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Integrable Lattices: Random Matrices and Random Permutations

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Introduction

The purpose of this article is to survey recent interactions between statistical questions and integrable theory. Two types of questions will be tackled here:

(i) Consider a random ensemble of matrices, with certain symmetry conditions to guarantee the reality of the spectrum and subjected to a given statistics. What is the probability that all its eigenvalues belong to a given subset E? What happens, when the size of the matrices gets very large? The probabilities here are functions of the boundary points c_i of E.

(ii) What is the statistics of the length of the largest increasing sequence in a random permutation, assuming each permutation is equally probable? Here, one considers generating functions (over the size of the permutations) for the probability distributions, depending on the variable x.

The main emphasis of this article is to show that integrable theory serves as a useful tool for finding equations satisfied by these functions of x, and conversely the probabilities point the way to new integrable systems.

These questions are all related to integrals over spaces of matrices. Such spaces can be classical Lie groups or algebras, symmetric spaces or their tangent spaces. In infinite-dimensional situations, the "∞-fold" integrals get replaced by Fredholm determinants.

During the last decade, astonishing discoveries have been made in a variety of directions. A first striking feature is that these probabilities are all related to Painlevé equations or interesting generalizations. In this way, new and unusual distributions have entered the statistical world.

Another feature is that each of these problems is related to some integrable hierarchy. Indeed, by inserting an infinite set of time variables $t_1, t_2, t_3, \dots$ in the integrals or Fredholm determinants — e.g., by introducing appropriate exponentials $e^{\sum_1^\infty t_i y^i}$ in the integral — this probability, as a function of $t_1, t_2, t_3, \dots$, satisfies an integrable hierarchy. Korteweg-de Vries, KP, Toda lattice equations are only a few examples of such integrable equations.

Typically integrable systems can be viewed as isospectral deformations of differential or difference operators $\mathcal{L}$. Perhaps, one of the most startling discoveries of integrable theory is that $\mathcal{L}$ can be expressed in terms of a single "τ-function" $\tau(t_1, t_2, \dots)$ (or vector of τ-functions), which satisfy an infinite set of nonlinear equations, encapsulated in a single "*bilinear identity*". The t_i account for the commuting flows of this integrable hierarchy. In this way, many interesting classical functions live under the same hat: characters of representations, Θ-functions of algebraic geometry, hypergeometric functions, certain integrals over classical Lie algebras or groups, Fredholm determinants, arising in statistical mechanics, in scattering and random matrix theory! They are all special instances of "*τ-functions*".

The point is that the probabilities or generating functions above, as functions of $t_1, t_2, \dots$ (after some minor renormalization) are precisely such τ-functions for the corresponding integrable hierarchy and thus automatically satisfy a large set of equations.

These probabilities are very special τ-functions: they happen to be a solution of yet another hierarchy of (linear) equations in the variables t_i and the boundary points c_i, namely $\mathbb{J}_k^{(2)}\tau(t;c) = 0$, where the $\mathbb{J}_k^{(2)}$ form — roughly speaking — a Virasoro-like algebra:

$$\left[\mathbb{J}_k^{(2)}, \mathbb{J}_l^{(2)}\right] = (k-l)\,\mathbb{J}_{k+l}^{(2)} + \cdots.$$

Each integrable hierarchy has a natural "*vertex operator*", which automatically leads to a natural Virasoro algebra. Then, eliminating the partial derivatives in t from the two hierarchy of equations, the integrable and the Virasoro hierarchies, and finally setting $t = 0$, lead to PDEs or ODEs satisfied by the probabilities.

Table 1 gives an overview of the different problems discussed in this article, the relevant integrals in the second column and the different hierarchies satisfied by the integrals. To fix notation, $\mathcal{H}_l$, $\mathcal{S}_l$, $\mathcal{T}_l$ refer to the Hermitian, symmetric and symplectic ensembles, populated respectively by $l \times l$ Hermitian matrices, symmetric matrices and self-dual Hermitian matrices, with quaternionic entries. $\mathcal{H}_l(E)$, $\mathcal{S}_l(E)$, $\mathcal{T}_l(E)$ are the corresponding set of matrices, with all spectral points belonging to E. $U(l)$ and $O(l)$ are the unitary and orthogonal groups respectively. In Table 1 we have $V_t(z) := V_0(z) + \sum t_i z^i$, where $V_0(z)$ stands for the unperturbed problem; in the last integral $\tilde{V}_t(z)$ is a more complicated function of $t_1, t_2, \ldots$ and z, to be specified later.

1. Matrix Integrals, Random Matrices and Permutations

1.1. Tangent Space to Symmetric Spaces and Associated Random Matrix Ensembles. Random matrices provided a model for excitation spectra of heavy nuclei at high excitations (Wigner [74], Dyson [27] and Mehta [49]), based on the nuclear experimental data by Porter and Rosenzweig [56]; they observed that the occurrence of two levels, close to each other, is a rare event (level repulsion), showing that the spacing is not Poissonian, as one might expect from a naive point of view.

Random matrix ideas play an increasingly prominent role in mathematics: not only have they come up in the spacings of the zeroes of the Riemann zeta function, but their relevance has been observed in the chaotic Sinai billiard and, more generally, in chaotic geodesic flows. Chaos seems to lead to the "spectral rigidity", typical of the spectral distributions of random matrices, whereas the spectrum of an integrable system is random (Poisson)! (e.g., see Odlyzko [53] and Sarnak [59]).

All these problems have led to three very natural random matrix ensembles: Hermitian, symmetric and symplectic ensembles. The purpose of this section is to show that these three examples appear very naturally as tangent spaces to symmetric spaces.

Probability problem	underlying t-perturbed integral, τ-function of $\longrightarrow$	corresponding integrable hierarchies
$P(M \in \mathcal{H}_n(E))$	$\int_{\mathcal{H}_n(E)} e^{\operatorname{Tr}(-V(M)+\sum_1^\infty t_i M^i)}\, dM$	Toda lattice KP hierarchy
$P(M \in \mathcal{S}_n(E))$	$\int_{\mathcal{S}_n(E)} e^{\operatorname{Tr}(-V(M)+\sum_1^\infty t_i M^i)}\, dM$	Pfaff lattice Pfaff-KP hierarchy
$P(M \in \mathcal{T}_n(E))$	$\int_{\mathcal{T}_n(E)} e^{\operatorname{Tr}(-V(M)+\sum_1^\infty t_i M^i)}\, dM$	Pfaff lattice Pfaff-KP hierarchy
$P((M_1, M_2) \in$ $\mathcal{H}_n(E_1) \times \mathcal{H}_n(E_2))$	$\int_{\mathcal{H}_n^2(E)} dM_1\, dM_2$ $e^{-\operatorname{Tr}(V_t(M_1)-V_s(M_2)-cM_1M_2)}$	2d-Toda lattice KP-hierarchy
$P(M \in \mathcal{H}_\infty(E))$	$\det(I - K_t(y,z) I_{E^c}(z))$ (Fredholm determinant)	KdV equation
longest increasing sequence in random permutations	$\int_{U(l)} e^{\operatorname{Tr}\sum_1^\infty (t_i M^i - s_i \bar{M}^i)}\, dM$	Toeplitz lattice 2d-Toda lattice
longest increasing sequence in random involutions	$\int_{O(l)} e^{\operatorname{Tr}(xM + \tilde{V}_t(M))}\, dM$	Toda lattice KP-hierarchy

Table 1. Overview of the article: problems discussed, relevant integrals (second column), and the hierarchies satisfied by the integrals (last column).

A symmetric space G/K is given by a semisimple Lie group G and a Lie group involution $\sigma : G \to G$ such that

$$K = \{x \in G,\, \sigma(x) = x\}.$$

Then the following identification holds:

$$G/K \cong \{g\sigma(g)^{-1} \quad \text{with } g \in G\},$$

and the involution σ induces a map of the Lie algebra,

$$\sigma_* : \mathfrak{g} \longrightarrow \mathfrak{g}, \quad \text{such that } {\sigma_*}^2 = 1,$$

where

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \quad \text{with } \mathfrak{k} = \{a \in \mathfrak{g} \mid \sigma_*(a) = a\} \text{ and } \mathfrak{p} = \{a \in \mathfrak{g} \mid \sigma_*(a) = -a\},$$

and

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

Then K acts on $\mathfrak{p}$ by conjugation: $k\mathfrak{p}k^{-1} \subset \mathfrak{p}$ for all $k \in K$ and $\mathfrak{p}$ is the tangent space to G/K at the identity. The action of K on $\mathfrak{p}$ induces a root space decomposition, with $\mathfrak{a}$ being a maximal abelian subalgebra in $\mathfrak{p}$:

$$\mathfrak{p} = \mathfrak{a} + \sum_{\alpha \in \Delta} \mathfrak{p}_\alpha, \quad \text{with } m_\alpha = \dim \mathfrak{p}_\alpha.$$

Then, according to Helgason [35], the volume element on $\mathfrak{p}$ is

$$dV = \left(\prod_{\alpha \in \Delta_+} \alpha(z)^{m_\alpha} \right) dz_1 \dots dz_n,$$

where Δ_+ is the set of positive roots; see [35; 36; 60; 61]. This will subsequently be worked out for the three so-called A_n-symmetric spaces. I like to thank Chuu-Lian Terng for very helpful conversations on these matters.

Examples

(i) Hermitian ensemble. Consider the *non-compact symmetric space*[1]

$$\mathrm{SL}(n, \mathbb{C}) / \mathrm{SU}(n)$$

with $\sigma(g) = \bar{g}^{\top -1}$. Then

$$\begin{aligned} \mathrm{SL}(n, \mathbb{C}) / \mathrm{SU}(n) &= \{ g\bar{g}^\top \mid g \in \mathrm{SL}(n, \mathbb{C}) \} \\ &= \{\text{positive definite matrices with } \det = 1\} \end{aligned}$$

with

$$K = \{ g \in \mathrm{SL}(n, \mathbb{C}) \mid \sigma(g) = g \} = \{ g \in \mathrm{SL}(n, \mathbb{C}) \mid g^{-1} = \bar{g}^\top \} = \mathrm{SU}(n).$$

Then $\sigma_*(a) = -\bar{a}^\top$ and the tangent space to G/K is then given by the space $\mathfrak{p} = \mathcal{H}_n$ of Hermitian matrices

$$\mathrm{sl}(n, \mathbb{C}) = \mathfrak{k} \oplus \mathfrak{p} = \mathrm{su}(n) \oplus \mathcal{H}_n \quad (\text{i.e., } a = a_1 + a_2 \text{ with } a_1 \in \mathrm{su}(n),\ a_2 \in \mathcal{H}_n).$$

If $M \in \mathcal{H}_n$, then the M_{ii}, $\operatorname{Re} M_{ij}$ and $\operatorname{Im} M_{ij}$ $(1 \le i < j \le n)$ are free variables, so that Haar measure on $M \in \mathcal{H}_n$ takes on the form

$$dM := \prod_1^n dM_{ii} \prod_{1 \le i < j \le n} (d \operatorname{Re} M_{ij}\, d \operatorname{Im} M_{ij}). \tag{1.1.1}$$

A maximal abelian subalgebra $a \subset \mathfrak{p} = \mathcal{H}_n$ is given by real diagonal matrices $z = \operatorname{diag}(z_1, \dots, z_n)$. Each $M \in \mathfrak{p} = \mathcal{H}_n$ can be written as

$$M = e^A z\, e^{-A}, \quad e^A \in K = \mathrm{SU}(n),$$

with[2]

$$A = \sum_{1 \le k \le l \le n} (a_{kl}(e_{kl} - e_{lk}) + i b_{kl}(e_{kl} + e_{lk})) \in \mathfrak{k} = \mathrm{su}(n), \quad a_{ll} = 0. \tag{1.1.2}$$

Notice that $e_{kl} - e_{lk}$ and $i(e_{kl} + e_{lk})$ belong to $\mathfrak{k} = \mathrm{su}(n)$ and that

$$\begin{aligned} [e_{kl} - e_{lk},\, z] &= (z_l - z_k)(e_{kl} + e_{lk}) \in \mathfrak{p} = \mathcal{H}_n, \\ [i(e_{kl} + e_{lk}),\, z] &= (z_l - z_k) i (e_{kl} - e_{lk}) \in \mathfrak{p} = \mathcal{H}_n. \end{aligned} \tag{1.1.3}$$

[1] The corresponding compact symmetric space is $(\mathrm{SU}(n) \times \mathrm{SU}(n)) / \mathrm{SU}(n)$.

[2] e_{kl} is the $n \times n$ matrix with all zeroes, except for 1 at the (k, l)-th entry.

Incidentally, this implies that $e_{kl}+e_{lk}$ and $i(e_{kl}+e_{lk})$ are two-dimensional eigenspaces of $(\mathrm{ad}\ z)^2$ (where ad is defined by $\mathrm{ad}\ x(y):=[x,y]$) with eigenvalue $(z_l-z_k)^2$. From (1.1.2) and (1.1.3) it follows that

$$[A,z]=(z_l-z_k)\sum_{1\le k<l\le n}(a_{kl}(e_{kl}+e_{lk})+ib_{kl}(e_{kl}-e_{lk}))\in\mathfrak{p}=\mathcal{H}_n \tag{1.1.4}$$

and thus, for small A, we have[3]

$$\begin{aligned} dM&=d(e^A z\, e^{-A})=d(z+[A,z]+\cdots)\\ &=\prod_1^n dz_i\prod_{1\le k<l\le n}d((z_l-z_k)a_{kl})\,d((z_l-z_k)b_{kl})\qquad\text{using (1.1.4) and (1.1.1)}\\ &=\prod_1^n dz_i\Delta_n^2(z)\prod_{1\le k<l\le n}da_{kl}\,db_{kl}. \end{aligned}\tag{1.1.5}$$

Therefore $\Delta^2(z)$ is also the Jacobian determinant of the map $M\to(z,U)$, such that $M=UzU^{-1}\in\mathcal{H}_n$, and thus dM admits the decomposition in polar coordinates:

$$dM=\Delta_n^2(z)\,dz_1\dots dz_n\,dU,\quad U\in\mathrm{SU}(n). \tag{1.1.6}$$

In random matrix theory, $\mathcal{H}_n$ is endowed with the following probability,

$$P(M\in dM)=c_ne^{-trV(M)}\,dM,\quad \rho(dz)=e^{-V(z)}\,dz, \tag{1.1.7}$$

where dM is Haar measure (1.1.6) on $\mathcal{H}_n$ and c_n is the normalizing factor. Since dM as in (1.1.6) contains dU and since the probability measure (1.1.7) only depends on the trace of $V(M)$, dU completely integrates out. Given $E\subset\mathbb{R}$, define

$$\mathcal{H}_n(E):=\{M\in\mathcal{H}_n\text{with all spectral points}\in E\subset\mathbb{R}\}\subset\mathcal{H}_n. \tag{1.1.8}$$

Then

$$P(M\in\mathcal{H}_n(E))=\int_{\mathcal{H}_n(E)}c_ne^{-\operatorname{Tr}V(M)}\,dM=\frac{\int_{E^n}\Delta^2(z)\prod_1^n\rho(dz_k)}{\int_{\mathbb{R}^n}\Delta^2(z)\prod_1^n\rho(dz_k)}. \tag{1.1.9}$$

As explained in the excellent book by Mehta [49], $V(z)$ is quadratic (Gaussian ensemble) if the probability $P(M\in dM)$ satisfies

(i) invariance under conjugation by unitary transformations $M\mapsto UMU^{-1}$, and
(ii) the condition that the random variables M_{ii}, $\operatorname{Re}M_{ij}$, and $\operatorname{Im}M_{ij}$, for $1\le i<j\le n$, be independent.

[3] $\Delta_n(z)=\prod_{1\le i<j\le n}(z_i-z_j)$ is the Vandermonde determinant.

(ii) Symmetric ensemble. Here we consider the *non-compact symmetric space*[4] $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ with $\sigma(g) = g^{\top -1}$. Then

$$\begin{aligned}\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n) &= \{gg^\top \mid g \in \mathrm{SL}(n,\mathbb{R})\} \\ &= \{\text{positive definite matrices with } \det = 1\}\end{aligned}$$

with

$$K = \{g \in \mathrm{SL}(n,\mathbb{R}) \mid \sigma(g) = g\} = \{g \in \mathrm{SL}(n,\mathbb{R}) \mid g^\top = g^{-1}\} = \mathrm{SO}(n).$$

Then $\sigma_*(a) = -a^\top$ and the tangent space to G/K is then given by the space $\mathfrak{p} = \mathcal{S}_n$ of symmetric matrices appearing in the decomposition of $\mathrm{sl}(n,\mathbb{R})$,

$$\mathrm{sl}(n,\mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p} = \mathrm{so}(n) \oplus \mathcal{S}_n \quad (\text{i.e., } a = a_1 + a_2 \text{ with } a_1 \in \mathrm{so}(n),\ a_2 \in \mathcal{S}_n),$$

with Haar measure $dM = \prod\limits_{1\le i\le j\le n} dM_{ij}$ on $\mathcal{S}_n$.

A maximal abelian subalgebra $a \subset \mathfrak{p} = \mathcal{S}_n$ is given by real traceless diagonal matrices $z = \mathrm{diag}(z_1, \ldots, z_n)$. Each $M \in \mathfrak{p} = \mathcal{S}_n$ conjugates to a diagonal matrix z:

$$M = e^A z\, e^{-A}, \quad e^A \in K = \mathrm{SO}(n), \quad A \in \mathrm{so}(n).$$

A calculation analogous to example (1.1.5)(i) leads to

$$dM = \left|\Delta_n(z)\right| dz_1 \ldots dz_n\, dU, \quad U \in \mathrm{SO}(n).$$

Random matrix theory deals with the following probability on $\mathcal{S}_n$:

$$P(M \in dM) = c_n e^{-trV(M)}\, dM, \quad \rho(dz) = e^{-V(z)}\, dz, \tag{1.1.10}$$

with normalizing factor c_n. Setting as in (1.1.8): $\mathcal{S}_n(E) \subset \mathcal{S}_n$ is the subset of matrices with spectrum $\in E$. Then

$$P(M \in \mathcal{S}_n(E)) = \int_{\mathcal{S}_n(E)} c_n e^{-\operatorname{Tr} V(M)}\, dM = \frac{\int_{E^n} |\Delta(z)| \prod_1^n \rho(dz_k)}{\int_{\mathbb{R}^n} |\Delta(z)| \prod_1^n \rho(dz_k)}. \tag{1.1.11}$$

As in the Hermitian case, $P(M \in dM)$ is Gaussian, if $P(M \in dM)$ satisfies

(i) invariance under conjugation by orthogonal conjugation $M \to OMO^{-1}$, and
(ii) the condition that M_{ii}, M_{ij} $(i < j)$ be independent random variables.

[4]The compact version is given by $\mathrm{SU}(n)/\mathrm{SO}(n)$.

(iii) Symplectic ensemble. Consider the *non-compact symmetric space*[5]

$$\mathrm{SU}^*(2n)/\,\mathrm{USp}(n)$$

with $\sigma(g) = Jg^{\top -1}J^{-1}$, where J is the $2n \times 2n$ matrix

$$J := \begin{pmatrix} \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & & & \\ & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & & \Large 0 \\ & & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & \\ \Large 0 & & & \ddots \end{pmatrix} \quad \text{with } J^2 = -I, \tag{1.1.12}$$

and

$$\begin{aligned} G &= \mathrm{SU}^*(2n) = \{g \in \mathrm{SL}(2n, \mathbb{C}) \mid g = J\bar{g}J^{-1}\}, \\ K &= \{g \in \mathrm{SU}^*(2n) \mid \sigma(g) = g\} := \mathrm{Sp}(n, \mathbb{C}) \cap U(2n) \\ &= \big\{g \in \mathrm{SL}(2n, \mathbb{C}) \bigm| g^\top J g = J\big\} \cap \big\{g \in \mathrm{SL}(2n, \mathbb{C}) \bigm| g^{-1} = \bar{g}^\top\big\} \\ &= \big\{g \in \mathrm{SL}(2n, \mathbb{C}) \bigm| g^{-1} = \bar{g}^\top \quad \text{and} \quad g = J\bar{g}J^{-1}\big\} \\ &=: \mathrm{USp}(n). \end{aligned}$$

Then $\sigma_*(a) = -Ja^\top J^{-1}$ and

$$\begin{aligned} \mathfrak{k} &= \{a \in \mathrm{su}^*(2n) \bigm| \sigma_*(a) = a\} = \mathrm{sp}(n, \mathbb{C}) \cap u(2n) \\ &= \{a \in \mathbb{C}^{2n \times 2n} \bigm| a^\top = -\bar{a},\, a = J\bar{a}J^{-1}\}, \end{aligned}$$

$$\begin{aligned} \mathfrak{p} &= \{a \in \mathrm{su}^*(2n) \bigm| \sigma_*(a) = -a\} = \mathrm{su}^*(2n) \cap iu(2n) \\ &= \{a \in \mathbb{C}^{2n \times 2n} \bigm| a^\top = \bar{a},\, a = J\bar{a}J^{-1}\} \\ &= \left\{ M = (M_{kl})_{1 \le k,l \le n}, M_{kl} = \begin{pmatrix} M_{kl}^{(0)} & M_{kl}^{(1)} \\ -\bar{M}_{kl}^{(1)} & \bar{M}_{kl}^{(0)} \end{pmatrix} \text{ with } M_{lk} = \bar{M}_{kl}^\top \in \mathbb{C}^{2\times 2} \right\} \\ &\cong \{\text{self-dual } n \times n \text{ Hermitian matrices, with quaternionic entries}\} \\ &=: \mathcal{T}_{2n}. \end{aligned}$$

The condition on the 2×2 matrices M_{kl} implies that $M_{kk} = M_k I$, with $M_k \in \mathbb{R}$ and the 2×2 identity I. Notice that $\mathrm{USp}(n)$ acts naturally by conjugation on the tangent space $\mathfrak{p}$ to G/K. Haar measure on $\mathcal{T}_{2n}$ is given by

$$dM = \prod_1^n dM_k \prod_{1 \le k < l \le n} dM_{kl}^{(0)}\, d\bar{M}_{kl}^{(0)}\, dM_{kl}^{(1)}\, d\bar{M}_{kl}^{(1)}, \tag{1.1.13}$$

[5]The corresponding compact symmetric space is $\mathrm{SU}(2n)/\,\mathrm{Sp}(n)$.

since these M_{ij} are the only free variables in the matrix $M \in \mathcal{T}_{2n}$. A maximal abelian subalgebra in $\mathfrak{p}$ is given by real diagonal matrices of the form $z = \operatorname{diag}(z_1, z_1, z_2, z_2, \ldots, z_n, z_n)$. Each $M \in \mathfrak{p} = \mathcal{T}_{2n}$ can be written as

$$M = e^A z e^{-A}, \quad e^A \in K = \mathrm{USp}(n), \tag{1.1.14}$$

with $(a_{kl}, b_{kl}, c_{kl}, d_{kl} \in \mathbb{R})$

$$A = \sum_{1 \le k \le l \le n} a_{kl}(e^{(0)}_{kl} - e^{(0)}_{lk}) + b_{kl}(e^{(1)}_{kl} + e^{(1)}_{lk}) + c_{kl}(e^{(2)}_{kl} - e^{(2)}_{lk}) + d_{kl}(e^{(3)}_{kl} + e^{(3)}_{lk}) \in \mathfrak{k} \tag{1.1.15}$$

in terms of the four 2×2 matrices[6]

$$e^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e^{(1)} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e^{(2)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e^{(3)} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Since

$$\begin{aligned}
\left[e^{(0)}_{kl} - e^{(0)}_{lk}, z\right] &= (z_l - z_k)(e^{(0)}_{kl} + e^{(0)}_{lk}) \in \mathfrak{p}, \\
\left[e^{(1)}_{kl} + e^{(1)}_{lk}, z\right] &= (z_l - z_k)(e^{(1)}_{kl} - e^{(1)}_{lk}) \in \mathfrak{p}, \\
\left[e^{(2)}_{kl} - e^{(2)}_{lk}, z\right] &= (z_l - z_k)(e^{(2)}_{kl} + e^{(2)}_{lk}) \in \mathfrak{p}, \\
\left[e^{(3)}_{kl} + e^{(3)}_{lk}, z\right] &= (z_l - z_k)(e^{(3)}_{kl} - e^{(3)}_{lk}) \in \mathfrak{p},
\end{aligned} \tag{1.1.16}$$

$[A, z] \in \mathfrak{p}$ has 2×2 zero blocks along the diagonal, and from (1.1.16) and (1.1.15),

$$\big((k,l)\text{-th block in } [A,z]\big) = (z_l - z_k) \begin{pmatrix} a_{kl} + ib_{kl} & c_{kl} + id_{kl} \\ -c_{kl} + id_{kl} & a_{kl} - ib_{kl} \end{pmatrix}, \quad k < l. \tag{1.1.17}$$

Therefore, using (1.1.17), Haar measure dM on $\mathcal{T}_{2n}$ equals

$$\begin{aligned}
dM &= d(e^A z e^{-A}) = d(I + A + \cdots) z (I - A + \cdots) = d(z + [A,z] + \cdots) \\
&= \prod_{1 \le k \le n} dz_k \prod_{1 \le k < l \le n} d((z_l - z_k)(a_{kl} + ib_{kl})) d((z_l - z_k)(a_{kl} - ib_{kl})) \\
&\quad \times d((z_l - z_k)(c_{kl} + id_{kl})) d((z_l - z_k)(-c_{kl} + id_{kl})) \\
&= \Delta^4(z)\, dz_1 \cdots dz_n \prod_{1 \le k < l \le n} 4\, da_{kl}\, db_{kl}\, dc_{kl}\, dd_{kl}.
\end{aligned}$$

As before, define $\mathcal{T}_{2n}(E) \subset \mathcal{T}_{2n}$ as the subset of matrices with spectrum $\in E$ and define the probability

$$P(M \in \mathcal{T}_{2n}(E)) = \int_{\mathcal{T}_{2n}(E)} c_n e^{-\operatorname{Tr} V(M)}\, dM = \frac{\int_{E^n} \Delta^4(z) \prod_1^n \rho(dz_k)}{\int_{\mathbb{R}^n} \Delta^4(z) \prod_1^n \rho(dz_k)}. \tag{1.1.18}$$

Remark. $\mathcal{T}_{2n}$ is called the symplectic ensemble, although the matrices in $\mathfrak{p} = \mathcal{T}_{2n}$ are not at all symplectic; rather, it's the matrices in $\mathfrak{k}$ that are.

[6] The notation $e^{(i)}_{kl}$ in (1.1.15) refers to putting the 2×2 matrix $e^{(i)}$ at place (k, l).

1.2. Infinite Hermitian Matrix Ensembles. Now consider the limit of the probability

$$P(M \in \mathcal{H}_n(E)) = \frac{\int_{E^n} \Delta^2(z) \prod_1^n \rho(dz_k)}{\int_{\mathbb{R}^n} \Delta^2(z) \prod_1^n \rho(dz_k)} \quad \text{when } n \nearrow \infty. \tag{1.2.1}$$

Dyson [27] (see also Mehta [49]) used the following trick to circumvent the problem of dealing with ∞-fold integrals. Using the orthogonality of the *monic orthogonal polynomials* $p_k = p_k(z)$ for the weight $\rho(dz)$ on $\mathbb{R}$, and the L^2-norms $h_k = \int_{\mathbb{R}} p_k^2(z)\rho(dz)$ of the p_k's, one finds, using $(\det A)^2 = \det(AA^\top)$,

$$\begin{aligned}\int_{\mathbb{R}^n} \Delta^2(z) \prod_1^n \rho(dz_i) &= \int_{\mathbb{R}^n} \det(p_{i-1}(z_j))_{1\le i,j\le n} \det(p_{k-1}(z_l))_{1\le k,l\le n} \prod_{k=1}^n \rho(dz_k)\\ &= \sum_{\pi,\pi'\in\sigma_n} (-1)^{\pi+\pi'} \prod_{k=1}^n \int_{\mathbb{R}} p_{\pi(k)-1}(z_k) p_{\pi'(k)-1}(z_k)\rho(dz_k)\\ &= n! \prod_0^{n-1} \int_{\mathbb{R}} p_k^2(z)\rho(dz) = n! \prod_0^{n-1} h_k. \end{aligned}\tag{1.2.2}$$

For the integral over an arbitrary subset $E \subset \mathbb{R}$, one stops at the second equality, since the p_n's are not necessarily orthogonal over E. This leads to the probability (1.2.1),

$$\begin{aligned}P(M \in \mathcal{H}_n(E)) &= \frac{1}{n! \prod_1^n h_{i-1}} \int_{E^n} \det\Biggl(\sum_{1\le j\le n} p_{j-1}(z_k)p_{j-1}(z_l)\Biggr)_{1\le k,l\le n} \prod_1^n \rho(dz_i)\\ &= \frac{1}{n!} \int_{E^n} \det(K_n(z_k,z_l))_{1\le k,l\le n} \prod_1^n \rho(dz_i), \end{aligned}\tag{1.2.3}$$

in terms of the kernel

$$K_n(y,z) := \sum_{j=1}^n \frac{p_{j-1}(y)}{\sqrt{h_{j-1}}} \frac{p_{j-1}(z)}{\sqrt{h_{j-1}}}. \tag{1.2.4}$$

The orthonormality relations of the $p_k(y)/\sqrt{h_k}$ lead to the reproducing property for the kernel $K_n(y,z)$:

$$\int_{\mathbb{R}} K_n(y,z)K_n(z,u)\rho(dz) = K_n(y,u), \qquad \int_{\mathbb{R}} K_n(z,z)\rho(dz) = n. \tag{1.2.5}$$

Upon replacing E^n by $\prod_1^k dz_i \times \mathbb{R}^{n-k}$ in (1.2.3), integrating out all the remaining variables $z_{k+1},\dots,z_n$ and using the reproducing property (1.2.5), one finds the n-point correlation function

$$\begin{aligned}&P(\text{one eigenvalue in each } [z_i,\, z_i+dz_i], \text{ for } i=1,\dots,k)\\ &\qquad = c_n \det\bigl(K_n(z_i,z_j)\bigr)_{1\le i,j\le k} \prod_1^k \rho(dz_i). \end{aligned}\tag{1.2.6}$$

Finally, by Poincaré's formula for the probability $P(\cup E_i)$, the probability that no spectral point of M belongs to E is given by a Fredholm determinant

$$\begin{aligned} P(M \in \mathcal{H}_n(E^c)) &= \det(I - \lambda K_n^E) \\ &= 1 + \sum_{k=1}^{\infty} (-\lambda)^k \int_{z_1 \leq \ldots \leq z_k} \det\left(K_n^E(z_i, z_j)\right)_{1 \leq i,j \leq k} \prod_1^k \rho(dz_i), \end{aligned}$$

for the kernel $K_n^E(y,z) = K_n(y,z) I_E(z)$.

• *Wigner's semicircle law*: For this ensemble (defined by a large class of ρ's, in particular for the Gaussian ensemble) and for very large n, the density of eigenvalues tends to Wigner's semicircle distribution on the interval $[-\sqrt{2n}, \sqrt{2n}]$:

$$\text{density of eigenvalues} = \begin{cases} \dfrac{1}{\pi}\sqrt{2n - z^2}\, dz & \text{for } |z| \leq \sqrt{2n}, \\ 0 & \text{for } |z| > \sqrt{2n} \end{cases}$$

• *Bulk scaling limit*: From the formula above, it follows that the average number of eigenvalues per unit length near $z = 0$ ("the bulk") is given by $\sqrt{2n}/\pi$ and thus the average distance between two consecutive eigenvalues is given by $\pi/\sqrt{2n}$. Upon using this rescaling, one shows (see [43; 48; 52; 55; 39]) that

$$\lim_{n \nearrow \infty} \frac{\pi}{\sqrt{2n}} K_n\left(\frac{\pi x}{\sqrt{2n}}, \frac{\pi y}{\sqrt{2n}}\right) = \frac{\sin \pi(x-y)}{\pi(x-y)} \qquad \text{(sine kernel)}$$

and

$$P(\text{exactly } k \text{ eigenvalues in } [0,a]) = \frac{(-1)^k}{k!}\left(\frac{\partial}{\partial \lambda}\right)^k \det(I - \lambda K I_{[0,a]})\Big|_{\lambda=1}$$

with

$$\det(I - \lambda K I_{[0,a]}) = \exp \int_0^{\pi a} \frac{f(x;\lambda)}{x}\, dx, \tag{1.2.7}$$

where $f(x,\lambda)$ is a solution to the following differential equation (where the prime stands for differentiation with respect to x), due to the pioneering work of Jimbo, Miwa, Mori, and Sato [39]:

$$(xf'')^2 = 4(xf' - f)(-f'^2 - xf' + f), \quad \text{with } f(x;\lambda) \cong -\frac{\lambda}{\pi} x \text{ for } x \simeq 0. \qquad \textbf{(Painlevé V)} \tag{1.2.8}$$

• *Soft edge scaling limit*: Near the edge $\sqrt{2n}$ of the Wigner semicircle, the scaling is $\sqrt{2}n^{1/6}$ and thus the scaling is more subtle (see [21; 30; 51; 49; 63]):

$$y = \sqrt{2n} + \frac{u}{\sqrt{2}n^{1/6}}, \tag{1.2.9}$$

and so for the kernel K_n as in (1.2.4), with the p_n's being Hermite polynomials,

$$\lim_{n \nearrow \infty} \frac{1}{\sqrt{2}n^{1/6}} K_n\left(\sqrt{2n} + \frac{u}{\sqrt{2}n^{1/6}},\ \sqrt{2n} + \frac{v}{\sqrt{2}n^{1/6}}\right) = K(u,v),$$

where

$$K(u,v)=\int_0^\infty A(x+u)A(x+v)\,dx,\quad A(u)=\int_{-\infty}^\infty e^{iux-x^3/3}\,dx.$$

Relating y and u by (1.2.9), the statistics of the largest eigenvalue for very large n is governed by the function

$$\begin{aligned}P(\lambda_{\max}\le y)&=P\left(2n^{2/3}\Big(\frac{\lambda_{\max}}{\sqrt{2n}}-1\Big)\le u\right),\ \text{ for } n\nearrow\infty,\\ &=\det(I-K(y,z)I_{(-\infty,u]}(z))=\exp\left(-\int_u^\infty(\alpha-u)g^2(\alpha)d\alpha\right),\end{aligned}$$

with $g(x)$ a solution of

$$\begin{cases} g''=xg+2g^3\\ g(x)\cong-\dfrac{e^{-(2/3)x^{3/2}}}{2\sqrt{\pi}x^{1/4}}\ \text{ for } x\nearrow\infty.\end{cases}\qquad\textbf{(Painlevé II)}\qquad(1.2.10)$$

The latter is essentially the asymptotics of the Airy function. In Section 5, I shall derive, via Virasoro constraints, not only this result, due to Tracy and Widom [63], but also a PDE for the probability that the eigenvalues belong to several intervals, due to Adler, Shiota, and van Moerbeke [11; 12].

• *Hard edge scaling limit*: Consider the ensemble of $n\times n$ random matrices for the Laguerre probability distribution, thus corresponding to (1.1.9) with $\rho(dz)=z^{\nu/2}e^{-z/2}\,dz$. One shows the density of eigenvalues near $z=0$ is given by $4n$ for very large n. At this edge, one computes for the kernel (1.2.4) with Laguerre polynomials p_n [52; 30]:

$$\lim_{n\nearrow\infty}\frac{1}{4n}K_n^{(\nu)}\left(\frac{u}{4n},\frac{v}{4n}\right)=K^{(\nu)}(u,v),\qquad(1.2.11)$$

where $K^{(\nu)}(u,v)$ is the *Bessel kernel*, with Bessel functions J_ν:

$$\begin{aligned}K^{(\nu)}(u,v)&=\frac12\int_0^1 xJ_\nu(xu)J_\nu(xv)\,dx\\ &=\frac{J_\nu(u)\sqrt{u}J'_\nu(v)-J_\nu(\sqrt{v})\sqrt{v}J'_\nu(\sqrt{u})}{2(u-v)}.\end{aligned}\qquad(1.2.12)$$

Then

$$P(\text{no eigenvalues in } [0,x])=\exp\left(-\int_0^x\frac{f(u)}{u}\,du\right),$$

with f satisfying

$$(xf'')^2-4\big(xf'-f\big)f'^2+\big((x-\nu^2)f'-f\big)f'=0.\quad\textbf{(Painlevé V)}\qquad(1.2.13)$$

This result of Tracy and Widom [64] and a more general statement from [11; 12] will be shown using Virasoro constraints in Section 5.

1.3. Integrals over Classical Groups. Integration on a compact, semisimple, simply connected Lie group G is given by the formula

$$\int_G f(M)\,dM = \frac{1}{|W|}\int_T \left|\prod_{\alpha\in\Delta} 2\sin\frac{\alpha(iH)}{2}\right| dt \int_U f(utu^{-1})\,du, \quad t = e^H \quad (1.3.1)$$

(see Helgason [36]), where $A \subset G$ is a maximal subgroup, du and dt are Haar measures on G and A respectively, satisfying $\int_A dt = \int_U du = 1$, the symbol Δ denotes the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$ ($\mathfrak{g}$ and $\mathfrak{a}$ being the Lie algebras of G and A), and $|W|$ is the order of the Weyl group of G.

Integration formula (1.3.1) *will be applied to integrals of* $f = e^{\sum_1^\infty t_i \operatorname{Tr} M^i}$ *over the groups* $\mathrm{SO}(2n)$, $\mathrm{SO}(2n+1)$ *and* $\mathrm{Sp}(n)$. Their Lie algebras (over $\mathbb{C}$) are given respectively by $\mathfrak{d}_n$, $\mathfrak{b}_n$, $\mathfrak{c}_n$, with the following sets of roots (see [20], for example):

$$\Delta_n = \{\pm\varepsilon e_i \mid 1 \le i \le k\} \cup \{\pm(e_i+e_j), \pm(e_i-e_j) \mid 1 \le i < j \le n\},$$

where

$$\varepsilon = \begin{cases} 0 & \text{for } \mathfrak{d}_n = \mathrm{so}(2n), \\ 1 & \text{for } \mathfrak{b}_n = \mathrm{so}(2n+1), \\ 2 & \text{for } \mathfrak{c}_n = \mathrm{sp}(n). \end{cases}$$

Setting $H = i\theta$, we have, in view of formula (1.3.1),

$$\left|\prod_{\alpha\in\Delta} 2\sin\frac{\alpha(iH)}{2}\right| dt$$

$$= \begin{cases} c_n\left(\displaystyle\prod_{1\le j<k\le n} \sin\frac{\theta_j-\theta_k}{2}\sin\frac{\theta_j+\theta_k}{2}\right)^2 \prod_1^n d\theta_j & \text{for } \mathfrak{d}_n \\ c_n\left(\displaystyle\prod_{1\le j<k\le n} \sin\frac{\theta_j-\theta_k}{2}\sin\frac{\theta_j+\theta_k}{2}\right)^2 \prod_1^n \sin^2\frac{\varepsilon\theta_j}{2}\,d\theta_j & \text{for } \mathfrak{b}_n,\ \mathfrak{c}_n \end{cases}$$

$$= c_n' \prod_{1\le j<k'\le n} (\cos\theta_j - \cos\theta_k)^2 \begin{cases} \displaystyle\prod_{1\le j\le n} d\theta_j & \text{for } \mathfrak{d}_n \\ \displaystyle\prod_{1\le j\le n} \left(\frac{1-\cos\theta_j}{2}\right) d\theta_j & \text{for } \mathfrak{b}_n \\ \displaystyle\prod_{1\le j\le n} (1-\cos^2\theta_j)\,d\theta_j & \text{for } \mathfrak{c}_n \end{cases}$$

$$= \begin{cases} c_n'\Delta^2(z) \displaystyle\prod_{1\le j\le n} \frac{dz_j}{\sqrt{1-z_j^2|}} & \text{for } \mathfrak{d}_n \\ c_n'\Delta^2(z) \displaystyle\prod_{1\le j\le n} (1-z_j)\frac{dz_j}{\sqrt{1-z_j^2|}} & \text{for } \mathfrak{b}_n \\ c_n'\Delta^2(z) \displaystyle\prod_{1\le j\le n} (1-z_j^2)\frac{dz_j}{\sqrt{1-z_j^2|}} & \text{for } \mathfrak{c}_n \end{cases}$$

$$= c_n'' \Delta^2(z) \prod_{1\le j\le n} (1-z_j)^{\alpha}(1+z_j)^{\beta}\, dz_j \text{ with } \begin{cases} \alpha=\beta=-\frac{1}{2} & \text{for } \mathfrak{d}_n \\ \alpha=\frac{1}{2},\ \beta=-\frac{1}{2} & \text{for } \mathfrak{b}_n \\ \alpha=\beta=\frac{1}{2} & \text{for } \mathfrak{c}_n \end{cases}$$

For $M \in \mathrm{SO}(2n)$, $\mathrm{Sp}(n)$, the eigenvalues are given by $e^{i\theta_j}$ and $e^{-i\theta_j}$, for $1 \le j \le n$; therefore, setting $f = \exp(\sum t_k \mathrm{tr} M^k)$ in formula (1.3.1) leads to

$$\begin{aligned} e^{\sum_1^\infty t_k \operatorname{Tr} M^k} &= e^{\sum_1^\infty t_k \sum_{j=1}^n (e^{ik\theta_j}+e^{-ik\theta_j})} = \prod_{j=1}^n e^{2\sum_{k=1}^\infty t_k \cos k\theta_j} \\ &= \prod_{j=1}^n e^{2\sum t_k T_k(z_j)}, \end{aligned} \tag{1.3.2}$$

where $T_n(z)$ are the Chebyshev polynomials, defined by $T_n(\cos\theta) := \cos n\theta$; in particular $T_1(z) = z$.

For $M \in \mathrm{SO}(2n+1)$, the eigenvalues are given by 1, $e^{i\theta_j}$ and $e^{-i\theta_j}$, for $1 \le j \le n$, which is responsible for the extra-exponential $e^{\sum t_i}$ appearing in (1.3.2).

Before listing various integrals, define the Jacobi weight

$$\rho_{\alpha\beta}(z)\, dz := (1-z)^{\alpha}(1+z)^{\beta} dz, \tag{1.3.3}$$

and the formal sum

$$g(z) := 2\sum_1^\infty t_i T_i(z).$$

The arguments above lead to the following integrals, originally due to H. Weyl [73], and in its present form, due to Johansson [40]; besides the integrals over $\mathrm{SO}(k) = O_+(k)$, the integrals over $O_-(k)$ and $U(n)$ will also be of interest in the theory of random permutations:

$$\begin{aligned}
\int_{O(2n)_+} e^{\sum_1^\infty t_i \mathrm{tr} M^i}\, dM &= \int_{[-1,1]^n} \Delta_n(z)^2 \prod_{k=1}^n e^{g(z_k)} \rho_{(-\frac{1}{2},-\frac{1}{2})}(z_k)\, dz_k, \\
\int_{O(2n+1)_+} e^{\sum_1^\infty t_i \mathrm{tr} M^i}\, dM &= e^{\sum_1^\infty t_i} \int_{[-1,1]^n} \Delta_n(z)^2 \prod_{k=1}^n e^{g(z_k)} \rho_{(\frac{1}{2},-\frac{1}{2})}(z_k)\, dz_k, \\
\int_{\mathrm{Sp}(n)} e^{\sum_1^\infty t_i \mathrm{tr} M^i}\, dM &= \int_{[-1,1]^n} \Delta_n(z)^2 \prod_{k=1}^n e^{g(z_k)} \rho_{(\frac{1}{2},\frac{1}{2})}(z_k)\, dz_k, \\
\int_{O(2n)_-} e^{\sum_1^\infty t_i \mathrm{tr} M^i}\, dM &= e^{\sum_1^\infty 2t_{2i}} \int_{[-1,1]^{n-1}} \Delta_{n-1}(z)^2 \prod_{k=1}^{n-1} e^{g(z_k)} \rho_{(\frac{1}{2},\frac{1}{2})}(z_k)\, dz_k, \\
\int_{O(2n+1)_-} e^{\sum_1^\infty t_i \mathrm{tr} M^i}\, dM &= e^{\sum_1^\infty (-1)^i t_i} \int_{[-1,1]^n} \Delta_n(z)^2 \prod_{k=1}^n e^{g(z_k)} \rho_{(-\frac{1}{2},\frac{1}{2})}(z_k)\, dz_k, \\
\int_{U(n)} e^{\sum_1^\infty tr(t_i M^i - s_i \bar{M}^i)}\, dM &= \frac{1}{n!} \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n e^{\sum_1^\infty (t_i z_k^i - s_i z_k^{-i})} \frac{dz_k}{2\pi i z_k},
\end{aligned} \tag{1.3.4}$$

1.4. Permutations and Integrals over Groups. Let S_n be the group of permutations π and S^0_{2n} the subset of fixed-point free involutions π^0 (this means that $(\pi^0)^2 = I$ and $\pi^0(k) \neq k$ for $1 \leq k \leq 2n$). Put the uniform distribution on S_n and S^0_{2n} — that is, give all permutations $\pi_n \in S_n$ and all involutions $\pi^0_{2n} \in S^0_{2n}$ equal probability:

$$P(\pi_n) = 1/n! \quad \text{and} \quad P(\pi^0_{2n}) = \frac{2^n n!}{(2n)!}. \tag{1.4.1}$$

An *increasing subsequence* of $\pi \in S_n$ or S^0_n is a sequence $1 \leq j_1 < \cdots < j_k \leq n$ such that $\pi(j_1) < \cdots < \pi(j_k)$. Define

$$L(\pi_n) = \text{ length of the longest increasing subsequence of } \pi_n\ . \tag{1.4.2}$$

Example: for $\pi = (\underline{3}, 1, \underline{4}, 2, \underline{6}, \underline{7}, 5)$, we have $L(\pi_7) = 4$.

Around 1960 and based on Monte Carlo methods, Ulam [68] conjectured that

$$\lim_{n\to\infty} \frac{E(L_n)}{\sqrt{n}} = c \text{ exists.}$$

An argument of Erdös and Szekeres [28], dating back from 1935 showed that $E(L_n) \geq \frac{1}{2}\sqrt{n-1}$, and thus $c \geq \frac{1}{2}$. In 1972, Hammersley [33] showed rigorously that the limit exists. Logan and Shepp [46] showed the limit $c \geq 2$, and finally Vershik and Kerov [72] that $c = 2$. In 1990, I. Gessel [31] showed that the following generating function is the determinant of a Toeplitz matrix:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} P(L_n \leq l) = \det \left(\int_0^{2\pi} e^{2\sqrt{t}\cos\theta} e^{i(k-m)\theta}\, d\theta \right)_{0\leq k,m\leq l-1}. \tag{1.4.3}$$

The next major contribution was due to Johansson [41] and Baik, Deift, and Johansson [17], who prove that for arbitrary $x \in \mathbb{R}$, we have a "*law of large numbers*" and a "*central limit theorem*", where $F(x)$ is the statistics (1.2.10),

$$\lim_{n\to\infty} \frac{L_n}{2\sqrt{n}} = 1 \quad \text{and} \quad P\left(\frac{L_n - 2\sqrt{n}}{n^{1/6}} \leq x \right) \longrightarrow F(x) \text{ for } n \longrightarrow \infty.$$

The next set of ideas is due to Diaconis and Shashahani [26], Rains [57; 58], and Baik and Rains [18]. For a nice state-of-the-art account, see Aldous and Diaconis [14]. An illustration is contained in the following proposition; the first part is essentially Gessel's and the second can be found in [26; 58; 18].

PROPOSITION 1.1.

(i)
$$\sum_{n=0}^{\infty} \frac{t^n}{n!} P(L(\pi_n) \leq l) = \int_{U(l)} e^{\sqrt{t}\,\mathrm{Tr}(M+\bar{M})}\, dM \tag{1.4.4}$$
$$= \int_{[0,2\pi]^l} \prod_{1\leq j<k\leq l} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{1\leq k\leq l} e^{2\sqrt{t}\cos\theta_k} \frac{d\theta_k}{2\pi}.$$

$$\text{(ii)} \qquad \sum_{n=0}^{\infty} \frac{(t^2/2)^n}{n!} P(L(\pi_{2n}^0) \le l) = \int_{O(l)} e^{t \operatorname{Tr} M} \, dM. \tag{1.4.5}$$

The proof of this statement will be sketched later. The connection with integrable systems goes via this chain of ideas:

$$\begin{array}{c} \text{Combinatorics} \\ \downarrow \\ \text{Robinson–Schensted–Knuth correspondence} \\ \downarrow \\ \text{Theory of symmetric polynomials} \\ \downarrow \\ \text{Integrals over classical groups} \\ \downarrow \\ \text{Integrable systems} \end{array}$$

All arrows but the last will be explained in this section; the last arrow will be discussed in Sections 7 and 8. We briefly sketch a few of the basic well known facts going into these arguments. They can be found in MacDonald [47], Knuth [45], and Aldous and Diaconis [14]. Useful facts on symmetric functions, applicable to integrable theory, can be found in the appendix to [1]. To mention a few:

- A *Young diagram* λ is a finite sequence of non-increasing, non-negative integers $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_l \ge 0$; also called a *partition* of $n = |\lambda| := \lambda_1 + \cdots + \lambda_l$, with $|\lambda|$ being the weight. It can be represented by a diagram, having λ_1 boxes in the first row, λ_2 boxes in the second row, etc., all aligned to the left. A *dual Young diagram* $\hat\lambda = (\hat\lambda_1 \ge \hat\lambda_2 \ge \cdots)$ is the diagram obtained by flipping the diagram λ about its diagonal.
- A *Young tableau* of shape λ is an array of positive integers a_{ij} (at place (i,j) in the Young diagram) placed in the Young diagram λ, which are non-decreasing from left to right *and* strictly increasing from top to bottom.
- A *standard Young tableau* of shape λ is an array of integers $1, \dots, n$ placed in the Young diagram, which are strictly increasing from left to right *and* from top to bottom. There are several formulae for the number of Young tableaux of a given shape $\lambda = (\lambda_1 \ge \cdots \ge \lambda_m)$:

$$\begin{aligned} f^\lambda &= \#\{\text{standard tableaux of shape } \lambda\} \\ &= \text{coefficient of } x_1 x_2 \dots x_n \text{ in the Schur polynomial } s_\lambda(x) \quad \text{(see next entry)} \\ &= \frac{|\lambda|!}{\prod_{\text{all } i,j} h_{ij}^\lambda} \\ &= |\lambda|! \det\left(\frac{1}{(\lambda_i - i + j)!}\right) \quad (\text{with } h_{ij}^\lambda := \lambda_i + \hat\lambda_j - i - j + 1 = \text{hook length}) \\ &= |\lambda|! \prod_{1 \le i < j \le m} (h_i - h_j) \prod_1^m \frac{1}{h_i!} \quad (\text{with } h_i := \lambda_i - i + m, \ m := \hat\lambda_1). \end{aligned} \tag{1.4.6}$$

- The *Schur polynomial* s_λ associated with a Young diagram λ is a symmetric function in the variables $x_1, x_2, \dots$ (finite or infinite), defined by

$$s_\lambda(x_1, x_2, \dots) := \sum_{\{a_{ij}\} \text{ tableaux of } \lambda} \prod_{ij} x_{a_{ij}}. \tag{1.4.7}$$

- The linear *space* Λ_n *of symmetric polynomials* in $x_1, \dots, x_n$ with rational coefficients comes equipped with the inner product

$$\begin{aligned}\langle f, g\rangle &= \frac{1}{n!}\int_{(S_1)^n} f(z_1, \dots, z_n) g(\bar z_1, \dots, \bar z_n) \prod_{1\le k<l\le n} |z_k - z_l|^2 \prod_1^n \frac{dz_k}{2\pi i z_k} \\ &= \int_{U(n)} f(M) g(\bar M)\, dM. \end{aligned}\tag{1.4.8}$$

- An *orthonormal basis of the space* Λ_n is given by the Schur polynomials $s_\lambda(x_1, \dots, x_n)$, in which the numbers a_{ij} are restricted to $1, \dots, n$. Therefore, each symmetric function admits a "*Fourier series*"

$$f(x_1, \dots, x_n) = \sum_{\substack{\lambda \text{ with} \\ \hat\lambda_1 \le n}} \langle f, s_\lambda\rangle\, s_\lambda(x_1, \dots, x_n), \quad \text{with } \langle s_\lambda, s_{\lambda'}\rangle = \delta_{\lambda\lambda'}. \tag{1.4.9}$$

In particular, one proves (see (1.4.6) for the definition of f^λ)

$$(x_1 + \dots + x_n)^k = \sum_{\substack{|\lambda|=k \\ \hat\lambda_1\le n}} f^\lambda s_\lambda, \tag{1.4.10}$$

If $\lambda = (\lambda_1 \ge \dots \ge \lambda_l > 0)$, with[7] $\hat\lambda_1 = l > n$, then obviously $s_\lambda = 0$.

- *Robinson–Schensted–Knuth correspondence:* There is a 1-1 correspondence

$$S_n \longrightarrow \left\{\begin{matrix} (P,Q), \text{two standard Young} \\ \text{tableaux from } 1, \dots, n, \text{ where} \\ P \text{ and } Q \text{ have the same shape} \end{matrix}\right\}$$

Given a permutation $i_1, \dots, i_n$, the correspondence constructs two standard Young tableaux P, Q having the same shape λ. This construction is inductive. Namely, having obtained two equally shaped Young diagrams P_k, Q_k from $i_1, \dots, i_k$, with the numbers $(i_1, \dots, i_k)$ in the boxes of P_k and the numbers $(1, \dots, k)$ in the boxes of Q_k, one creates a new diagram Q_{k+1}, by putting the *next number* i_{k+1} *in the first row of* P, according to the following rule:

(i) if $i_{k+1} \ge$ all numbers appearing in the first row of P_k, then one creates a new box with i_{k+1} in that box to the right of the first column,

[7] Remember from the definition of the dual Young diagram that $\hat\lambda_1$ is the length of the first column of λ.

(ii) if not, place i_{k+1} in the box (of the first row) with the smallest higher number. That number then gets pushed down to the second row of P_k according to the rule (i) or (ii), as if the first row had been removed.

The diagram Q is a bookkeeping device; namely, add a box (with the number $k+1$ in it) to Q_k exactly at the place, where the new box has been added to P_k. This produces a new diagram Q_{k+1} of same shape as P_{k+1}.

The inverse of this map is constructed essentially by reversing the steps above.

Example. We take $\pi = (5,1,4,3,2) \in S_5$ and follow the construction rules for P and Q:

$$\begin{array}{lllll} 5 & 1 & 1\,4 & 1\,3 & 1\,2 \\ & 5 & 5 & 4 & 3 \\ & & & 5 & 4 \\ & & & & 5 \end{array} \qquad \begin{array}{lllll} 1 & 1 & 1\,3 & 1\,3 & 1\,3 \\ & 2 & 2 & 2 & 2 \\ & & & 4 & 4 \\ & & & & 5 \end{array}$$

Hence $\pi \longrightarrow (P(\pi), Q(\pi)) = \left(\begin{pmatrix} 1\,2 \\ 3 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 1\,3 \\ 2 \\ 4 \\ 5 \end{pmatrix} \right)$ and so $L_5(\pi) = 2 =$ number of columns of P or Q.

The Robinson–Schensted–Knuth correspondence has these properties:

- if $\pi \mapsto (P,Q)$, then $\pi^{-1} \mapsto (Q,P)$;
- length (longest increasing subsequence of π) $= \#$ (columns in P);
- length (longest decreasing subsequence of π) $= \#$ (rows in P);
- if $\pi^2 = I$, then $\pi \mapsto (P,P)$;
- if $\pi^2 = I$ with k fixed points, then P has exactly k columns of odd length. (1.4.11)

From representation theory (see Weyl [73] and especially Rains [57]), one proves:

Lemma 1.2. *The following perpendicularity relations hold*:

(i) $\displaystyle\int_{U(n)} s_\lambda(M) s_\mu(\bar{M}) dM = \langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$.

(ii) $\displaystyle\int_{O(n)} s_\lambda(M)\, dM = \begin{cases} 1 & \text{for } \lambda = (\lambda_1 \geq \cdots \geq \lambda_k \geq 0),\ k \leq n,\ \lambda_i \text{ even}, \\ 0 & \text{otherwise}. \end{cases}$

(iii) $\displaystyle\int_{\mathrm{Sp}(n)} s_\lambda(M)\, dM = \begin{cases} 1 & \text{for } \hat{\lambda}_i \text{ even},\ \hat{\lambda}_1 \leq 2n\ , \\ 0 & \text{otherwise}. \end{cases}$ (1.4.12)

PROOF OF PROPOSITION 1.1. On the one hand,

$$
\begin{aligned}
&\big\langle (x_1+\cdots+x_n)^k,\, (x_1+\cdots+x_n)^k \big\rangle \\
&\qquad = \sum_{\substack{|\lambda|=|\mu|=k\\ \hat\lambda_1,\hat\mu_1\le n}} f^\lambda f^\mu \langle s_\lambda, s_\mu\rangle = \sum_{\substack{|\lambda|=k\\ \hat\lambda_1\le n}} (f^\lambda)^2 = \sum_{\substack{|\lambda|=k\\ \lambda_1\le n}} (f^\lambda)^2 \\
&\qquad = \#\{(P,Q) \text{ standard Young tableaux, each of arbitrary} \\
&\qquad\qquad\qquad \text{shape } \lambda \text{ with } |\lambda|=k,\ \lambda_1\le n\} \\
&\qquad = \#\{\pi_k\in S_k \text{ such that } L(\pi_k)\le n\}. \qquad (1.4.13)
\end{aligned}
$$

On the other hand, notice that, upon setting $\theta_j=\theta_j'+\theta_1$ for $2\le j\le n$, the expression $\prod_{1\le j<k\le n}|e^{i\theta_j}-e^{i\theta_k}|^2$ is independent of θ_1. Then, setting $z_k=e^{i\theta_k}$, one computes:

$$
\begin{aligned}
&\big\langle (x_1+\cdots+x_n)^k,\, (x_1+\cdots+x_n)^l \big\rangle \\
&\quad = \frac{1}{n!}\int_{[0,2\pi]^n} (z_1+\cdots+z_n)^k(\bar z_1+\cdots+\bar z_n)^l \prod_{1\le j<k\le n}|e^{i\theta_j}-e^{i\theta_k}|^2\, d\theta_1\dots d\theta_n \\
&\quad = \frac{1}{n!}\int_{[0,2\pi]^n} e^{ik\theta_1}(1+z_2'+\cdots+z_n')^k e^{-il\theta_1}(1+\bar z_2'+\cdots+\bar z_n')^l \\
&\qquad\qquad \prod_{1\le j<k\le n}|e^{i\theta_j}-e^{i\theta_k}|^2\, d\theta_1\dots d\theta_n \\
&\qquad\qquad \text{and upon setting } \theta_j=\theta_j'+\theta_1, \text{ for } j\ge 2 \text{ and } z_k'=e^{i\theta_k'}, \\
&\quad = \frac{1}{n!}\int_0^{2\pi} e^{i(k-l)\theta_1}\, d\theta_1 \times \text{ an } (n-1)\text{-fold integral} \\
&\quad = \delta_{kl}\langle (x_1+\cdots+x_n)^k,\, (x_1+\cdots+x_n)^k\rangle = \delta_{kl}\int_{U(n)} |\operatorname{Tr} M|^{2k}\, dM. \qquad (1.4.14)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\int_{U(n)} (\operatorname{tr}(M+\bar M))^k\, dM &= \sum_{0\le j\le k}\binom{k}{j}\int_{U(n)} (\operatorname{tr}M)^j(\overline{\operatorname{tr}M})^{k-j}\, dM \\
&= \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \binom{k}{k/2}\int_{U(n)} |\operatorname{tr}M|^k\, dM & \text{if } k \text{ is even.}\end{cases} \qquad (1.4.15)
\end{aligned}
$$

(The equality for k odd follows because then $j\ne k-j$ for all $0\le j\le k$.) Combining the three identities (1.4.13), (1.4.14) and (1.4.15) leads to

$$
\#\{\pi_k\in S_k \text{ such that } L(\pi_k)\le n\} = \binom{2k}{k}^{-1}\int_{U(n)} (\operatorname{Tr}(M+\bar M))^{2k}\, dM. \qquad (1.4.16)
$$

Finally

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{t^n}{n!} P(L(\pi_n) \le l) &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\#\{\pi_n \in S_n \mid L(\pi_n) \le l\}}{n!} \\
&= \sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} \binom{2n}{n}^{-1} \int_{U(l)} (\operatorname{tr}(M+\bar{M}))^{2n}\, dM \\
&= \sum_{n=0}^{\infty} \frac{(\sqrt{t})^{2n}}{(2n)!} \int_{U(l)} (\operatorname{tr}(M+\bar{M}))^{2n}\, dM \\
&= \int_{U(l)} e^{\sqrt{t}\operatorname{Tr}(M+\bar{M})}\, dM \\
&= \frac{1}{l!} \int_{[0,2\pi]^l} e^{\sqrt{t}(z_1+z_1^{-1}+\cdots+z_l+z_l^{-1})} \prod_{1\le j<k\le l} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{1\le k\le l} \frac{d\theta_k}{2\pi}, \\
&\qquad\qquad (\text{where } z_k = e^{i\theta_k}) \\
&= \frac{1}{l!} \int_{[0,2\pi]^l} \prod_{1\le j<k\le l} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{k=1}^{l} e^{2\sqrt{t}\cos\theta_k} \frac{d\theta_k}{2\pi},
\end{aligned}$$

showing (1.4.4) of Proposition 1.1. The latter also equals:

$$\begin{aligned}
&= \frac{1}{l!} \int_{(S^1)^l} \Delta_l(z)\Delta_l(\bar{z}) \prod_{k=1}^{l} \left(e^{\sqrt{t}(z_k+\bar{z}_k)} \frac{dz_k}{2\pi i z_k} \right) \\
&= \frac{1}{l!} \int_{(S^1)^l} \sum_{\sigma\in S_l} \det\left(z_{\sigma(m)}^{k-1} \bar{z}_{\sigma(m)}^{m-1} \right)_{1\le k,m\le l} \prod_{k=1}^{l} \left(e^{\sqrt{t}(z_k+\bar{z}_k)} \frac{dz_k}{2\pi i z_k} \right) \\
&= \frac{1}{l!} \sum_{\sigma\in S_l} \det\left(\int_{S^1} z_k^{k-1} \bar{z}_k^{m-1} e^{\sqrt{t}(z_k+\bar{z}_k)} \frac{dz_k}{2\pi i z_k} \right)_{1\le k,m\le l} \\
&= \det\left(\int_0^{2\pi} e^{2\sqrt{t}\cos\theta} e^{i(k-m)\theta} d\theta \right)_{1\le k,m\le l},
\end{aligned}$$

confirming Gessel's result (1.4.3).

The proof of the second relation (1.4.5) of Proposition 1.1 is based on the following computation:

$$\begin{aligned}
\int_{O(n)} (\operatorname{Tr} M)^k\, dM &= \sum_{\substack{|\lambda|=k, \\ \hat{\lambda}_1 \le n}} f^\lambda \int_{O(n)} s_\lambda(M)\, dM && \text{using (1.4.10)} \\
&= \sum_{\substack{|\lambda|=k,\ \hat{\lambda}_1\le n \\ \lambda_i \text{ even}}} f^\lambda && \text{using Lemma 1.2}
\end{aligned}$$

$$= \sum_{\substack{|\lambda|=k,\ \lambda_1\le n,\\ \hat\lambda_i \text{ even}}} f^\lambda \qquad\qquad \text{using duality}$$

$$= \# \left\{ \begin{array}{l} (P,P) \mid P \text{ standard Young tableau of} \\ \text{shape } \lambda \text{ with } |\lambda| = k,\ \lambda_1 \le n,\ \hat\lambda_i \text{ even} \end{array} \right\}$$

$$= \# \left\{ \pi_k^0 \in S_k^0,\ \text{no fixed points and } L(\pi_k^0) \le n \right\}. \tag{1.4.17}$$

In the last equality, we have used property (1.4.11): an involution has no fixed points if and only if all columns of P have even length. Since all columns $\hat\lambda_i$ have even length, it follows that $|\lambda| = k$ is even and then only is $\int_{O(n)} (\operatorname{Tr} M)^k\, dM > 0$; otherwise this integral equals 0. Finally, one computes

$$\begin{aligned}
\sum_{k=0}^\infty \frac{(t^2/2)^k}{k!} & P\left(L(\pi_{2k}^0) \le n,\ \pi_{2k}^0 \in S_{2k}^0\right) \\
&= \sum_{k=0}^\infty \frac{t^{2k}}{2^k k!} \frac{2^k k!}{(2k)!} \#\{\pi_{2k}^0 \in S_{2k}^0,\ L(\pi_{2k}^0) \le n\} \quad \text{using (1.4.1)} \\
&= \sum_{k=0}^\infty \frac{t^k}{k!} \#\{\pi_k^0 \in S_k^0,\ L(\pi_k^0) \le n\} \\
&= \sum_{k=0}^\infty \frac{t^k}{k!} \int_{O(n)} (\operatorname{Tr} M)^k \, dM \qquad \text{using (1.4.17)} \\
&= \int_{O(n)} e^{t \operatorname{Tr} M} dM,
\end{aligned}$$

ending the proof of Proposition 1.1. □

2. Integrals, Vertex Operators and Virasoro Relations

In Section 1, we discussed random matrix problems over different finite and infinite matrix ensembles, generating functions for the statistics of the length of longest increasing sequences in random permutations and involutions. One can also consider two Hermitian random matrix ensembles, coupled together. All those problems lead to matrix integrals or Fredholm determinants, which we list in the following formulas (where $\beta = 2, 1, 4$):

- $\displaystyle\int_{\mathcal{H}_n(E),\ \mathcal{S}_n(E) \text{ or } \mathcal{T}_n(E)} e^{-\operatorname{Tr} V(M)}\, dM = c_n \int_{E^n} |\Delta_n(z)|^\beta \prod_1^n \rho(z_k)\, dz_k$
- $\displaystyle\int\int_{\mathcal{H}_n^2(E_1 \times E_2)} dM_1\, dM_2\, e^{-\frac12 \operatorname{Tr}(M_1^2 + M_2^2 - 2cM_1M_2)}$
- $\displaystyle\int_{O(n)} e^{x \operatorname{Tr} M} dM$

- $\displaystyle\int_{U(n)} e^{\sqrt{x}\,\mathrm{Tr}(M+\bar M)}\,dM$
- $\det\bigl(I-\lambda K(y,z)I_E(z)\bigr)$, with $K(y,z)$ as in (1.2.4). (2.0.1)

Each of these quantities admits a natural deformation, by inserting time variables $t_1, t_2, \dots$ and possibly a second set $s_1, s_2, \dots$, seemingly *ad hoc.* Each of these integrals or Fredholm determinant is then a fixed point for a natural *vertex operator*, which generates a Virasoro-like algebra. These new integrals in $t_1, t_2, \dots$ are all annihilated by the precise subalgebra of the Virasoro generators, which annihilates τ_0. This will be the topic of this section.

2.1. β-Integrals

2.1.1. Virasoro constraints for β-integrals. Consider weights of the form

$$\rho(z)\,dz := e^{-V(z)}\,dz$$

on an interval $F=[A,B]\subseteq\mathbb{R}$, with rational logarithmic derivative and subjected to the boundary conditions

$$-\frac{\rho'}{\rho}=V'=\frac{g}{f}=\frac{\sum_0^\infty b_i z^i}{\sum_0^\infty a_i z^i},\qquad \lim_{z\to A,B} f(z)\rho(z)z^k=0\ \text{ for } k\geq 0, \tag{2.1.1}$$

and a disjoint union of intervals,

$$E=\bigcup_1^r [c_{2i-1},c_{2i}]\subset F\subset\mathbb{R}. \tag{2.1.2}$$

These data define an algebra of differential operators

$$\mathcal{B}_k=\sum_1^{2r} c_i^{k+1} f(c_i)\frac{\partial}{\partial c_i}. \tag{2.1.3}$$

Take the first type of integrals in the list (2.0.1) for general $\beta>0$, thus generalizing the integrals appearing in the probabilities (1.1.9), (1.1.11) and (1.1.18). Consider t-deformations of such integrals, for general (fixed) $\beta>0$; they can be written as follows, with $t:=(t_1,t_2,\dots)$, $c=(c_1,c_2,\dots,c_{2r})$ and $z=(z_1,\dots,z_n)$:

$$I_n(t,c;\beta):=\int_{E^n}|\Delta_n(z)|^\beta\prod_{k=1}^n\bigl(e^{\sum_1^\infty t_i z_k^i}\rho(z_k)\,dz_k\bigr)\quad\text{for } n>0. \tag{2.1.4}$$

The main statement of this section is Theorem 2.1, whose proof will be outlined in the next subsection. The central charge (2.1.9) has already appeared in the work of Awata et al. [16].

THEOREM 2.1 (Adler and van Moerbeke [3; 6]). *The multiple integrals*

$$I_n(t,c;\beta):=\int_{E^n}|\Delta_n(z)|^\beta\prod_{k=1}^n\left(e^{\sum_1^\infty t_i z_k^i}\rho(z_k)\,dz_k\right),\quad \textit{for } n>0, \tag{2.1.5}$$

and

$$I_n\Big(t,c;\frac{4}{\beta}\Big) := \int_{E^n} |\Delta_n(z)|^{4/\beta} \prod_{k=1}^{n} \Big(e^{\sum_1^\infty t_i z_k^i}\rho(z_k)\,dz_k\Big), \quad \textit{for } n>0, \tag{2.1.6}$$

with $I_0=1$, *satisfy respectively the Virasoro constraints*[8]

$$\begin{gathered}\Big(-\mathcal{B}_k+\sum_{i\geq 0} a_i\,{}^{\beta}\mathbb{J}^{(2)}_{k+i,n}(t,n)-b_i\ {}^{\beta}\mathbb{J}^{(1)}_{k+i+1,n}(t,n)\Big)I_n(t,c;\beta)=0,\\ \Big(-\mathcal{B}_k+\sum_{i\geq 0} a_i\,{}^{\beta}\mathbb{J}^{(2)}_{k+i,n}\Big(-\frac{\beta t}{2},-\frac{2n}{\beta}\Big)+\frac{\beta b_i}{2}\ {}^{\beta}\mathbb{J}^{(1)}_{k+i+1,n}\Big(-\frac{\beta t}{2},-\frac{2n}{\beta}\Big)\Big)I_n\Big(t,c;\frac{4}{\beta}\Big)=0,\end{gathered} \tag{2.1.7}$$

for all $k\geq -1$, *in terms of the coefficients* a_i, b_i *of the rational function* $(-\log\rho)'$ *and the end points* c_i *of the subset* E, *as in* (2.1.1) *to* (2.1.2). *For all* $n\in\mathbb{Z}$, *the* ${}^{\beta}\mathbb{J}^{(2)}_{k,n}(t,n)$ *and* ${}^{\beta}\mathbb{J}^{(1)}_{k,n}(t,n)$ *form a Virasoro and a Heisenberg algebra respectively, interacting via the formulas*

$$\begin{aligned} \big[{}^{\beta}\mathbb{J}^{(2)}_{k,n},\ {}^{\beta}\mathbb{J}^{(2)}_{l,n}\big] &= (k-l)\,{}^{\beta}\mathbb{J}^{(2)}_{k+l,n}+c\Big(\frac{k^3-k}{12}\Big)\delta_{k,-l}\\ \big[{}^{\beta}\mathbb{J}^{(2)}_{k,n},\ {}^{\beta}\mathbb{J}^{(1)}_{l,n}\big] &= -l\,{}^{\beta}\mathbb{J}^{(1)}_{k+l,n}+c'k(k+1)\delta_{k,-l}.\\ \big[{}^{\beta}\mathbb{J}^{(1)}_{k,n},\ {}^{\beta}\mathbb{J}^{(1)}_{l,n}\big] &= \frac{k}{\beta}\delta_{k,-l}, \end{aligned} \tag{2.1.8}$$

with central charge

$$c=1-6\bigg(\Big(\frac{\beta}{2}\Big)^{1/2}-\Big(\frac{\beta}{2}\Big)^{-1/2}\bigg)^2 \quad \textit{and} \quad c'=\Big(\frac{1}{\beta}-\frac{1}{2}\Big). \tag{2.1.9}$$

REMARK 1. The ${}^{\beta}\mathbb{J}^{(2)}_{k,n}$'s are defined by

$${}^{\beta}\mathbb{J}^{(2)}_{k,n}=\frac{\beta}{2}\sum_{i+j=k} :\ {}^{\beta}\mathbb{J}^{(1)}_{i,n}\ {}^{\beta}\mathbb{J}^{(1)}_{j,n}\ :+\Big(1-\frac{\beta}{2}\Big)\Big((k+1)\,{}^{\beta}\mathbb{J}^{(1)}_{k,n}-k\mathbb{J}^{(0)}_{k,n}\Big). \tag{2.1.10}$$

Componentwise, we have

$${}^{\beta}\mathbb{J}^{(1)}_{k,n}(t,n)=\ {}^{\beta}J^{(1)}_k+nJ^{(0)}_k \quad \text{and} \quad {}^{\beta}\mathbb{J}^{(0)}_{k,n}=nJ^{(0)}_k=n\delta_{0k}$$

and

$${}^{\beta}\mathbb{J}^{(2)}_{k,n}(t,n)=\frac{\beta}{2}\,{}^{\beta}J^{(2)}_k+\Big(n\beta+(k+1)\Big(1-\frac{\beta}{2}\Big)\Big)\,{}^{\beta}J^{(1)}_k+n\Big((n-1)\frac{\beta}{2}+1\Big)J^{(0)}_k,$$

---

[8] When E equals the whole range F, then the $\mathcal{B}_k$'s are absent in the formulae (2.1.7).

where

$$
\begin{aligned}
{}^{\beta}J_k^{(1)} &= \frac{\partial}{\partial t_k} + \frac{1}{\beta}(-k)t_{-k},\\
{}^{\beta}J_k^{(2)} &= \sum_{i+j=k} \frac{\partial^2}{\partial t_i \partial t_j} + \frac{2}{\beta} \sum_{-i+j=k} i t_i \frac{\partial}{\partial t_j} + \frac{1}{\beta^2} \sum_{-i-j=k} i t_i j t_j.
\end{aligned}
\tag{2.1.11}
$$

We put n explicitly in ${}^{\beta}\mathbb{J}_{l,n}^{(2)}(t,n)$ to indicate the n-th component contains n explicitly, besides t. For $\beta = 2$, (2.1.10) becomes particularly simple:

$$
{}^{\beta}\mathbb{J}_{k,n}^{(2)}\big|_{\beta=2} = \sum_{i+j=k} : {}^{2}\mathbb{J}_{i,n}^{(1)}\, {}^{2}\mathbb{J}_{j,n}^{(1)} : .
$$

Remark 2. The Heisenberg and Virasoro generators satisfy the following *duality* properties:

$$
\begin{aligned}
{}^{\frac{4}{\beta}}\mathbb{J}_{l,n}^{(2)}(t,n) &= {}^{\beta}\mathbb{J}_{l,n}^{(2)}\Big(-\frac{\beta t}{2}, -\frac{2n}{\beta}\Big), \quad n \in \mathbb{Z},\\
{}^{\frac{4}{\beta}}\mathbb{J}_{l,n}^{(1)}(t,n) &= -\frac{\beta}{2}\, {}^{\beta}\mathbb{J}_{l,n}^{(1)}\Big(-\frac{\beta t}{2}, -\frac{2n}{\beta}\Big), \quad n > 0.
\end{aligned}
\tag{2.1.12}
$$

In (2.1.7), ${}^{\beta}\mathbb{J}_{l,n}^{(2)}(-\beta t/2, -2n/\beta)$ means that the variable n, which appears in the n-th component, gets replaced by $2n/\beta$ and t by $-\beta t/2$.

Remark 3. Theorem 2.1 states that the integrals (2.1.5) and (2.1.6) satisfy two sets of differential equations (2.1.7) respectively. Of course, the second integral also satisfies the first set of equations, with β replaced by $4/\beta$.

2.1.2. Proof: β-integrals as fixed points of vertex operators. Theorem 2.1 can be established by using the invariance of the integral under the transformation $z_i \mapsto z_i + \varepsilon f(z_i) z_i^{k+1}$ of the integration variables. However, the most transparent way to prove Theorem 2.1 is via vector vertex operators, for which the β-integrals are fixed points. This is a technique that we have used already in [2]. Indeed, define the (vector) vertex operator, for $t = (t_1, t_2, \dots) \in \mathbb{C}^\infty$, $u \in \mathbb{C}$, and setting $\chi(z) := (1, z, z^2, \dots)$:

$$
\mathbb{X}_\beta(t,u) = \Lambda^{-1} e^{\sum_1^\infty t_i u^i} e^{-\beta \sum_1^\infty \frac{u^{-i}}{i} \frac{\partial}{\partial t_i}} \chi(|u|^\beta).
\tag{2.1.13}
$$

It acts on vectors $f(t) = (f_0(t), f_1(t), \dots)$ of functions, as follows:[9]

$$
\big(\mathbb{X}_\beta(t,u) f(t)\big)_n = e^{\sum_1^\infty t_i u^i} \big(|u|^\beta\big)^{n-1} f_{n-1}(t - \beta[u^{-1}]).
$$

For the sake of these arguments, it is convenient to introduce the vector Virasoro generators ${}^{\beta}\mathbb{J}_k^{(i)}(t) := ({}^{\beta}\mathbb{J}_{k,n}^{(i)}(t,n))_{n \in \mathbb{Z}}$.

[9] For $\alpha \in \mathbb{C}$, define $[\alpha] := (\alpha, \frac{1}{2}\alpha^2, \frac{1}{3}\alpha^3, \dots) \in \mathbb{C}^\infty$. The operator Λ is the shift matrix with zeroes everywhere except for 1's just above the diagonal: $(\Lambda v)_n = v_{n+1}$.

PROPOSITION 2.2. *The multiplication operator z^k and the differential operators $\frac{\partial}{\partial z} z^{k+1}$ with $z \in \mathbb{C}^*$, acting on the vertex operator $\mathbb{X}_\beta(t,z)$, have realizations as commutators, in terms of the Heisenberg and Virasoro generators ${}^\beta\mathbb{J}_k^{(1)}(t,n)$ and ${}^\beta\mathbb{J}_k^{(2)}(t,n)$:*

$$
\begin{aligned}
z^k \mathbb{X}_\beta(t,z) &= \left[{}^\beta\mathbb{J}_k^{(1)}(t), \mathbb{X}_\beta(t,z)\right] \\
\frac{\partial}{\partial z} z^{k+1} \mathbb{X}_\beta(t,z) &= \left[{}^\beta\mathbb{J}_k^{(2)}(t), \mathbb{X}_\beta(t,z)\right].
\end{aligned}
\tag{2.1.14}
$$

PROOF. By explicit computation; see [3]. □

COROLLARY 2.3. *Given a weight $\rho(z)\,dz$ on $\mathbb{R}$ satisfying* (2.1.1), *we have*

$$
\frac{\partial}{\partial z} z^{k+1} f(z) \mathbb{X}_\beta(t,z)\rho(z) = \left[\sum_{i\geq 0} \left(a_i\, {}^\beta\mathbb{J}_{k+i}^{(2)}(t) - b_i\, {}^\beta\mathbb{J}_{k+i+1}^{(1)}(t)\right),\ \mathbb{X}_\beta(t,z)\rho(z)\right].
\tag{2.1.15}
$$

PROOF. Using (2.1.14) on the last line below, compute

$$
\begin{aligned}
&\frac{\partial}{\partial z} z^{k+1} f(z) \mathbb{X}_\beta(t,z)\rho(z) \\
&\quad = \left(\frac{\rho'(z)}{\rho(z)} f(z)\right) z^{k+1} \mathbb{X}_\beta(t,z)\rho(z) + \rho(z)\frac{\partial}{\partial z}\left(z^{k+1} f(z) \mathbb{X}_\beta(t,z)\right) \\
&\quad = -\left(\sum_0^\infty b_i z^{k+i+1} \mathbb{X}_\beta(t,z)\right)\rho(z) + \rho(z)\frac{\partial}{\partial z}\left(\sum_0^\infty a_i z^{k+i+1} \mathbb{X}_\beta(t,z)\right) \\
&\quad = -\left[\sum_0^\infty b_i\, {}^\beta\mathbb{J}_{k+i+1}^{(1)}, \mathbb{X}_\beta(t,z)\rho(z)\right] + \left[\sum_0^\infty a_i\, {}^\beta\mathbb{J}_{k+i}^{(2)}, \mathbb{X}_\beta(t,z)\rho(z)\right],
\end{aligned}
\tag{2.1.16}
$$

establishing (2.1.15). □

Giving the weight $\rho_E(u)\,du = \rho(u) I_E(u)\,du$, with ρ and E as before, define the integrated vector vertex operator

$$
\mathbb{Y}_\beta(t,\rho_E) := \int_E du\, \rho(u) \mathbb{X}_\beta(t,u),
\tag{2.1.17}
$$

and the vector operator

$$
\begin{aligned}
\mathcal{D}_k &:= \mathcal{B}_k \ - \ \mathcal{V}_k \\
&:= \sum_1^{2r} c_i^{k+1} f(c_i) \frac{\partial}{\partial c_i} - \sum_{i\geq 0}\left(a_i\, {}^\beta\mathbb{J}_{k+i}^{(2)}(t) - b_i\, {}^\beta\mathbb{J}_{k+i+1}^{(1)}(t)\right),
\end{aligned}
\tag{2.1.18}
$$

consisting of a c-dependent boundary part $\mathcal{B}_k$ and a (t,n)-dependent Virasoro part $\mathcal{V}_k$.

PROPOSITION 2.4. *The following commutation relation holds*:

$$
\left[\mathcal{D}_k,\ \mathbb{Y}_\beta(t,\rho_E)\right] = 0.
\tag{2.1.19}
$$

PROOF. Integrating both sides of (2.1.15) over E, one computes:

$$\begin{aligned}\int_E dz\frac{\partial}{\partial z}\left(z^{k+1}f(z)\mathbb{X}_\beta(t,z)\rho(z)\right) &= \sum_1^{2r}(-1)^i c_i^{k+1}f(c_i)\mathbb{X}_\beta(t,c_i)\rho(c_i)\\ &= \sum_1^{2r} c_i^{k+1}f(c_i)\frac{\partial}{\partial c_i}\int_E \mathbb{X}_\beta(t,z)\rho(z)\,dz\\ &= \left[\mathcal{B}_k,\mathbb{Y}_\beta(t,\rho_E)\right], \qquad (2.1.20)\end{aligned}$$

while, on the other hand,

$$\begin{aligned}&\int_E dz\left[\sum_{i\geq 0}\left(a_i\,{}^\beta\mathbb{J}^{(2)}_{k+i}-b_i\,{}^\beta\mathbb{J}^{(1)}_{k+i+1}\right),\mathbb{X}_\beta(t,z)\rho(z)\right]\\ &\qquad = \left[\sum_{i\geq 0}\left(a_i\,{}^\beta\mathbb{J}^{(2)}_{k+i}-b_i\,{}^\beta\mathbb{J}^{(1)}_{k+i+1}\right),\int_{\mathbb{R}} dz\rho_E(z)\mathbb{X}_\beta(t,z)\right]\\ &\qquad = \left[\mathcal{V}_k,\mathbb{Y}_\beta(t,\rho_E)\right]. \qquad (2.1.21)\end{aligned}$$

Subtracting both expressions (2.1.20) and (2.1.21) yields

$$0=\left[\mathcal{B}_k-\mathcal{V}_k,\ \mathbb{Y}_\beta(t,\rho_E)\right]=\left[\mathcal{D}_k,\ \mathbb{Y}_\beta(t,\rho_E)\right]. \qquad \square$$

PROPOSITION 2.5. *The column vector*

$$I(t):=\left(\int_{E^n}|\Delta_n(z)|^\beta\prod_{k=1}^n e^{\sum_1^\infty t_iz_k^i}\rho(z_k)\,dz_k\right)_{n\geq 0}$$

is a fixed point for the vertex operator $\mathbb{Y}_\beta(t,\rho_E)$ (see definition (2.1.17)):

$$\left(\mathbb{Y}_\beta(t,\rho_E)I\right)_n=I_n,\quad n\geq 1. \qquad (2.1.22)$$

PROOF. We have

$$\begin{aligned}I_n(t)&=\int_{\mathbb{R}^n}|\Delta_n(z)|^\beta\prod_{k=1}^n\left(e^{\sum_1^\infty t_iz_k^i}I_E(z_k)\rho(z_k)\,dz_k\right)\\ &=\int_{\mathbb{R}}du\,\rho_E(u)e^{\sum_1^\infty t_iu^i}|u|^{\beta(n-1)}\\ &\qquad\times\int_{\mathbb{R}^{n-1}}\prod_{k=1}^{n-1}\left|1-\frac{z_k}{u}\right|^\beta|\Delta_{n-1}(z)|^\beta\prod_{k=1}^{n-1}\left(e^{\sum_1^\infty t_iz_k^i}\rho_E(z_k)\,dz_k\right)\\ &=\int_{\mathbb{R}}du\,\rho_E(u)e^{\sum_1^\infty t_iu^i}|u|^{\beta(n-1)}\\ &\qquad\times e^{-\beta\sum_1^\infty\frac{u^{-i}}{i}\frac{\partial}{\partial t_i}}\int_{\mathbb{R}^{n-1}}|\Delta_{n-1}(z)|^\beta\prod_{k=1}^{n-1}\left(e^{\sum_1^\infty t_iz_k^i}\rho_E(z_k)\,dz_k\right)\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} du\, \rho_E(u) |u|^{\beta(n-1)} e^{\sum_1^\infty t_i u^i} e^{-\beta \sum_1^\infty \frac{u^{-i}}{i} \frac{\partial}{\partial t_i}} I_{n-1}(t) \\
&= \big(\mathbb{Y}_\beta(t, \rho_E) I(t)\big)_n. &(2.1.23)
\end{aligned}
$$

□

Proof of Theorem 2.1. Proposition 2.4 implies that, for $n \geq 1$,

$$
\begin{aligned}
0 &= [\mathcal{D}_k,\ (\mathbb{Y}_\beta(t, \rho_E))^n]\, I \\
&= \mathcal{D}_k \mathbb{Y}_\beta(t, \rho_E)^n I - \mathbb{Y}_\beta(t, \rho_E)^n \mathcal{D}_k I. &(2.1.24)
\end{aligned}
$$

Taking the n-th component for $n \geq 1$ and $k \geq -1$, setting

$$
X_\beta(t, u) = e^{\sum t_i u^i} e^{-\beta \sum \frac{u^{-i}}{i} \frac{\partial}{\partial t_i}},
$$

and using (2.1.22), we get

$$
\begin{aligned}
0 &= \big(\mathcal{D}_k I - \mathbb{Y}_\beta(t, \rho_E)^n \mathcal{D}_k I\big)_n \\
&= (\mathcal{D}_k I)_n - \int du\, \rho_E(u) X_\beta(t; u) (|u|^\beta)^{n-1} \dots \int du\, \rho_E(u) X_\beta(t; u) (\mathcal{D}_k I)_0 \\
&= (\mathcal{D}_k I)_n.
\end{aligned}
$$

Indeed $(\mathcal{D}_k I)_0 = 0$ for $k \geq -1$, since $\tau_0 = 1$ and $\mathcal{D}_k$ involves ${}^\beta J_k^{(2)}$, ${}^\beta J_k^{(1)}$ and $J_k^{(0)}$ for $k \geq -1$:

${}^\beta J_k^{(2)}$ is pure differentiation for $k \geq -1$;

${}^\beta J_k^{(1)}$ is pure differentiation, except for $k = -1$; but

${}^\beta J_{-1}^{(1)}$ appears with coefficient $n\beta$, which vanishes for $n = 0$;

$J_k^{(0)}$ appears with coefficient $n\big((n-1)\frac{\beta}{2} + 1\big)$, vanishing for $n = 0$. □

2.1.3. Examples

Example 1: Gaussian β-integrals. The weight and the a_i and b_i, as in (2.1.1), are given by

$$
\rho(z) = e^{-V(z)} = e^{-z^2}, \quad V' = g/f = 2z,
$$

$a_0 = 1$, $b_0 = 0$, $b_1 = 2$, and all other $a_i, b_i = 0$.

From (2.1.7), the integrals

$$
I_n = \int_{E^n} |\Delta_n(z)|^\beta \prod_{k=1}^n e^{-z_k^2 + \sum_{i=1}^\infty t_i z_k^i}\, dz_k \tag{2.1.25}
$$

satisfy the Virasoro constraints, for $k \geq -1$,

$$
-\mathcal{B}_k I_n = -\sum_1^{2r} c_i^{k+1} \frac{\partial}{\partial c_i} I_n = \big(-{}^\beta \mathbb{J}_{k,n}^{(2)} + 2\, {}^\beta \mathbb{J}_{k+2,n}^{(1)}\big) I_n. \tag{2.1.26}
$$

Introducing the notation

$$\sigma_i = \Big(n - \frac{i+1}{2}\Big)\beta + i + 1 - b_0 = \Big(n - \frac{i+1}{2}\Big)\beta + i + 1, \tag{2.1.27}$$

and setting $F_n = \log I_n$, the first three constraints have the form

$$\begin{aligned}
-\mathcal{B}_{-1}F &= \Big(2\frac{\partial}{\partial t_1} - \sum_{i\geq 2} it_i \frac{\partial}{\partial t_{i-1}}\Big)F - nt_1,\\
-\mathcal{B}_0 F &= \Big(2\frac{\partial}{\partial t_2} - \sum_{i\geq 1} it_i \frac{\partial}{\partial t_i}\Big)F - \frac{n}{2}\sigma_1,\\
-\mathcal{B}_1 F &= \Big(2\frac{\partial}{\partial t_3} - \sigma_1\frac{\partial}{\partial t_1} - \sum_{i\geq 1} it_i \frac{\partial}{\partial t_{i+1}}\Big)F.
\end{aligned} \tag{2.1.28}$$

For later use, take the linear combinations

$$\mathcal{D}_1 = -\tfrac{1}{2}\mathcal{B}_{-1}, \quad \mathcal{D}_2 = -\tfrac{1}{2}\mathcal{B}_0, \quad \mathcal{D}_3 = -\tfrac{1}{2}\big(\mathcal{B}_1 + \tfrac{1}{2}\sigma_1\mathcal{B}_{-1}\big), \tag{2.1.29}$$

such that each $\mathcal{D}_i$ contains the pure term $\partial F/\partial t_i$, i.e., $\mathcal{D}_i F = (\partial F/\partial t_i) + \cdots$.

Example 2: Laguerre β-integrals. Here the weight and the a_i and b_i, as in (2.1.1), are given by

$$e^{-V} = z^a e^{-z}, \quad V' = \frac{g}{f} = \frac{z-a}{z},$$

$a_0 = 0$, $a_1 = 1$, $b_0 = -a$, $b_1 = 1$, and all other $a_i, b_i = 0$. Thus, from Theorem 2.1, the integrals

$$I_n = \int_{E^n} \big|\Delta_n(z)\big|^\beta \prod_{k=1}^n z_k^a e^{-z_k + \sum_{i=1}^\infty t_i z_k^i}\, dz_k \tag{2.1.30}$$

satisfy the Virasoro constraints, for $k \geq -1$:

$$-\mathcal{B}_k I_n = -\sum_1^{2r} c_i^{k+2}\frac{\partial}{\partial c_i} I_n = \big(-\,{}^\beta\mathbb{J}_{k+1,n}^{(2)} - a\,{}^\beta\mathbb{J}_{k+1,n}^{(1)} + {}^\beta\mathbb{J}_{k+2,n}^{(1)}\big)I_n. \tag{2.1.31}$$

Introducing the notation

$$\sigma_i == \Big(n - \frac{i+1}{2}\Big)\beta + i + 1 - b_0 = \Big(n - \frac{i+1}{2}\Big)\beta + i + 1 + a,$$

and setting $F = F_n = \log I_n$, the first three have the form

$$\begin{aligned}
-\mathcal{B}_{-1}F &= \Big(\frac{\partial}{\partial t_1} - \sum_{i\geq 1} it_i \frac{\partial}{\partial t_i}\Big)F - \frac{n}{2}(\sigma_1 + a)\\
-\mathcal{B}_0 F &= \Big(\frac{\partial}{\partial t_2} - \sigma_1\frac{\partial}{\partial t_1} - \sum_{i\geq 1} it_i \frac{\partial}{\partial t_{i+1}}\Big)F\\
-\mathcal{B}_1 F &= \Big(\frac{\partial}{\partial t_3} - \sigma_2\frac{\partial}{\partial t_2} - \sum_{i\geq 1} it_i \frac{\partial}{\partial t_{i+2}} - \frac{\beta}{2}\frac{\partial^2}{\partial t_1^2}\Big)F - \frac{\beta}{2}\Big(\frac{\partial F}{\partial t_1}\Big)^2.
\end{aligned}$$

Again, replace the operators $\mathcal{B}_i$ by linear combinations $\mathcal{D}_i$, such that $\mathcal{D}_i F = (\partial F/\partial t_i) + \cdots$:

$$\mathcal{D}_1 = -\mathcal{B}_{-1}, \quad \mathcal{D}_2 = -\mathcal{B}_0 - \sigma_1 \mathcal{B}_{-1}, \quad \mathcal{D}_3 = -\mathcal{B}_1 - \sigma_2 \mathcal{B}_0 - \sigma_1\sigma_2 \mathcal{B}_{-1}. \tag{2.1.32}$$

Example 3: Jacobi β-integral. This case is particularly important, because it covers not only the first integral, but also the third integral in the list (2.0.1), used in the problem of random permutations. The weight and the a_i and b_i, as in (2.1.1), are given by

$$\rho(z) := e^{-V} = (1-z)^a(1+z)^b, \quad V' = \frac{g}{f} = \frac{a-b+(a+b)z}{1-z^2},$$

$a_0 = 1$, $a_1 = 0$, $a_2 = -1$, $b_0 = a-b$, $b_1 = a+b$, and all other $a_i, b_i = 0$. The integrals

$$I_n = \int_{E^n} \left|\Delta_n(z)\right|^\beta \prod_{k=1}^n (1-z_k)^a (1+z_k)^b e^{\sum_{i=1}^\infty t_i z_k^i}\, dz_k \tag{2.1.33}$$

satisfy the Virasoro constraints ($k \geq -1$):

$$\begin{aligned} -\mathcal{B}_k I_n &= -\sum_1^{2r} c_i^{k+1}(1-c_i^2)\frac{\partial}{\partial c_i} I_n \\ &= \left({}^\beta\mathbb{J}^{(2)}_{k+2,n} - {}^\beta\mathbb{J}^{(2)}_{k,n} + b_0\, {}^\beta\mathbb{J}^{(1)}_{k+1,n} + b_1\, {}^\beta\mathbb{J}^{(1)}_{k+2,n}\right) I_n. \end{aligned} \tag{2.1.34}$$

Introducing $\sigma_i = \left(n - \dfrac{i+1}{2}\right)\beta + i + 1 + b_1$, the first four ($k = -1, 0, 1, 2$) have the form

$$\begin{aligned}
-\mathcal{B}_{-1}F &= \left(\sigma_1 \frac{\partial}{\partial t_1} + \sum_{i\geq 1} i t_i \frac{\partial}{\partial t_{i+1}} - \sum_{i\geq 2} i t_i \frac{\partial}{\partial t_{i-1}}\right) F + n(b_0 - t_1), \\
-\mathcal{B}_0 F &= \left(\sigma_2 \frac{\partial}{\partial t_2} + b_0 \frac{\partial}{\partial t_1} + \sum_{i\geq 1} i t_i \left(\frac{\partial}{\partial t_{i+2}} - \frac{\partial}{\partial t_i}\right) + \frac{\beta}{2}\frac{\partial^2}{\partial t_1^2}\right) F + \frac{\beta}{2}\left(\frac{\partial F}{\partial t_1}\right)^2 \\
&\qquad - \frac{n}{2}(\sigma_1 - b_1), \\
-\mathcal{B}_1 F &= \left(\sigma_3 \frac{\partial}{\partial t_3} + b_0 \frac{\partial}{\partial t_2} - (\sigma_1 - b_1)\frac{\partial}{\partial t_1} + \sum_{i\geq 1} i t_i \left(\frac{\partial}{\partial t_{i+3}} - \frac{\partial}{\partial t_{i+1}}\right) + \beta \frac{\partial^2}{\partial t_1 \partial t_2}\right) F \\
&\qquad + \beta \frac{\partial F}{\partial t_1}\frac{\partial F}{\partial t_2}, \\
-\mathcal{B}_2 F &= \left(\sigma_4 \frac{\partial}{\partial t_4} + b_0 \frac{\partial}{\partial t_3} - (\sigma_2 - b_1)\frac{\partial}{\partial t_2} + \sum_{i\geq 1} i t_i \left(\frac{\partial}{\partial t_{i+4}} - \frac{\partial}{\partial t_{i+2}}\right)\right. \\
&\qquad \left. + \frac{\beta}{2}\left(\frac{\partial^2}{\partial t_2^2} - \frac{\partial^2}{\partial t_1^2} + 2\frac{\partial^2}{\partial t_1 \partial t_3}\right)\right) F + \frac{\beta}{2}\left(\left(\frac{\partial F}{\partial t_2}\right)^2 - \left(\frac{\partial F}{\partial t_1}\right)^2 + 2\frac{\partial F}{\partial t_1}\frac{\partial F}{\partial t_3}\right).
\end{aligned} \tag{2.1.35}$$

2.2. Double Matrix Integrals. Consider now weights of the form

$$\rho(x,y)=e^{\sum_{i,j\geq 1} r_{ij}x^i y^j}\rho(x)\tilde\rho(y), \tag{2.2.1}$$

defined on a product of intervals $F_1\times F_2\subset\mathbb{R}^2$, with rational logarithmic derivative

$$-\frac{\rho'}{\rho}=\frac{g}{f}=\frac{\sum_{i\geq 0} b_i x^i}{\sum_{i\geq 0} a_i x^i} \quad\text{and}\quad -\frac{\tilde\rho'}{\tilde\rho}=\frac{\tilde g}{\tilde f}=\frac{\sum_{i\geq 0}\tilde b_i y^i}{\sum_{i\geq 0}\tilde a_i y^i},$$

satisfying

$$\lim_{x\to\partial F_1} f(x)\rho(x)x^k=\lim_{y\to\partial F_2}\tilde f(y)\tilde\rho(y)y^k=0 \quad\text{for all } k\geq 0. \tag{2.2.2}$$

Consider subsets of the form

$$E=E_1\times E_2:=\bigcup_{i=1}^{r}[c_{2i-1},c_{2i}]\times\bigcup_{i=1}^{s}[\tilde c_{2i-1},\tilde c_{2i}]\subset F_1\times F_2\subset\mathbb{R}^2. \tag{2.2.3}$$

A natural deformation of the second integral in the list (2.0.1) is given by the following integrals:

$$I_n(t,s,r;E)=\iint_{E^n}\Delta_n(x)\Delta_n(y)\prod_{k=1}^{n} e^{\sum_{i=1}^{\infty}(t_i x_k^i - s_i y_k^i)}\rho(x_k,y_k)\,dx_k\,dy_k \tag{2.2.4}$$

In the theorem below, $\mathbb{J}_{k,n}^{(i)}$ and $\tilde{\mathbb{J}}_{k,n}^{(i)}$ are vectors of operators, whose components are given by the operators (2.1.10) for $\beta=1$; i.e.,

$$\mathbb{J}_{k,n}^{(i)}(t)={}^{\beta}\mathbb{J}_{k,n}^{(i)}(t)\big|_{\beta=1},\quad \tilde{\mathbb{J}}_{k,n}^{(i)}(s):={}^{\beta}\mathbb{J}_{k,n}^{(i)}(t)\big|_{\beta=1,\,t\mapsto -s};$$

thus, from (2.1.10) and (2.1.11), one finds

$$\mathbb{J}_{k,n}^{(2)}(t)=\frac{1}{2}\left(J_k^{(2)}(t)+(2n+k+1)J_k^{(1)}(t)+n(n+1)J_k^{(0)}\right), \tag{2.2.5}$$

satisfying the Heisenberg and Virasoro relations (2.1.8), with *central charge* $c=-2$ and $c'=\frac{1}{2}$.

The $a_i,\tilde a_i,b_i,\tilde b_i,c_i,\tilde c_i,r_{ij}$ given by (2.2.1), (2.2.2) and (2.2.3) define differential operators

$$\begin{aligned}
\mathcal{D}_{k,n}&:=\sum_1^{2r} c_i^{k+1}f(c_i)\frac{\partial}{\partial c_i}-\sum_{i\geq 0}\left(a_i\left(\mathbb{J}_{k+i,n}^{(2)}+\sum_{m,l\geq 1} m r_{ml}\frac{\partial}{\partial r_{m+k+i,l}}\right)-b_i\mathbb{J}_{k+i+1,n}^{(1)}\right),\\
\tilde{\mathcal{D}}_{k,n}&:=\sum_1^{2r}\tilde c_i^{k+1}\tilde f(\tilde c_i)\frac{\partial}{\partial\tilde c_i}-\sum_{i\geq 0}\left(\tilde a_i\left(\tilde{\mathbb{J}}_{k+i,n}^{(2)}+\sum_{m,l\geq 1} l r_{ml}\frac{\partial}{\partial r_{m,l+k+i}}\right)-\tilde b_i\tilde{\mathbb{J}}_{k+i+1,n}^{(1)}\right).
\end{aligned} \tag{2.2.6}$$

THEOREM 2.6 (Adler and van Moerbeke [3; 4]). *Given* $\rho(x,y)$ *as in* (2.2.1), *the integrals*

$$I_n(t,s,r;E) := \iint_{E^n} \Delta_n(x)\Delta_n(y) \prod_{k=1}^{n} e^{\sum_{i=1}^{\infty}(t_i x_k^i - s_i y_k^i)} \rho(x_k,y_k)\,dx_k\,dy_k \quad (2.2.7)$$

satisfy two families of Virasoro equations for $k \geq -1$:

$$\mathcal{D}_{k,n} I_n(t,s,r;E) = 0 \quad \textit{and} \quad \tilde{\mathcal{D}}_{k,n} I_n(t,s,r;E) = 0. \quad (2.2.8)$$

PROOF. The proof of this statement is very similar to the one for β-integrals. Namely, define the vector vertex operator,

$$\mathbb{X}_{12}(t,s;u,v) = \Lambda^{-1} e^{\sum_1^\infty (t_i u^i - s_i v^i)} e^{-\sum_1^\infty \left(\frac{u^{-i}}{i}\frac{\partial}{\partial t_i} - \frac{v^{-i}}{i}\frac{\partial}{\partial s_i}\right)} \chi(uv), \quad (2.2.9)$$

which, as a consequence of Proposition 2.2 for $\beta = 1$, interacts with the operators $\mathbb{J}_k^{(i)}(t) = \left(\mathbb{J}_{k,n}^{(i)}(t,n)\right)_{n\in\mathbb{Z}}$ as follows:

$$\begin{aligned} u^k \mathbb{X}_{12}(t,s;u,v) &= \left[\mathbb{J}_k^{(1)}(t),\ \mathbb{X}_{12}(t,s;u,v)\right] \\ \frac{\partial}{\partial u} u^{k+1} \mathbb{X}_{12}(t,s;u,v) &= \left[\mathbb{J}_k^{(2)}(t),\ \mathbb{X}_{12}(t,s;u,v)\right]. \end{aligned} \quad (2.2.10)$$

A similar statement can be made, upon replacing the operators u^k and $\frac{\partial}{\partial u}u^{k+1}$ by v^k and $\frac{\partial}{\partial v}v^{k+1}$, and upon using the $\tilde{\mathbb{J}}_k^{(i)}(s)$'s.

Finally, one checks that the integral vertex operator

$$\mathbb{Y}(t,s;\rho_E) := \iint_E dx\,dy\,\rho(x,y)\mathbb{X}_{12}(t,s;x,y) \quad (2.2.11)$$

commutes with the two vectors of differential operators $\mathcal{D}_k = (\mathcal{D}_{k,n})_{n\in\mathbb{Z}}$, as in (2.2.6):

$$\left[\mathcal{D}_k, \mathbb{Y}(t,s;\rho_E)\right] = \left[\tilde{\mathcal{D}}_k, \mathbb{Y}(t,s;\rho_E)\right] = 0,$$

and that the vector $I = (I_0 = 1, I_1, \dots)$ of integrals (2.2.7) is a fixed point for $\mathbb{Y}(t,s;\rho_E)$,

$$\mathbb{Y}(t,s;\rho_E)I(t,s,r;E) = I(t,s,r;E).$$

Then, as before, the proof of Theorem 2.6 hinges ultimately on the fact that $\mathcal{D}_{k,0}$ annihilates $I_0 = 1$. $\square$

2.3. Integrals over the Unit Circle. We now deal with the fourth type of integral in the list (2.0.1), which we deform, this time, by inserting two sequences of times $t_1, t_2, \dots$ and $s_1, s_2, \dots$. The following theorem holds:

THEOREM 2.7 (Adler and van Moerbeke [7]). *The multiple integrals over the unit circle* S^1,

$$I_n(t,s) = \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^{n} e^{\sum_1^\infty (t_i z_k^i - s_i z_k^{-i})} \frac{dz_k}{2\pi i z_k}, \quad n > 0, \quad (2.3.1)$$

with $I_0 = 1$, satisfy an $\mathrm{SL}(2,\mathbb{Z})$*-algebra of Virasoro constraints*:

$$\mathcal{D}^{\theta}_{k,n} I_n(t,s) = 0, \quad \textit{only for} \begin{cases} k=-1, & \theta = 0, \\ k=0, & \theta \textit{ arbitrary}, \\ k=1, & \theta=1, \end{cases} \tag{2.3.2}$$

where the operators $\mathcal{D}^{\theta}_{k,n} := \mathcal{D}^{\theta}_{k,n}(t,s,n), \quad k \in \mathbb{Z},\ n \geq 0$ *are given by*

$$\mathcal{D}^{\theta}_{k,n} := \mathbb{J}^{(2)}_{k,n}(t,n) - \mathbb{J}^{(2)}_{-k,n}(-s,n) - k\big(\theta \mathbb{J}^{(1)}_{k,n}(t,n) + (1-\theta)\mathbb{J}^{(1)}_{-k,n}(-s,n)\big), \tag{2.3.3}$$

with $\mathbb{J}^{(i)}_{k,n}(t,n) := {}^{\beta}\mathbb{J}^{(i)}_{k,n}(t,n)\big|_{\beta=1}$, *as in* (2.1.11).

The explicit expressions are

$$\begin{aligned} \mathcal{D}_{-1} I_n &= \left(\sum_{i\geq 1} (i+1) t_{i+1} \frac{\partial}{\partial t_i} - \sum_{i\geq 2} (i-1) s_{i-1} \frac{\partial}{\partial s_i} + n\left(t_1 + \frac{\partial}{\partial s_1}\right)\right) I_n = 0 \\ \mathcal{D}_0 I_n &= \sum_{i\geq 1} \left(i t_i \frac{\partial}{\partial t_i} - i s_i \frac{\partial}{\partial s_i}\right) I_n = 0 \\ \mathcal{D}_1 I_n &= \left(-\sum_{i\geq 1} (i+1) s_{i+1} \frac{\partial}{\partial s_i} + \sum_{i\geq 2} (i-1) t_{i-1} \frac{\partial}{\partial t_i} + n\left(s_1 + \frac{\partial}{\partial t_1}\right)\right) I_n = 0. \end{aligned} \tag{2.3.4}$$

Here the key vertex operator is a reduction of $\mathbb{X}_{12}(t,s;u,v)$, defined in the previous section (formula (2.2.9)). For all $k \in \mathbb{Z}$, the vector of operators

$$\mathcal{D}^{\theta}_{k}(t,s) = \big(\mathcal{D}^{\theta}_{k,n}(t,s,n)\big)_{n\in\mathbb{Z}}$$

forms a realization of the first order differential operators $(d/du)u^{k+1}$, using the vertex operator $\mathbb{X}_{12}(t,s;u,u^{-1})$, namely

$$\frac{d}{du} u^{k+1} \frac{\mathbb{X}_{12}(t,s;u,u^{-1})}{u} = \left[\mathcal{D}^{\theta}_{k}(t,s),\ \frac{\mathbb{X}_{12}(t,s;u,u^{-1})}{u}\right]. \tag{2.3.5}$$

Indeed,

$$\begin{aligned} & u\frac{d}{du} u^k \mathbb{X}_{12}(t,s;u,u^{-1}) \\ &\qquad = \Big(\frac{\partial}{\partial u} u^{k+1} - \frac{\partial}{\partial v} v^{1-k} - k\theta u^k - k(1-\theta) v^{-k}\Big) \mathbb{X}_{12}(t,s;u,v)\big|_{v=-u} \\ &\qquad = \big[\mathbb{J}^{(2)}_k(t) - \mathbb{J}^{(2)}_{-k}(-s) - k\big(\theta \mathbb{J}^{(1)}_k(t) + (1-\theta)\mathbb{J}^{(1)}_k(-s)\big),\ \mathbb{X}_{12}(t,s;u,-u)\big] \\ &\qquad = \big[\mathcal{D}^{\theta}_k(t,s),\ \mathbb{X}_{12}(t,s;u,u^{-1})\big]. \end{aligned}$$

The $\mathcal{D}^{\theta}_k := \mathcal{D}^{\theta}_k(t,s)$ satisfy Virasoro relations with central charge zero:

$$\big[\mathcal{D}^{\theta}_k,\ \mathcal{D}^{\theta}_l\big] = (k-l)\mathcal{D}^{\theta}_{k+l}; \tag{2.3.6}$$

thus, from (2.3.5) we have the commutation relation

$$\big[\mathcal{D}^{\theta}_k(t,s), \mathbb{Y}(t,s)\big] = 0, \quad \text{with } \mathbb{Y}(t,s) := \int_{S^1} \frac{du}{2\pi i u} \mathbb{X}_{12}(t,s;u,u^{-1}). \tag{2.3.7}$$

The point is that the column vector $I(t,s) = (I_0, I_1, \dots)$ of integrals (2.3.1) is a fixed point for $\mathbb{Y}(t,s)$:

$$(\mathbb{Y}(t,s)I)_n = I_n, \quad n \geq 1, \tag{2.3.8}$$

which is shown in a way similar to Proposition 2.5.

Proof of Theorem 2.7. Here again the proof is similar to the one of Theorem 2.1. Taking the n-th component and the n-th power of $\mathbb{Y}(t,s)$, with $n \geq 1$, and noticing from the explicit formulae (2.3.4) that $\left(\mathcal{D}_k^\theta(t,s)I\right)_0 = 0$, we have, by means of a calculation similar to the proof of Theorem 2.1,

$$\begin{aligned} 0 &= \left([\mathcal{D}_k^\theta, \mathbb{Y}(t,s)^n]I\right)_n \\ &= \left(\mathcal{D}_k^\theta \mathbb{Y}(t,s)^n I - \mathbb{Y}(t,s)^n \mathcal{D}_k^\theta I\right)_n \\ &= \left(\mathcal{D}_k^\theta I - \mathbb{Y}(t,s)^n \mathcal{D}_k^\theta I\right)_n = \left(\mathcal{D}_k^\theta I\right)_n. \end{aligned}$$ □

3. Integrable Systems and Associated Matrix Integrals

3.1. Toda lattice and Hermitian matrix integrals

3.1.1. Toda lattice, factorization of symmetric matrices and orthogonal polynomials. Given a weight $\rho(z) = e^{-V(z)}$ defined as in (2.1.1), the inner product over $E \subseteq \mathbb{R}$,

$$\langle f, g\rangle_t = \int_E f(z)g(z)\rho_t(z)\,dz, \quad \text{with } \rho_t(z) := e^{\sum t_i z^i}\rho(z), \tag{3.1.1}$$

leads to a moment matrix

$$m_n(t) = (\mu_{ij}(t))_{0\leq i,j<n} = (\langle z^i, z^j\rangle_t)_{0\leq i,j<n}, \tag{3.1.2}$$

which is a *Hänkel matrix*[10], thus symmetric. This is tantamount to $\Lambda m_\infty = m_\infty \Lambda^\top$, where Λ denotes the shift matrix; see footnote 9. As easily seen, the semi-infinite moment matrix m_∞ evolves in t according to the equations

$$\frac{\partial \mu_{ij}}{\partial t_k} = \mu_{i+k,j}, \text{ and thus } \frac{\partial m_\infty}{\partial t_k} = \Lambda^k m_\infty. \quad \begin{pmatrix}\text{commuting}\\ \text{vector fields}\end{pmatrix} \tag{3.1.3}$$

Another important ingredient is the factorization of m_∞ into a lower-triangular times an upper-triangular matrix[11]

$$m_\infty(t) = S(t)^{-1} S(t)^{\top -1},$$

where $S(t)$ is lower triangular with non-zero diagonal elements.

The main ideas of the following theorem can be found in [2; 5]. Remember that $c = (c_1, \dots, c_{2r})$ denotes the boundary points of the set E; further, dM refers to properly normalized Haar measure on $\mathcal{H}_n$.

[10]Hänkel means that μ_{ij} depends on $i+j$ only.

[11]This factorization is possible for those t's for which $\tau_n(t) := \det m_n(t) \neq 0$ for all $n > 0$.

THEOREM 3.1. *The determinants of the moment matrices*

$$\tau_n(t,c) := \det m_n(t,c) = \frac{1}{n!}\int_{E^n} \Delta_n^2(z) \prod_{k=1}^{n} \rho_t(z_k)\, dz_k$$
$$= \int_{\mathcal{H}_n(E)} e^{tr(-V(M)+\sum_1^\infty t_i M^i)}\, dM, \tag{3.1.4}$$

satisfy the following relations:

(i) *Virasoro constraints* (2.1.7) *for* $\beta = 2$,

$$\Big(-\sum_1^{2r} c_i^{k+1} f(c_i)\frac{\partial}{\partial c_i} + \sum_{i\ge 0}\Big(a_i\, \mathbb{J}^{(2)}_{k+i,n} - b_i\, \mathbb{J}^{(1)}_{k+i+1,n}\Big)\Big)\tau_n(t,c) = 0. \tag{3.1.5}$$

(ii) *The KP-hierarchy*[12] $(k = 0,1,2,\dots)$

$$\Big(\mathbf{s}_{k+4}\Big(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, \dots\Big) - \frac{1}{2}\frac{\partial^2}{\partial t_1 \partial t_{k+3}}\Big)\, \tau_n \circ \tau_n = 0,$$

of which the first equation reads:

$$\Big(\Big(\frac{\partial}{\partial t_1}\Big)^4 + 3\Big(\frac{\partial}{\partial t_2}\Big)^2 - 4\frac{\partial^2}{\partial t_1 \partial t_3}\Big)\log\tau_n + 6\Big(\frac{\partial^2}{\partial t_1^2}\log\tau_n\Big)^2 = 0. \tag{3.1.6}$$

(iii) *The standard Toda lattice, i.e., the symmetric tridiagonal matrix*

$$L(t) := S(t)\Lambda S(t)^{-1} = \begin{pmatrix} \frac{\partial}{\partial t_1}\log\frac{\tau_1}{\tau_0} & \Big(\frac{\tau_0\tau_2}{\tau_1^2}\Big)^{1/2} & 0 & \\ \Big(\frac{\tau_0\tau_2}{\tau_1^2}\Big)^{1/2} & \frac{\partial}{\partial t_1}\log\frac{\tau_2}{\tau_1} & \Big(\frac{\tau_1\tau_3}{\tau_2^2}\Big)^{1/2} & \\ 0 & \Big(\frac{\tau_1\tau_3}{\tau_2^2}\Big)^{1/2} & \frac{\partial}{\partial t_1}\log\frac{\tau_3}{\tau_2} & \\ & & & \ddots \end{pmatrix} \tag{3.1.7}$$

satisfies the commuting equations[13]

$$\frac{\partial L}{\partial t_k} = \big[\tfrac{1}{2}(L^k)_{\mathfrak{s}},\, L\big]. \tag{3.1.8}$$

(iv) *Eigenvectors of L: The tridiagonal matrix L admits two independent eigenvectors*:

[12] Given a polynomial $p(t_1, t_2, \dots)$, define the customary Hirota symbol
$$p(\partial_t) f\circ g := p\Big(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \dots\Big) f(t+y)g(t-y)\Big|_{y=0}.$$
The $\mathbf{s}_l$'s are the elementary Schur polynomials $e^{\sum_1^\infty t_i z^i} := \sum_{i\ge 0}\mathbf{s}_i(t)z^i$. For later use, set
$$\mathbf{s}_l(\tilde\partial) := \mathbf{s}_l\Big(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \dots\Big).$$

[13] The notation $(\ )_{\mathfrak{s}}$ means the skew-symmetric part of $(\ )$; for details, see Section 3.1.2.

- $p(t;z) = (p_n(t;z))_{n\geq 0}$ *satisfying* $(L(t)p(t;z))_n = zp_n(t;z)$, $n \geq 0$, *where* $p_n(t;z)$ *are* n*-th degree polynomials in* z, *depending on* $t \in \mathbb{C}^\infty$, *orthonormal with respect to the* t*-dependent inner product*[14] (3.1.1)

$$\langle p_k(t;z), p_l(t;z)\rangle_t = \delta_{kl};$$

they are eigenvectors of L, *i.e.*, $L(t)p(t;z) = zp(t;z)$, *and enjoy the following representations, where* $\chi(z) = (1, z, z^2, \dots)^\top$:

$$\begin{aligned} p_n(t;z) := (S(t)\chi(z))_n &= \frac{1}{\sqrt{\tau_n(t)\tau_{n+1}(t)}} \det \left(\begin{array}{ccc|c} & m_n(t) & & \begin{matrix} 1 \\ z \\ \vdots \end{matrix} \\ \hline \mu_{n,0} & \dots & \mu_{n,n-1} & z^n \end{array} \right) \\ &= z^n h_n^{-1/2} \frac{\tau_n(t - [z^{-1}])}{\tau_n(t)}, \quad \textit{with } h_n := \frac{\tau_{n+1}(t)}{\tau_n(t)}. \end{aligned} \tag{3.1.9}$$

- $q(t,z) = (q_n(t;z))_{n\geq 0}$, *with*

$$q_n(t;z) := z \int_{\mathbb{R}^n} \frac{p_n(t;u)}{z-u} \rho_t(u)\, du,$$

satisfying $(L(t)q(t;z))_n = zq_n(t;z)$ *for* $n \geq 1$; *the function* $q_n(t;z)$ *enjoys the representations*

$$\begin{aligned} q_n(t;z) &= \left(S^{\top -1}(t)\chi(z^{-1})\right)_n = \left(S(t)m_\infty(t)\chi(z^{-1})\right)_n \\ &= z^{-n} h_n^{-1/2} \frac{\tau_{n+1}(t + [z^{-1}])}{\tau_n(t)}. \end{aligned} \tag{3.1.10}$$

In the case $\beta = 2$, the Virasoro generators (2.1.11) take on a particularly elegant form, namely for $n \geq 0$,

$$\begin{aligned} \mathbb{J}^{(2)}_{k,n}(t) &= \sum_{i+j=k} : \mathbb{J}^{(1)}_{i,n}(t)\, \mathbb{J}^{(1)}_{j,n}(t) := J^{(2)}_k(t) + 2nJ^{(1)}_k(t) + n^2\delta_{0k} \\ \mathbb{J}^{(1)}_{k,n}(t) &= J^{(1)}_k(t) + n\delta_{0k}, \end{aligned}$$

with[15]

$$\begin{aligned} J^{(1)}_k &= \frac{\partial}{\partial t_k} + \tfrac{1}{2}(-k)t_{-k}, \\ J^{(2)}_k &= \sum_{i+j=k} \frac{\partial^2}{\partial t_i \partial t_j} + \sum_{-i+j=k} it_i \frac{\partial}{\partial t_j} + \tfrac{1}{4} \sum_{-i-j=k} it_i jt_j. \end{aligned} \tag{3.1.11}$$

Statement (i) is already contained in Section 2, whereas statement (ii) will be established in Section 3.1.2, using elementary methods.

---

[14] The explicit dependence on the boundary points c will be omitted in this point (iv).

[15] The expression $J^{(1)}_k$ vanishes for $k = 0$.

3.1.2. Sketch of Proof

Orthogonal polynomials and τ-function representation: Representation (3.1.4) of the determinants of moment matrices as integrals follows immediately from the fact that the square of a Vandermonde determinant can be represented as a sum of determinants

$$\Delta^2(u_1,\dots,u_n)=\sum_{\sigma\in S_n}\det\left(u_{\sigma(k)}^{l+k-2}\right)_{1\le k,l\le n}.$$

Indeed,

$$\begin{aligned}
n!\tau_n(t)=n!\det m_n(t)&=\sum_{\sigma\in S_n}\det\left(\int_E z_{\sigma(k)}^{l+k-2}\rho_t(z_{\sigma(k)})dz_{\sigma(k)}\right)_{1\le k,l\le n}\\
&=\sum_{\sigma\in S_n}\int_{E^n}\det\left(z_{\sigma(k)}^{l+k-2}\right)_{1\le k,l\le n}\rho_t(z_{\sigma(k)})dz_{\sigma(k)}\\
&=\int_{E^n}\Delta_n^2(z)\prod_{k=1}^n\rho_t(z_k)\,dz_k,
\end{aligned}$$

whereas the representation (3.1.4) in terms of integrals over Hermitian matrices follows from Section 1.1.

The Borel factorization of m_∞ is responsible for the orthonormality of the polynomials $p_n(t;z)=(S(t)\chi(z))_n$; indeed,

$$\langle p_k(t;z),p_l(t;z)\rangle_{0\le k,l<\infty}=\int_E S\chi(z)(S\chi(z))^\top\rho_t(z)\,dz=Sm_\infty S^\top=I.$$

Note that $S\chi(z)(S\chi(z))^\top$ should be viewed as a semi-infinite matrix obtained by multiplying a semi-infinite column and row. The determinantal representation (3.1.9) follows at once from noticing that $\langle p_n(t;z),z^k\rangle=0$ for $0\le k\le n-1$, because taking that inner product produces two identical columns in the matrix thus obtained. From the same representation (3.1.9), one has $p_n(t;z)=h_n^{-1/2}z^n+\cdots$, where $h_n:=\tau_{n+1}/\tau_n(t)$.

The "Sato" representation (3.1.9) of $p_n(t;z)$ in terms of the determinant $\tau_n(t)$ of the moment matrix can be shown by first proving the Heine representation of the orthogonal polynomials, which goes as follows:

$$\begin{aligned}
h_n^{1/2}p_n(t;z)&=\frac{1}{\tau_n}\det\left(\begin{array}{ccc|c} & & & 1\\ & m_n(t) & & z\\ & & & \vdots\\ \hline \mu_{n,0} & \cdots & \mu_{n,n-1} & z^n\end{array}\right)\\
&=\frac{1}{\tau_n}\int_{E^n}\det\left(\begin{array}{cccc|c} u_1^0 & u_2^1 & \cdots & u_n^{n-1} & 1\\ u_1^1 & u_2^2 & \cdots & u_n^n & z\\ \vdots & & & \vdots & \vdots\\ u_1^{n-1} & u_2^n & \cdots & u_n^{2n-2} & z^{n-1}\\ \hline u_1^n & u_2^{n+1} & \cdots & u_n^{2n-1} & z^n\end{array}\right)\prod_1^n\rho_t(u_i)\,du_i
\end{aligned}$$

$$= \frac{1}{\tau_n}\int_{E^n}\det\left(\begin{array}{cccc|c} u_1^0 & u_2^0 & \cdots & u_n^0 & 1 \\ u_1^1 & u_2^1 & \cdots & u_n^1 & z \\ \vdots & & & \vdots & \vdots \\ u_1^{n-1} & u_2^{n-1} & \cdots & u_n^{n-1} & z^{n-1} \\ \hline u_1^n & u_2^n & \cdots & u_n^n & z^n \end{array}\right) u_1^0 u_2^1 \cdots u_n^{n-1} \prod_1^n \rho_t(u_i)\,du_i$$

$$= \frac{1}{\tau_n}\int_{E^n}\det\left(\begin{array}{cccc|c} u_{\sigma(1)}^0 & u_{\sigma(2)}^0 & \cdots & u_{\sigma(n)}^0 & 1 \\ u_{\sigma(1)}^1 & u_{\sigma(2)}^1 & \cdots & u_{\sigma(n)}^1 & z \\ \vdots & & & \vdots & \vdots \\ u_{\sigma(1)}^{n-1} & u_{\sigma(2)}^{n-1} & \cdots & u_{\sigma(n)}^{n-1} & z^{n-1} \\ \hline u_{\sigma(1)}^n & u_{\sigma(2)}^n & \cdots & u_{\sigma(n)}^n & z^n \end{array}\right) u_{\sigma(1)}^0 u_{\sigma(2)}^1 \cdots u_{\sigma(n)}^{n-1} \prod_1^n \rho_t(u_{\sigma(i)})du_{\sigma(i)}$$

for any permutation $\sigma \in S_n$

$$= \frac{1}{\tau_n}\int_{E^n}\det\left(\begin{array}{cccc|c} u_1^0 & u_2^0 & \cdots & u_n^0 & 1 \\ u_1^1 & u_2^1 & \cdots & u_n^1 & z \\ \vdots & & & \vdots & \vdots \\ u_1^{n-1} & u_2^{n-1} & \cdots & u_n^{n-1} & z^{n-1} \\ \hline u_1^n & u_2^n & \cdots & u_n^n & z^n \end{array}\right) (-1)^\sigma u_{\sigma(1)}^0 u_{\sigma(2)}^1 \cdots u_{\sigma(n)}^{n-1} \prod_1^n \rho_t(u_i)du_i$$

$$= \frac{1}{n!\,\tau_n}\int_{E^n} \Delta_n^2(u) \prod_{k=1}^n (z-u_k)\rho_t(u_k)\,du_k, \qquad \text{upon summing over all } \sigma.$$

Therefore, using again the representation of $\Delta^2(z)$ as a sum of determinants, Heine's formula leads to

$$\begin{aligned} h_n^{1/2}p_n(t,z) &= \frac{z^n}{n!\,\tau_n}\int_{E^n}\sum_{\sigma\in S_n}\det\left(u_{\sigma(k)}^{l+k-2}\right)_{1\le k,l\le n}\prod_{k=1}^n\left(1-\frac{u_{\sigma(k)}}{z}\right)\rho_t(u_{\sigma(k)})\,du_{\sigma(k)}, \\ &= \frac{z^n}{n!\,\tau_n}\int_{E^n}\sum_{\sigma\in S_n}\det\left(u_{\sigma(k)}^{l+k-2}-\frac{1}{z}u_{\sigma(k)}^{l+k-1}\right)_{1\le k,l\le n}\rho_t(u_{\sigma(k)})\,du_{\sigma(k)} \\ &= \frac{z^n}{\tau_n}\det\left(\mu_{ij}-\frac{1}{z}\mu_{i,j+1}\right)_{0\le i,j\le n-1} \\ &= \frac{z^n}{\tau_n}\det\left(\mu_{ij}(t-[z^{-1}])\right)_{0\le i,j\le n-1} \\ &= z^n\frac{\tau_n(t-[z^{-1}])}{\tau_n(t)}, \end{aligned} \tag{3.1.12}$$

invoking the fact that

$$\mu_{ij}(t-[z^{-1}]) = \int u^{i+j}e^{\sum_1^\infty\left(t_i-\frac{z^{-i}}{i}\right)u^i}\rho(u)\,du = \int u^{i+j}\left(1-\frac{u}{z}\right)\rho(u)e^{\sum_1^\infty t_iu^i}\,du$$
$$= \mu_{i+j}(t)-\frac{1}{z}\mu_{i+j+1}(t).$$

Formula (3.1.10) follows from computing on the one hand $S(t)m_\infty\chi(z)$ using the explicit moments μ_{ij}, together with (3.1.12), and on the other hand the equivalent expression $S^{\top-1}(t)\chi(z^{-1})$. Indeed, using $(S(t)\chi(z))_n = p_n(t;z) = \sum_0^n p_{nk}(t)z^k$,

$$\begin{aligned}
\sum_{j\geq 0} (Sm_\infty)_{nj}\, z^{-j} &= \sum_{j\geq 0} z^{-j} \sum_{l\geq 0} p_{nl}(t)\mu_{lj} \\
&= \sum_{j\geq 0} z^{-j} \sum_{l\geq 0} p_{nl}(t) \int_E u^{l+j}\rho_t(u)\,du \\
&= \int_E \sum_{l\geq 0} p_{nl}(t)u^l \sum_{j\geq 0} \left(\frac{u}{z}\right)^j \rho_t(u)\,du \\
&= z\int_E \frac{p_n(t,u)\rho_t(u)}{z-u}\,du.
\end{aligned}$$

Mimicking computation (3.1.12), one shows that

$$h_n^{1/2}\sum_{j\geq 0}\left(S^{\top-1}(t)\right)_{nj} z^{-j} = \frac{\tau_{n+1}(t+[z^{-1}])}{\tau_n(t)} z^{-n},$$

from which (3.1.10) follows, upon using $Sm_\infty = S^{\top-1}$. Details of this and subsequent derivation can be found in [5; 6].

The vectors p and q are eigenvectors of L. Indeed, remembering $\chi(z) = (1, z, z^2, \dots)^\top$, and the shift $(\Lambda v)_n = v_{n+1}$, we have

$$\Lambda\chi(z) = z\chi(z) \quad \text{and} \quad \Lambda^\top\chi(z^{-1}) = z\chi(z^{-1}) - ze_1, \quad \text{with } e_1 = (1,0,0,\dots)^\top.$$

Therefore, $p(z) = S\chi(z)$ and $q(z) = S^{\top-1}\chi(z^{-1})$ are eigenvectors, in the sense that

$$\begin{aligned}
&Lp = S\Lambda S^{-1}S\chi(z) = zS\chi(z) = zp, \\
&L^\top q = S^{\top-1}\Lambda^\top S^\top S^{\top-1}\chi(z^{-1}) = zS^{\top-1}\chi(z^{-1}) - zS^{\top-1}e_1 = zq - zS^{\top-1}e_1.
\end{aligned}$$

Then, using $L = L^\top$, one is led to

$$\begin{aligned}
\big((L-zI)p\big)_n &= 0 \quad \text{for } n \geq 0, \\
\big((L-zI)q\big)_n &= 0 \quad \text{for } n \geq 1.
\end{aligned}$$

Toda lattice and Lie algebra splitting: The Lie algebra splitting of semi-infinite matrices and the corresponding projections (used in (3.1.8)), denoted by $(\ )_\mathfrak{s}$ and $(\ )_\mathfrak{b}$, are defined as follows:

$$\mathrm{gl}(\infty) = \mathfrak{s}\oplus\mathfrak{b} \quad \text{with} \quad \begin{cases} \mathfrak{s} = \{\text{skew-symmetric matrices}\}, \\ \mathfrak{b} = \{\text{lower-triangular matrices}\}. \end{cases}$$

Conjugating the shift matrix Λ by $S(t)$ yields a matrix

$$\begin{aligned} L(t) &= S(t)\Lambda S(t)^{-1} \\ &= S\Lambda S^{-1}S^{\top -1}S^{\top} \\ &= S\Lambda m_\infty S^{\top} && \text{using (3.1.3)} \\ &= Sm_\infty \Lambda^{\top} S^{\top} && \text{using } \Lambda m_\infty = m_\infty \Lambda^{\top} \\ &= S(S^{-1}S^{\top -1})\Lambda^{\top}S^{\top} && \text{using (3.1.3) again} \\ &= (S\Lambda S^{-1})^{\top} = L(t)^{\top}, \end{aligned}$$

which is symmetric and thus tridiagonal. Moreover, from (3.1.3) one computes

$$\begin{aligned} 0 = S\Big(\Lambda^k m_\infty - \frac{\partial m_\infty}{\partial t_k}\Big)S^{\top} &= S\Lambda^k S^{-1} - S\frac{\partial}{\partial t_k}(S^{-1}S^{\top -1})S^{\top} \\ &= L^k + \frac{\partial S}{\partial t_k}S^{-1} + S^{\top -1}\frac{\partial S^{\top}}{\partial t_k}. \end{aligned}$$

Taking the $(\ )_-$ and $(\ )_0$ parts of this equation (A_- means the lower-triangular part of the matrix A, including the diagonal and A_0 the diagonal part) leads to

$$(L^k)_- + \frac{\partial S}{\partial t_k}S^{-1} + \left(S^{\top -1}\frac{\partial S^{\top}}{\partial t_k}\right)_0 = 0 \quad \text{and} \quad \left(\frac{\partial S}{\partial t_k}S^{-1}\right)_0 = -\tfrac{1}{2}(L^k)_0.$$

Upon observing that, for any symmetric matrix

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix}_{\mathfrak{b}} = \begin{pmatrix} a & 0 \\ 2c & b \end{pmatrix} = 2\begin{pmatrix} a & c \\ c & b \end{pmatrix}_- - \begin{pmatrix} a & c \\ c & b \end{pmatrix}_0,$$

it follows that the matrices $L(t)$, $S(t)$ and the vector $p(t;z) = (p_n(t;z))_{n\geq 0} = S(t)\chi(z)$ satisfy the (commuting) differential equations and the eigenvalue problem

$$\frac{\partial S}{\partial t_k} = -\tfrac{1}{2}(L^k)_{\mathfrak{b}}S, \quad L(t)p(t;z) = zp(t;z), \tag{3.1.13}$$

and thus

$$\frac{\partial L}{\partial t_k} = -\big[\tfrac{1}{2}(L^k)_{\mathfrak{b}}, L\big], \quad \frac{\partial p}{\partial t_k} = -\tfrac{1}{2}(L^k)_{\mathfrak{b}}p \qquad \text{(Standard Toda lattice).}$$

The bilinear identity: The functions $\tau_n(t)$ satisfy the following identity, for $n \geq m+1$, $t, t' \in \mathbb{C}$, where one integrates along a small circle about ∞,

$$\oint_{z=\infty} \tau_n(t - [z^{-1}])\tau_{m+1}(t' + [z^{-1}])e^{\sum(t_i - t_i')z^i} z^{n-m-1}\, dz = 0. \tag{3.1.14}$$

An elementary proof can be given by expressing the left hand side of (3.1.14), in terms of $p_n(t;z)$ and $p_m(t,z)$, using (3.1.9) and (3.1.10). One uses below the following identity (see [2]):

$$\int_{\mathbb{R}} f(z)g(z)\, dz = \left\langle f, \int_{\mathbb{R}} \frac{g(u)}{z-u}\, du \right\rangle_\infty, \tag{3.1.15}$$

involving the residue pairing[16]. So, modulo terms depending on t and t' only, the left-hand side of (3.1.14) equals

$$
\begin{aligned}
\oint_{z=\infty} & dz\, z^{-n} p_n(t;z) e^{\sum_1^\infty (t_i - t_i')z^i} z^{n-m-1} z^{m+1} \int_{\mathbb{R}} \frac{p_m(t';u)}{z-u} e^{\sum_1^\infty t_i' u^i} \rho(u)\, du \\
&= \int_{\mathbb{R}} p_n(t;z) e^{\sum (t_i - t_i')z^i} p_m(t';z) e^{\sum t_i' z^i} \rho(z)\, dz \qquad \text{using (3.1.15)} \\
&= \int_{\mathbb{R}} p_n(t;z) p_m(t';z) e^{\sum t_i z^i} \rho(z)\, dz = 0 \quad \text{when } m \le n-1. \qquad (3.1.16)
\end{aligned}
$$

The KP-hierarchy: Setting $n = m+1$, shifting $t \mapsto t-y, t' \mapsto t+y$, evaluating the residue and Taylor expanding in y_k and using the elementary Schur polynomials $\mathbf{s}_n$, leads to (see footnote 12 for the definition of $p(\partial_t) f \circ g$ and $\tilde{\partial}$)

$$
\begin{aligned}
0 &= \frac{1}{2\pi i} \oint dz\, e^{-\sum_1^\infty 2y_i z^i} \tau_n(t - y - [z^{-1}]) \tau_n(t + y + [z^{-1}]) \\
&= \frac{1}{2\pi i} \oint dz \left(\sum_0^\infty z^i \mathbf{s}_i(-2y) \right) \left(\sum_0^\infty z^{-j} \mathbf{s}_j(\tilde{\partial}) \right) e^{\sum_1^\infty y_k \frac{\partial}{\partial t_k}} \tau_n \circ \tau_n \\
&= e^{\sum_1^\infty y_k \frac{\partial}{\partial t_k}} \sum_0^\infty \mathbf{s}_i(-2y) \mathbf{s}_{i+1}(\tilde{\partial}) \tau_n \circ \tau_n \\
&= \left(1 + \sum_1^\infty y_j \frac{\partial}{\partial t_j} + O(y^2) \right) \left(\frac{\partial}{\partial t_1} + \sum_1^\infty \mathbf{s}_{i+1}(\tilde{\partial})(-2y_i + O(y^2)) \right) \tau_n \circ \tau_n \\
&= \left(\frac{\partial}{\partial t_1} + \sum_1^\infty y_k \left(\frac{\partial}{\partial t_k} \frac{\partial}{\partial t_1} - 2\mathbf{s}_{k+1}(\tilde{\partial}) \right) \right) \tau_n \circ \tau_n + O(y^2),
\end{aligned}
$$

thus yielding (3.1.6), taking into account that

$$
\frac{\partial}{\partial t_1} \tau \circ \tau = 0
$$

and that the coefficient of y_k is trivial for $k = 1, 2$.

The Riemann–Hilbert problem: As a function of z, the integral (3.1.10) has a jump across the real axis:

$$
\frac{1}{2\pi i} \lim_{\substack{z' \to z \\ \Im z' < 0}} \int_{\mathbb{R}} \frac{p_n(t;u)}{z'-u} \rho_t(u)\, du = p_n(t,z)\rho_t(z) + \frac{1}{2\pi i} \lim_{\substack{z' \to z \\ \Im z' > 0}} \int_{\mathbb{R}} \frac{p_n(t;u)}{z'-u} \rho_t(u)\, du.
$$

Thus we have (see [29; 19; 5]):

---

[16]The residue pairing about $z = \infty$ between $f = \sum_{i \ge 0} a_i z^i \in \mathcal{H}^+$ and $g = \sum_{j \in \mathbb{Z}} b_j z^{-j-1} \in \mathcal{H}$ is defined as

$$
\langle f, g \rangle_\infty = \oint_{z=\infty} f(z) g(z) \frac{dz}{2\pi i} = \sum_{i \ge 0} a_i b_i.
$$

Corollary 3.2. *The matrix*

$$Y_n(z) = \begin{pmatrix} \dfrac{\tau_n(t-[z^{-1}])}{\tau_n(t)} z^n & \dfrac{\tau_{n+1}(t+[z^{-1}])}{\tau_n(t)} z^{-n-1} \\ \dfrac{\tau_{n-1}(t-[z^{-1}])}{\tau_n(t)} z^{n-1} & \dfrac{\tau_n(t+[z^{-1}])}{\tau_n(t)} z^{-n} \end{pmatrix}$$

satisfies the Riemann–Hilbert problem:

1. $Y_n(z)$ *holomorphic on the upper and lower half-planes* $\mathbb{C}_+$ *and* $\mathbb{C}_-$;
2. $Y_{n-}(z) = Y_{n+}(z) \begin{pmatrix} 1 & 2\pi i \rho_t(z) \\ 0 & 1 \end{pmatrix}$ (*jump condition*);
3. $Y_n'(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} = 1 + O(z^{-1})$, *when* $z \to \infty$.

3.2. Pfaff Lattice and symmetric/symplectic matrix integrals

3.2.1. Pfaff lattice, factorization of skew-symmetric matrices and skew-orthogonal polynomials. Consider an inner product with a skew-symmetric weight $\tilde\rho(y,z)$:

$$\langle f,g\rangle_t = \iint_{\mathbb{R}^2} f(y)g(z)e^{\sum t_i(y^i+z^i)}\tilde\rho(y,z)\,dy\,dz, \text{ with } \tilde\rho(z,y) = -\tilde\rho(y,z). \tag{3.2.1}$$

Since $\langle f,g\rangle_t = -\langle g,f\rangle_t$, the moment matrix, depending on $t=(t_1,t_2,\dots)$,

$$m_n(t) = (\mu_{ij}(t))_{0\le i,j\le n-1} = (\langle y^i, z^j\rangle_t)_{0\le i,j\le n-1}$$

is skew-symmetric. It is clear from formula (3.2.1) that the semi-infinite matrix m_∞ evolves in t according to the *commuting vector fields*:

$$\frac{\partial \mu_{ij}}{\partial t_k} = \mu_{i+k,j} + \mu_{i,j+k}, \quad \text{i.e., } \frac{\partial m_\infty}{\partial t_k} = \Lambda^k m_\infty + m_\infty \Lambda^{\top k}. \tag{3.2.2}$$

Since m_∞ is skew-symmetric, m_∞ does not admit a Borel factorization in the standard sense, but m_∞ admits a unique factorization, with an inserted semi-infinite, skew-symmetric matrix J, with $J^2=-I$, of the form (1.1.12) (see [2]):

$$m_\infty(t) = Q^{-1}(t) J\, Q^{\top -1}(t),$$

where

$$Q(t) = \left(\begin{array}{ccc} \ddots & & \\ & \boxed{\begin{matrix} Q_{2n,2n} & 0 \\ 0 & Q_{2n,2n}\end{matrix}} & 0 \\ & & \boxed{\begin{matrix} Q_{2n+2,2n+2} & 0 \\ 0 & Q_{2n+2,2n+2}\end{matrix}} \\ & * & & \ddots \end{array}\right) \in K. \tag{3.2.3}$$

K is the group of lower-triangular invertible matrices of the form above, with Lie algebra $\mathfrak{k}$. Consider the Lie algebra splitting, given by

$$\mathrm{gl}(\infty) = \mathfrak{k} \oplus \mathfrak{n} \quad \begin{cases} \mathfrak{k} = \{\text{lower-triangular matrices of the form (3.2.3)}\} \\ \mathfrak{n} = \mathrm{sp}(\infty) = \{a \text{ such that } Ja^\top J = a\}, \end{cases} \tag{3.2.4}$$

with unique decomposition[17]

$$\begin{aligned} a &= (a)_{\mathfrak{k}} + (a)_{\mathfrak{n}} \\ &= \big((a_- - J(a_+)^\top J) + \tfrac{1}{2}(a_0 - J(a_0)^\top J)\big) \\ &\qquad\qquad + \big((a_+ + J(a_+)^\top J) + \tfrac{1}{2}(a_0 + J(a_0)^\top J)\big). \end{aligned} \tag{3.2.5}$$

Consider as a special skew-symmetric weight (3.2.1) (see [13]):

$$\tilde\rho(y,z) = 2D^\alpha \delta(y-z)\tilde\rho(y)\tilde\rho(z) \text{ with } \alpha = \mp 1, \quad \tilde\rho(y) = e^{-\tilde V(y)}, \tag{3.2.6}$$

together with the associated inner product[18] of type (3.2.1):

$$\langle f,g\rangle_t = \iint_{\mathbb{R}^2} f(y)g(z)e^{\sum t_i(y^i+z^i)} 2D^\alpha \delta(y-z)\tilde\rho(y)\tilde\rho(z)\,dy\,dz \tag{3.2.7}$$

$$= \begin{cases} \displaystyle\iint_{\mathbb{R}^2} f(y)g(z)e^{\sum_1^\infty t_i(y^i+z^i)}\varepsilon(y-z)\tilde\rho(y)\tilde\rho(z)\,dy\,dz & \text{for } \alpha=-1, \\ \displaystyle\int_{\mathbb{R}} \{f,g\}(y)e^{\sum_1^\infty 2t_iy^i}\tilde\rho(y)^2\,dy & \text{for } \alpha=+1, \end{cases}$$

in terms of the Wronskian $\{f,g\} := \frac{\partial f}{\partial y}g - f\frac{\partial g}{\partial y}$. The moments with regard to these inner products (with that precise definition of time t!) satisfy the differential equations $\partial\mu_{ij}/\partial t_k = \mu_{i+k,j} + \mu_{i,j+k}$, as in (3.2.2).

Now recall that the determinant of an odd skew-symmetric matrix equals 0, whereas the determinant of an even skew-symmetric matrix is the square of a polynomial in the entries, the *Pfaffian*, which is defined by this property up to sign.[19] Now introduce the *Pfaffian τ-functions*, defined with regard to the inner products (3.2.7):

$$\tau_{2n}(t) := \begin{cases} \mathrm{pf}\left(\displaystyle\iint_{\mathbb{R}^2} y^k z^l \varepsilon(y-z)e^{\sum_1^\infty t_i(y^i+z^i)}\tilde\rho(y)\tilde\rho(z)\,dy\,dz\right)_{0\le k,l\le 2n-1} & \text{if } \alpha=-1, \\ \mathrm{pf}\left(\displaystyle\int_{\mathbb{R}} \{y^k,y^l\}e^{\sum_1^\infty 2t_iy^i}\tilde\rho^2(y)\,dy\right)_{0\le k,l\le 2n-1} & \text{if } \alpha=+1. \end{cases} \tag{3.2.8}$$

Setting

$$\tilde\rho(z) = \begin{cases} \rho(z)I_E(z) & \text{for } \alpha=-1, \\ \rho^{1/2}(z)I_E(z), \quad t\mapsto t/2 & \text{for } \alpha=+1 \end{cases}$$

---

[17] $a_\pm$ refers to projection onto strictly upper (strictly lower) triangular matrices, with all 2×2 diagonal blocks equal zero. a_0 refers to projection onto the "diagonal", consisting of 2×2 blocks.

[18] We set $\varepsilon(x) = \operatorname{sign} x$, so that $\varepsilon' = 2\delta(x)$.

[19] We have $(\det m_{2n}(t))^{1/2} = \mathrm{pf}(m_{2n}(t)) = \dfrac{1}{n!}\dfrac{\left(\sum_{0\le i<j\le 2n-1}\mu_{ij}(t)\,dx_i\wedge dx_j\right)^n}{dx_0\wedge dx_1\wedge\cdots\wedge dx_{2n-1}}$.

in the identities (3.2.8) leads to the identities (3.2.9) between integrals and Pfaffians, spelled out in Theorem 3.3 below. Recall that $c = (c_1, \dots, c_{2r})$ stands for the boundary points of the disjoint union $E \subset \mathbb{R}$. Denote by $\mathcal{S}_n(E)$ and $\mathcal{T}_n(E)$ the set of matrices in $\mathcal{S}_n$ and $\mathcal{T}_n$ with spectrum in E.

THEOREM 3.3 (Adler, Horozov, and van Moerbeke [9]; Adler and van Moerbeke [7]). *Consider the integral*

$$I_n = I_n(t,c) := \int_{E^n} |\Delta_n(z)|^\beta \prod_{k=1}^n \left(e^{\sum_1^\infty t_i z_k^i} \rho(z_k)\, dz_k\right).$$

Then I_n is a Pfaffian in certain cases:

- $\beta = 1$, *n even:*

$$\begin{aligned} I_n &= \int_{\mathcal{S}_n(E)} e^{\operatorname{Tr}(-V(X)+\sum_1^\infty t_i X^i)}\, dX \\ &= n!\, \operatorname{pf}\left(\iint_{E^2} y^k z^l \varepsilon(y-z) e^{\sum_1^\infty t_i(y^i+z^i)} \rho(y)\rho(z)\, dy\, dz\right)_{0\le k,l\le n-1} \\ &= n!\, \tau_n(t,c); \end{aligned} \tag{3.2.9a}$$

- $\beta = 4$, *n arbitrary:*

$$\begin{aligned} I_n &= \int_{\mathcal{T}_{2n}(E)} e^{\operatorname{Tr}(-V(X)+\sum_1^\infty t_i X^i)}\, dX \\ &= n!\, \operatorname{pf}\left(\int_E \{y^k, y^l\} e^{\sum_1^\infty t_i y^i} \rho(y)\, dy\right)_{0\le k,l\le 2n-1} \\ &= n!\, \tau_{2n}(t/2, c). \end{aligned} \tag{3.2.9b}$$

The I_n and τ_n's satisfy satisfy the following relations:

(i) *The Virasoro constraints*[20] (2.1.7) *for $\beta = 1, 4$:*

$$\left(-\sum_1^{2r} c_i^{k+1} f(c_i) \frac{\partial}{\partial c_i} + \sum_{i\ge 0} \left(a_i\, {}^\beta\mathbb{J}^{(2)}_{k+i,n} - b_i\, {}^\beta\mathbb{J}^{(1)}_{k+i+1,n}\right)\right) I_n(t,c) = 0. \tag{3.2.10}$$

(ii) *The Pfaff-KP hierarchy* (*see footnote* 12 *for notation*)

$$\left(\mathbf{s}_{k+4}(\tilde\partial) - \frac{1}{2}\frac{\partial^2}{\partial t_1 \partial t_{k+3}}\right) \tau_n \circ \tau_n = \mathbf{s}_k(\tilde\partial)\, \tau_{n+2} \circ \tau_{n-2} \tag{3.2.11}$$

for n even and $k = 0, 1, 2, \dots$. The first relation in this hierarchy reads (*for n even*)

$$\left(\left(\frac{\partial}{\partial t_1}\right)^4 + 3\left(\frac{\partial}{\partial t_2}\right)^2 - 4\frac{\partial^2}{\partial t_1 \partial t_3}\right) \log \tau_n + 6\left(\frac{\partial^2}{\partial t_1^2} \log \tau_n\right)^2 = 12 \frac{\tau_{n-2}\tau_{n+2}}{\tau_n^2}.$$

[20] Here the a_i's and b_i's are defined in the usual way, in terms of $\rho(z)$; namely, $-\rho'/\rho = (\sum b_i z^i)/(\sum a_i z^i)$.

(iii) *The Pfaff Lattice: The time-dependent matrix $L(t)$, zero above the first superdiagonal, obtained by dressing up Λ and having the general form*

$$L(t)=Q(t)\Lambda Q(t)^{-1}=\begin{pmatrix} 0 & 1 & & & & \\ & -d_1 & (h_2/h_0)^{1/2} & & 0 & \\ & & d_1 & 1 & & \\ & & & -d_2 & (h_4/h_2)^{1/2} & \\ & & * & & d_2 & \\ & & & & & \ddots \end{pmatrix} \tag{3.2.12}$$

satisfies the Hamiltonian commuting equations

$$\frac{\partial L}{\partial t_i}=[-(L^i)_{\mathfrak{k}},L]. \qquad (\textbf{Pfaff lattice}) \tag{3.2.13}$$

(iv) *Skew-orthogonal polynomials: The vector of time-dependent polynomials*

$$q(t;z):=(q_n(t;z))_{n\geq 0}=Q(t)\chi(z)$$

in z satisfies the eigenvalue problem

$$L(t)q(t,z)=zq(t,z) \tag{3.2.14}$$

and enjoy the following representations (with $h_{2n}=\tau_{2n+2}(t)/\tau_{2n}(t)$)

$$\begin{aligned} q_{2n}(t;z)&=\frac{h_{2n}^{-1/2}}{\tau_{2n}(t)}\operatorname{pf}\left(\begin{array}{c|c} & 1\\ m_{2n+1}(t) & z\\ & \vdots\\ & z^{2n}\\ \hline -1\quad -z\quad \dots\quad -z^{2n} & 0\end{array}\right)\\ &=z^{2n}h_{2n}^{-1/2}\frac{\tau_{2n}(t-[z^{-1}])}{\tau_{2n}(t)}=z^{2n}h_{2n}^{-1/2}+\cdots,\\ q_{2n+1}(t;z)&=\frac{h_{2n}^{-1/2}}{\tau_{2n}(t)}\operatorname{pf}\left(\begin{array}{ccc|cc} & & & 1 & \mu_{0,2n+1}\\ & m_{2n}(t) & & z & \mu_{1,2n+1}\\ & & & \vdots & \vdots\\ & & & z^{2n-1} & \mu_{2n-1,2n+1}\\ \hline -1 & \dots & -z^{2n-1} & 0 & -z^{2n+1}\\ \mu_{2n+1,0} & \dots & \mu_{2n+1,2n-1} & z^{2n+1} & 0\end{array}\right).\\ &=z^{2n}h_{2n}^{-1/2}\frac{1}{\tau_{2n}(t)}\left(z+\frac{\partial}{\partial t_1}\right)\tau_{2n}(t-[z^{-1}])=z^{2n+1}h_{2n}^{-1/2}+\cdots. \end{aligned} \tag{3.2.15}$$

They are skew-orthogonal polynomials in z; that is, $\langle q_i(t;z),q_j(t;z)\rangle_t=J_{ij}$.

The hierarchy (3.2.11) already appears in the work of Kac and van de Leur [42] in the context of what they call the DKP-hierarchy. Interesting further work has been done by van de Leur [69].

3.2.2. Sketch of Proof.

Skew-orthogonal polynomials and the Pfaff Lattice: The equalities (3.2.9) between the Pfaffians and the matrix integrals are based on two identities [49]. The first, due to de Bruyn, reads

$$\frac{1}{n!}\int_{\mathbb{R}^n}\prod_1^n dy_i \,\det\big(F_i(y_1)\ \ G_i(y_1)\ \ \dots\ \ F_i(y_n)\ \ G_i(y_n)\big)_{0\le i\le 2n-1}$$
$$= \det{}^{1/2}\left(\int_{\mathbb{R}}(G_i(y)F_j(y)-F_i(y)G_j(y))\,dy\right)_{0\le i,j\le 2n-1};$$

the second (Mehta [50]) is

$$\Delta_n^4(x) = \det\left(x_1^i\ \ (x_1^i)'\ \ x_2^i\ \ (x_2^i)'\ \ \dots\ \ x_n^i\ \ (x_n^i)'\right)_{0\le i\le 2n-1}.$$

On the one hand (see Mehta [49]), setting in the calculation below $\rho_{t,E}(z) = \rho(z)e^{\sum t_i z^i}I_E(z)$, $F_i(x) := \int_{-\infty}^x y^i\rho_{t,E}(y)\,dy$, and $G_i(x) := F_i'(x) = x^i\rho_{t,E}(x)$, one computes:

$$\frac{1}{(2n)!}\int_{\mathbb{R}^{2n}}|\Delta_{2n}(z)|\prod_{i=1}^{2n}\rho_{t,E}(z_i)\,dz_i$$
$$= \int_{\substack{-\infty<z_1<z_2<\\ \cdots<z_{2n}<\infty}}\det\left(z_{j+1}^i\rho_{t,E}(z_{j+1})\right)_{0\le i,j\le 2n-1}\prod_{i=1}^{2n}dz_i,$$
$$= \int_{\substack{-\infty<z_2<z_4<\\ \cdots<z_{2n}<\infty}}\prod_{k=1}^n\rho_{t,E}(z_{2k})\,dz_{2k}$$
$$\det\left(\int_{-\infty}^{z_2}z_1^i\rho_{t,E}(z_1)\,dz_1, z_2^i,\dots,\int_{z_{2n-2}}^{z_{2n}}z_{2n-1}^i\rho_{t,E}(z_{2n-1})\,dz_{2n-1}, z_{2n}^i\right)_{0\le i\le 2n-1}$$
$$= \int_{\substack{-\infty<z_2<z_4<\\ \cdots<z_{2n}<\infty}}\prod_{k=1}^n\rho_{t,E}(z_{2k})\,dz_{2k}$$
$$\det\big(F_i(z_2), z_2^i, F_i(z_4)-F_i(z_2), z_4^i,\dots,F_i(z_{2n})-F_i(z_{2n-2}), z_{2n}^i\big)_{0\le i\le 2n-1}$$
$$= \int_{\substack{-\infty<z_2<z_4<\\ \cdots<z_{2n}<\infty}}\prod_1^n dz_{2i}\,\det\big(F_i(z_2),G_i(z_2),\dots,F_i(z_{2n}),G_i(z_{2n})\big)_{0\le i\le 2n-1},$$
$$= \frac{1}{n!}\int_{\mathbb{R}^n}\prod_1^n dy_i\,\det\big(F_i(y_1),G_i(y_1),\dots,F_i(y_n),G_i(y_n)\big)_{0\le i\le 2n-1},$$

$$
\begin{aligned}
&= \det{}^{1/2}\left(\int_{\mathbb{R}}(G_i(y)F_j(y)-F_i(y)G_j(y))\,dy\right)_{0\le i,j\le 2n-1}\\
&= \operatorname{pf}\left(\iint_{E^2} y^k z^l \varepsilon(y-z)e^{\sum_1^\infty t_i(y^i+z^i)}\rho(y)\rho(z)dydz\right)_{0\le k,l\le 2n-1} = \tau_{2n}(t),
\end{aligned}
$$

which establishes (3.2.9a).

On the other hand, upon setting

$$
F_j(x) = x^j\rho(x)e^{\sum t_i x^i} \quad \text{and} \quad G_j(x) := F_j'(x) = \left(x^j\rho(x)e^{\sum t_i x^i}\right)',
$$

one computes

$$
\begin{aligned}
&\frac{1}{n!}\int_E \prod_{1\le i,j\le n}(x_i-x_j)^4 \prod_{k=1}^n \left(\rho^2(x_k)e^{2\sum_{i=1}^\infty t_i x_k^i}\,dx_k\right)\\
&\quad= \frac{1}{n!}\int_E \prod_{k=1}^n \left(\rho^2(x_k)e^{2\sum t_i x_k^i}\,dx_k\right)\det\left(x_1^i\ (x_1^i)'\ x_2^i\ (x_2^i)'\ \dots\ x_n^i\ (x_n^i)'\right)_{0\le i\le 2n-1}\\
&\quad= \frac{1}{n!}\int_E \prod_1^n dy_i \quad \det\left(F_i(y_1)\ \ G_i(y_1)\ \ \dots\ \ F_i(y_n)\ \ G_i(y_n)\right)_{0\le i\le 2n-1}\\
&\quad= \det{}^{1/2}\left(\int_E (G_i(y)F_j(y)-F_i(y)G_j(y))\,dy\right)_{0\le i,j\le 2n-1}\\
&\quad= \operatorname{pf}\left(\int_E \{y^k,y^l\}e^{\sum_1^\infty 2t_i y^i}\rho^2(y)\,dy\right)_{0\le k,l\le 2n-1} = \tau_{2n}(t),
\end{aligned}
$$

establishing (3.2.9b).

The skew-orthogonality of the polynomials $q_k(t;z)$ follows immediately from the skew-Borel decomposition of m_∞:

$$
\langle q_k(t,y), q_l(t,z)\rangle_{k,l\ge 0} = Q(\langle y^i, z^j\rangle)_{i,j\ge 0}Q^\top = Q\,m_\infty Q^\top = J,
$$

with the q_n's admitting the representation (3.2.15) in terms of the moments.

Using $L = Q\Lambda Q^{-1}$, $m_\infty = Q^{-1}JQ^{\top -1}$ and $J^2 = -I$, one computes from the differential equations (3.2.2)

$$
\begin{aligned}
0 &= Q\Big(\Lambda^k m_\infty + m_\infty\Lambda^{\top k} - \frac{\partial m_\infty}{\partial t_k}\Big)Q^\top\\
&= (Q\Lambda^k Q^{-1})J - (JQ^{\top -1}\Lambda^{\top k}Q^\top J)J + \frac{\partial Q}{\partial t_k}Q^{-1}J - \Big(JQ^{-1\top}\frac{\partial Q^\top}{\partial t_k}J\Big)J\\
&= \Big(L^k + \frac{\partial Q}{\partial t_k}Q^{-1}\Big) - J\Big(L^k + \frac{\partial Q}{\partial t_k}Q^{-1}\Big)^\top J.
\end{aligned}
$$

Then computing the $+$, $-$ and the diagonal part (in the sense of footnote 17) of the expression leads to commuting Hamiltonian differential equations for Q, and thus for L and $q(t;z)$, confirming (3.2.13):

$$\frac{\partial Q}{\partial t_i} = -(L^i)_{\mathfrak{k}} Q, \quad \frac{\partial L}{\partial t_i} = [(L^i)_{\mathfrak{n}}, L], \quad \frac{\partial q}{\partial t_i} = -(L^i)_{\mathfrak{k}} q \quad \text{(Pfaff lattice)}. \tag{3.2.16}$$

The bilinear identities: For all $n, m \geq 0$, the τ_{2n}'s satisfy the bilinear identity

$$\oint_{z=\infty} \tau_{2n}(t-[z^{-1}])\tau_{2m+2}(t'+[z^{-1}])e^{\sum(t_i-t_i')z^i} z^{2n-2m-2}\frac{dz}{2\pi i} + \oint_{z=0} \tau_{2n+2}(t+[z])\tau_{2m}(t'-[z])e^{\sum(t_i'-t_i)z^{-i}} z^{2n-2m}\frac{dz}{2\pi i} = 0. \tag{3.2.17}$$

The differential equation (3.2.2) on the moment matrix m_∞ admits the following solution, which upon using the Borel decomposition $m_\infty = Q^{-1}J\,Q^{\top -1}$, leads to

$$\begin{aligned} m_\infty(0) &= e^{-\sum_1^\infty t_k\Lambda^k} m_\infty(t) e^{-\sum_1^\infty t_k \Lambda^{\top k}} \\ &= \big(Q(t)e^{\sum_1^\infty t_k\Lambda^k}\big)^{-1} J \big(Q(t)e^{\sum_1^\infty t_k\Lambda^k}\big)^{\top -1}, \end{aligned} \tag{3.2.18}$$

so the right-hand side of (3.2.18) is independent of t; equal, say, to the same expression with t replaced by t'. Upon rearrangement, one finds

$$\big(Q(t)e^{\sum t_k\Lambda^k}\big)\big(JQ(t')e^{\sum t_k'\Lambda^k}\big)^{-1} = \big(JQ(t)e^{\sum t_k\Lambda^k}\big)^{\top -1}\big(Q(t')e^{\sum t_k'\Lambda^k}\big)^{\top},$$

and therefore[21]

$$\oint_{z=\infty} \big(Q(t)\chi(z)\otimes(JQ(t'))^{\top -1}\chi(z^{-1})\big)\, e^{\sum_1^\infty(t_k-t_k')z^k}\frac{dz}{2\pi i z} = \oint_{z=0}\big((JQ(t))^{\top -1}\chi(z)\otimes Q(t')\chi(z^{-1})\big)\, e^{\sum_1^\infty(t_k'-t_k)z^{-k}}\frac{dz}{2\pi i z}. \tag{3.2.19}$$

Setting $t - t' = [z_1^{-1}] + [z_2^{-1}]$ in the exponential leads to

$$e^{\sum_1^\infty(t_k-t_k')z^k} = \Big(1-\frac{z}{z_1}\Big)^{-1}\Big(1-\frac{z}{z_2}\Big)^{-1}, \quad e^{\sum_1^\infty(t_k'-t_k)z^{-k}} = \Big(1-\frac{1}{zz_1}\Big)\Big(1-\frac{1}{zz_2}\Big),$$

and somewhat enlarging the integration circle about $z=\infty$ to include the points z_1 and z_2, the integrand on the left-hand side has poles at $z = z_1$ and z_2, whereas the integrand on the right-hand side is holomorphic. Combining the identity obtained and the one, with $z_2 \nearrow \infty$, one finds a functional relation involving a function $\varphi(t;z) = 1 + O(z^{-1})$:

$$\frac{\varphi(t-[z_2^{-1}];z_1)}{\varphi(t;z_1)} = \frac{\varphi(t-[z_1^{-1}];z_2)}{\varphi(t;z_2)}, \quad t\in\mathbb{C}^\infty,\ z\in\mathbb{C}.$$

[21] We use $\Lambda\chi(z) = z\chi(z)$, $\Lambda^\top\chi(z) = z^{-1}\chi(z)$ and the matrix identities (see [25])
$$U_1V_1 = \oint_{z=\infty} U_1\chi(z)\otimes V_1^\top\chi(z^{-1})\frac{dz}{2\pi i z}, \quad U_2V_2 = \oint_{z=0} U_2\chi(z)\otimes V_2^\top\chi(z^{-1})\frac{dz}{2\pi i z}.$$

Such an identity leads, by a standard argument (see [8, Appendix], for example) to the existence of a function $\tau(t)$ such that

$$\varphi(t;z)=\frac{\tau(t-[z^{-1}])}{\tau(t)}.$$

This, combined with the bilinear identity (3.2.19), yields the bilinear identity (3.2.17).

The Pfaff-KP hierarchy: Shifting

$$t\mapsto t-y,\quad t'\mapsto t+y$$

in (3.2.17), evaluating the residue and Taylor expanding in y_k leads to (for $\tilde{\partial}$, see footnote 12)

$$\begin{aligned}
&\frac{1}{2\pi i}\oint_{z=\infty} e^{-\sum_1^\infty 2y_iz^i}\tau_{2n}(t-y-[z^{-1}])\tau_{2m+2}(t+y+[z^{-1}])z^{2n-2m-2}\,dz\\
&\qquad+\frac{1}{2\pi i}\oint_{z=0} e^{\sum_1^\infty 2y_iz^{-i}}\tau_{2n+2}(t-y+[z])\tau_{2m}(t+y-[z])z^{2n-2m}\,dz\\
&=\frac{1}{2\pi i}\oint_{z=\infty}\sum_{j=0}^\infty z^j\mathbf{s}_j(-2y)e^{\sum -y_i\frac{\partial}{\partial t_i}}\sum_{k=0}^\infty z^{-k}\mathbf{s}_k(-\tilde{\partial})\tau_{2n}\circ\tau_{2m+2}z^{2n-2m-2}\,dz\\
&\qquad+\frac{1}{2\pi i}\oint_{z=0}\sum_{j=0}^\infty z^{-j}\mathbf{s}_j(2y)e^{\sum -y_i\frac{\partial}{\partial t_i}}\sum_{k=0}^\infty z^k\mathbf{s}_k(\tilde{\partial})\tau_{2n+2}\circ\tau_{2m}z^{2n-2m}\,dz\\
&=\sum_{j-k=-2n+2m+1}\mathbf{s}_j(-2y)e^{\sum_1^\infty -y_i\frac{\partial}{\partial t_i}}\mathbf{s}_k(-\tilde{\partial})\tau_{2n}\circ\tau_{2m+2}\\
&\qquad+\sum_{k-j=-2n+2m-1}\mathbf{s}_j(2y)e^{\sum_1^\infty -y_i\frac{\partial}{\partial t_i}}\mathbf{s}_k(\tilde{\partial})\tau_{2n+2}\circ\tau_{2m}\\
&=\cdots+y_k\left(\left(\frac{1}{2}\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_k}-\mathbf{s}_{k+1}(\tilde{\partial})\right)\tau_{2n}\circ\tau_{2n}+\mathbf{s}_{k-3}(\tilde{\partial})\tau_{2n+2}\circ\tau_{2n-2}\right)+\cdots,
\end{aligned}$$

establishing the Pfaff-KP hierarchy (3.2.11), different from the usual KP hierarchy, because of the presence of a right-hand side.

REMARK. L admits the following representation in terms of τ, much in the style of (3.1.7):

$$L=h^{-1/2}\begin{pmatrix}\hat{L}_{00}&\hat{L}_{01}&0&0&\\ \hat{L}_{10}&\hat{L}_{11}&\hat{L}_{12}&0&\\ *&\hat{L}_{21}&\hat{L}_{22}&\hat{L}_{23}&\\ *&*&\hat{L}_{32}&\hat{L}_{33}&\\ &&&&\ddots\end{pmatrix}h^{1/2},$$

with the 2×2 entries $\hat{L}_{ij}$ and

$$h=\operatorname{diag}(h_0,h_0,h_2,h_2,h_4,h_4,\dots),\quad h_{2n}=\tau_{2n+2}/\tau_{2n}.$$

For example, using $\cdot$ to denote partial differentiation with respect to t_1:

$$\hat{L}_{nn} := \begin{pmatrix} -(\log \tau_{2n})^{\cdot} & 1 \\ -\dfrac{s_2(\tilde{\partial})\tau_{2n}}{\tau_{2n}} - \dfrac{s_2(-\tilde{\partial})\tau_{2n+2}}{\tau_{2n+2}} & (\log \tau_{2n+2})^{\cdot} \end{pmatrix},$$

$$\hat{L}_{n,n+1} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \hat{L}_{n+1,n} := \begin{pmatrix} * & (\log \tau_{2n+2})^{\cdot\cdot} \\ * & * \end{pmatrix}.$$

3.3. 2d-Toda Lattice and Coupled Hermitian Matrix Integrals

3.3.1. 2d-Toda lattice, factorization of moment matrices and bi-orthogonal polynomials. Consider the inner product

$$\langle f, g\rangle_{t,s} = \iint_{E\subset\mathbb{R}^2} f(y)g(z)e^{\sum_1^\infty (t_i y^i - s_i z^i)+cyz}dy\,dz, \tag{3.3.1}$$

on a subset $E = E_1 \times E_2 := \bigcup_{i=1}^r [c_{2i-1}, c_{2i}] \times \bigcup_{i=1}^s [\tilde{c}_{2i-1}, \tilde{c}_{2i}] \subset F_1 \times F_2 \subset \mathbb{R}^2$. Define the customary moment matrix, depending on $t = (t_1, t_2, \dots)$ and $s = (s_1, s_2, \dots)$:

$$m_n(t, s) = \big(\mu_{ij}(t, s)\big)_{0\le i,j\le n-1} = \big(\langle y^i, z^j\rangle_{t,s}\big)_{0\le i,j\le n-1},$$

and let its factorization in lower-triangular times upper-triangular matrices be

$$m_\infty(t, s) = S_1^{-1}(t, s)S_2(t, s). \tag{3.3.2}$$

Then m_∞ evolves in t, s according to the equations

$$\frac{\partial \mu_{ij}}{\partial t_k} = \mu_{i+k,j}, \qquad \frac{\partial \mu_{ij}}{\partial s_k} = -\mu_{i,j+k},$$
$$\text{i.e.,}\quad \frac{\partial m_\infty}{\partial t_k} = \Lambda^k m_\infty, \qquad \frac{\partial m_\infty}{\partial s_k} = -m_\infty \Lambda^{\top k}. \tag{3.3.3}$$

In the next integral (3.3.4), dM denotes properly normalized Haar measure on $\mathcal{H}_n$.

THEOREM 3.4 (Adler and van Moerbeke [4; 3]). *The integrals $I_n(t, s; c, \tilde{c})$, with $I_0 = 1$,*

$$\tau_n = \det m_n = \frac{1}{n!}I_n = \frac{1}{n!}\iint_{E^n} \Delta_n(x)\Delta_n(y)\prod_{k=1}^n e^{\sum_1^\infty (t_i x_k^i - s_i y_k^i)+cx_k y_k}dx_k\,dy_k$$
$$= \iint_{\mathcal{H}_n^2(E)} e^{c\,\mathrm{Tr}(M_1M_2)}e^{\mathrm{Tr}\sum_1^\infty (t_i M_1^i - s_i M_2^i)}\,dM_1\,dM_2, \tag{3.3.4}$$

satisfy the following relations:

(i) Virasoro constraints[22] (2.2.8) *for* $k \geq -1$:

$$\left(-\sum_{i=1}^{r} c_i^{k+1}\frac{\partial}{\partial c_i} + J_{k,n}^{(2)}\right)\tau_n^E + c\,\mathbf{s}_{k+n}(\tilde{\partial}_t)\mathbf{s}_n(-\tilde{\partial}_s)\tau_1^E \circ \tau_{n-1}^E = 0,$$
$$\left(-\sum_{i=1}^{s} \tilde{c}_i^{k+1}\frac{\partial}{\partial \tilde{c}_i} + \tilde{J}_{k,n}^{(2)}\right)\tau_n^E + c\,\mathbf{s}_{n}(\tilde{\partial}_t)\mathbf{s}_{k+n}(-\tilde{\partial}_s)\tau_1^E \circ \tau_{n-1}^E = 0, \tag{3.3.5}$$

with

$$J_{k,n}^{(2)} = \tfrac{1}{2}\big(J_k^{(2)} + (2n+k+1)J_k^{(1)} + n(n+1)J_k^{(0)}\big),$$
$$\tilde{J}_{k,n}^{(2)} = \tfrac{1}{2}\big(\tilde{J}_k^{(2)} + (2n+k+1)\tilde{J}_k^{(1)} + n(n+1)J_k^{(0)}\big).$$

(ii) *A* Wronskian identity:[23]

$$\left\{\frac{\partial^2 \log\tau_n}{\partial t_1\partial s_2}, \frac{\partial^2 \log\tau_n}{\partial t_1\partial s_1}\right\}_{t_1} + \left\{\frac{\partial^2 \log\tau_n}{\partial s_1\partial t_2}, \frac{\partial^2 \log\tau_n}{\partial t_1\partial s_1}\right\}_{s_1} = 0. \tag{3.3.6}$$

(iii) *The* 2d-Toda lattice: *Given the factorization* (3.3.2), *the matrices* $L_1 := S_1\Lambda S_1^{-1}$ *and* $L_2 := S_2\Lambda^\top S_2^{-1}$, *with* $h_n = \tau_{n+1}/\tau_n$, *have the following form, where the* $(k-l)$*-th subdiagonal is given by the diagonal matrix in front of* Λ^{k-l}:

$$L_1^k = \sum_{l=0}^{\infty} \operatorname{diag}\left(\frac{\mathbf{s}_l(\tilde{\partial}_t)\tau_{n+k-l+1}\circ\tau_n}{\tau_{n+k-l+1}\tau_n}\right)_{n\in\mathbb{Z}}\Lambda^{k-l}$$
$$hL_2^{\top k}h^{-1} = \sum_{l=0}^{\infty} \operatorname{diag}\left(\frac{\mathbf{s}_l(-\tilde{\partial}_s)\tau_{n+k-l+1}\circ\tau_n}{\tau_{n+k-l+1}\tau_n}\right)_{n\in\mathbb{Z}}\Lambda^{k-l}, \tag{3.3.7}$$

and satisfy the 2**d-Toda Lattice**[24]

$$\frac{\partial L_i}{\partial t_n} = [(L_1^n)_+, L_i] \quad \text{and} \quad \frac{\partial L_i}{\partial s_n} = [(L_2^n)_-, L_i], \quad i = 1, 2. \tag{3.3.8}$$

(iv) Bi-orthogonal polynomials: *The expressions*

$$p_n^{(1)}(t,s;y) := (S_1(t,s)\chi)y))_n = y^n\frac{\tau_n(t-[y^{-1}],s)}{\tau_n(t,s)},$$
$$p_n^{(2)}(t,s;z) := \big(hS_2^{\top-1}(t,s)\chi(z)\big)_n = z^n\frac{\tau_n(t,s+[z^{-1}])}{\tau_n(t,s)} \tag{3.3.9}$$

---

[22]For the Hirota symbol, see footnote 12. The $J_k^{(i)}$'s are as in remark 1 at the end of Theorem 2.1, for $\beta = 1$ and $\tilde{J}_k^{(i)} = J_k^{(i)}|_{t\to -s}$, with

$$J_k^{(1)} = \frac{\partial}{\partial t_k} + (-k)t_{-k}, \quad J_k^{(2)} = \sum_{i+j=k}\frac{\partial^2}{\partial t_i\partial t_j} + 2\sum_{-i+j=k} it_i\frac{\partial}{\partial t_j} + \sum_{-i-j=k} it_ijt_j.$$

[23]in terms of the Wronskian $\{f,g\}_t = \dfrac{\partial f}{\partial t}g - f\dfrac{\partial g}{\partial t}$.

[24]P_+ and P_- denote the upper (including diagonal) and strictly lower triangular parts of the matrix P, respectively.

form a system of monic bi-orthogonal polynomials in z:

$$\langle p_n^{(1)}(t,s;y), p_m^{(2)}(t,s;z)\rangle_{t,s} = \delta_{n,m} h_n \quad \textit{with } h_n = \frac{\tau_{n+1}}{\tau_n}, \tag{3.3.10}$$

which are also eigenvectors of L_1 *and* L_2:

$$\begin{aligned} z p_n^{(1)}(t,s;z) &= L_1(t,s) p_n^{(1)}(t,s;z), \\ z p_n^{(2)}(t,s;z) &= L_2^\top(t,s) p_n^{(2)}(t,s;z). \end{aligned} \tag{3.3.11}$$

REMARK. Each statement can be dualized via the duality $t \leftrightarrow -s$, $L_1 \leftrightarrow hL_2^\top h^{-1}$.

3.3.2. Sketch of proof. Identity (3.3.4) follows from the fact that the product of the two Vandermonde appearing in the integral (3.3.4) can be expressed as a sum of determinants:

$$\Delta_n(u)\Delta_n(v) = \sum_{\sigma\in S_n} \det\big(u_{\sigma(k)}^{l-1} v_{\sigma(k)}^{k-1}\big)_{1\le l,k\le n}, \tag{3.3.12}$$

together with the Harish-Chandra, Itzykson and Zuber formula [34; 38]

$$\int_{U(n)} dU\, e^{c\operatorname{Tr} xUy\bar{U}^\top} = \frac{(2\pi)^{n(n-1)/2}}{n!} \frac{\det(e^{cx_iy_j})_{1\le i,j\le n}}{\Delta_n(x)\Delta_n(y)}. \tag{3.3.13}$$

Moreover the τ_n's satisfy the following bilinear identities, for all integer $m,n\ge 0$ and $t,s\in\mathbb{C}^\infty$:

$$\begin{aligned} &\oint_{z=\infty} \tau_n(t-[z^{-1}],s)\tau_{m+1}(t'+[z^{-1}],s') e^{\sum_1^\infty (t_i-t_i')z^i} z^{n-m-1}\,dz \\ &\qquad = \oint_{z=0} \tau_{n+1}(t,s-[z])\tau_m(t',s'+[z]) e^{\sum_1^\infty (s_i-s_i')z^{-i}} z^{n-m-1}\,dz. \end{aligned} \tag{3.3.14}$$

Again, the bi-orthogonal nature (3.3.10) of the polynomials (3.3.9) is tantamount to the Borel decomposition, written in the form $S_1 m_\infty (hS_2^{\top -1})^\top = h$. These polynomials satisfy the eigenvalue problem (3.3.11) and evolve in t,s according to the differential equations

$$\begin{aligned} \frac{\partial p^{(1)}}{\partial t_n} &= -(L_1^n)_- p^{(1)}, & \frac{\partial p^{(1)}}{\partial s_n} &= -(L_2^n)_- p^{(1)}, \\ \frac{\partial p^{(2)}}{\partial t_n} &= -\big((h^{-1}L_1h)^{\top n}\big)_- p^{(2)}, & \frac{\partial p^{(2)}}{\partial s_n} &= \big((h^{-1}L_2h)^{\top n}\big)_- p^{(2)}. \end{aligned} \tag{3.3.15}$$

From the representation (3.3.7) and the bilinear identity (3.3.14), it follows that

$$\frac{p_{k-1}(\tilde{\partial}_t)\tau_{n+2}\circ\tau_n}{\tau_{n+1}^2} = -\frac{\partial^2}{\partial s_1\partial t_k}\log\tau_{n+1}, \tag{3.3.16}$$

and so, for $k=1$,

$$\frac{\tau_n\tau_{n+2}}{\tau_{n+1}^2} = -\frac{\partial^2}{\partial s_1\partial t_1}\log\tau_{n+1}. \tag{3.3.17}$$

Thus, using (3.3.7), (3.3.16) and (3.3.17), we have

$$\left(L_1^k\right)_{n,n+1} = \frac{p_{k-1}(\tilde{\partial}_t)\tau_{n+2}\circ\tau_n}{\tau_{n+2}\tau_n} = \frac{\dfrac{\partial^2 \log\tau_{n+1}}{\partial s_1\partial t_k}}{\dfrac{\partial^2 \log\tau_{n+1}}{\partial s_1\partial t_1}},$$

$$\left(hL_2^{\top k}h^{-1}\right)_{n,n+1} = \frac{p_{k-1}(-\tilde{\partial}_s)\tau_{n+2}\circ\tau_n}{\tau_{n+2}\tau_n} = \frac{\dfrac{\partial^2 \log\tau_{n+1}}{\partial t_1\partial s_k}}{\dfrac{\partial^2 \log\tau_{n+1}}{\partial s_1\partial t_1}}. \tag{3.3.18}$$

Combining (3.3.18) with (3.3.17) for $k=2$ yields

$$\begin{aligned}\left(L_1^2\right)_{n,n+1} = \frac{\dfrac{\partial^2}{\partial s_1\partial t_2}\log\tau_{n+1}}{\dfrac{\partial^2}{\partial s_1\partial t_1}\log\tau_{n+1}} &= \frac{\partial}{\partial t_1}\log\left(-\frac{\tau_{n+2}}{\tau_n}\right) \\ &= \frac{\partial}{\partial t_1}\log\left(\left(\frac{\tau_{n+1}}{\tau_n}\right)^2\frac{\partial^2}{\partial s_1\partial t_1}\log\tau_{n+1}\right).\end{aligned} \tag{3.3.19}$$

Then, subtracting $\partial/\partial s_1$ of (3.3.19) from $\partial/\partial t_1$ of the dual of the same equation (see remark at the end of Theorem 3.4) leads to (3.3.6).

3.4. The Toeplitz Lattice and Unitary Matrix Integrals

3.4.1. Toeplitz lattice, factorization of moment matrices and bi-orthogonal polynomials. Recall that a Toeplitz matrix is one whose (i,j)-th entry depends only on $i-j$. Consider the inner product

$$\langle f(z), g(z)\rangle_{t,s} := \oint_{S^1}\frac{dz}{2\pi i z}f(z)g(z^{-1})e^{\sum_1^\infty(t_iz^i - s_iz^{-i})},\quad t,s\in\mathbb{C}^\infty, \tag{3.4.1}$$

where the integral is taken over the unit circle $S^1\subset\mathbb{C}$ around the origin. It has the property

$$\langle z^k f, g\rangle_{t,s} = \langle f, z^{-k}g\rangle_{t,s}. \tag{3.4.2}$$

The t,s-dependent semi-infinite moment matrix $m_\infty(t,s)$, where

$$\begin{aligned}m_n(t,s) := \left(\langle z^k, z^l\rangle_{t,s}\right)_{0\le k,l\le n-1} &= \left(\oint_{S^1}\frac{\rho(z)\,dz}{2\pi i z}z^{k-l}e^{\sum_1^\infty(t_iz^i-s_iz^{-i})}\right)_{0\le k,l\le n-1} \\ &= \text{Toeplitz matrix}\end{aligned} \tag{3.4.3}$$

satisfies the same differential equations as in (3.3.3):

$$\frac{\partial m_\infty}{\partial t_n} = \Lambda^n m_\infty \quad\text{and}\quad \frac{\partial m_\infty}{\partial s_n} = -m_\infty\Lambda^{\top n}. \qquad \text{(2-Toda Lattice)} \tag{3.4.4}$$

As before, define

$$\tau_n(t,s) := \det m_n(t,s).$$

Also, consider the factorization $m_\infty(t,s) = S_1^{-1}(t,s)S_2(t,s)$, as in (3.3.2), from which one defines $L_1 := S_1 \Lambda S_1^{-1}$ and $L_2 := S_2 \Lambda^\top S_2^{-1}$ and the bi-orthogonal polynomials $p_i^{(k)}(t,s;z)$ for $k = 1,2$. Since m_∞ satisfies the same equations (3.3.3), the matrices L_1 and L_2 satisfy the 2-Toda lattice equations; the Toeplitz nature of m_∞ implies a peculiar "rank 2"-structure, with $h_i/h_{i-1} = 1 - x_i y_i$ and $x_0 = y_0 = 1$:

$$h^{-1}L_1 h = \begin{pmatrix} -x_1y_0 & 1-x_1y_1 & 0 & 0 & \\ -x_2y_0 & -x_2y_1 & 1-x_2y_2 & 0 & \\ -x_3y_0 & -x_3y_1 & -x_3y_2 & 1-x_3y_3 & \\ -x_4y_0 & -x_4y_1 & -x_4y_2 & -x_4y_3 & \\ & & & & \ddots \end{pmatrix}$$

and

$$L_2 = \begin{pmatrix} -x_0y_1 & -x_0y_2 & -x_0y_3 & -x_0y_4 & \\ 1-x_1y_1 & -x_1y_2 & -x_1y_3 & -x_1y_4 & \\ 0 & 1-x_2y_2 & -x_2y_3 & -x_2y_4 & \\ 0 & 0 & 1-x_3y_3 & -x_3y_4 & \\ & & & & \ddots \end{pmatrix}. \tag{3.4.5}$$

Some of the ideas in the next theorem are inspired by the work of Hisakado [37].

THEOREM 3.5 (Adler and van Moerbeke [7]). *The integrals $I_n(t,s)$, with $I_0 = 1$,*

$$\begin{aligned} \tau_n(t,s) = \det m_n = \frac{1}{n!} I_n :&= \frac{1}{n!} \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n \left(e^{\sum_1^\infty (t_i z_k^i - s_i z_k^{-i})} \frac{dz_k}{2\pi i z_k} \right) \\ &= \int_{U(n)} e^{\sum_1^\infty \operatorname{Tr}(t_i M^i - s_i \bar{M}^i)}\, dM \\ &= \sum_{\{\text{Young diagrams } \lambda \mid \hat{\lambda}_1 \le n\}} \mathbf{s}_\lambda(t) \mathbf{s}_\lambda(-s), \end{aligned} \tag{3.4.6}$$

satisfy the following relations:

(i) *An* $\mathrm{SL}(2,\mathbb{Z})$*-algebra of three* *Virasoro constraints* (2.3.2):

$$\mathbb{J}_{k,n}^{(2)}(t,n) - \mathbb{J}_{-k,n}^{(2)}(-s,n) - k\left(\theta \mathbb{J}_{k,n}^{(1)}(t,n) + (1-\theta)\mathbb{J}_{-k,n}^{(1)}(-s,n)\right) I_n(t,s) = 0,$$

$$\text{for } \begin{cases} k=-1, & \theta = 0, \\ k=0, & \theta \text{ arbitrary}, \\ k=1, & \theta=1. \end{cases} \tag{3.4.7}$$

(ii) *2d-Toda identities*: *The matrices L_1 and L_2 defined above satisfy the 2-Toda lattice equations* (3.3.8); *in particular*,

$$\frac{\partial^2}{\partial s_1 \partial t_1} \log \tau_n = -\frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2}$$

and

$$\frac{\partial^2}{\partial s_2 \partial t_1} \log \tau_n = -2 \frac{\partial}{\partial s_1} \log \frac{\tau_n}{\tau_{n-1}} \cdot \frac{\partial^2}{\partial s_1 \partial t_1} \log \tau_n - \frac{\partial^3}{\partial s_1^2 \partial t_1} \log \tau_n, \tag{3.4.8}$$

the first being equivalent to the discrete sinh-Gordon equation

$$\frac{\partial^2 q_n}{\partial t_1 \partial s_1} = e^{q_n - q_{n-1}} - e^{q_{n+1} - q_n}, \quad \textit{where } q_n = \log \frac{\tau_{n+1}}{\tau_n}.$$

(iii) *The Toeplitz lattice: The 2-Toda lattice solution is a very special one — the matrices L_1 and L_2 have a "rank 2" structure, given by* (3.4.5), *whose x_n's and y_n's equal*[25]

$$\begin{aligned} x_n(t,s) &= \frac{1}{\tau_n} \int_{U(n)} \mathbf{s}_n\big(-\operatorname{Tr} M, -\tfrac{1}{2} \operatorname{Tr} M^2, -\tfrac{1}{3} \operatorname{Tr} M^3, \dots \big)\, e^{\sum_1^\infty \operatorname{Tr}(t_i M^i - s_i \bar{M}^i)}\, dM \\ &= \frac{\mathbf{s}_n(-\frac{\partial}{\partial t_1}, -\frac{1}{2}\frac{\partial}{\partial t_2}, -\frac{1}{3}\frac{\partial}{\partial t_3}, \dots) \tau_n(t,s)}{\tau_n(t,s)} = p_n^{(1)}(t,s;0), \\ y_n(t,s) &= \frac{1}{\tau_n} \int_{U(n)} \mathbf{s}_n\big(-\operatorname{Tr} \bar{M}, -\tfrac{1}{2} \operatorname{Tr} \bar{M}^2, -\tfrac{1}{3} \operatorname{Tr} \bar{M}^3, \dots \big)\, e^{\sum_1^\infty \operatorname{Tr}(t_i M^i - s_i \bar{M}^i)}\, dM \\ &= \frac{\mathbf{s}_n(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, \dots) \tau_n(t,s)}{\tau_n(t,s)} = p_n^{(2)}(t,s;0), \end{aligned} \tag{3.4.9}$$

and satisfy the integrable Hamiltonian system

$$\begin{aligned} \frac{\partial x_n}{\partial t_i} &= (1 - x_n y_n) \frac{\partial H_i^{(1)}}{\partial y_n}, \quad \frac{\partial y_n}{\partial t_i} = -(1 - x_n y_n) \frac{\partial H_i^{(1)}}{\partial x_n}, \\ \frac{\partial x_n}{\partial s_i} &= (1 - x_n y_n) \frac{\partial H_i^{(2)}}{\partial y_n}, \quad \frac{\partial y_n}{\partial s_i} = -(1 - x_n y_n) \frac{\partial H_i^{(2)}}{\partial x_n} \end{aligned} \tag{3.4.10}$$

(**Toeplitz lattice**), *with initial condition $x_n(0,0) = y_n(0,0) = 0$ for $n \geq 1$ and boundary condition $x_0(t,s) = y_0(t,s) = 1$. The traces*

$$H_i^{(k)} = -\frac{1}{i} \operatorname{Tr} L_k^i, \quad i = 1,2,3,\dots, \ k = 1,2$$

of the matrices L_i in (3.4.5) *are integrals in involution with regard to the symplectic structure $\omega := \sum_0^\infty (1 - x_k y_k)^{-1}\, dx_k \wedge dy_k$. The Toeplitz nature of m_∞ leads to identities between τ's, the simplest* (due to Hisakado [37]) *being:*

$$\Big(1 + \frac{\partial^2}{\partial s_1 \partial t_1} \log \tau_{n+1}\Big)\Big(1 + \frac{\partial^2}{\partial s_1 \partial t_1} \log \tau_n\Big) = -\frac{\partial}{\partial t_1} \log \frac{\tau_{n+1}}{\tau_n} \frac{\partial}{\partial s_1} \log \frac{\tau_{n+1}}{\tau_n}. \tag{3.4.11}$$

[25] Remember that the $\mathbf{s}(t_1, t_2, \dots)$ are elementary Schur polynomials.

REMARK. The first equation in the hierarchy above reads:

$$\frac{\partial x_n}{\partial t_1} = x_{n+1}(1 - x_n y_n), \quad \frac{\partial y_n}{\partial t_1} = -y_{n-1}(1 - x_n y_n),$$
$$\frac{\partial x_n}{\partial s_1} = x_{n-1}(1 - x_n y_n), \quad \frac{\partial y_n}{\partial s_1} = -y_{n+1}(1 - x_n y_n).$$

3.4.2. Sketch of Proof. The identity (3.4.6) between the determinant and the moment matrix uses again the Vandermonde identity (3.3.12),

$$\begin{aligned}
&\int_{U(n)} e^{\sum_1^\infty \operatorname{Tr}(t_i M^i - s_i \bar{M}^i)}\, dM \\
&\quad = \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n \left(e^{\sum_1^\infty (t_i z_k^i - s_i z_k^{-i})} \frac{dz_k}{2\pi i z_k} \right) \\
&\quad = \int_{(S^1)^n} \Delta_n(z)\Delta_n(\bar z) \prod_{k=1}^n \left(e^{\sum_1^\infty (t_i z_k^i - s_i z_k^{-i})} \frac{dz_k}{2\pi i z_k} \right) \\
&\quad = \int_{(S^1)^n} \sum_{\sigma\in S_n} \det \left(z_{\sigma(m)}^{l-1} \bar z_{\sigma(m)}^{m-1} \right)_{1\le l,m\le n} \prod_{k=1}^n \left(e^{\sum_1^\infty (t_i z_k^i - s_i z_k^{-i})} \frac{dz_k}{2\pi i z_k} \right) \\
&\quad = \sum_{\sigma\in S_n} \det \left(\oint_{S^1} z_k^{l-1} \bar z_k^{m-1} e^{\sum_1^\infty (t_i z_k^i - s_i z_k^{-i})} \frac{dz_k}{2\pi i z_k} \right)_{1\le l,m\le n} \\
&\quad = n! \det \left(\oint_{S^1} z^{l-m} e^{\sum_1^\infty (t_i z^i - s_i z^{-i})} \frac{dz}{2\pi i z} \right)_{1\le l,m\le n} \\
&\quad = n! \det m_n(t,s) = n!\, \tau_n(t).
\end{aligned}$$

The last equation of (3.4.6) follows from the fact that the solution to equations (3.4.4) with initial condition $m_\infty(0,0)$ is given by

$$m_\infty(t,s) = e^{\sum_1^\infty t_i \Lambda^i} m_\infty(0,0) e^{-\sum_1^\infty s_i \Lambda^{\top i}}.$$

Since (3.4.3) gives the initial condition $m_\infty(0,0) = I_\infty$, the Cauchy–Binet formula implies

$$\begin{aligned}
\tau_n(t,s) &= \det m_n(t,s) \\
&= \sum_{\substack{\lambda,\nu \\ \hat\lambda_1, \hat\nu_1 \le n}} s_\lambda(t) s_\nu(-s) \det \left(\mu_{\lambda_i - i + n, \nu_j - j + n} \right)_{1\le i,j\le n} = \sum_{\{\lambda \mid \hat\lambda \le n\}} \mathbf{s}_\lambda(t) \mathbf{s}_\lambda(-s),
\end{aligned}$$

establishing the last equation of (3.4.6); details can be found in [7].

Using the equality $z^{k\top} = z^{-k}$ (see (3.4.2)), one shows that the polynomials

$$p_{n+1}^{(1)}(z) - z p_n^{(1)}(z) \quad \text{and} \quad p_{n+1}^{(1)}(0) z^n p_n^{(2)}(z^{-1})$$

are perpendicular to the monomials $z^0, z^1, \dots, z^n$ and that they have the same z^0-term; one makes a similar argument, by dualizing $1 \leftrightarrow 2$. Therefore, we have the Hisakado identities between the polynomials

$$\begin{aligned} p_{n+1}^{(1)}(z) - zp_n^{(1)}(z) &= p_{n+1}^{(1)}(0) z^n p_n^{(2)}(z^{-1}) \\ p_{n+1}^{(2)}(z) - zp_n^{(2)}(z) &= p_{n+1}^{(2)}(0) z^n p_n^{(1)}(z^{-1}). \end{aligned} \tag{3.4.12}$$

The rank 2 structure (3.4.5) of L_1 and L_2, with $x_n = p_n^{(1)}(t,s;0)$ and $y_n = p_n^{(2)}(t,s;0)$, is obtained by taking the inner product of $p_{n+1}^{(1)}(z) - zp_n^{(1)}(z)$ with itself, for different n and m, and using the fact that $zp_n^{(1)}(z) = L_1 p_n^{(1)}(z)$.

To check the first equation in the hierarchy (see remark at the end of Theorem 3.5), consider, from (3.4.9),

$$\begin{aligned} \frac{\partial x_n}{\partial t_1} &= \frac{\partial p_n^{(1)}(t,s;z)}{\partial t_1}\Big|_{z=0} = -\big((L_1)_- p^{(1)}\big)_n\big|_{z=0} && \text{using (3.3.15)} \\ &= h_n p_{n+1}^{(1)}(t,s;0) \sum_{i=0}^{n-1} \frac{p_i^{(1)}(t,s;0) p_i^{(2)}(t,s;0)}{h_i} && \text{using (3.4.5)} \\ &= h_n x_{n+1} \sum_{i=0}^{n-1} \frac{x_i y_i}{h_i} \\ &= h_n x_{n+1} \sum_{i=0}^{n-1} \left(\frac{1}{h_i} - \frac{1}{h_{i-1}} \right) && \text{using } \frac{h_i}{h_{i-1}} = 1 - x_i y_i \\ &= x_{n+1} \frac{h_n}{h_{n-1}} = x_{n+1}(1 - x_n y_n), \end{aligned}$$

and similarly for the other coordinates. From (3.3.7) and (3.4.5), upon making the products of the corresponding diagonal entries of L_1 and $hL_2^\top h^{-1}$, one finds (3.4.11):

$$\begin{aligned} \frac{\partial}{\partial t_1} \log \frac{\tau_{n+1}}{\tau_n} \frac{\partial}{\partial s_1} \log \frac{\tau_{n+1}}{\tau_n} &= -x_{n+1} y_n x_n y_{n+1} = -x_n y_n x_{n+1} y_{n+1} \\ &= -\left(1 - \frac{h_n}{h_{n-1}}\right)\left(1 - \frac{h_{n+1}}{h_n}\right). \end{aligned}$$

4. Ensembles of Finite Random Matrices

4.1. PDEs Defined by the Probabilities in Hermitian, Symmetric and Symplectic Random Ensembles. As used earlier, the disjoint union $E = \bigcup_1^r [c_{2i-1}, c_{2i}] \subset \mathbb{R}$, and the weight $\rho(z) = e^{-V(z)}$, with $-\rho'/\rho = V' = g/f$ define an algebra of differential operators

$$\mathcal{B}_k = \sum_1^{2r} c_i^{k+1} f(c_i) \frac{\partial}{\partial c_i}, \quad k \in \mathbb{Z}.$$

The aim of this section is to find PDEs for the following probabilities in terms of the boundary points c_i of E (see (1.1.9), (1.1.11) and (1.1.18)), i.e.

$$\begin{aligned} P_n(E) :&= P_n(\text{ all spectral points of } M \in E) \\ &= \frac{\int_{\mathcal{H}_n(E),\ \mathcal{S}_n(E)\ \text{or}\ \mathcal{T}_n(E)} e^{-tr\ V(M)}\, dM}{\int_{\mathcal{H}_n(\mathbb{R}),\ \mathcal{S}_n(\mathbb{R})\ \text{or}\ \mathcal{T}_n(\mathbb{R})} e^{-tr\ V(M)}\, dM} \\ &= \frac{\int_{E^n} |\Delta_n(z)|^\beta \prod_{k=1}^n e^{-V(z_k)}\, dz_k}{\int_{\mathbb{R}^n} |\Delta_n(z)|^\beta \prod_{k=1}^n e^{-V(z_k)}\, dz_k}, \quad \beta = 2, 1, 4 \text{ respectively,} \end{aligned} \tag{4.1.1}$$

involving the classical weights below. In anticipation, the equations obtained in Theorems 4.1, 4.2 and 4.3 are closely related to three of the six Painlevé differential equations:

weight	$\rho(z)$	Painlevé
Gauss	e^{-bz^2}	IV
Laguerre	$z^a e^{-bz}$	V
Jacobi	$(1-z)^a(1+z)^b$	VI

For $\beta = 2$, the probabilities satisfy partial differential equations in the boundary points of E, whereas in the case $\beta = 1, 4$, the equations are inductive. Namely, for $\beta = 1$ (resp. $\beta = 4$) , the probabilities P_{n+2} (resp. P_{n+1}) are given in terms of P_{n-2} (resp. P_{n-1}) and a differential operator acting on P_n. The weights above involve the parameters β, a, b and

$$\delta_{1,4}^\beta := 2\left(\left(\frac{\beta}{2}\right)^{1/2} - \left(\frac{\beta}{2}\right)^{-1/2}\right)^2 = \begin{cases} 0 & \text{for } \beta = 2, \\ 1 & \text{for } \beta = 1, 4. \end{cases}$$

As a consequence of the duality (2.1.12) between β-Virasoro generators under the map $\beta \mapsto 4/\beta$, and the equations (2.1.7), the PDEs obtained have a remarkable property: the coefficients Q and Q_i of the PDEs are functions in the variables n, β, a, b, having the invariance property under the map

$$n \to -2n, \quad a \to -\frac{a}{2}, \quad b \to -\frac{b}{2};$$

to be precise,

$$Q_i\Big(-2n, \beta, -\frac{a}{2}, -\frac{b}{2}\Big)\Big|_{\beta=1} = Q_i(n, \beta, a, b)\big|_{\beta=4}. \tag{4.1.2}$$

The results in this section are mainly due to Adler, Shiota, and van Moerbeke [11] for $\beta = 2$ and to Adler and van Moerbeke [6] for $\beta = 1, 4$. For more detailed references, see the end of Section 4.2.

4.1.1. Gaussian Hermitian, symmetric and symplectic ensembles. Given the disjoint union E and the weight e^{-bz^2}, the differential operators $\mathcal{B}_k$ take on the form

$$\mathcal{B}_k = \sum_1^{2r} c_i^{k+1} \frac{\partial}{\partial c_i}.$$

Define the *invariant* polynomials

$$Q = 12b^2 n\left(n+1-\frac{2}{\beta}\right) \quad \text{and} \quad Q_2 = 4(1+\delta_{1,4}^{\beta})b\left(2n+\delta_{1,4}^{\beta}\left(1-\frac{2}{\beta}\right)\right).$$

THEOREM 4.1. *The probabilities*

$$P_n(E) = \frac{\int_{E^n} |\Delta_n(z)|^\beta \prod_{k=1}^n e^{-bz_k^2}\, dz_k}{\int_{\mathbb{R}^n} |\Delta_n(z)|^\beta \prod_{k=1}^n e^{-bz_k^2}\, dz_k}, \tag{4.1.3}$$

with $\beta = 2, 1, 4$, *satisfy the following PDE, where we put* $F := F_n = \log P_n$:

$$\delta_{1,4}^{\beta} Q\left(\frac{P_{n-\substack{2\\1}} P_{n+\substack{2\\1}}}{P_n^2} - 1\right) \qquad \textit{with index}\begin{cases} 2 & \textit{when } n \textit{ even and } \beta = 1,\\ 1 & \textit{when } n \textit{ arbitrary and } \beta = 4\end{cases}$$
$$= \left(\mathcal{B}_{-1}^4 + (Q_2 + 6\mathcal{B}_{-1}^2 F)\mathcal{B}_{-1}^2 + 4(2-\delta_{1,4}^{\beta})\frac{b^2}{\beta}(3\mathcal{B}_0^2 - 4\mathcal{B}_{-1}\mathcal{B}_1 + 6\mathcal{B}_0)\right)F. \tag{4.1.4}$$

4.1.2. Laguerre Hermitian, symmetric and symplectic ensembles. Given the disjoint union $E \subset \mathbb{R}^+$ and the weight $z^a e^{-bz}$, the $\mathcal{B}_k$ take on the form

$$\mathcal{B}_k = \sum_1^{2r} c_i^{k+2} \frac{\partial}{\partial c_i}.$$

Define the polynomials, also respecting the duality (4.1.2),

$$Q = \begin{cases} \frac{3}{4}n(n-1)(n+2a)(n+2a+1) & \text{for } \beta = 1,\\ \frac{3}{2}n(2n+1)(2n+a)(2n+a-1) & \text{for } \beta = 4,\end{cases}$$
$$Q_2 = \left(3\beta n^2 - \frac{a^2}{\beta} + 6an + 4\left(1-\frac{\beta}{2}\right)a + 3\right)\delta_{1,4}^{\beta} + (1-a^2)(1-\delta_{1,4}^{\beta})$$
$$Q_1 = \left(\beta n^2 + 2an + \left(1-\frac{\beta}{2}\right)a\right), \quad Q_0 = b(2-\delta_{1,4}^{\beta})\left(n+\frac{a}{\beta}\right).$$

THEOREM 4.2. *The probabilities*

$$P_n(E) = \frac{\int_{E^n} |\Delta_n(z)|^\beta \prod_{k=1}^n z_k^a e^{-bz_k}\, dz_k}{\int_{\mathbb{R}_+^n} |\Delta_n(z)|^\beta \prod_{k=1}^n z_k^a e^{-bz_k}\, dz_k} \tag{4.1.5}$$

satisfy the following PDE, where we put $F := F_n = \log P_n$ *and use the same convention on the indices* $n \pm 2$ *and* $n \pm 1$ *as in* (4.1.4):

$$\delta_{1,4}^{\beta} Q\left(\frac{P_{n-\frac{2}{1}} P_{n+\frac{2}{1}}}{P_n^2} - 1\right) = \Big(\mathcal{B}_{-1}^4 - 2(\delta_{1,4}^{\beta}+1)\mathcal{B}_{-1}^3$$
$$+ (Q_2 + 6\mathcal{B}_{-1}^2 F - 4(\delta_{1,4}^{\beta}+1)\mathcal{B}_{-1}F)\mathcal{B}_{-1}^2 - 3\delta_{1,4}^{\beta}(Q_1 - \mathcal{B}_{-1}F)\mathcal{B}_{-1}$$
$$+ \frac{b^2}{\beta}(2-\delta_{1,4}^{\beta})(3\mathcal{B}_0^2 - 4\mathcal{B}_1\mathcal{B}_{-1} - 2\mathcal{B}_1) + Q_0(2\mathcal{B}_0\mathcal{B}_{-1} - \mathcal{B}_0)\Big)F. \quad (4.1.6)$$

4.1.3. Jacobi Hermitian, symmetric and symplectic ensembles. In terms of $E \subset [-1,+1]$ and the Jacobi weight $(1-z)^a(1+z)^b$, the differential operators $\mathcal{B}_k$ take on the form

$$\mathcal{B}_k = \sum_1^{2r} c_i^{k+1}(1-c_i^2)\frac{\partial}{\partial c_i}.$$

With $b_0 = a-b$, $b_1 = a+b$, introduce the variables

$$r = \frac{4}{\beta}(b_0^2 + (b_1+2-\beta)^2), \quad s = \frac{4}{\beta}b_0(b_1+2-\beta), \quad q_n = \frac{4}{\beta}(\beta n + b_1 + 2 - \beta)(\beta n + b_1),$$

which themselves have the invariance property (4.1.2). Introduce also the following *invariant* polynomials in q, r, s:

$$\begin{aligned}
Q &= \tfrac{3}{16}\left((s^2 - qr + q^2)^2 - 4(rs^2 - 4qs^2 - 4s^2 + q^2 r)\right), \\
Q_1 &= 3s^2 - 3qr - 6r + 2q^2 + 23q + 24 \\
Q_2 &= 3qs^2 + 9s^2 - 4q^2 r + 2qr + 4q^3 + 10q^2, \\
Q_3 &= 3qs^2 + 6s^2 - 3q^2 r + q^3 + 4q^2 \\
Q_4 &= 9s^2 - 3qr - 6r + q^2 + 22q + 24 = Q_1 + (6s^2 - q^2 - q). \qquad (4.1.7)
\end{aligned}$$

THEOREM 4.3. *The probabilities*

$$P_n(E) = \frac{\int_{E^n} |\Delta_n(z)|^\beta \prod_{k=1}^n (1-z_k)^a (1+z_k)^b \, dz_k}{\int_{[-1,1]^n} |\Delta_n(z)|^\beta \prod_{k=1}^n (1-z_k)^a (1+z_k)^b \, dz_k} \qquad (4.1.8)$$

satisfy the following PDE, where we put $F = F_n = \log P_n$:

for $\beta = 2$:

$$\big(2\mathcal{B}_{-1}^4 + (q-r+4)\mathcal{B}_{-1}^2 - (4\mathcal{B}_{-1}F - s)\mathcal{B}_{-1} + 3q\mathcal{B}_0^2 - 2q\mathcal{B}_0 + 8\mathcal{B}_0\mathcal{B}_{-1}^2$$
$$- 4(q-1)\mathcal{B}_1\mathcal{B}_{-1} + (4\mathcal{B}_{-1}F - s)\mathcal{B}_1 + 2(4\mathcal{B}_{-1}F - s)\mathcal{B}_0\mathcal{B}_{-1} + 2q\mathcal{B}_2\big)F$$
$$+ 4\mathcal{B}_{-1}^2 F\left(2\mathcal{B}_0 F + 3\mathcal{B}_{-1}^2 F\right) = 0 \quad (4.1.9)$$

for $\beta = 1, 4$:

$$
\begin{aligned}
&Q\left(\frac{P_{n+{2\atop 1}}P_{n-{2\atop 1}}}{P_n^2}-1\right)\\
&\quad=(q+1)\big(4q\mathcal{B}_{-1}^4+12(4\mathcal{B}_{-1}F-s)\mathcal{B}_{-1}^3+2\,(q+12)\,(4\mathcal{B}_{-1}F-s)\mathcal{B}_0\mathcal{B}_{-1}\\
&\qquad\qquad+3q^2\mathcal{B}_0^2-4(q-4)q\mathcal{B}_1\mathcal{B}_{-1}+q(4\mathcal{B}_{-1}F-s)\mathcal{B}_1+20q\mathcal{B}_0\mathcal{B}_{-1}^2+2q^2\mathcal{B}_2\big)F\\
&\qquad+\big(Q_2\mathcal{B}_{-1}^2-sQ_1\mathcal{B}_{-1}+Q_3\mathcal{B}_0\big)F+48(\mathcal{B}_{-1}F)^4-48s(\mathcal{B}_{-1}F)^3+2Q_4(\mathcal{B}_{-1}F)^2\\
&\qquad+12\,q^2(\mathcal{B}_0F)^2+16\,q\,\left(2\,q-1\right)(\mathcal{B}_{-1}^2F)(\mathcal{B}_0F)+24\,(q-1)\,q(\mathcal{B}_{-1}^2F)^2\\
&\qquad+24\big(2\mathcal{B}_{-1}F-s\big)\big((q+2)\mathcal{B}_0F+(q+3)\mathcal{B}_{-1}^2F\big)\mathcal{B}_{-1}F. \qquad (4.1.10)
\end{aligned}
$$

The proof of these three theorems will be sketched in Sections 4.3, 4.4 and 4.5.

4.2. ODEs When E Has One Boundary Point. Assume the set E consists of one boundary point $c = x$, besides the boundary of the full range; thus, setting respectively $E = [-\infty, x]$, $E = [0, x]$, $E = [-1, x]$ in the PDEs (4.1.4), (4.1.6) and (4.1.9), (4.1.10), leads to the equations in x below. Notice that, for $\beta = 2$, the equations obtained are ODEs and, for $\beta = 1, 4$, these equations express P_{n+2} in terms of P_{n-2} and a differential operator acting on P_n:

(1) *Gauss ensemble* ($\beta = 2, 1, 4$) : $f_n(x) = \dfrac{d}{dx}\log P_n(\max_i\ \lambda_i \le x)$ satisfies

$$
\begin{aligned}
&\delta_{1,4}^{\beta}Q\left(\frac{P_{n-{2\atop 1}}P_{n+{2\atop 1}}}{P_n^2}-1\right)\\
&\qquad=f_n'''+6f_n'^2+\left(4\frac{b^2x^2}{\beta}(\delta_{1,4}^{\beta}-2)+Q_2\right)f_n'-4\frac{b^2x}{\beta}(\delta_{1,4}^{\beta}-2)f_n. \quad (4.2.1)
\end{aligned}
$$

(2) *Laguerre ensemble* ($\beta = 2, 1, 4$): $f_n(x) = x\dfrac{d}{dx}\log P_n(\max_i\ \lambda_i \le x)$ (with all eigenvalues $\lambda_i \ge 0$) satisfies:

$$
\begin{aligned}
&\delta_{1,4}^{\beta}Q\left(\frac{P_{n-{2\atop 1}}P_{n+{2\atop 1}}}{P_n^2}-1\right)-\left(3\delta_{1,4}^{\beta}f_n-\frac{b^2x^2}{\beta}(\delta_{1,4}^{\beta}-2)-Q_0x-3\delta_{1,4}^{\beta}Q_1\right)f_n\\
&\quad=x^3f_n'''-(2\delta_{1,4}^{\beta}-1)x^2f_n''+6x^2f_n'^2\\
&\qquad-x\left(4(\delta_{1,4}^{\beta}+1)f_n-\frac{b^2x^2}{\beta}(\delta_{1,4}^{\beta}-2)-2Q_0x-Q_2+2\delta_{1,4}^{\beta}+1\right)f_n'. \quad (4.2.2)
\end{aligned}
$$

(3) *Jacobi ensemble*: $f := f_n(x) = (1-x^2)\dfrac{d}{dx}\log P_n(\max_i \lambda_i \le x)$ (with all eigenvalues $-1 \le \lambda_i \le 1$) satisfies:

- for $\underline{\beta = 2}$:

$$
\begin{aligned}
2(x^2-1)^2f'''+4(x^2-1)\left(xf''-3f'^2\right)+\big(16xf-q(x^2-1)-2sx-r\big)\,f'&\\
-f\,(4f-qx-s)=0,&\quad (4.2.3)
\end{aligned}
$$

- for $\underline{\beta = 1, 4}$:

$$Q\left(\frac{P_{n+\frac{2}{1}}P_{n-\frac{2}{1}}}{P_n^2} - 1\right)$$
$$= 4(q+1)(x^2-1)^2\big(-q(x^2-1)f''' + (12f - qx - 3s)f'' + 6q(q-1)f'^2\big)$$
$$-(x^2-1)f'\big(24f(q+3)(2f-s) + 8fq(5q-1)x - q(q+1)(qx^2+2sx+8) + Q_2\big)$$
$$+ f\big(48f^3 + 48f^2(qx+2x-s) + 2f(8q^2x^2 + 2qx^2 - 12qsx - 24sx + Q_4)$$
$$- q(q+1)x(3qx^2 + sx - 2qx - 3q) + Q_3x - Q_1 s\big). \tag{4.2.4}$$

For $\beta = 2$, the term containing the ratio $(P_{n+\frac{2}{1}}P_{n-\frac{2}{1}}/P_n^2) - 1$ on the left-hand side of (4.2.1), (4.2.2) and (4.2.4) vanishes, and one thus obtains the following ODEs:

- **Gauss**: $f_n(x) := \dfrac{d}{dx}\log P_n(\max_i\ \lambda_i \le x)$ satisfies
$$f''' + 6f'^2 + 4b(2n - bx^2)f' + 4b^2xf = 0.$$
- **Laguerre**: $f_n(x) := x\dfrac{d}{dx}\log P_n(\max_i\ \lambda_i \le x)$ satisfies
$$x^2f''' + xf'' + 6xf'^2 - 4ff' - ((a-bx)^2 - 4nbx)f' - b(2n + a - bx)f = 0.$$
- **Jacobi**: $f_n(x) = (1-x^2)\dfrac{d}{dx}\log P_n(\max_i\lambda_i \le x)$ satisfies
$$2(x^2-1)^2f''' + 4(x^2-1)\left(xf'' - 3f'^2\right) + \left(16xf - q(x^2-1) - 2sx - r\right)f'$$
$$- f\,(4f - qx - s) = 0.$$

Each of these three equations is of the Chazy form (see the Appendix on Chazy classes)

$$f''' + \frac{P'}{P}f'' + \frac{6}{P}f'^2 - \frac{4P'}{P^2}ff' + \frac{P''}{P^2}f^2 + \frac{4Q}{P^2}f' - \frac{2Q'}{P^2}f + \frac{2R}{P^2} = 0, \tag{4.2.5}$$

with $c = 0$ and P, Q, R having the following form:

Gauss	$P(x) = 1$	$4Q(x) = -4b^2x^2 + 8bn$	$R = 0$
Laguerre	$P(x) = x$	$4Q(x) = -(bx-a)^2 + 4bnx$	$R = 0$
Jacobi	$P(x) = 1 - x^2$	$4Q(x) = -\frac{1}{2}(q(x^2-1) + 2sx + r)$	$R = 0$

Cosgrove shows that such a third order equation (4.2.5) in $f(x)$, with $P(x)$, $Q(x)$, $R(x)$ of respective degrees 3, 2, 1, has a first integral (9.0.2), which is second order in f and quadratic in f'', with an integration constant c. Equation (9.0.2) is a master Painlevé equation, containing the 6 Painlevé equations. If $f(x)$ satisfies the equations above, then the new (renormalized) function $g(z)$ defined by

Gauss	$g(z) = b^{-1/2}f(zb^{-1/2}) + \frac{2}{3}nz$
Laguerre	$g(z) = f(z) + \frac{1}{4}b(2n+a)z + \frac{1}{4}a^2$
Jacobi	$g(z) := -\frac{1}{2}f(x)\big\vert_{x=2z-1} - \frac{1}{8}qz + \frac{1}{16}(q+s)$

satisfies the canonical equations, which then can be transformed into the standard Painlevé equations; these canonical equations are respectively

- $g''^2 = -4g'^3 + 4(zg' - g)^2 + A_1 g' + A_2,$ **(Painlevé IV)**
- $(zg'')^2 = (zg' - g)\big(-4g'^2 + A_1(zg' - g) + A_2\big) + A_3 g' + A_4,$ **(Painlevé V)**
- $(z(z-1)g'')^2 = (zg' - g)\big(4g'^2 - 4g'(zg' - g) + A_2\big) + A_1 g'^2 + A_3 g' + A_4,$ **(Painlevé VI)**

with respective coefficients

- $A_1 = 3\big(\frac{4}{3}n\big)^2$, $A_2 = -\big(\frac{4}{3}n\big)^3$,
- $A_1 = b^2$, $A_2 = b^2\big((n+\frac{1}{2}a)^2 + \frac{1}{2}a^2\big)$, $A_3 = -a^2 b(n + \frac{1}{2}a)$, $A_4 = \frac{1}{2}(ab)^2 \times \big((n+\frac{1}{2}a)^2 + \frac{1}{8}a^2\big)$,
- $A_1 = \frac{1}{8}(2q + r)$, $A_2 = \frac{1}{16}qs$, $A_3 = \frac{1}{64}\big((q-s)^2 + 2qr\big)$, $A_4 = \frac{1}{512}q(2s^2 + qr)$.

For $\beta = 1$ and 4, the inductive partial differential equations (4.1.4), (4.1.6), (4.1.10), and the derived differential equations (4.2.1), (4.2.2) and (4.2.4) are due to Adler and van Moerbeke [6]. For $\beta = 2$ and for general E, they were first computed by Adler, Shiota, and van Moerbeke [11], using the method of the present paper. For $\beta = 2$ and for E having one boundary point, the equations obtained here coincide with the ones first obtained by Tracy and Widom in [63], who recognized them to be Painlevé IV and V for the Gaussian and Laguerre distribution respectively. In his Louvain doctoral dissertation, J. P. Semengue, together with L. Haine [32], were lead to Painlevé VI for the Jacobi ensemble, for $\beta = 2$ and E having one boundary point, upon subtracting the Tracy and Widom differential equation [63] from the one computed with the method of Adler, Shiota, and van Moerbeke [11]. The classification by Cosgrove [23] and Cosgrove and Scoufis [24, (A.3)] leads directly to these results.

4.3. Proof of Theorems 4.1, 4.2 and 4.3

4.3.1. Gaussian and Laguerre ensembles. The three first Virasoro equations, as in (2.1.29) and (2.1.32), are differential equations, involving partials in $t \in \mathbb{C}^\infty$ and partials $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ in $c = (c_1, \dots, c_{2r}) \in \mathbb{R}^{2r}$, for $F := F_n(t, c) = \log I_n$; they have the general form:

$$\mathcal{D}_k F = \frac{\partial F}{\partial t_k} + \sum_{-1 \le j < k} \gamma_{kj} V_j(F) + \gamma_k + \delta_k t_1, \quad k = 1, 2, 3, \tag{4.3.1}$$

with first $V_j(F)$'s given by

$$V_j(F) = \sum_{i, i+j \ge 1} i t_i \frac{\partial F}{\partial t_{i+j}} + \frac{\beta}{2}\delta_{2,j}\left(\frac{\partial^2 F}{\partial t_1^2} + \left(\frac{\partial F}{\partial t_1}\right)^2\right), \quad -1 \le j \le 2. \tag{4.3.2}$$

In (4.3.1) and (4.3.2), $\beta > 0, \gamma_{kj}, \gamma_k, \delta_k$ are arbitrary parameters; also $\delta_{2j} = 0$ for $j \ne 2$ and $= 1$ for $j = 2$. The claim is that the equations (4.3.1) enable one to express all partial derivatives,

$$\left.\frac{\partial^{i_1+\cdots+i_k}F(t,c)}{\partial t_1^{i_1}\dots\partial t_k^{i_k}}\right|_{\mathcal{L}}, \quad \text{along } \mathcal{L} := \{\text{all } t_i = 0,\ c = (c_1,\dots,c_{2r}) \text{ arbitrary}\}, \tag{4.3.3}$$

uniquely in terms of polynomials in

$$\mathcal{D}_{j_1}\dots\mathcal{D}_{j_r}F(0,c).$$

Indeed, the method consists of expressing $\partial F/\partial t_k\big|_{t=0}$ in terms of $\mathcal{D}_k f\big|_{t=0}$, using (4.3.1). Second derivatives are obtained by acting on $\mathcal{D}_kF$ with $\mathcal{D}_l$, by noting that $\mathcal{D}_l$ commutes with all t-derivatives, by using the equation for $\mathcal{D}_lF$, and by setting in the end $t=0$:

$$\begin{aligned}
\mathcal{D}_l\mathcal{D}_kF &= \mathcal{D}_l\frac{\partial F}{\partial t_k} + \sum_{-1\le j<k}\gamma_{kj}\mathcal{D}_l(V_j(F))\\
&= \Big(\frac{\partial}{\partial t_k} + \sum_{-1\le j<k}\gamma_{kj}V_j\Big)\mathcal{D}_l(F), \quad \begin{array}{l}\text{provided } V_j(F) \text{ does not}\\ \text{contain nonlinear terms}\end{array}\\
&= \Big(\frac{\partial}{\partial t_k} + \sum_{-1\le j<k}\gamma_{kj}V_j\Big)\Big(\frac{\partial F}{\partial t_l} + \sum_{-1\le j<l}\gamma_{lj}V_j(F) + \delta_l t_1\Big)\\
&= \frac{\partial^2 F}{\partial t_k\partial t_l} + \text{lower-weight terms.}
\end{aligned}$$

When the nonlinear term is present, it is taken care of as follows:

$$\mathcal{D}_l\Big(\frac{\partial F}{\partial t_1}\Big)^2 = 2\frac{\partial F}{\partial t_1}\mathcal{D}_l\frac{\partial F}{\partial t_1} = 2\frac{\partial F}{\partial t_1}\frac{\partial}{\partial t_1}\Big(\frac{\partial F}{\partial t_l} + \sum_{-1\le j<l}\gamma_{lj}V_j(F) + \gamma_l + \delta_l t_1\Big).$$

Higher derivatives are obtained in the same way. We only record here, for future use, the few partials appearing in the KP equation (3.1.6):

$$\begin{aligned}
\left.\frac{\partial^2 F}{\partial t_1^2}\right|_{\mathcal{L}} &= \big(\mathcal{D}_1^2 - \gamma_{10}\mathcal{D}_1\big)F + \gamma_{10}\gamma_1 - \delta_1\\
\left.\frac{\partial^4 F}{\partial t_1^4}\right|_{\mathcal{L}} &= \big(\mathcal{D}_1^4 - 6\gamma_{10}\mathcal{D}_1^3 + 11\gamma_{10}^2\mathcal{D}_1^2 - 6\gamma_{10}^3\mathcal{D}_1\big)F - 6\gamma_{10}^2(\delta_1 - \gamma_1\gamma_{10})\\
\left.\frac{\partial^2 F}{\partial t_2^2}\right|_{\mathcal{L}} &= \big(\mathcal{D}_2^2 - 2\gamma_{20}\mathcal{D}_2 + \beta\gamma_{21}\gamma_{32}\mathcal{D}_1^2 - ((2\gamma_1+\gamma_{10})\gamma_{21}\gamma_{32}\beta + 2\gamma_{2,-1})\mathcal{D}_1 - 2\gamma_{21}\mathcal{D}_3\big)F\\
&\quad + \beta\gamma_{21}\gamma_{32}(\mathcal{D}_1F)^2 + \beta\gamma_{21}\gamma_{32}(\gamma_1^2 + \gamma_{10}\gamma_1 - \delta_1) + 2(\gamma_{21}\gamma_3 + \gamma_{20}\gamma_2 + \gamma_1\gamma_{2,-1})\\
\left.\frac{\partial^2 F}{\partial t_1\partial t_3}\right|_{\mathcal{L}} &= \Big(\mathcal{D}_1\mathcal{D}_3 - \frac{\beta}{2}\gamma_{32}\mathcal{D}_1^3 + \beta\gamma_{32}(\gamma_1 + 2\gamma_{10})\mathcal{D}_1^2 - \frac{3\beta}{2}\gamma_{10}\gamma_{32}(2\gamma_1 + \gamma_{10})\mathcal{D}_1\\
&\quad - 3\gamma_{1,-1}\mathcal{D}_2 - 3\gamma_{10}\mathcal{D}_3\Big)F + \frac{3\beta}{2}\gamma_{10}\gamma_{32}(\mathcal{D}_1F)^2 - \beta\gamma_{32}(\mathcal{D}_1F)(\mathcal{D}_1^2F)\\
&\quad + \tfrac{3}{2}\big(2\gamma_{10}\gamma_3 + \beta\gamma_{32}\gamma_{10}(\gamma_1^2 + \gamma_{10}\gamma_1 - \delta_1) + 2\gamma_{1,-1}\gamma_2\big).
\end{aligned}$$

4.3.2. Jacobi ensemble. Here, from the Virasoro constraints (2.1.35), one proceeds in the same way as before, by forming $\mathcal{B}_i F\big|_{t=0}$, $\mathcal{B}_i\mathcal{B}_j F\big|_{t=0}$, etc., in terms of t_i partials. For example, from the expressions $\mathcal{B}_{-1}F\big|_{t=0}$, $\mathcal{B}_{-1}^2 F\big|_{t=0}$, $\mathcal{B}_0 F\big|_{t=0}$, one extracts

$$\frac{\partial F}{\partial t_1}\Big|_{t=0},\quad \frac{\partial^2 F}{\partial t_1^2}\Big|_{t=0},\quad \frac{\partial F}{\partial t_2}\Big|_{t=0}.$$

From the expressions $\mathcal{B}_{-1}^3 F\big|_{t=0}$, $\mathcal{B}_0\mathcal{B}_{-1}F\big|_{t=0}$, $\mathcal{B}_1 F\big|_{t=0}$, and using the previous information, one extracts

$$\frac{\partial F}{\partial t_3}\Big|_{t=0},\quad \frac{\partial^2 F}{\partial t_1^3}\Big|_{t=0},\quad \frac{\partial^2 F}{\partial t_1\partial t_2}\Big|_{t=0}.$$

Finally, from the expressions $\mathcal{B}_2 F\big|_{t=0}$, $\mathcal{B}_1\mathcal{B}_{-1}F\big|_{t=0}$, $\mathcal{B}_0^2 F\big|_{t=0}$, $\mathcal{B}_0\mathcal{B}_{-1}^2 F\big|_{t=0}$, $\mathcal{B}_{-1}^4 F\big|_{t=0}$, one deduces

$$\frac{\partial^4 F}{\partial t_1^4}\Big|_{t=0},\quad \frac{\partial F}{\partial t_4}\Big|_{t=0},\quad \frac{\partial^3 F}{\partial t_1^2\partial t_2}\Big|_{t=0},\quad \frac{\partial^2 F}{\partial t_1\partial t_3}\Big|_{t=0},\quad \frac{\partial^2 F}{\partial t_2^2}\Big|_{t=0}. \tag{4.3.4}$$

This provides all the partials, appearing in the KP equation (3.1.6).

4.3.3. Inserting partials into the integrable equation. From Theorem 3.3, the integrals $I_n(t,c)$, depending on $\beta = 2,1,4$, on $t = (t_1, t_2, \dots)$ and on the boundary points $c = (c_1, \dots, c_{2r})$ of E, relate to τ-functions, as follows:

$$\begin{aligned} I_n(t,c) &= \int_{E^n} |\Delta_n(z)|^\beta \prod_{k=1}^n \left(e^{\sum_1^\infty t_i z_k^i}\rho(z_k)\,dz_k\right) \\ &= \begin{cases} n!\,\tau_n(t,c), & n \text{ arbitrary}, \beta = 2, \\ n!\,\tau_n(t,c), & n \text{ even}, \quad\ \ \beta = 1, \\ n!\tau_{2n}(t/2,c), & n \text{ arbitrary}, \beta = 4, \end{cases} \end{aligned} \tag{4.3.5}$$

where $\tau_n(t,c)$ satisfies the KP-like equation

$$12\frac{\tau_{n-2}(t,c)\tau_{n+2}(t,c)}{\tau_n(t,c)^2}\delta_{1,4}^{\beta} = (KP)_t \log\tau_n(t,c), \quad \begin{cases} n \text{ arbitrary for } \beta = 2, \\ n \text{ even for } \beta = 1,4, \end{cases} \tag{4.3.6}$$

with

$$(KP)_t F := \left(\left(\frac{\partial}{\partial t_1}\right)^4 + 3\left(\frac{\partial}{\partial t_2}\right)^2 - 4\frac{\partial^2}{\partial t_1\partial t_3}\right)F + 6\left(\frac{\partial^2}{\partial t_1^2}F\right)^2.$$

<u>*Evaluating the left hand side of* (4.3.6)*:*</u> Here $I_n(t)$ will refer to the integral (4.3.5) over the full range. For <u>$\beta = 2$</u>, the left-hand side is zero. For <u>$\beta = 1$</u>, the left-hand side can be evaluated in terms of the probability $P_n(E)$, as follows: taking into account $P_n := P_n(E) = I_n(0,c)/I_n(0)$,

$$12\frac{\tau_{n-2}(t,c)\tau_{n+2}(t,c)}{\tau_n(t,c)^2}\Big|_{t=0} = 12\frac{(n!)^2}{(n-2)!(n+2)!}\ \frac{I_{n-2}(t,c)I_{n+2}(t,c)}{I_n(t,c)^2}\Big|_{t=0}$$

$$= 12 \frac{n(n-1)}{(n+1)(n+2)} \frac{I_{n-2}(0) I_{n+2}(0)}{I_n(0)^2} \frac{P_{n-2} P_{n+2}}{P_n^2}$$
$$= 12 b_n^{(1)} \frac{P_{n-2}(E) P_{n+2}(E)}{P_n^2(E)},$$

with $b_n^{(1)}$ given by[26]

$$b_n^{(1)} = \frac{n(n-1)}{(n+2)(n+1)} \frac{I_{n-2}(0) I_{n+2}(0)}{I_n(0)^2} = \begin{cases} \dfrac{n(n-1)}{16b^2} & \text{(Gauss)}, \\ \dfrac{n(n-1)(n+2a)(n+2a+1)}{16b^4} & \text{(Laguerre)}, \\ \dfrac{Q}{Q_6^{\pm}} & \text{(Jacobi)}. \end{cases} \tag{4.3.7}$$

For $\underline{\beta = 4}$, we have

$$12 \frac{\tau_{2n-2}(t/2,c)\tau_{2n+2}(t/2,c)}{\tau_{2n}(t/2,c)^2}\Big|_{t=0} = 12 \frac{(n!)^2}{(n-1)!(n+1)!} \frac{I_{n-1}(t,c) I_{n+1}(t,c)}{I_n(t,c)^2}\Big|_{t=0}$$
$$= 12 \frac{n}{(n+1)} \frac{I_{n-1}(0) I_{n+1}(0)}{I_n(0)^2} \frac{P_{n-1} P_{n+1}}{P_n^2}$$
$$= 12 b_n^{(4)} \frac{P_{n-1}(E) P_{n+1}(E)}{P_n^2(E)},$$

with

$$b_n^{(4)} = \frac{(n!)^2}{(n-1)!(n+1)!} \frac{I_{n-1}(0) I_{n+1}(0)}{I_n^2(0)} = \begin{cases} \dfrac{2n(2n+1)}{4b^2} & \text{(Gauss)}, \\ \dfrac{2n(2n+1)(2n+a)(2n+a-1)}{b^4} & \text{(Laguerre)}, \\ \dfrac{Q}{Q_6^{\pm}} & \text{(Jacobi)}, \end{cases} \tag{4.3.8}$$

where Q is precisely the expression appearing in (4.1.7) and where

$$Q_6^{\pm} = 3q\,(q+1)\,(q-3)\left(q+4 \pm 4\sqrt{q+1}\right) \quad \begin{cases} + \text{ for } \beta = 1, \\ - \text{ for } \beta = 4. \end{cases} \tag{4.3.9}$$

The exact formulae $b_n^{(4)}$ and $b_n^{(1)}$ show they satisfy the duality property (4.1.2):

$$b_n^{(4)}(a,b,n) = b_n^{(1)}\left(-\tfrac{1}{2}a, -\tfrac{1}{2}b, -2n\right).$$

[26] This calculation is based on Selberg's integrals: see Mehta [49, p. 340]. For instance, in the Jacobi case, one uses

$$I_n^{(\beta)} = \int_{[-1,1]^n} \Delta_n(x)^\beta \prod_{j=1}^{n} (1-x_j)^a (1+x_j)^b \, dx_j$$
$$= 2^{n(2a+2b+\beta(n-1)+2)/2} \prod_{j=0}^{n-1} \frac{\Gamma(a+j\beta/2+1)\Gamma(b+j\beta/2+1)\Gamma((j+1)\beta/2+1)}{\Gamma(\beta/2+1)\Gamma(a+b+(n+j-1)\beta/2+2)}.$$

Evaluating the right-hand side of (4.3.6): From Section 2.4, it also follows that $F_n(t;c) = \log I_n(t;c)$ satisfies Virasoro constraints, corresponding precisely to the situation (4.3.1), with
Gaussian ensemble:[27]

$$\begin{aligned}
&\gamma_{1,-1} = -\tfrac{1}{2},\ \gamma_{1,0} = \gamma_1 = 0, \delta_1 = -\tfrac{1}{2}n;\\
&\gamma_{2,-1} = 0,\ \gamma_{2,0} = -\tfrac{1}{2},\ \gamma_{2,1} = 0, \gamma_2 = -\tfrac{1}{4}n\sigma_1,\ \delta_2 = 0;\\
&\gamma_{3,-1} = -\tfrac{1}{4}\sigma_1,\ \gamma_{3,0} = 0,\ \gamma_{3,1} = -\tfrac{1}{2},\ \gamma_{3,2} = \gamma_3 = 0,\ \delta_3 = -\tfrac{1}{4}n\sigma_1.
\end{aligned}$$

Laguerre ensemble:[28] $\delta_1 = \delta_2 = \delta_3 = 0$, and

$$\begin{aligned}
&\gamma_{1,-1} = 0,\ \gamma_{1,0} = -1,\ \gamma_1 = -\tfrac{1}{2}n(\sigma_1 + a),\\
&\gamma_{2,-1} = 0,\ \gamma_{2,0} = -\sigma_1,\ \gamma_{2,1} = -1,\ \gamma_2 = -\tfrac{1}{2}n\sigma_1(\sigma_1 + a);\\
&\gamma_{3,-1} = 0,\ \gamma_{3,0} = -\sigma_1\sigma_2,\ \gamma_{3,1} = -\sigma_2,\ \gamma_{3,2} = -1,\ \gamma_3 = -\tfrac{1}{2}n\sigma_1\sigma_2(\sigma_1 + a).
\end{aligned}$$

Jacobi ensemble: see (4.3.4).

They lead to expressions for

$$\left.\frac{\partial^4 F}{\partial t_1^4}\right|_{t=0},\quad \left.\frac{\partial^2 F}{\partial t_2^2}\right|_{t=0},\quad \left.\frac{\partial^2 F}{\partial t_1 \partial t_3}\right|_{t=0},\quad \left.\frac{\partial^2 F}{\partial t_1^2}\right|_{t=0},$$

in terms of $\mathcal{D}_k$ and $\mathcal{B}_k$, which substituted in the right-hand side of (4.3.6) — i.e., in the KP-expressions — leads to the right-hand side of (4.1.4),(4.1.6), (4.1.9) and (4.1.10). In the Jacobi case, the right-hand side of (4.3.6) contains the same coefficient $1/Q_6^{\pm}$ as in (4.3.9), which therefore cancels with the one appearing on the left-hand side; see the expression $b_n^{1,4}$ in (4.3.7) and (4.3.8).

5. Ensembles of Infinite Random Matrices: Fredholm Determinants As τ-Functions of the KdV Equation

Infinite Hermitian matrix ensembles typically relate to the Korteweg-de Vries hierarchy, itself a reduction of the KP hierarchy; a brief sketch will be necessary. The *KP-hierarchy* is given by t_n-deformations of a pseudo-differential operator[29] L: (commuting vector fields)

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L], \quad L = D + a_{-1}D^{-1} + \cdots, \quad \text{with} \quad D = \frac{\partial}{\partial x}. \tag{5.0.1}$$

Wave and adjoint wave functions are eigenfunctions $\Psi^+(x,t;z)$ and $\Psi^-(x,t;z)$, depending on $x \in \mathbb{R}$, $t \in \mathbb{C}^\infty$, $z \in \mathbb{C}$, behaving asymptotically like (5.0.3) below

[27] Remember from Section 2.1 that $\sigma_1 = \beta(n-1) + 2$.

[28] Remember from Section 2.1 that $\sigma_1 = \beta(n-1) + a + 2$ and $\sigma_2 = \beta(n - \frac{3}{2}) + a + 3$.

[29] In this section, given P a pseudo-differential operator, P_+ and P_- denote the differential and the (strictly) smoothing part of P respectively.

and satisfying

$$z\Psi^{+}=L\Psi^{+},\quad \frac{\partial\Psi^{+}}{\partial t_n}=(L^n)_{+}\Psi^{+},\quad z\Psi^{-}=L^{\top}\Psi^{-},\quad \frac{\partial\Psi^{-}}{\partial t_n}=-(L^{\top n})_{+}\Psi^{-}. \tag{5.0.2}$$

According to Sato's theory, Ψ^{+} and Ψ^{-} have the following representation in terms of a τ-function (see [25]):

$$\begin{aligned}\Psi^{\pm}(x,t;z)&=e^{\pm(xz+\sum_1^\infty t_i z^i)}\frac{\tau(t\mp[z^{-1}])}{\tau(t)}\\ &=e^{\pm(xz+\sum_1^\infty t_i z^i)}(1+O(z^{-1})),\quad \text{for } z\nearrow\infty,\end{aligned} \tag{5.0.3}$$

where τ satisfies and is characterized by the following bilinear relation

$$\oint e^{\sum_1^\infty (t_i-t_i')z^i}\tau(t-[z^{-1}])\tau(t'+[z^{-1}])\,dz=0 \quad \text{for all } t,t'\in\mathbb{C}^\infty; \tag{5.0.4}$$

the integral is taken over a small circle around $z=\infty$. From the bilinear relation, one derives the KP-hierarchy, already mentioned in Theorem 3.1, of which the first equation reads as in (3.1.6).

We consider the p-reduced KP hierarchy, i.e., the reduction to pseudo-differential L's such that $L^p=D^p+\cdots$ is a differential operator for some fixed $p\geq 2$. Then $(L^{kp})_{+}=L^{kp}$ for all $k\geq 1$ and thus $\partial L/\partial t_{kp}=0$, in view of the deformation equations (5.0.1) on L. Therefore the variables $t_p,t_{2p},t_{3p},\ldots$ are not active and can thus be set $=0$. The case $p=2$ is particularly interesting and leads to the KdV equation, upon setting all even $t_i=0$.

For the time being, take the integer $p\geq 2$ arbitrary. The arbitrary linear combinations[30]

$$\Phi^{\pm}(x,t;z):=\sum_{\omega\in\zeta_p}a_\omega^{\pm}\Psi^{\pm}(x,t;\omega z) \tag{5.0.5}$$

are the most general solution of the spectral problems $L^p\Phi^{+}=z^p\Phi^{+}$ and $L^{\top p}\Phi^{-}=z^p\Phi^{-}$ respectively, leading to the definition of the kernels:

$$\begin{aligned}k_{x,t}(y,z)&:=\int^x dx\,\Phi^{-}(x,t;y)\Phi^{+}(x,t;z),\\ k_{x,t}^E(y,z)&:=k_{x,t}(y,z)I_E(z),\end{aligned} \tag{5.0.6}$$

where the integral is taken from a fixed, but arbitrary origin in $\mathbb{R}$. In the same way that $\Psi^{\pm}(x,t,z)$ has a τ-function representation, so also does $k_{x,t}^E(y,z)$ have a similar representation, involving the vertex operator

$$Y(x,t;y,z):=\sum_{\omega,\omega'\in\zeta_p}a_\omega^{-}a_{\omega'}^{+}X(x,t;\omega y,\omega' z), \tag{5.0.7}$$

where (see [25; 11; 12])

$$X(x,t;y,z):=\frac{1}{z-y}e^{(z-y)x+\sum_1^\infty(z^i-y^i)t_i}e^{\sum_1^\infty(y^{-i}-z^{-i})\frac{1}{i}\frac{\partial}{\partial t_i}}. \tag{5.0.8}$$

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[30] Here $\zeta_p:=\{\omega \text{ such that } \omega^p=1\}$.

A condition $\sum_{\omega\in\zeta_p} a_\omega^+ a_\omega^-/\omega = 0$ is needed to guarantee that the right-hand side of (5.0.7) is free of singularities in the positive quadrant $\{y_i \geq 0$ and $z_j \geq 0$ with $i, j = 1, \dots, n\}$ and $\lim_{y\to z} Y(x,t;y,z)$ exists. Indeed, using Fay identities and higher degree Fay identities, one shows stepwise the following three statements, the last one being a statement about a Fredholm determinant:[31]

$$\begin{aligned} k_{x,t}(y,z) &= \frac{1}{\tau(t)} Y(x,t;y,z)\tau(t), \\ \det\bigl(k_{x,t}(y_i,z_j)\bigr)_{1\leq i,j\leq n} &= \frac{1}{\tau}\prod_{i=1}^{k} Y(x,t;y_i,z_i)\tau, \\ \det(I-\lambda k_{x,t}^E) &= \frac{1}{\tau} e^{-\lambda\int_E dz\, Y(x,t;z,z)}\tau =: \frac{\tau(t,E)}{\tau(t)}. \end{aligned} \tag{5.0.9}$$

The kernel (5.0.12) at $t = 0$ will define the statistics of a random Hermitian ensemble, when the size $n \nearrow \infty$. The next theorem is precisely a statement about Fredholm determinants of kernels of the form (5.0.12); it will be identified at $t = 0$ with the probability that no eigenvalue belongs to a subset E; see Section 1.2. The initial condition that Virasoro annihilates τ_0, as in Sections 2.1.2 (Proof of Theorem 2.1), is now replaced by the *initial condition* (5.0.11) below.

THEOREM 5.1 (Adler, Shiota, and van Moerbeke [11; 12]). *Consider Virasoro generators $J_l^{(2)}$ satisfying*

$$\frac{\partial}{\partial z} z^{l+1} Y(x,t;z,z) = \bigl[\tfrac{1}{2} J_l^{(2)}(t),\ Y(x,t;z,z)\bigr], \tag{5.0.10}$$

where $Y(x,t;z,z)$ is defined in (5.0.7), *and a τ-function satisfying the Virasoro constraint, with an arbitrary constant c_{kp}:*

$$\Bigl(J_{kp}^{(2)} - c_{kp}\Bigr)\tau = 0 \quad \textit{for a fixed } k \geq -1. \tag{5.0.11}$$

Then, given the disjoint union $E \subset \mathbb{R}^+$, the Fredholm determinant of

$$K_{x,t}^E(\lambda,\lambda') := \frac{1}{p}\frac{k_{x,t}(z,z')}{z^{(p-1)/2} z'^{(p-1)/2}} I_E(\lambda'), \quad \lambda = z^p,\ \lambda' = z'^p, \tag{5.0.12}$$

satisfies the following constraint for that same $k \geq -1$:

$$\Biggl(-\sum_{i=1}^{2r} c_i^{k+1}\frac{\partial}{\partial c_i} + \frac{1}{2p}(J_{kp}^{(2)} - c_{kp})\Biggr)\tau\det(I-\mu K_{x,t}^E) = 0. \tag{5.0.13}$$

[31] The Fredholm determinant of a kernel $A(y,z)$ is defined by

$$\det(I-\lambda A) = 1 + \sum_{m=1}^{\infty}(-\lambda)^m \int_{z_1\leq\cdots\leq z_m}\!\!\cdots\int \det\bigl(A(z_i,z_j)\bigr)_{1\leq i,j\leq m}\, dz_1\dots dz_m.$$

The generators $J_n^{(2)}$ take on the following precise form:

$$\begin{aligned} J_n^{(1)} &:= \frac{\partial}{\partial t_n} + (-n)t_{-n}, \\ J_n^{(2)} &:= \sum_{i+j=n} :J_i^{(1)} J_j^{(1)}: - (n+1)J_n^{(1)} \\ &= \sum_{i+j=n} \frac{\partial^2}{\partial t_i \partial t_j} + 2 \sum_{-i+j=n} it_i \frac{\partial}{\partial t_j} + \sum_{-i-j=n} it_i jt_j - (n+1)J_n^{(1)}. \end{aligned} \tag{5.0.14}$$

REMARK. For KdV (i.e., $p = 2$), we have $(L^2)^\top = L^2 = D^2 - q(x)$, so the adjoint wave function has the simple expression $\Psi^-(x,t;z) = \Psi^+(x,t;-z)$. In the next two examples, which deal with KdV, set

$$\Psi(x,t;z) := \Psi^+(x,t;z).$$

Example 1: Eigenvalues of large random Hermitian matrices near the "soft edge" and the Airy kernel. Remember from Section 1.2, the spectrum of the Gaussian Hermitian matrix ensemble has, for large size n, its edge at $\pm\sqrt{2n}$, near which the scaling is given by $\sqrt{2}n^{1/6}$. Therefore, the eigenvalues in Theorem 5.2 must be expressed in that new scaling. Define the disjoint union $E = \bigcup_1^r [c_{2i-1}, c_{2i}]$, with c_{2r} possibly ∞.

THEOREM 5.2. *Given the spectrum $z_1 \geq z_2 \geq \cdots$ of the large random Hermitian matrix M, define the "eigenvalues" in the new scale*:

$$u_i = 2n^{2/3}\left(\frac{z_i}{\sqrt{2n}} - 1\right) \quad \textit{for } n \nearrow \infty. \tag{5.0.15}$$

The probability of the "eigenvalues"

$$P(E^c) := P(\textit{all "eigenvalues" } u_i \in E^c) \tag{5.0.16}$$

satisfies the partial differential equation (*setting* $\mathcal{B}_k := \sum_{i=1}^{2r} c_i^{k+1}\partial/\partial c_i$)[32]

$$\left(\mathcal{B}_{-1}^3 - 4(\mathcal{B}_0 - \tfrac{1}{2})\right)\mathcal{B}_{-1} \log P(E^c) + 6(\mathcal{B}_{-1}^2 \log P(E^c))^2 = 0. \tag{5.0.17}$$

In particular, *the statistics of the largest "eigenvalue" u_1* (*in the new scale*) *is given by*

$$P(u_1 \leq x) = \exp\left(-\int_x^\infty (\alpha - x) g^2(\alpha)\, d\alpha\right), \tag{5.0.18}$$

with

$$\begin{cases} g'' = xg + 2g^3 & (\textbf{Painlevé II}) \\ g(x) \cong -\dfrac{e^{-(2/3)x^{3/2}}}{2\sqrt{\pi}x^{1/4}} & \textit{for } x \nearrow \infty. \end{cases} \tag{5.0.19}$$

[32] When $c_{2r} = \infty$, that term in $\mathcal{B}_k$ is absent.

The partial differential equation (5.0.17) is due to Adler, Shiota, and van Moerbeke [11; 12]. The equation (5.0.19) for the largest eigenvalue is a special case of (5.0.17), but was first derived by Tracy and Widom [63], by methods of functional analysis.

PROOF. Remember from Section 1.2 that the statistics of the eigenvalues is governed by the Fredholm determinant of the kernel (1.2.4), for the Hermite polynomials. In the limit,

$$\lim_{n\nearrow\infty}\frac{1}{\sqrt{2}n^{1/6}}K_n\left(\sqrt{2n}+\frac{u}{\sqrt{2}n^{1/6}},\sqrt{2n}+\frac{v}{\sqrt{2}n^{1/6}}\right)=K(u,v),$$

where

$$K(u,v)=\int_0^\infty A(x+u)A(x+v)\,dx,\quad A(u)=\int_{-\infty}^\infty e^{iux-x^3/3}\,dx. \tag{5.0.20}$$

Then

$$P(E^c):=P(\text{all eigenvalues } u_i\in E^c)=\det\big(I-K(u,v)I_E(v)\big). \tag{5.0.21}$$

In order to compute the PDEs of this expression, with regard to the endpoints c_i of the disjoint union E, one proceeds as follows:

Consider the KdV wave function $\Psi(x,t;z)$, as in (5.0.2), with initial condition

$$\Psi(x,t_0;z)=z^{1/2}A(x+z^2)=e^{xz+(2/3)z^3}(1+O(z^{-1})),$$
$$z\to\infty,\ t_0=\big(0,0,\tfrac{2}{3},0,\dots\big), \tag{5.0.22}$$

in terms of the Airy function[33], which, by stationary phase, has the asymptotics

$$A(u):=\frac{1}{\sqrt{\pi}}\int_{-\infty}^\infty e^{-y^3/3+yu}\,dy=u^{-1/4}e^{(2/3)u^{3/2}}\big(1+O(u^{-3/2})\big).$$

The definition of $A(u)$ is slightly changed, compared to (5.0.20). $A(u)$ satisfies the differential equation $A(y)''=yA(y)$, and thus the wave function $\Psi(x,t_0;z)$ satisfies $(D^2-x)\Psi(x,t_0;z)=z^2\Psi(x,t_0;z)$. Therefore $L^2|_{t=t_0}=SD^2S^{-1}|_{t=t_0}=D^2-x$, so that L^2 is a differential operator, and Ψ is a KdV wave function, with $\tau(t)$ satisfying[34] the Virasoro constraints (5.0.11) with $c_{2k}=-\frac14\delta_{k0}$. The argument to prove these constraints is based on the fact that the linear span (a point in an infinite-dimensional Grassmannian)

$$\mathcal{W}=\operatorname{span}_{\mathbb{C}}\left\{\psi_n(z):=e^{-(2/3)z^3}\sqrt{z}\frac{\partial^n}{\partial u^n}A(u)\Big|_{u=z^2},\ n=0,1,2,\dots\right\}$$

[33] The i in the definition of the Airy function is omitted here.

[34] Although not used here, the τ-function is Kontsevich's integral [44; 10]:
$\tau(t)=\dfrac{\int_{\mathcal{H}}dY\,e^{-\operatorname{Tr}(Y^3/3+Y^2Z)}}{\int_{\mathcal{H}}dY\,e^{-\operatorname{Tr}Y^2Z}}$ with $t_n=-\dfrac{1}{n}\operatorname{Tr}(Z^{-n})+\dfrac{2}{3}\delta_{n,3}$, $Z=$ diagonal matrix.

is invariant under multiplication by z^2 and under the operator

$$\frac{1}{2z}\Big(\frac{\partial}{\partial z}+2z^2\Big)-\frac{1}{4}z^{-2}.$$

Define, for $\lambda = z^2$ and $\lambda' = z'^2$, the kernel

$$K_t(\lambda,\lambda') := \frac{1}{2z^{1/2}z'^{1/2}}\int_0^\infty \Psi(x,t;z)\Psi(x,t;z')\,dx, \tag{5.0.23}$$

which flows off the Airy kernel, by (5.0.22),

$$K_{t_0}(\lambda,\lambda') = \frac{1}{2}\int_0^\infty A(x+\lambda)A(x+\lambda')\,dx.$$

Thus $\tau\det(I-K^E_{x,t})$ satisfies (5.0.13), with that same constant c_{kp}, for $k = -1,0,1,\dots$:

$$\left(-\sum_{i=1}^{2r} c_i^{k+1}\frac{\partial}{\partial c_i}+\frac{1}{4}J^{(2)}_{pk}+\frac{1}{16}\delta_{k,0}\right)\tau\det(I-K^E_t)=0. \tag{5.0.24}$$

Upon shifting $t_3 \mapsto t_3+2/3$, in view of (5.0.22), the two first Virasoro constraints for $k=-1$ and $k=0$ read as follows, with $\mathcal{B}_k := \sum_{i=1}^{2r} c_i^{k+1}\partial/\partial c_i$:

$$\begin{aligned}\mathcal{B}_{-1}\log\tau(t,E) &= \left(\frac{\partial}{\partial t_1}+\frac{1}{2}\sum_{i\geq 3} it_i\frac{\partial}{\partial t_{i-2}}\right)\log\tau(t,E)+\frac{t_1^2}{4}\\ \mathcal{B}_0\log\tau(t,E) &= \left(\frac{\partial}{\partial t_3}+\frac{1}{2}\sum_{i\geq 1} it_i\frac{\partial}{\partial t_i}\right)\log\tau(t,E)+\frac{1}{16}.\end{aligned} \tag{5.0.25}$$

The same method as in Section 4 enables one to express all the t-partials, appearing in the KdV equation,

$$\Big(\frac{\partial^4}{\partial t_1^4}-4\frac{\partial^2}{\partial t_1\partial t_3}\Big)\log\tau(t,E)+6\Big(\frac{\partial^2}{\partial t_1^2}\log\tau(t,E)\Big)^2=0,$$

in terms of c-partials, which upon substitution leads to the partial differential equation $\big(\mathcal{B}_{-1}^3-4(\mathcal{B}_0-\frac{1}{2})\big)f+6(\mathcal{B}_{-1}f)^2=0$ (announced in (5.0.17)) for

$$f := \mathcal{B}_{-1}\log P(E^c)=\sum_1^{2r}\frac{\partial}{\partial c_i}\log P(E^c), \text{ where } P(E^c)=\det(I-K^E)=\frac{\tau(t,E)}{\tau(t)}.$$

When $E=(-\infty,x)$, this PDE reduces to an ODE:

$$f'''-4xf'+2f+6f'^2=0, \quad \text{with } f=\frac{d}{dx}\log P(\max_i \lambda_i\leq x). \tag{5.0.26}$$

According to Appendix on Chazy classes (section 9) , this equation can be reduced to

$$f''^2+4f'(f'^2-xf'+f)=0, \qquad \textbf{(Painlevé II)} \tag{5.0.27}$$

which can be solved by setting

$$f' = -g^2 \text{ and } f = g'^2 - xg^2 - g^4.$$

An easy computation shows g satisfies the equation $g'' = 2g^3 + xg$ (Painlevé II), thus leading to (5.0.19). ☐

Example 2: Eigenvalues of large random Laguerre Hermitian matrices near the "hard edge" and the Bessel kernel. Consider the ensemble of $n \times n$ random matrices for the Laguerre probability distribution, thus corresponding to (1.1.9) with $\rho(dz) = z^{\nu/2} e^{-z/2}\, dz$. Remember from Section 1.2, the density of eigenvalues near the "hard edge" $z = 0$ is given by $4n$ for very large n. At this edge, the kernel (1.2.4) with Laguerre polynomials p_n tends to the Bessel kernel [52; 30]:

$$\lim_{n \nearrow \infty} \frac{1}{4n} K_n^{(\nu)} \left(\frac{u}{4n}, \frac{v}{4n} \right) = K^{(\nu)}(u, v) := \frac{1}{2} \int_0^1 x J_\nu(xu) J_\nu(xv)\, dx. \tag{5.0.28}$$

Therefore, the eigenvalues in the theorem below will be expressed in that new scaling. Define, as before, the disjoint union $E = \bigcup_1^r [c_{2i-1}, c_{2i}]$.

THEOREM 5.3. *Given the spectrum $0 \le z_1 \le z_2 \le \dots$ of the large random Laguerre-distributed Hermitian matrix M, define the "eigenvalues" in the new scale*:

$$u_i = 4nz_i \quad \textit{for } n \nearrow \infty. \tag{5.0.29}$$

The statistics of the "eigenvalues"

$$P(E^c) := P(\textit{all "eigenvalues" } u_i \in E^c) \tag{5.0.30}$$

leads to the following PDE for $F = \log P(E^c)$, where $\mathcal{B}_k := \sum_{i=1}^{2r} c_i^{k+1} \partial / \partial c_i$):

$$\left(\mathcal{B}_0^4 - 2\mathcal{B}_0^3 + (1-\nu^2)\mathcal{B}_0^2 + \mathcal{B}_1 \left(\mathcal{B}_0 - \tfrac{1}{2} \right) \right) F - 4(\mathcal{B}_0 F)(\mathcal{B}_0^2 F) + 6(\mathcal{B}_0^2 F)^2 = 0. \tag{5.0.31}$$

In particular, for very large n, the statistics of the smallest eigenvalue is governed by

$$P(u_1 \ge x) = \exp \left(- \int_0^x \frac{f(u)}{u} du \right), \quad u_1 \sim 4nz_1,$$

with f satisfying

$$(xf'')^2 - 4\big(xf' - f\big) f'^2 + \big((x - \nu^2) f' - f\big) f' = 0. \quad (\textbf{Painlevé V}) \tag{5.0.32}$$

Equation (5.0.32) for the smallest eigenvalue, first derived by Tracy and Widom [63], by methods of functional analysis, is a special case of the partial differential equation (5.0.31), originating in [11; 12].

REMARK. This same theorem would hold for the Jacobi ensemble, near the "hard edges" $z = \pm 1$.

PROOF. Define a wave function $\Psi(x,t;z)$, flowing off

$$\Psi(x,0;z) = e^{xz}B(-xz) = e^{xz}\big(1+O(z^{-1})\big),$$

where $B(z)$ is the Bessel function[35]

$$B(z) = \varepsilon\sqrt{z}e^{z}H_\nu(iz) = \frac{e^{z}2^{\nu+1/2}}{\Gamma(-\nu+1/2)}\int_1^\infty \frac{z^{-\nu+1/2}e^{-uz}}{(u^2-1)^{\nu+1/2}}\,du = 1+O(z^{-1}).$$

As the operator

$$L^2|_{t=0} = D^2 - \frac{\nu^2-1/4}{x^2}$$

is a differential operator, we are in the KdV situation; again one may assume $t_2 = t_4 = \ldots = 0$ and we have

$$\Psi^-(x,t;-z) = \Psi^+(x,t;z) = e^{xz+\sum t_i z^i}\frac{\tau(t-[z^{-1}])}{\tau(t)},$$

in terms of a τ-function[36] satisfying the Virasoro constraints

$$J^{(2)}_{2k}\tau = \big((2\nu)^2-1\big)\delta_{k0}\tau. \tag{5.0.33}$$

Set $p=2$, $a_1^- = a_{-1}^+ = (1/4\pi)ie^{i\pi\nu/2}$ and $a_{-1}^- = a_1^+ = (1/4\pi)e^{-i\pi\nu/2}$ in (5.0.5); this defines the kernel (5.0.6) and so (5.0.12), which in terms of $\lambda = z^2$ and $\lambda' = z'^2$, takes on the form:

$$\begin{aligned}K^{(\nu)}_{x,t}(\lambda,\lambda') = \frac{1}{4\pi\sqrt{zz'}}\int^x &\big(ie^{i\pi\nu/2}\Psi^*(x,t,z) + e^{-i\pi\nu/2}\Psi^*(x,t,-z)\big)\\ &\cdot\big(e^{-i\pi\nu/2}\Psi(x,t,z') + ie^{i\pi\nu/2}\Psi(x,t,-z')\big)\,dx,\end{aligned}$$

which flows off the *Bessel kernel*

$$\begin{aligned}K^{(\nu)}_{x,0}(\lambda,\lambda') &= \frac{1}{2}\int_0^x xJ_\nu(x\sqrt{\lambda})J_\nu(x\sqrt{\lambda'})\,dx.\\ &= \frac{J_\nu(\sqrt{\lambda})\sqrt{\lambda'}J'_\nu(\sqrt{\lambda'}) - J_\nu(\sqrt{\lambda'})\sqrt{\lambda}J'_\nu(\sqrt{\lambda})}{2(\lambda-\lambda')} \quad \text{for } x=1.\end{aligned}$$

The Fredholm determinant satisfies for $E\subset\mathbb{R}_+$ and for $k=0,1,\ldots$:

$$\left(-\sum_1^{2r} c_i^{k+1}\frac{\partial}{\partial c_i} + \tfrac14 J^{(2)}_{2k} + \big(\tfrac14-\nu^2\big)\delta_{k,0}\right)\big(\tau\det(I-K^{(\nu)E}_{x,t})\big) = 0. \tag{5.0.34}$$

[35] $\varepsilon = i\sqrt{\pi/2}\,e^{i\pi\nu/2}$, $-\frac12 < \nu < \frac12$.

[36] $\tau(t)$ is given by the Adler–Morozov–Shiota–van Moerbeke double Laplace matrix transform, with t_n given in a similar way as in footnote 34 (see [10]):

$$\tau(t) = c(t)\int_{\mathcal{H}_N^+} dX\,\det X^{\nu-1/2}e^{-\operatorname{Tr}(Z^2X)}\int_{\mathcal{H}_N^+} dY\,S_0(Y)e^{-\operatorname{Tr}(XY^2)}.$$

Upon shifting $t_1 \mapsto t_1 + \sqrt{-1}$ and using the same $\mathcal{B}_i$ as in (5.0.25), the equations for $k=0$ and $k=1$ read

$$\mathcal{B}_0 \log\tau(t,E) = \frac{1}{2}\Big(\sum_{i\geq 1} it_i \frac{\partial}{\partial t_i} + \sqrt{-1}\frac{\partial}{\partial t_1}\Big)\log\tau(t,E) + \tfrac{1}{4}\big(\tfrac{1}{4}-\nu^2\big),$$
$$\mathcal{B}_1 \log\tau(t,E) = \frac{1}{2}\Big(\sum_{i\geq 1} it_i \frac{\partial}{\partial t_{i+2}} + \frac{1}{2}\frac{\partial^2}{\partial t_1^2} + \sqrt{-1}\frac{\partial}{\partial t_3} + \frac{1}{2}\frac{\partial}{\partial t_1}\Big)\log\tau(t,E). \quad (5.0.35)$$

Expressing the t-partials (5.0.20), appearing in the KdV-equation at $t=0$ (see formula just below (5.0.25)) in terms of the c-partials applied to $\log\tau(0,E)$, leads to the following PDE for $F = \log P(E^c)$:

$$\big(\mathcal{B}_0^4 - 2\mathcal{B}_0^3 + (1-\nu^2)\mathcal{B}_0^2 + \mathcal{B}_1(\mathcal{B}_0 - \tfrac{1}{2})\big)F - 4(\mathcal{B}_0 F)(\mathcal{B}_0^2 F) + 6(\mathcal{B}_0^2 F)^2 = 0. \quad (5.0.36)$$

Specializing this equation to the interval $E=(0,x)$ leads to an ODE for $f := -x\,\partial F/\partial x$, namely

$$f''' + \frac{1}{x}f'' - \frac{6}{x}f'^2 + \frac{4}{x^2}ff' + \frac{(x-\nu^2)}{x^2}f' - \frac{1}{2x^2}f = 0, \quad (5.0.37)$$

which is an equation of the type (9.0.1); changing $x \curvearrowright -x$ and $f \curvearrowright -f$ leads again to an equation of type (9.0.1), with $P(x) = x$, $4Q(x) = -x-\nu^2$ and $R=0$. According to Cosgrove and Scoufis [24] (see the Appendix on Chazy classes), this equation can be reduced to the equation (9.0.2), with the same P, Q, R and with $c = 0$. Since $P(x) = x$, this equation is already in one of the canonical forms (9.0.3), which upon changing back x and f, leads to

$$(xf'')^2 + 4\big(-xf'+f\big)f'^2 + \big((x-\nu^2)f' - f\big)f' = 0. \quad \textbf{(Painlevé V)} \quad \square$$

Example 3: Eigenvalues of large random Gaussian Hermitian matrices in the bulk and the sine kernel. Setting $\nu = \pm\frac{1}{2}$ yields kernels related to the sine kernel:

$$K_{x,0}^{(+1/2)}(y^2,z^2) = \frac{1}{\pi}\int_0^x \frac{\sin xy\ \sin xz}{y^{1/2}z^{1/2}}\,dx = \frac{1}{2\pi}\left(\frac{\sin x(y-z)}{y-z} - \frac{\sin x(y+z)}{y+z}\right),$$
$$K_{x,0}^{(-1/2)}(y^2,z^2) = \frac{1}{\pi}\int_0^x \frac{\cos xy\ \cos xz}{y^{1/2}z^{1/2}}\,dx = \frac{1}{2\pi}\left(\frac{\sin x(y-z)}{y-z} + \frac{\sin x(y+z)}{y+z}\right).$$

Therefore the sine-kernel obtained in the context of the bulk-scaling limit (see (1.2.7)) is the sum $K_{x,0}^{(+1/2)} + K_{x,0}^{(-1/2)}$. Expressing the Fredholm determinant of this sum in terms of the Fredholm determinants of each of the parts, leads to the Painlevé V equation (1.2.8).

6. Coupled Random Hermitian Ensembles

Consider a product ensemble $(M_1, M_2) \in \mathcal{H}_n^2 := \mathcal{H}_n \times \mathcal{H}_n$ of $n \times n$ Hermitian matrices, equipped with a Gaussian probability measure,

$$c_n \, dM_1 \, dM_2 \, e^{-(1/2)\,\mathrm{Tr}(M_1^2 + M_2^2 - 2cM_1M_2)}, \tag{6.0.1}$$

where $dM_1 \, dM_2$ is Haar measure on the product $\mathcal{H}_n^2$, with each dM_i,

$$dM_1 = \Delta_n^2(x) \prod_1^n dx_i \, dU \quad \text{and} \quad dM_2 = \Delta_n^2(y) \prod_1^n dy_i \, dU \tag{6.0.2}$$

decomposed into radial and angular parts. In terms of the coupling constant c, appearing in (6.0.1), and the boundary of the set

$$E = E_1 \times E_2 := \bigcup_{i=1}^r [a_{2i-1}, a_{2i}] \times \bigcup_{i=1}^s [b_{2i-1}, b_{2i}] \subset \mathbb{R}^2, \tag{6.0.3}$$

define differential operators $\mathcal{A}_k$, $\mathcal{B}_k$ of "weight" k,

$$\begin{aligned} \mathcal{A}_1 &= \frac{1}{c^2-1}\Big(\sum_1^r \frac{\partial}{\partial a_j} + c\sum_1^s \frac{\partial}{\partial b_j}\Big), & \mathcal{B}_1 &= \frac{1}{1-c^2}\Big(c\sum_1^r \frac{\partial}{\partial a_j} + \sum_1^s \frac{\partial}{\partial b_j}\Big), \\ \mathcal{A}_2 &= \sum_{j=1}^r a_j \frac{\partial}{\partial a_j} - c\frac{\partial}{\partial c}, & \mathcal{B}_2 &= \sum_{j=1}^s b_j \frac{\partial}{\partial b_j} - c\frac{\partial}{\partial c}, \end{aligned}$$

forming a closed Lie algebra.[37] The following theorem can be derived, via similar methods, from the Virasoro constraints (3.3.5) and the 2-Toda equation (3.3.6):

THEOREM 6.1 (**Gaussian probability**) (Adler and van Moerbeke [3]). *The joint statistics*

$$\begin{aligned} P_n(M \in \mathcal{H}_n^2(E_1 \times E_2)) &= \frac{\displaystyle\iint_{\mathcal{H}_n^2(E_1\times E_2)} dM_1\, dM_2\, e^{-\frac12 \mathrm{Tr}(M_1^2+M_2^2-2cM_1M_2)}}{\displaystyle\iint_{\mathcal{H}_n^2} dM_1\, dM_2\, e^{-\frac12 \mathrm{Tr}(M_1^2+M_2^2-2cM_1M_2)}} \\ &= \frac{\displaystyle\iint_{E^n} \Delta_n(x)\Delta_n(y) \prod_{k=1}^n e^{-\frac12 (x_k^2+y_k^2-2cx_ky_k)}\, dx_k dy_k}{\displaystyle\iint_{\mathbb{R}^{2n}} \Delta_n(x)\Delta_n(y) \prod_{k=1}^n e^{-\frac12 (x_k^2+y_k^2-2cx_ky_k)}\, dx_k dy_k} \end{aligned}$$

---

[37] We have

$$\begin{aligned} &[\mathcal{A}_1, \mathcal{B}_1] = 0, \quad [\mathcal{A}_1, \mathcal{A}_2] = \frac{1+c^2}{1-c^2}\mathcal{A}_1, \quad [\mathcal{A}_2, \mathcal{B}_1] = \frac{2c}{1-c^2}\mathcal{A}_1, \\ &[\mathcal{A}_2, \mathcal{B}_2] = 0, \quad [\mathcal{A}_1, \mathcal{B}_2] = \frac{-2c}{1-c^2}\mathcal{B}_1, \quad [\mathcal{B}_1, \mathcal{B}_2] = \frac{1+c^2}{1-c^2}\mathcal{B}_1. \end{aligned}$$

satisfies the following nonlinear third-order partial differential equation[38] (*independent of* n), *where* $F_n := (1/n)\log P_n(E)$:

$$\left\{\mathcal{B}_2\mathcal{A}_1F_n,\, \mathcal{B}_1\mathcal{A}_1F_n+\frac{c}{c^2-1}\right\}_{\mathcal{A}_1}-\left\{\mathcal{A}_2\mathcal{B}_1F_n,\, \mathcal{A}_1\mathcal{B}_1F_n+\frac{c}{c^2-1}\right\}_{\mathcal{B}_1}=0. \quad (6.0.4)$$

7. Random Permutations

The purpose of this section is to show that the generating function of the probability

$$P(L(\pi_n)\le l)=\frac{1}{n!}\#\{\pi_n\in S_n \mid L(\pi_n)\le l\}$$

is closely related to a special solution of the Painlevé V equation, with peculiar initial condition. Remember from Section 1.4 that $L(\pi_n)$ is the length of the longest increasing sequence in the permutation π_n.

THEOREM 7.1 (Tracy and Widom [66]). *For every* $l\ge 0$,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{x^n}{n!}P(L(\pi_n)\le l)&=\int_{U(l)}e^{\sqrt{x}\ tr(M+\bar M)}\,dM\\ &=\exp\int_0^x\log\left(\frac{x}{u}\right)g_l(u)\,du,\end{aligned} \quad (7.0.1)$$

with g_l *satisfying the initial value problem for* **Painlevé V**:

$$\begin{cases} g''-\dfrac{g'^2}{2}\left(\dfrac{1}{g-1}+\dfrac{1}{g}\right)+\dfrac{g'}{u}+\dfrac{2}{u}g(g-1)-\dfrac{l^2}{2u^2}\dfrac{g-1}{g}=0,\\ g_l(u)=1-\dfrac{u^l}{l!^2}+O(u^{l+1}),\quad \textit{near } u=0.\end{cases} \quad (7.0.2)$$

The systematic derivation below is due to Adler and van Moerbeke [7].

PROOF. The first identity in (7.0.1) follows from Proposition 1.1. Upon inserting $(t_1,t_2,\dots)$ and $(s_1,s_2,\dots)$ variables in the $U(n)$-integral (7.0.1), the integral

$$\begin{aligned}I_n(t,s)&=\int_{U(n)}e^{\operatorname{Tr}\sum_1^\infty(t_iM^i-s_i\bar M^i)}\,dM \qquad (7.0.3)\\ &=n!\det\left(\int_{S^1}z^{k-l}e^{\sum_1^\infty(t_iz^i-s_iz^{-i})}\frac{dz}{2\pi iz}\right)_{0\le k,l\le n-1}=n!\,\tau_n(t,s)\end{aligned}$$

puts us in the conditions of Theorem 3.5. It deals with semi-infinite matrices L_1 and $hL_2^\top h^{-1}$ of "rank 2", having diagonal elements

$$b_n:=\frac{\partial}{\partial t_1}\log\frac{\tau_n}{\tau_{n-1}}=(L_1)_{n-1,n-1},\qquad b_n^*:=-\frac{\partial}{\partial s_1}\log\frac{\tau_n}{\tau_{n-1}}=(hL_2^\top h^{-1})_{n-1,n-1}.$$

To summarize Theorem 3.5, $I_n(t,s)$ satisfies three types of identities:

[38] in terms of the Wronskian $\{f,g\}_X=(Xf)g-f(Xg)$, with regard to a first order differential operator X.

(i) **Virasoro** (see (3.4.7)); we set $F := \log \tau_n$:

$$
\begin{aligned}
0 &= \frac{\mathcal{V}_{-1}\tau_n}{\tau_n} = \left(\sum_{i\geq 1}(i+1)t_{i+1}\frac{\partial}{\partial t_i} - \sum_{i\geq 2}(i-1)s_{i-1}\frac{\partial}{\partial s_i} + n\frac{\partial}{\partial s_1} \right) F + nt_1,\\
0 &= \frac{\mathcal{V}_0\tau_n}{\tau_n} = \sum_{i\geq 1}\left(it_i\frac{\partial}{\partial t_i} - is_i\frac{\partial}{\partial s_i}\right) F,\\
0 &= \frac{\mathcal{V}_1\tau_n}{\tau_n} = \left(-\sum_{i\geq 1}(i+1)s_{i+1}\frac{\partial}{\partial s_i} + \sum_{i\geq 2}(i-1)t_{i-1}\frac{\partial}{\partial t_i}\frac{\partial}{\partial t_1} \right) F + ns_1,\\
0 &= \frac{\partial}{\partial t_1}\frac{\mathcal{V}_{-1}\tau_n}{\tau_n}\\
&= \left(\sum_{i\geq 1}(i+1)t_{i+1}\frac{\partial^2}{\partial t_1\partial t_i} - \sum_{i\geq 2}(i-1)s_{i-1}\frac{\partial^2}{\partial t_1\partial s_i} + n\frac{\partial^2}{\partial t_1\partial s_1} \right) F + n. \qquad (7.0.4)
\end{aligned}
$$

(ii) **two-Toda** (see (3.4.8)):

$$
\begin{aligned}
\frac{\partial^2 \log\tau_n}{\partial s_2\partial t_1} &= -2\frac{\partial}{\partial s_1}\log\frac{\tau_n}{\tau_{n-1}}\frac{\partial^2}{\partial s_1\partial t_1}\log\tau_n - \frac{\partial^3}{\partial s_1^2\partial t_1}\log\tau_n\\
&= 2b_n^*\frac{\partial^2}{\partial s_1\partial t_1}\log\tau_n - \frac{\partial^3}{\partial s_1^2\partial t_1}\log\tau_n. \qquad (7.0.5)
\end{aligned}
$$

(iii) **Toeplitz** (see (3.4.11)):

$$
\begin{aligned}
\mathcal{T}(\tau)_n &= \frac{\partial}{\partial t_1}\log\frac{\tau_n}{\tau_{n-1}}\frac{\partial}{\partial s_1}\log\frac{\tau_n}{\tau_{n-1}}\\
&\quad + \left(1 + \frac{\partial^2}{\partial s_1\partial t_1}\log\tau_n\right)\left(1 + \frac{\partial^2}{\partial s_1\partial t_1}\log\tau_n - \frac{\partial}{\partial s_1}\left(\frac{\partial}{\partial t_1}\log\frac{\tau_n}{\tau_{n-1}}\right)\right)\\
&= -b_nb_n^* + \left(1 + \frac{\partial^2}{\partial s_1\partial t_1}\log\tau_n\right)\left(1 + \frac{\partial^2}{\partial s_1\partial t_1}\log\tau_n - \frac{\partial}{\partial s_1}b_n\right) = 0. \qquad (7.0.6)
\end{aligned}
$$

Defining the locus $\mathcal{L} = \{$ all $t_i = s_i = 0$, except $t_1, s_1 \neq 0\}$, and using the second relation (7.0.4), we have on $\mathcal{L}$

$$
\frac{\mathcal{V}_0\tau_n}{\tau_n}\Big|_{\mathcal{L}} = \left(t_1\frac{\partial}{\partial t_1} - s_1\frac{\partial}{\partial s_1}\right)\log\tau_n\Big|_{\mathcal{L}} = 0,
$$

implying that $\tau_n(t,s)\big|_{\mathcal{L}}$ is a function of $x := -t_1s_1$ only. Therefore we may write $\tau_n\big|_{\mathcal{L}} = \tau_n(x)$, and so, along $\mathcal{L}$, we have

$$
\frac{\partial}{\partial t_1} = -s_1\frac{\partial}{\partial x},\quad \frac{\partial}{\partial s_1} = -t_1\frac{\partial}{\partial x},\quad \frac{\partial^2}{\partial t_1\partial s_1} = -\frac{\partial}{\partial x}x\frac{\partial}{\partial x}.
$$

Setting

$$
f_n(x) = \frac{\partial}{\partial x}x\frac{\partial}{\partial x}\log\tau_n(x) = -\frac{\partial^2}{\partial t_1\partial s_1}\log\tau_n(t,s)\Big|_{\mathcal{L}}, \qquad (7.0.7)
$$

and using $x=-t_1s_1$, the two-Toda relation (7.0.5) takes on the form

$$s_1\frac{\partial^2\log\tau_n}{\partial s_2\partial t_1}\Big|_{\mathcal{L}}=s_1\left(2b_n^*\frac{\partial^2}{\partial s_1\partial t_1}\log\tau_n-\frac{\partial}{\partial s_1}\Big(\frac{\partial^2\log\tau_n}{\partial s_1\partial t_1}\Big)\right)=x\Big(2\frac{b_n^*}{t_1}f_n+f_n'\Big).$$

Setting relation (7.0.7) into the Virasoro relations (7.0.4) yields

$$\begin{aligned}0&=\frac{\mathcal{V}_0\tau_n}{\tau_n}-\frac{\mathcal{V}_0\tau_{n-1}}{\tau_{n-1}}\Big|_{\mathcal{L}}=\Big(t_1\frac{\partial}{\partial t_1}-s_1\frac{\partial}{\partial s_1}\Big)\log\frac{\tau_n}{\tau_{n-1}}\Big|_{\mathcal{L}}=t_1b_n+s_1b_n^*\\
0&=\frac{\partial}{\partial t_1}\frac{\mathcal{V}_{-1}\tau_n}{\tau_n}\Big|_{\mathcal{L}}=\Big(-s_1\frac{\partial^2}{\partial s_2\partial t_1}+n\frac{\partial^2}{\partial t_1\partial s_1}\Big)\log\tau_n\Big|_{\mathcal{L}}+n\\
&\qquad=-x\Big(2\frac{b_n^*}{t_1}f_n(x)+f_n'(x)\Big)+n(-f_n(x)+1).\end{aligned}$$

This is a system of two linear relations in b_n and b_n^*, whose solution, together with its derivatives, are given by

$$\begin{aligned}\frac{b_n^*}{t_1}&=-\frac{b_n}{s_1}=-\frac{n(f_n-1)+xf_n'}{2xf_n},\\
\frac{\partial b_n}{\partial s_1}&=\frac{\partial}{\partial x}x\frac{b_n}{s_1}=\frac{x(f_nf_n''-f_n'^2)+(f_n+n)f_n'}{2f_n^2}.\end{aligned}$$

Substituting the result into the Toeplitz relation (7.0.6), namely

$$b_nb_n^*=(1-f_n)\Big(1-f_n-\frac{\partial}{\partial s_1}b_n\Big),$$

leads to f_n satisfying Painlevé equation (7.0.2), with $g=f_n$, as in (7.0.7) and $u=x$. Note, along the locus $\mathcal{L}$, we may set $t_1=\sqrt{x}$ and $s_1=-\sqrt{x}$, since it respects $t,s_1=-x$. Thus, $I_n(t,s)|_{\mathcal{L}}$ equals (7.0.1).

The initial condition (7.0.2) follows from the fact that as long as $0\le n\le l$, the inequality $L(\pi_n)\le l$ is always verified, and so

$$\begin{aligned}\sum_0^\infty\frac{x^n}{(n!)^2}\#\{\pi\in S_n\mid L(\pi_n)\le l\}&=\sum_0^l\frac{x^n}{n!}+\frac{x^{l+1}}{(l+1)!^2}((l+1)!-1)+O(x^{l+2})\\
&=\exp\left(x-\frac{x^{l+1}}{(l+1)!^2}+O(x^{l+2})\right),\end{aligned}$$

thus proving Theorem 7.1. □

Remark. Setting

$$f_n(x)=\frac{g(x)}{g(x)-1}$$

leads to standard Painlevé V, with $\alpha=\delta=0$, $\beta=-n^2/2$, $\gamma=-2$.

8. Random Involutions

This section deals with a generating function for the distribution of the length of the longest increasing sequence of a fixed-point free random involution π^0_{2k}, with the uniform distribution:

$$P\left(L(\pi^0_{2k}) \le l+1,\ \pi^0_{2k} \in S^0_{2k}\right) = \frac{2^k k!}{(2k)!} \#\left\{\pi^0_{2k} \in S^0_{2k} \mid L(\pi^0_{2k}) \le l+1\right\}.$$

PROPOSITION 8.1 (Adler and van Moerbeke [7]). *The generating function*

$$\begin{aligned} &2\sum_{k=0}^{\infty} \frac{(x^2/2)^k}{k!} P(L(\pi^0_{2k}) \le l+1) \\ &= E_{O(l+1)_-} e^{x \operatorname{Tr} M} + E_{O(l+1)_+} e^{x \operatorname{Tr} M} \\ &= \exp\left(\int_0^x \frac{f_l^-(u)}{u}\,du\right) + \exp\left(\int_0^x \frac{f_l^+(u)}{u}\,du\right), \end{aligned} \tag{8.0.1}$$

where $f = f_l^{\pm}$, satisfies the initial value problem for **Painlevé V**:

$$\begin{cases} f''' + \dfrac{1}{u} f'' + \dfrac{6}{u} f'^2 - \dfrac{4}{u^2} f f' - \dfrac{16u^2 + l^2}{u^2} f' + \dfrac{16}{u} f + \dfrac{2(l^2-1)}{u} = 0, \\ f_l^{\pm}(u) = u^2 \pm \dfrac{u^{l+1}}{l!} + O(u^{l+2}) \quad \textit{near } u = 0. \end{cases} \tag{8.0.2}$$

PROOF. The first equality in (8.0.1), due to Rains [57; 58], follows immediately from Proposition 1.1. The results of Section 1.3 lead to

$$\int_{O(2n+1)_{\pm}} e^{x \operatorname{Tr} M}\, dM = e^{\pm x} \int_{[-1,1]^n} \Delta_n(z)^2 \prod_{k=1}^{n} e^{2xz_k}(1-z_k)^a (1+z_k)^b dz_k, \tag{8.0.3}$$

with $a = \pm\frac{1}{2}$, $b = \mp\frac{1}{2}$, (with corresponding signs). Inserting t_i's in the integral, the perturbed integral, with $e^{\pm x}$ removed and with $t_1 = 2x$, reads

$$I_n(t) = \int_{[-1,1]^n} \Delta_n(z)^2 \prod_{k=1}^{n} (1-z_k)^a (1+z_k)^b e^{\sum_1^\infty t_i z_k^i}\, dz_k = n!\,\tau_n(t); \tag{8.0.4}$$

this is precisely integral (3.1.4) of Section 3.1.1 and thus it satisfies the Virasoro constraints (3.1.5), but without boundary contribution $\mathcal{B}_i F$. Explicit Virasoro expressions appear in (2.1.35), upon setting $\beta = 2$. Also, $\tau_n(t)$, as in (3.1.4), (see Theorem 3.1) satisfies the KP equation (3.1.6). Differentiating the Virasoro constraints in t_1 and t_2, and restricting to the locus

$$\mathcal{L} := \{t_1 = x,\ \text{all other } t_i = 0\},$$

lead to a linear system of five equations, with $b_0 = a - b$, $b_1 = a + b$,

$$\frac{1}{I_n}\left(\mathbb{J}^{(2)}_{k+2} - \mathbb{J}^{(2)}_{k} + b_0\mathbb{J}^{(1)}_{k+1} + b_1\mathbb{J}^{(1)}_{k+2}\right) I_n\Big|_{\mathcal{L}} = 0 \quad for k = -1, 0,$$
$$\frac{\partial}{\partial t_1}\frac{1}{I_n}\left(\mathbb{J}^{(2)}_{k+2} - \mathbb{J}^{(2)}_{k} + b_0\mathbb{J}^{(1)}_{k+1} + b_1\mathbb{J}^{(1)}_{k+2}\right) I_n\Big|_{\mathcal{L}} = 0 \quad for k = -1, 0,$$
$$\frac{\partial}{\partial t_2}\frac{1}{I_n}\left(\mathbb{J}^{(2)}_{k+2} - \mathbb{J}^{(2)}_{k} + b_0\mathbb{J}^{(1)}_{k+1} + b_1\mathbb{J}^{(1)}_{k+2}\right) I_n\Big|_{\mathcal{L}} = 0 \quad for k = -1,$$

in five unknowns ($F_n = \log\tau_n$)

$$\frac{\partial F_n}{\partial t_2}\Big|_{\mathcal{L}}, \quad \frac{\partial F_n}{\partial t_3}\Big|_{\mathcal{L}}, \quad \frac{\partial^2 F_n}{\partial t_1\partial t_2}\Big|_{\mathcal{L}}, \quad \frac{\partial^2 F_n}{\partial t_1\partial t_3}\Big|_{\mathcal{L}}, \quad \frac{\partial^2 F_n}{\partial t_2^2}\Big|_{\mathcal{L}}.$$

Setting $t_1 = x$ and $F_n' = \partial F_n/\partial x$, the solution is given by the expressions

$$\begin{aligned}
\frac{\partial F_n}{\partial t_2}\Big|_{\mathcal{L}} &= -\frac{1}{x}\big((2n+b_1)F_n' + n(b_0 - x)\big),\\
\frac{\partial F_n}{\partial t_3}\Big|_{\mathcal{L}} &= -\frac{1}{x^2}\big(x\left(F_n'' + F_n'^2 + (b_0 - x)F_n' + n(n+b_1)\right)\\
&\qquad - (2n+b_1)\left((2n+b_1)F_n' + b_0 n\right)\big),\\
\frac{\partial^2 F_n}{\partial t_1\partial t_2}\Big|_{\mathcal{L}} &= -\frac{1}{x^2}\big((2n+b_1)(xF_n'' - F_n') - b_0 n\big),\\
\frac{\partial^2 F_n}{\partial t_1\partial t_3}\Big|_{\mathcal{L}} &= -\frac{1}{x^3}\,\big(x^2(F_n''' + 2F_n'F_n'') - x\big((x^2 - b_0 x + 1)F_n'' + F_n'^2 + b_0 F_n'\\
&\qquad + (2n+b_1)^2 F_n'' + n(n+b_1)\big) + 2(2n+b_1)^2 F_n' + 2b_0 n(2n+b_1)\big),\\
\frac{\partial^2 F_n}{\partial t_2^2}\Big|_{\mathcal{L}} &= \frac{1}{x^3}\big(x\big(2F_n'^2 + 2b_0F_n' + ((2n+b_1)^2 + 2)F_n'' + 2n(n+b_1)\big)\\
&\qquad - 3(2n+b_1)^2 F_n' - 3b_0 n(2n+b_1)\big).
\end{aligned}$$

Putting these expressions into KP and setting $t_1 = x$, one finds

$$\begin{aligned}
0 &= \left(\left(\frac{\partial}{\partial t_1}\right)^4 + 3\left(\frac{\partial}{\partial t_2}\right)^2 - 4\frac{\partial^2}{\partial t_1\partial t_3}\right)F_n + 6\left(\frac{\partial^2}{\partial t_1^2}F_n\right)^2\\
&= \frac{1}{x^3}\big(x^3F'''' + 4x^2F''' + x(-4x^2 + 4b_0x + 2 - (2n+b_1)^2)F'' + 8x^2F'F''\\
&\qquad + 6x^3F''^2 + 2xF'^2 + (2b_0x - (2n+b_1)^2)F' + n(2x - b_0)(n+b_1) - b_0n^2\big).
\end{aligned}$$

Finally, the function

$$H(x) := x\frac{d}{dx}F(x) = x\frac{d}{dx}\log\tau_n(x)$$

satisfies

$$\begin{aligned}
x^2H''' + xH'' + 6xH'^2 - \left(4H + 4x^2 - 4bx + (2n+a)^2\right)H' + (4x - 2b)H&\\
+ 2n(n+a)x - bn(2n+a) = 0.& \quad (8.0.5)
\end{aligned}$$

This third order equation is Cosgrove's [24; 23] equation, with $P = x$, $4Q = -4x^2 + 4bx - (2n+a)^2$, $2R = 2n(n+a)x - bn(2n+a)$. So, this third order equation can be transformed into the Painlevé V equation (9.0.3) in the appendix. The boundary condition $f(0) = 0$ follows from the definition of H above, whereas, after an elementary, but tedious computation, $f'(0) = f''(0) = 0$ follows from the differential equation (8.0.5) and the Aomoto extension [15] (see Mehta [49, p. 340]) of Selberg's integral:[39]

$$\frac{\int_0^1 \cdots \int_0^1 x_1 \dots x_m \, |\Delta(x)|^\beta \prod_{j=1}^n x_j^\gamma (1-x_j)^\delta \, dx_1 \dots dx_n}{\int_0^1 \cdots \int_0^1 |\Delta(x)|^\beta \prod_{j=1}^n x_j^\gamma (1-x_j)^\delta \, dx_1 \dots dx_n} = \prod_{j=1}^m \frac{\gamma + 1 + (n-j)\beta/2}{\gamma + \delta + 2 + (2n-j-1)\beta/2}.$$

However, the initial condition (8.0.2) is a much stronger statement, again stemming from the fact that as long as $0 \le n \le l$, the inequality $L(\pi_n) \le l$ is trivially verified, thus leading to

$$E_{O_\pm(l+1)} e^{x \operatorname{Tr} M} = \exp\left(\frac{x^2}{2} \pm \frac{x^{l+1}}{(l+1)!} + O(x^{l+2})\right),$$

ending the proof of Proposition 8.1. □

Appendix: Chazy Classes

Most of the differential equations encountered in this survey belong to the general Chazy class

$$f''' = F(z, f, f', f''), \text{ where } F \text{ is rational in } f, f', f'' \text{ and locally analytic in z,}$$

subjected to the requirement that the general solution be free of movable branch points; the latter is a branch point whose location depends on the integration constants. In his classification Chazy found thirteen cases, the first of which is given by

$$f''' + \frac{P'}{P} f'' + \frac{6}{P} f'^2 - \frac{4P'}{P^2} f f' + \frac{P''}{P^2} f^2 + \frac{4Q}{P^2} f' - \frac{2Q'}{P^2} f + \frac{2R}{P^2} = 0 \qquad (9.0.1)$$

with arbitrary polynomials $P(z), Q(z), R(z)$ of degree $3, 2, 1$ respectively. Cosgrove and Scoufis [24; 23, (A.3)] show that this third order equation has a first integral, which is second order in f and quadratic in f'',

$$f''^2 + \frac{4}{P^2}\Big((P f'^2 + Q f' + R) f' - (P' f'^2 + Q' f' + R') f + \frac{1}{2}(P'' f' + Q'') f^2 - \frac{1}{6} P''' f^3 + c\Big) = 0; \qquad (9.0.2)$$

c is the integration constant. Equations of the general form

$$f''^2 = G(x, f, f')$$

[39] Here $\operatorname{Re}\gamma$, $\operatorname{Re}\delta > -1$, and $\operatorname{Re}\beta > -2\min\left(\frac{1}{n}, \frac{\operatorname{Re}\gamma + 1}{n-1}, \frac{\operatorname{Re}\delta + 1}{n-1}\right)$.

are invariant under the map

$$x \mapsto \frac{a_1 z + a_2}{a_3 z + a_4} \quad \text{and} \quad f \mapsto \frac{a_5 f + a_6 z + a_7}{a_3 z + a_4}.$$

Using this map, the polynomial $P(z)$ can be normalized to

$$P(z) = z(z-1), \ z, \ \text{or } 1.$$

In this way, Cosgrove shows (9.0.2) is a master Painlevé equation, containing the 6 Painlevé equations. In each of the cases, the canonical equations are respectively:

- $g''^2 = -4g'^3 - 2g'(zg' - g) + A_1$ **(Painlevé II)**
- $g''^2 = -4g'^3 + 4(zg' - g)^2 + A_1 g' + A_2$ **(Painlevé IV)**
- $(zg'')^2 = (zg' - g)\big(-4g'^2 + A_1(zg' - g) + A_2\big) + A_3 g' + A_4$ **(Painlevé V)**
- $(z(z-1)g'')^2 = (zg' - g)\big(4g'^2 - 4g'(zg' - g) + A_2\big) + A_1 g'^2 + A_3 g' + A_4$ **(Painlevé VI)**

(9.0.3)

The Painlevé II equation above can be solved by setting

$$\begin{aligned} g(z) &= \tfrac{1}{2}(u')^2 - \tfrac{1}{2}(u^2 + \tfrac{1}{2}z)^2 - (\alpha + \tfrac{1}{2}\varepsilon_1)u, \\ g'(z) &= -\tfrac{1}{2}\varepsilon_1 u' - \tfrac{1}{2}(u^2 + \tfrac{1}{2}z), \\ A_1 &= \tfrac{1}{4}\big(\alpha + (u^2 + \tfrac{1}{2}z)^2\varepsilon_1\big)^2 \quad (\varepsilon = \pm 1). \end{aligned}$$

Then $u(z)$ satisfies yet another version of the Painlevé II equation

$$u'' = 2u^3 + zu + \alpha. \qquad \textbf{(Painlevé II)}$$

Now, each of these Painlevé II, IV, V, VI equations can be transformed into the standard Painlevé equations, which are all differential equations of the form

$$f'' = F(z, f, f'), \quad \text{rational in } f \text{ and } f' \text{ and analytic in } z,$$

whose general solution has no movable critical points. Painlevé showed that this requirement leads to 50 types of equations, six of which cannot be reduced to known equations.

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