θ to the new variable $z = \frac{1}{2}(1 - \sin 3\theta)$, which maps this interval $[\pi/6, \pi/2]$ on the interval $[0, 1]$. The function $F(z) := h(\theta)$ satisfies the hypergeometric differential equation

$$z(1-z)F'' + (1+\beta)\Big(\frac{1}{2} - z\Big)F' + \frac{L^2}{9}F = 0.$$

A careful examination shows that any solution of this equation has necessarily a singularity in the variable z, either at $z = 0$ or at $z = 1$, unless $L^2/9$ has the form $m(m+\beta)$, where m is an integer. This leads, using (6–5), to the only acceptable integer values of l, namely $l = 3m$, for m a nonnegative integer. The unique regular solution $h(\theta)$ is then a Gegenbauer polynomial in $\sin 3\theta$.

Finally,

$$\begin{aligned} f_K(X) &= \frac{1}{\sqrt{2\pi}} e^{iKX}, \\ g_{k,m}(r) &= (kr)^{-3\beta/2} J_{3m+3\beta/2}(kr), \\ h_m(\theta) &= \mathrm{const}\, F\big(-m,\, m+\beta;\, \tfrac{1}{2}(1+\beta);\, \tfrac{1}{2}(1-\sin 3\theta)\big) \\ &= \Gamma\Big(\frac{\beta}{2}\Big)\bigg(\frac{2^\beta(3m+3\beta/2)m!}{2\pi\Gamma(m+\beta)}\bigg)^{1/2} C_m^{(\beta/2)}(\sin 3\theta). \end{aligned}$$

The orthogonality and closure relations (3–5) for $g_{k,m}(r)h_m(\theta)$ read

$$\begin{aligned} &\int_0^\infty dr\, r^{3\beta+1} \int_{\pi/6}^{\pi/2} d\theta (-\cos 3\theta)^\beta g_{k,m}(r) h_m(\theta) g_{k',m'}(r) h_{m'}(\theta) \\ &\qquad\qquad = k^{-3\beta-1}\delta(k-k')\delta_{mm'}, \\ &\int_0^\infty dk\ k^{3\beta+1} \sum_{m=0}^\infty g_{k,m}(r)h_m(\theta)g_{k,m}(r')h_m(\theta') \\ &\qquad\qquad = r^{-3\beta-1}(-\cos 3\theta)^{-\beta}\delta(r-r')\delta(\theta-\theta'). \end{aligned}$$

Therefore

$$\begin{aligned} \langle \boldsymbol{x}| e^{-\mathcal{H}t/2} |\boldsymbol{x}'\rangle = \int_{-\infty}^\infty dK \int_0^\infty dk\ k^\beta \sum_{m=0}^\infty e^{-(K^2+k^2)t/2} \\ \times f_K(X) f_K^*(X') g_{k,m}(r) h_m(\theta) g_{k,m}(r') h_m(\theta'), \end{aligned}$$

where we recall that the variables are: X given by (6–1) and r and θ such that

$$r^2 = \tfrac{1}{3}\big((x_1-x_2)^2 + (x_2-x_3)^2 + (x_3-x_1)^2\big), \quad \tan\theta = \frac{x_1+x_2-2x_3}{\sqrt{3}(x_1-x_2)},$$

and X', r', θ' have the same expressions in terms of x_1', x_2', x_3'. The integrations over K and k can be done as before and yield

$$\int_{-\infty}^{\infty} dK\, e^{iK(X-X')-\frac{t}{2}K^2} = \sqrt{\frac{2\pi}{t}} \exp\Bigl(-\frac{1}{2t}(X-X')^2\Bigr),$$
$$\int_0^{\infty} dk\, k\, e^{-\frac{t}{2}k^2} J_\nu(kr) J_\nu(kr') = \frac{1}{t}\exp\Bigl(-\frac{1}{2t}(r^2+r'^2)\Bigr) I_\nu\Bigl(\frac{rr'}{t}\Bigr).$$

Collecting the constants one gets from (3–7)

$$\int dU \exp\Bigl(-\frac{1}{2t}\operatorname{tr}(A-UA'U^{-1})^2\Bigr) = 3(2t)^{3\beta/2}\,\Gamma(\beta)\,\Gamma\bigl(\tfrac{3}{2}\beta\bigr) \times (rr')^{3\beta/2}\exp\Bigl(-\frac{1}{2t}\bigl((X-X')^2+r^2+r'^2\bigr)\Bigr)\,\boldsymbol{S},$$

with

$$\boldsymbol{S} = \sum_{m=0}^{\infty} \frac{(m+\beta/2)m!}{\Gamma(m+\beta)}\, C_m^{(\beta/2)}(\sin 3\theta)\, C_m^{(\beta/2)}(\sin 3\theta')\, I_{3m+3\beta/2}\Bigl(\frac{rr'}{t}\Bigr). \tag{6–6}$$

Therefore the integration over the 3×3 orthogonal, unitary or symplectic matrices is effectively replaced by the infinite sum $\boldsymbol{S}$ of (6–6).

For $\beta = 2$, the Gegenbauer polynomials reduce to Chebyshev polynomials of the second kind:

$$C_m^{(1)}(\cos\theta) = U_m(\cos\theta) := \frac{\sin(m+1)\theta}{\sin\theta}.$$

Using the integral representation

$$I_n(z) = \frac{1}{2\pi}\int_0^{2\pi} d\phi\ \exp(z\cos\phi \pm in\phi) \tag{6–7}$$

one can write the sum (6–6) as

$$\begin{aligned}\boldsymbol{S} &= \sum_{m=0}^{\infty} \frac{\sin\bigl((m+1)\left(\frac{\pi}{2}-3\theta\right)\bigr)}{\sin\left(\frac{\pi}{2}-3\theta\right)}\, \frac{\sin\bigl((m+1)\left(\frac{\pi}{2}-3\theta'\right)\bigr)}{\sin\left(\frac{\pi}{2}-3\theta'\right)} \\ &\quad\times \frac{1}{2\pi}\int_0^{2\pi} d\phi\, \exp\Bigl(\frac{rr'}{t}\cos\phi \pm i(3m+3)\phi\Bigr) \\ &= \frac{1}{8\pi\cos 3\theta\cos 3\theta'} \sum_{m=-\infty}^{\infty}\int_0^{2\pi} d\phi\ e^{(rr'/t)\cos\phi}\bigl(e^{im(3\theta-3\theta'+3\phi)} - e^{im(\pi-3\theta-3\theta'+3\phi)}\bigr).\end{aligned}$$

Interchanging the order of summation and integration and using the identity

$$\sum_{m=-\infty}^{\infty} e^{imz} = 2\pi \sum_{m=-\infty}^{\infty} \delta(z+2m\pi),$$

one gets

$$\begin{aligned}\cos(3\theta)\cos(3\theta')\,\boldsymbol{S} = \tfrac{1}{12}\big(&e^{(rr'/t)\cos(\theta-\theta')} + e^{(rr'/t)\cos(\theta-\theta'+2\pi/3)}\\ &+ e^{(rr'/t)\cos(\theta-\theta'-2\pi/3)} - e^{-(rr'/t)\cos(\theta+\theta')}\\ &- e^{-(rr'/t)\cos(\theta+\theta'+2\pi/3)} - e^{-(rr'/t)\cos(\theta+\theta'-2\pi/3)}\big).\end{aligned}$$

The six terms in the right-hand side correspond to the six terms in the expansion of the 3×3 determinant $\det\big[\exp(-(1/2t)(x_j-y_k)^2)\big]_{j,k=1,2,3}$ of (1–1), and the correctness of the multiplying factors can be easily checked.

For $\beta = 1$ the Gegenbauer polynomials reduce to Legendre polynomials, $C_m^{(1/2)}(z) = P_m(z)$; while for $\beta = 4$ we have from [Bateman 1953, p. 176, § 10.9, eq. (23)]

$$\frac{d}{dz}C_{m+1}^{(1)}(z) = 2C_m^{(2)}(z),$$

or more explicitely

$$C_m^{(2)}(\cos\theta) = \frac{\sin\big((m+1)\theta\big)}{2\sin^3\theta} - (m+1)\frac{\cos\big((m+2)\theta\big)}{2\sin^2\theta}.$$

But we do not know how to evaluate the sum $\boldsymbol{S}$ in (6–6).

A sum similar to (6–6) is known [Watson 1952, p. 370, eq. (9)], namely

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{(\pm 1)^m(m+\nu)m!}{\Gamma(m+2\nu)}&C_m^{(\nu)}(\cos\theta)C_m^{(\nu)}(\cos\theta')I_{m+\nu}(z)\\ &= \frac{\sqrt{2\pi}}{2^{2\nu}\Gamma(\nu)^2}\frac{I_{\nu-1/2}(z\sin\theta\sin\theta')}{(z\sin\theta\sin\theta')^{\nu-1/2}}\exp(\pm z\cos\theta\cos\theta').\end{aligned}$$

The only significant difference with our sum is the index $m+\nu$ of the Bessel function I instead of $3(m+\nu)$.

7. The Case $n > 3$

When $n > 3$, following Calogero [1971], with slight modifications we can again change variables to

$$\begin{aligned}X &:= \frac{1}{\sqrt{n}}(x_1 + \ldots + x_n)\\ z_j &:= \frac{1}{\sqrt{j(j+1)}}(-x_1 - \cdots - x_j + jx_{j+1}), \quad j = 1, 2, \ldots, n-1.\end{aligned}$$

This is an orthogonal change of variables, so that

$$\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2} = \frac{\partial^2}{\partial X^2} + \sum_{j=1}^{n-1}\frac{\partial^2}{\partial z_j^2}.$$

The "linear derivative terms"

$$\sum_{1\le j<k\le n}(x_j - x_k)^{-1}\Big(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_k}\Big)$$

being independent of the "center of mass" X, one can separate the variable X. One can then change to "polar coordinates"

$$
\begin{aligned}
z_1 &:= r\cos\theta_1, \\
z_j &:= r\sin\theta_1 \ldots \sin\theta_{j-1}\cos\theta_j, \quad 2 \le j \le n-2, \\
z_{n-1} &:= r\sin\theta_1 \ldots \sin\theta_{n-2}, \\
\sum_{j=1}^{n-1}\frac{\partial^2}{\partial z_j^2} &= \frac{\partial^2}{\partial r^2} + \frac{n-2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\nabla^2_{(\theta)}, \\
r^2 &= \frac{1}{n}\sum_{1\le j<k\le n}(x_j - x_k)^2,
\end{aligned}
$$

and separate the variable r. The linear derivative terms having a complicated expression in terms of the remaining variables θ_j, it seems difficult to say something more.

A. Expression of the Laplacian. Proof of Equation (2–12)

For completeness we give here a derivation of the splitting of the laplacian of a matrix in terms of its eigenvalues and the "angle variables".

We first recall some well known results of the general tensor analysis on a riemannian manifold. The line element ds in terms of the nondegenerate positive definite metric tensor $g = [g_{jk}]$ is

$$ds^2 := \sum_{j,k} g_{jk}\, dx_j\, dx_k.$$

Then the volume element or measure $d\mu(\boldsymbol{x})$ and the laplacian ∇^2 read as follows (see [Gouyons 1963, p. 91, § 103], for example):

$$d\mu(\boldsymbol{x}) = \sqrt{\det g}\prod_j dx_j, \tag{A–1}$$

$$\nabla^2 = \sum_{j,k}\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x_j}\sqrt{\det g}\,(g^{-1})_{jk}\frac{\partial}{\partial x_k}. \tag{A–2}$$

As in (2–6), an $n\times n$ real symmetric, complex hermitian or quaternion self-dual matrix A can be diagonalized:

$$A = U\boldsymbol{X}U^{-1}, \tag{A–3}$$

with $\boldsymbol{X}$ real diagonal and U in $\mathcal{G}_\beta$, for $\beta = 1$, 2 or 4 respectively. These matrices A depend on the n real diagonal elements A_{jj} and the β real components of each nondiagonal element A_{jk} for $j<k$. These later parameters, a total of $\beta n(n-1)/2$, will be denoted by A_μ in what follows. The latin indices j, k, ...

will vary from 1 to n, while the greek indices μ, ν, ... will vary from 1 to $\beta n(n-1)/2$. The line element is

$$ds^2 := \operatorname{tr}\ dA^2 \tag{A–4}$$

where dA denotes the variation of a matrix A in the corresponding set of matrices according as $\beta = 1$, 2 or 4. In terms of the variables A_{jj}, A_μ, the line element reads

$$ds^2 = \sum_j dA_{jj}^2 + 2\sum_\mu dA_\mu^2. \tag{A–5}$$

Thus the metric tensor is diagonal and it follows from (A–1) and (A–2) that the measure and the laplacian are respectively

$$d\mu(A) = 2^{\beta n(n-1)/4}\Big(\prod_j dA_{jj}\Big)\Big(\prod_\mu dA_\mu\Big) \tag{A–6}$$

and

$$\nabla^2 = \sum_j \frac{\partial^2}{\partial A_{jj}^2} + \frac{1}{2}\sum_\mu \frac{\partial^2}{\partial A_\mu^2}.$$

Notice that the measure $d\mu(A)$ considered in (2–4) was denoted by dA for briefness (it must not be confused with dA in (A–4)), and furthermore the normalization is different.

Now to specify any matrix A of the corresponding set according as $\beta = 1$, 2 or 4, we take the n real eigenvalues x_j and $\beta n(n-1)/2$ additional real parameters p_μ, i.e., the "angle variables" entering the definition of U in (A–3). Notice that these matrices U, according as $\beta = 1$, 2 or 4, actually depend on $n(n-1)/2$, n^2 or $n(2n-1)$ real parameters respectively. Thus, for $\beta = 2$ or 4 — that is, for the unitary or symplectic group — U depends on n more real parameters than the collection of the variables p_μ. These n parameters correspond to the possibility of multiplying each column k of U by a phase factor $\exp(i\phi_k)$ without changing A in (A–3). From (A–3) the variation of the matrix A reads

$$dA = U\big(d\boldsymbol{X} + (U^{-1}dU)\boldsymbol{X} - \boldsymbol{X}(U^{-1}dU)\big)\,U^{-1}.$$

Then, using the cyclic invariance of the trace and the diagonal character of $\boldsymbol{X}$ and $d\boldsymbol{X}$, the line element (A–4) is

$$ds^2 = \sum_j dx_j^2 + \operatorname{tr}\big((U^{-1}dU)\boldsymbol{X} - \boldsymbol{X}(U^{-1}dU)\big)^2.$$

Notice that

$$dU = \begin{cases} \displaystyle\sum_\mu \frac{\partial U}{\partial p_\mu} dp_\mu & \text{for } \beta = 1, \\ \displaystyle\sum_\mu \frac{\partial U}{\partial p_\mu} dp_\mu + \sum_k \frac{\partial U}{\partial \phi_k} d\phi_k & \text{for } \beta = 2 \text{ or } 4, \end{cases}$$

where for the unitary and symplectic groups, the dependence of the matrix element U_{jk} on the parameter ϕ_k is through the phase factor $\exp(i\phi_k)$ already mentioned. It follows that

$$\left(\frac{\partial U}{\partial \phi_k}\right)_{jl} = iU_{jl}\delta_{lk}$$

and then, a straightforward calculation shows that the variations $d\phi_k$ do not contribute, as expected, to $(U^{-1}dU)\boldsymbol{X} - \boldsymbol{X}(U^{-1}dU)$. Finally, in terms of the variables x_j and p_μ the metric tensor has the block diagonal structure

$$\begin{aligned}
g_{jk} &= \delta_{jk}, \\
g_{j\mu} &= g_{\mu j} = 0, \\
g_{\mu\nu} &= \operatorname{tr}\left(\left(U^{-1}\frac{\partial U}{\partial p_\mu}\boldsymbol{X} - \boldsymbol{X}U^{-1}\frac{\partial U}{\partial p_\mu}\right)\left(U^{-1}\frac{\partial U}{\partial p_\nu}\boldsymbol{X} - \boldsymbol{X}U^{-1}\frac{\partial U}{\partial p_\nu}\right)\right) \\
&= -\sum_{j,k}(x_j - x_k)^2\left(U^{-1}\frac{\partial U}{\partial p_\mu}\right)_{jk}\left(U^{-1}\frac{\partial U}{\partial p_\nu}\right)_{kj}.
\end{aligned}$$

Consequently, the inverse matrix g^{-1} of g, occuring in (A–2), also has a block diagonal structure

$$\left.\begin{aligned}
\left(g^{-1}\right)_{jk} &= \delta_{jk}, \\
\left(g^{-1}\right)_{j\mu} &= \left(g^{-1}\right)_{\mu j} = 0 \\
\left(g^{-1}\right)_{\mu\nu} &= \text{a complicated expression not needed here.}
\end{aligned}\right\} \tag{A–7}$$

We now show that the determinant of the metric tensor, which is positive from (A–5), satisfies

$$\sqrt{\det g} = |\Delta(\boldsymbol{x})|^\beta f(\boldsymbol{p}), \tag{A–8}$$

where f depends only on the p_μ, denoted collectively $\boldsymbol{p}$. Indeed, one shows that the measure $d\mu(A)$, (A–6), is given by

$$d\mu(A) = \left|\frac{\partial(A_{jj}, A_\mu)}{\partial(x_j, p_\mu)}\right|\left(\prod_j dx_j\right)\left(\prod_\mu dp_\mu\right)$$

(see [Mehta 1991, Chapter 3]), where the absolute value of the jacobian is

$$\left|\frac{\partial(A_{jj}, A_\mu)}{\partial(x_j, p_\mu)}\right| = |\Delta(\boldsymbol{x})|^\beta f(\boldsymbol{p}).$$

Using (A–1) this ends the proof of (A–8). Finally, from (A–2), (A–7), and (A–8) one gets the expression (2–12) for the laplacian assuming we restrict our attention to one of the sectors where $\Delta(\boldsymbol{x}) \geq 0$ (the absolute value of $\Delta(\boldsymbol{x})$ no longer occurs).

B. The Schrödinger Equation with the Singular Calogero Potential

We consider the time independent Schrödinger equation (4–3), with Calogero's hamiltonian H defined in (1–2), interpreted as describing a system of n one-dimensional particles. Our aim is to investigate the validity of the assertion that two such particles cannot cross each other. For that, it is sufficient to study the simple case $n = 2$.

Separating the center of mass motion and using the notations of section 5, we write the wave function $\phi(\boldsymbol{x})$ solution of (4–3) as

$$\phi(\boldsymbol{x}) = f(X)\,u(x),$$

where $u(x)$ satisfies the equation

$$u''(x) + \left(\frac{\gamma}{4x^2} + E\right) u(x) = 0 \tag{B–1}$$

and $\gamma = \beta(2-\beta)$. When $\beta = 2$, the singular potential disappears ($\gamma = 0$); when $\beta = 1$, it is attractive ($\gamma = 1$), and when $\beta = 4$, it is repulsive ($\gamma = -8$). In the following, γ is allowed to take any real value.

We are interested in the neighbourhood of the singularity at $x = 0$, so in (B–1) we drop the energy term $E\,u(x)$, negligible in comparison to the potential term. Furthermore, following Landau [1958, § 35, p. 118], we regularize (B–1) by replacing the potential $\gamma/4x^2$ by the constant $\gamma/4x_0^2$ in a small interval $[-x_0, x_0]$ around the origin:

$$\begin{aligned} u''(x) + \frac{\gamma}{4x^2}\,u(x) = 0, &\quad \text{for } |x| > x_0, \\ u''(x) + \frac{\gamma}{4x_0^2}\,u(x) = 0, &\quad \text{for } |x| < x_0. \end{aligned} \tag{B–2}$$

Ultimately, we will let x_0 go to zero.

In the outer region $|x| > x_0$, when $\gamma \neq 1$, we define two linearly independent solutions $u_\pm(x)$ by setting

$$u_+(x) := |x|^{s_+} \quad \text{and} \quad u_-(x) := \pm|x|^{s_-} \quad \text{if } \pm x > x_0, \tag{B–3}$$

where the indices $s_\pm$ are equal to $(1 \pm \sqrt{1-\gamma})/2$ or $(1 \pm i\sqrt{\gamma-1})/2$, according to whether $\gamma < 1$ or $\gamma > 1$. When $\gamma = 1$, the two indices coincide, and we choose the following two linearly independent solutions

$$u_+(x) := |x|^{1/2} \quad \text{and} \quad u_-(x) := \pm|x|^{1/2}\log|x| \quad \text{if } \pm x > x_0, \tag{B–4}$$

(the signs $\pm$ in (B–3) and (B–4) ensure that the wronskian $W(u_+, u_-)$ takes on the same value for $x > x_0$ and $x < -x_0$).

In the inner region $|x| < x_0$, when $\gamma \neq 0$, we define two linearly independent solutions $v_\pm(x)$

$$v_\pm(x) := e^{\pm \kappa x/x_0}, \quad \text{where } \kappa = \begin{cases} \sqrt{-\gamma}/2 & \text{if } \gamma < 0, \\ i\sqrt{\gamma}/2 & \text{if } \gamma > 0. \end{cases}$$

The general solution of (B–2) reads

$$u(x) = \begin{cases} A_+ u_+(x) + A_- u_-(x) & \text{for } x > x_0, \\ B_+ v_+(x) + B_- v_-(x) & \text{for } -x_0 < x < x_0, \\ C_+ u_+(x) + C_- u_-(x) & \text{for } x < -x_0. \end{cases}$$

The constants $A_\pm$, $B_\pm$ and $C_\pm$ are related by the continuity conditions of $u(x)$ and $u'(x)$ at both points x_0 and $-x_0$. The only relations of interest are the two connecting $A_\pm$ to $C_\pm$, which we write

$$\begin{pmatrix} C_+ \\ C_- \end{pmatrix} = M \begin{pmatrix} A_+ \\ A_- \end{pmatrix}. \tag{B–5}$$

Here, M is a 2×2 unimodular matrix which depends on x_0. Straightforward calculations lead to the following expression, valid when $\gamma \neq 0$ and 1:

$$M = \begin{pmatrix} a & b\, x_0^{s_- - s_+} \\ c\, x_0^{s_+ - s_-} & a \end{pmatrix},$$

where a, b and c do not depend on x_0:

$$\begin{pmatrix} a & b \\ c & a \end{pmatrix} = \frac{1}{2\kappa(s_+ - s_-)} \begin{pmatrix} s_- + \kappa & s_- - \kappa \\ s_+ + \kappa & s_+ - \kappa \end{pmatrix} \begin{pmatrix} e^{-2\kappa} & 0 \\ 0 & e^{2\kappa} \end{pmatrix} \begin{pmatrix} -s_+ - \kappa & -s_- - \kappa \\ s_+ - \kappa & s_- - \kappa \end{pmatrix}. \tag{B–6}$$

Equation (B–5) can be rewritten conveniently as

$$\begin{pmatrix} A_- \\ C_- \end{pmatrix} = \frac{1}{b}\, x_0^{s_+ - s_-} \begin{pmatrix} -a & 1 \\ -1 & a \end{pmatrix} \begin{pmatrix} A_+ \\ C_+ \end{pmatrix}. \tag{B–7}$$

The x_0 dependence is now contained in a global factor $x_0^{s_+ - s_-}$. Indeed, a simple dimensional argument leads directly to this result.

When $\gamma = 1$, similar calculations lead to the following formula (we have dropped terms which are negligible when $x_0 \to 0$)

$$\begin{pmatrix} A_- \\ C_- \end{pmatrix} \simeq \begin{pmatrix} -1/\log x_0 & 1/(\log x_0)^2 \cos 1 \\ 1/\log x_0 & -1/(\log x_0)^2 \cos 1 \end{pmatrix} \begin{pmatrix} A_+ \\ C_+ \end{pmatrix}. \tag{B–8}$$

We now let x_0 go to zero.

When $\gamma < 1$, the difference $s_+ - s_- = \sqrt{1-\gamma}$ is positive, and the right-hand side of (B–7) goes to zero with x_0. Similarly, the right-hand side of (B–8) vanishes in the same limit. Thus, when $\gamma \leq 1$, the constants A_- and C_- vanish with x_0, and A_+ and C_+ are the two arbitrary integration constants of the problem. Consequently, two particular linearly independent solutions are, in this limit, $\theta(x)u_+(x)$ and $\theta(-x)u_+(x)$, which are localised respectively in the sectors $x \geq 0$

and $x \leq 0$. This proves that *when the singular interaction $\gamma/(x_1 - x_2)^2$ between two one-dimensional particles is either repulsive or weakly attractive ($\gamma \leq 1$), the particles cannot cross each other.*

When $\gamma > 1$, the difference $s_+ - s_- = i\sqrt{\gamma - 1}$ is pure imaginary, and the right-hand side of (B–7) has no limit when $x_0 \to 0$: it oscillates indefinitely. Indeed, in that case, namely *when the interaction $\gamma/(x_1 - x_2)^2$ is strongly attractive, the two particles collapse.* The argument is exactly the one developed in reference [Landau and Lifshitz 1958], for a three-dimensional particle in a central potential const$/r^2$, and it will not be reproduced here.

As a final remark, we raise the question of the independence of these results with respect to the regularization process. We just note that if, choosing a different regularization, we replace in the small interval $[-x_0, x_0]$ the potential $\gamma/4x^2$ by a constant τ^2 independent of x_0 (which induces discontinuities in the potential), (B–6) and (B–7) are still valid once κ has been replaced by τx_0. The quantities a, b and c now depend on x_0, but they have a finite limit when $x_0 \to 0$, and nothing is changed in the above conclusions.

Acknowledgements

We are thankful to R. Balian for indicating us the use of (6–7) to evaluate the sum in (6–6) for the case $\beta = 2$.

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