Random Matrices and Their Applications
MSRI Publications
Volume **40**, 2001

SL(2) and z-Measures

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ABSTRACT. We provide a representation-theoretic derivation of the determinantal formula of Borodin and Olshanski for the correlation functions of z-measures in terms of the hypergeometric kernel.

1. Introduction

This paper is about z-measures, a remarkable two-parameter family of measures on partitions introduced by S. Kerov, G. Olshanski and A. Vershik [Kerov et al. 1993] in the context of harmonic analysis on the infinite symmetric group. In a series of papers, A. Borodin and Olshanski obtained fundamental results on z-measures; see the survey [Borodin and Olshanski 2001] in this volume and also [Borodin and Olshanski 1998]. The culmination of this development is an exact determinantal formula for the correlation functions of the z-measures in terms of the hypergeometric kernel [Borodin and Olshanski 2000]. We mention [Borodin et al. 2000] as one of the applications of this formula. The main result of this paper is a representation-theoretic derivation of the formula of Borodin and Olshanski.

In the early days of z-measures, it was already noticed that they have some mysterious connection to the representation theory of SL(2). For example, a z-measure is in fact positive if its two parameters z and z' are either complex conjugate $z' = \bar{z}$ or $z, z' \in (n, n+1)$ for some $n \in \mathbb{Z}$. In these cases $z - z'$ is either imaginary or lies in $(-1, 1)$, which is reminiscent of the principal and complementary series of representations of SL(2).

Later, Kerov (private communication) constructed an SL(2)-action on partitions for which the z-measures are certain matrix elements. Finally, Borodin and Olshanski computed the correlation functions of the z-measures is in terms of the Gauss hypergeometric function, which appears in matrix elements of representations of SL(2). The aim of this paper is to put these pieces together.

The constructions of this paper were subsequently generalized beyond SL(2) and z-measures in [Okounkov 1999].

2. z-measures, Kerov Operators, and Correlation Functions

2.1. Definition of z-Measures. Let $z, z' \in \mathbb{C}$ be two parameters and consider the following measure on the set of all partitions λ of n:

$$\mathcal{M}_n(\lambda) = \frac{n!}{(zz')_n} \prod_{\square \in \lambda} \frac{(z + c(\square))(z' + c(\square))}{h(\square)^2}, \tag{2–1}$$

where

$$(x)_n = x(x+1)\dots(x+n-1),$$

the product is over all squares $\square$ in the diagram of λ, $h(\square)$ is the length of the corresponding hook, and $c(\square)$ stands for the content of the square $\square$. Recall that, by definition, the content of $\square$ is

$$c(\square) = \text{column}(\square) - \text{row}(\square),$$

where column($\square$) denotes the column number of the square $\square$. See [Macdonald 1995] for general facts about partitions.

It is not immediately obvious from the definition (2–1) that

$$\sum_{|\lambda|=n} \mathcal{M}_n(\lambda) = 1. \tag{2–2}$$

One possible proof of (2–2) uses the following operators on partitions, introduced by S. Kerov.

2.2. Kerov Operators. Consider the vector space with an orthonormal basis $\{\delta_\lambda\}$ indexed by all partitions of λ of any size. Introduce the operators

$$\begin{aligned} U\delta_\lambda &= \sum_{\mu=\lambda+\square} (z + c(\square))\,\delta_\mu, \\ L\delta_\lambda &= (zz' + 2|\lambda|)\,\delta_\lambda, \\ D\delta_\lambda &= \sum_{\mu=\lambda-\square} (z' + c(\square))\,\delta_\mu, \end{aligned}$$

where $\mu = \lambda + \square$ means that μ is obtained from λ by adding a square $\square$ and $c(\square)$ is the content of this square. The letters U and D here stand for "up" and "down".

These operators satisfy the commutation relations

$$[D, U] = L, \quad [L, U] = 2U, \quad [L, D] = -2D, \tag{2–3}$$

as does the basis of $\mathfrak{sl}(2)$ given by

$$U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

In particular, it is clear that if $|\lambda| = n$ then

$$(U^n \delta_\varnothing, \delta_\lambda) = \dim \lambda \prod_{\square \in \lambda} (z + c(\square)),$$

where

$$\dim \lambda = n! \prod_{\square \in \lambda} h(\square)^{-1}$$

is the number of standard tableaux on λ. It follows that

$$\mathcal{M}_n(\lambda) = \frac{1}{n!\,(zz')_n} (U^n \delta_\varnothing, \delta_\lambda)(L^n \delta_\lambda, \delta_\varnothing).$$

Using this presentation and the commutation relations (2–3) one proves (2–2) by induction on n.

2.3. The Measure $\mathcal{M}$ and Its Normalization. In a slightly different language, with induction on n replaced by the use of generating functions, this computation goes as follows.

As in [Borodin and Olshanski 2000], the sequence of the measures $\mathcal{M}_n$ can be conveniently assembled into one measure $\mathcal{M}$ on the set of all partitions of all numbers:

$$\mathcal{M} = (1-\xi)^{zz'} \sum_{n=0}^{\infty} \xi^n \frac{(zz')_n}{n!} \mathcal{M}_n,$$

where $\xi \in [0,1)$ is a new parameter. In other words, $\mathcal{M}$ is the mixture of the measures $\mathcal{M}_n$ by means of a negative binomial distribution on n with parameter ξ.

It is clear that (2–2) is now equivalent to $\mathcal{M}$ being a probability measure. It is also clear that

$$\mathcal{M}(\lambda) = (1-\xi)^{zz'} (e^{\sqrt{\xi}U} \delta_\varnothing, \delta_\lambda)(e^{\sqrt{\xi}D} \delta_\lambda, \delta_\varnothing). \tag{2–4}$$

Therefore

$$\sum_\lambda \mathcal{M}(\lambda) = (1-\xi)^{zz'} (e^{\sqrt{\xi}D} e^{\sqrt{\xi}U} \delta_\varnothing, \delta_\varnothing). \tag{2–5}$$

It follows from the definitions that

$$D\delta_\varnothing = 0, \quad L\delta_\varnothing = zz'\delta_\varnothing, \quad U^*\delta_\varnothing = 0, \tag{2–6}$$

where U^* is the operator adjoint to U. Therefore, in order to evaluate (2–5), it suffices to commute $e^{\sqrt{\xi}L}$ through $e^{\sqrt{\xi}U}$.

A computation in SL(2),

$$\begin{pmatrix} 1 & 0 \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \dfrac{\alpha}{1-\alpha\beta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \dfrac{1}{1-\alpha\beta} & 0 \\ 0 & 1-\alpha\beta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\dfrac{\beta}{1-\alpha\beta} & 1 \end{pmatrix},$$

implies that

$$\exp(\beta D)\exp(\alpha U) = \exp\left(\frac{\alpha}{1-\alpha\beta}U\right)(1-\alpha\beta)^{-L}\exp\left(\frac{\beta}{1-\alpha\beta}D\right), \quad (2\text{–}7)$$

provided $|\alpha\beta| < 1$. Therefore

$$\sum_\lambda \mathcal{M}(\lambda) = (1-\xi)^{zz'}\left(\exp\left(\frac{\sqrt{\xi}}{1-\xi}U\right)(1-\xi)^{-L}\exp\left(\frac{\sqrt{\xi}}{1-\xi}D\right)\delta_\varnothing, \delta_\varnothing\right)$$
$$= (1-\xi)^{zz'}\big((1-\xi)^{-L}\delta_\varnothing, \delta_\varnothing\big) = 1,$$

as was to be shown.

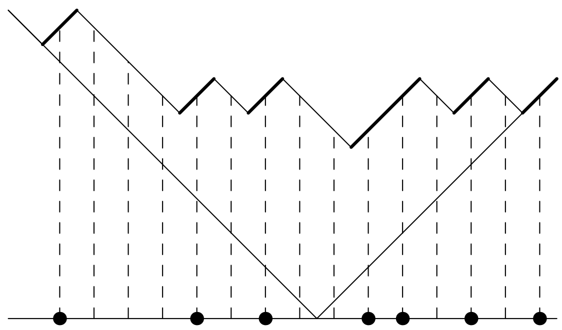
2.4. Correlation Functions. We now introduce coordinates on the set of partitions. To a partition λ we associate a subset

$$\mathfrak{S}(\lambda) = \left\{\lambda_i - i + \tfrac{1}{2}\right\} \subset \mathbb{Z} + \tfrac{1}{2}.$$

For example,

$$\mathfrak{S}(\varnothing) = \left\{-\tfrac{1}{2}, -\tfrac{3}{2}, -\tfrac{5}{2}, \dots\right\}$$

This set $\mathfrak{S}(\lambda)$ has a geometric interpretation. Take the diagram of λ and rotate it 135° thus:



The positive direction of the axis points to the left in the figure. The boundary of λ forms a zigzag path and the elements of $\mathfrak{S}(\lambda)$, which are marked by $\bullet$, correspond to moments when this zigzag goes up.

Subsets $S \subset \mathbb{Z} + \frac{1}{2}$ of the form $S = \mathfrak{S}(\lambda)$ can be characterized by

$$|S_+| = |S_-| < \infty,$$

where

$$S_+ = S \setminus \left(\mathbb{Z}_{\le 0} - \tfrac{1}{2}\right), \quad S_- = \left(\mathbb{Z}_{\le 0} - \tfrac{1}{2}\right) \setminus S.$$

The number $|\mathfrak{S}_+(\lambda)| = |\mathfrak{S}_-(\lambda)|$ is the number of squares in the diagonal of the diagram of λ and the finite set $\mathfrak{S}_+(\lambda) \cup \mathfrak{S}_-(\lambda) \subset \mathbb{Z} + \frac{1}{2}$ is known as the modified Frobenius coordinates of λ.

Given a finite subset $X \in \mathbb{Z} + \frac{1}{2}$, define the *correlation function* by

$$\rho(X) = \mathcal{M}\big(\{\lambda, X \subset \mathfrak{S}(\lambda)\}\big).$$

Borodin and Olshanski [2000] proved that

$$\rho(X) = \det \left[K(x_i, x_j)\right]_{x_i, x_j \in X}$$

where K the *hypergeometric kernel* introduced in [Borodin and Olshanski 2000]. This kernel involves the Gauss hypergeometric function and the explicit formula for K will be reproduced below.

It is our goal in the present paper to give a representation-theoretic derivation of the formula for correlation functions and, in particular, show how the kernel K arises from matrix elements of irreducible SL(2)-modules.

3. SL(2) and Correlation Functions

3.1. Matrix Elements of $\mathfrak{sl}(2)$-Modules and the Gauss Hypergeometric Function. It is well known that the hypergeometric function arises as matrix coefficients of SL(2) modules. A standard way to see this is to use a functional realization of these modules; the computation of matrix elements leads then to an integral representation of the hypergeometric function, see for example how matrix elements of SL(2)-modules are treated in [Vilenkin 1968]. An alternative approach is to use explicit formulas for the action of the Lie algebra $\mathfrak{sl}(2)$ and it goes as follows.

Consider the $\mathfrak{sl}(2)$-module V with the basis v_k indexed by all half-integers $k \in \mathbb{Z} + \frac{1}{2}$, and the action of $\mathfrak{sl}(2)$ given by

$$\begin{aligned} U v_k &= \left(z + k + \tfrac{1}{2}\right) v_{k+1}, \\ L v_k &= \left(2k + z + z'\right) v_k, \\ D v_k &= \left(z' + k - \tfrac{1}{2}\right) v_{k-1}. \end{aligned} \tag{3–1}$$

It is clear that

$$e^{\alpha U} v_k = \sum_{s=0}^{\infty} \frac{\alpha^s}{s!} \left(z + k + \tfrac{1}{2}\right)_s v_{k+s}.$$

Introduce the notation

$$(a)_{\downarrow s} = a(a-1)(a-2)\cdots(a-s+1),$$

so that

$$e^{\beta D} v_k = \sum_{s=0}^{\infty} \frac{\beta^s}{s!} \left(z' + k - \tfrac{1}{2}\right)_{\downarrow s} v_{k-s}.$$

Denote by $[i \to j]_{\alpha,\beta,z,z'}$ the coefficient of v_j in the expansion of $e^{\alpha U} e^{\beta D} v_i$:

$$e^{\alpha U} e^{\beta D} v_i = \sum_j \, [i \to j]_{\alpha,\beta,z,z'} \, v_j.$$

A direct computation yields

$$[i \to j]_{\alpha,\beta,z,z'} = \begin{cases} \dfrac{\alpha^{j-i}}{(j-i)!}\,(z+i+\tfrac12)_{j-i}\, F\left(\begin{matrix} -z-i+\frac12,\, -z'-i+\frac12 \\ j-i+1 \end{matrix}; \alpha\beta\right) & \text{if } i \le j\ , \\ \dfrac{\beta^{i-j}}{(i-j)!}\,(z'+j+\tfrac12)_{i-j}\, F\left(\begin{matrix} -z-j+\frac12,\, -z'-j+\frac12 \\ i-j+1 \end{matrix}; \alpha\beta\right) & \text{if } i \ge j\ , \end{cases} \tag{3–2}$$

where

$$F\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k\,k!}\, z^k$$

is the Gauss hypergeometric function.

Now consider the dual module V^* spanned by functionals v_j^* such that

$$\langle v_i^*, v_j \rangle = \delta_{ij}$$

and equipped with the dual action of $\mathfrak{sl}(2)$:

$$U\,v_k^* = -\bigl(z + k - \tfrac12\bigr)\, v_{k-1}^*, \quad D\,v_k^* = -\bigl(z' + k + \tfrac12\bigr)\, v_{k+1}^*.$$

Denote by $[i \to j]^*_{\alpha,\beta,z,z'}$ the coefficient of v_j^* in the expansion of $e^{\alpha U} e^{\beta D} v_i^*$:

$$e^{\alpha U} e^{\beta D}\, v_i^* = \sum_j\, [i \to j]^*_{\alpha,\beta,z,z'}\, v_j^*.$$

We have

$$[i \to j]^*_{\alpha,\beta,z,z'} = \begin{cases} \dfrac{(-\beta)^{j-i}}{(j-i)!}\,(z'+i+\tfrac12)_{j-i}\, F\left(\begin{matrix} z+j+\frac12,\, z'+j+\frac12 \\ j-i+1 \end{matrix}; \alpha\beta\right) & \text{if } i \le j, \\ \dfrac{(-\alpha)^{i-j}}{(i-j)!}\,(z+j+\tfrac12)_{i-j}\, F\left(\begin{matrix} z+i+\frac12,\, z'+i+\frac12 \\ i-j+1 \end{matrix}; \alpha\beta\right) & \text{if } i \ge j. \end{cases} \tag{3–3}$$

3.2. Remarks

3.2.1. Periodicity. Observe that representations whose parameters z and z' are related by the transformation

$$(z, z') \mapsto (z + m, z' + m), \quad m \in \mathbb{Z},$$

are equivalent. The above transformation amounts to just a renumeration of the vectors v_k. Olshanski has pointed out that this periodicity in (z, z') is reflected in a similar periodicity of various asymptotic properties of z-measures; see [Borodin and Olshanski 1998, Sections 10 and 11].

3.2.2. Unitarity. Recall that the z-measures are positive if either $z' = \bar{z}$ or $z, z' \in (n, n+1)$ for some n. By analogy with representation theory of SL(2), these cases were called the principal and the complementary series.

In these cases the representations above have a positive definite Hermitian form Q, invariant in the sense that

$$Q(Lu, v) = Q(u, Lv), \quad Q(Uu, v) = Q(u, Dv).$$

The form Q is given by

$$Q(v_k, v_k) = \begin{cases} 1 & \text{if } z' = \bar{z}, \\ \dfrac{\Gamma\left(z' + k + \frac{1}{2}\right)}{\Gamma\left(z + k + \frac{1}{2}\right)} & \text{if } z, z' \in (n, n+1), \end{cases}$$

and $Q(v_k, v_l) = 0$ if $k \neq l$. It follows that the operators

$$\frac{i}{2} L, \ \frac{1}{2}(U - D), \ \frac{i}{2}(U + D) \in \mathfrak{sl}(2),$$

which form a standard basis of $\mathfrak{su}(1,1)$, are skew-Hermitian; hence this representation of $\mathfrak{su}(1,1)$ can be integrated to a unitary representation of the universal covering group of SU(1,1). This group SU(1,1) is isomorphic to $\mathrm{SL}(2,\mathbb{R})$ and the above representations correspond to the principal and complementary series of unitary representations of the universal covering of $\mathrm{SL}(2,\mathbb{R})$; see [Pukánszky 1964].

3.3. The Infinite Wedge Module. Consider the module $\Lambda^{\frac{\infty}{2}} V$, which is, by definition, spanned by vectors

$$\delta_S = v_{s_1} \wedge v_{s_2} \wedge v_{s_3} \wedge \cdots,$$

where $S = \{s_1 > s_2 > \ldots\} \subset \mathbb{Z} + \frac{1}{2}$ is a subset such that both sets

$$S_+ = S \setminus \left(\mathbb{Z}_{\leq 0} - \tfrac{1}{2}\right), \quad S_- = \left(\mathbb{Z}_{\leq 0} - \tfrac{1}{2}\right) \setminus S$$

are finite. We equip this module with the inner product in which the basis $\{\delta_S\}$ is orthonormal. Introduce the operators

$$\psi_k, \psi_k^* : \Lambda^{\frac{\infty}{2}} V \to \Lambda^{\frac{\infty}{2}} V.$$

The operator ψ_k is the exterior multiplication by v_k

$$\psi_k(f) = v_k \wedge f.$$

The operator ψ_k^* is the adjoint operator; it can be also given by the formula

$$\psi_k^*(v_{s_1} \wedge v_{s_2} \wedge v_{s_3}) = \sum_i (-1)^{i+1} \langle v_k^*, v_{s_i} \rangle v_{s_1} \wedge v_{s_2} \wedge \cdots \wedge \widehat{v_{s_i}} \wedge \ldots.$$

These operators satisfy the canonical anticommutation relations

$$\psi_k \psi_k^* + \psi_k^* \psi_k = 1,$$

all other anticommutators being equal to 0. It is clear that

$$\psi_k \psi_k^* \delta_S = \begin{cases} \delta_S & \text{if } k \in S, \\ 0 & \text{if } k \notin S. \end{cases} \tag{3–4}$$

A general reference on the infinite wedge space is [Kac 1990, Chapter 14].

The Lie algebra $\mathfrak{sl}(2)$ acts on $\Lambda^{\frac{\infty}{2}} V$. The actions of U and D are the obvious extensions of the action on V. In terms of the fermionic operators ψ_k and ψ_k^* they can be written as

$$U = \sum_{k \in \mathbb{Z}+\frac{1}{2}} \left(z + k + \tfrac{1}{2}\right) \psi_{k+1} \psi_k^*,$$

$$D = \sum_{k \in \mathbb{Z}+\frac{1}{2}} \left(z' + k + \tfrac{1}{2}\right) \psi_k \psi_{k+1}^*.$$

The easiest way to define the action of L is to set it equal to $[D, U]$ by definition. We obtain

$$L = 2H + (z + z')C + zz',$$

where H is the energy operator

$$H = \sum_{k>0} k\, \psi_k \psi_k^* - \sum_{k<0} k\, \psi_k^* \psi_k$$

and C is the charge

$$C = \sum_{k>0} \psi_k \psi_k^* - \sum_{k<0} \psi_k^* \psi_k.$$

It is clear that

$$C\, \delta_S = \big(|S_+| - |S_-|\big)\, \delta_S;$$

similarly,

$$H\, \delta_S = \left(\sum_{k \in S_+} k - \sum_{k \in S_-} k \right) \delta_S.$$

The charge is preserved by the $\mathfrak{sl}(2)$ action.

Consider the zero charge subspace, that is, the kernel of C:

$$\Lambda_0 \subset \Lambda^{\frac{\infty}{2}} V.$$

It is spanned by vectors which, abusing notation, we shall denote by

$$\delta_\lambda = \delta_{S(\lambda)}, \quad S(\lambda) = \left\{ \lambda_1 - \tfrac{1}{2},\ \lambda_2 - \tfrac{3}{2},\ \lambda_3 - \tfrac{5}{2},\ \ldots \right\},$$

where λ is a partition. One immediately sees that the action of $\mathfrak{sl}(2)$ on $\{\delta_\lambda\}$ is identical with Kerov operators.

3.4. Correlation Functions. Recall that the correlation functions were defined by

$$\rho(X) = \mathcal{M}\big(\{\lambda, X \subset \mathfrak{S}(\lambda)\}\big),$$

where the finite set

$$X = \{x_1, \ldots, x_s\} \subset \mathbb{Z} + \tfrac{1}{2}$$

is arbitrary.

The important observation is that (2–4) and (3–4) imply the following expression for the correlation functions:

$$\rho(X) = (1-\xi)^{zz'} \left(e^{\sqrt{\xi} D} \prod_{x \in X} \psi_x \psi_x^* e^{\sqrt{\xi} U} \delta_\varnothing, \delta_\varnothing \right). \tag{3–5}$$

We apply to (3–5) the same strategy we applied to (2–5), which is to commute the operators $e^{\sqrt{\xi} D}$ and $e^{\sqrt{\xi} U}$ all the way to the right and left, respectively, and then use (2–6). From (2–7), we have for any operator A the identity

$$e^{\beta D} A e^{\alpha U} = e^{\frac{\alpha}{1-\alpha\beta} U} \big(e^{-\frac{\alpha}{1-\alpha\beta} U} e^{\beta D} A e^{-\beta D} e^{\frac{\alpha}{1-\alpha\beta} U} \big) (1-\alpha\beta)^{-L} e^{\frac{\beta}{1-\alpha\beta} D}.$$

We now apply this identity with $\alpha = \beta = \sqrt{\xi}$ and $A = \prod \psi_x \psi_x^*$ to obtain

$$\rho(X) = \left(G \prod_{x \in X} \psi_x \psi_x^* G^{-1} \delta_\varnothing,\ \delta_\varnothing \right), \tag{3–6}$$

where

$$G = \exp\left(\frac{\sqrt{\xi}}{\xi - 1} U \right) \exp\big(\sqrt{\xi} D\big).$$

Consider the operators

$$\begin{aligned} \Psi_k &:= G \psi_k G^{-1} = \sum_i [k \to i]\ \psi_i, \\ \Psi_k^* &:= G \psi_k^* G^{-1} = \sum_i [k \to i]^*\ \psi_i^*, \end{aligned} \tag{3–7}$$

with the understanding that matrix elements without parameters stand for the choice of parameters

$$[k \to i] = [k \to i]_{\xi^{1/2}(\xi-1)^{-1}, \xi^{1/2}, z, z'}\,, \tag{3–8}$$

and with same choice of parameters for $[k \to i]^*$. The first equality on either line of (3–7) is a definition and the second equality follows from the definition of the operators ψ_i and the definition of the matrix coefficients $[i \to j]_{\alpha,\beta,z,z'}$.

From (3–6) we obtain

$$\rho(X) = \left(\prod_{x \in X} \Psi_x \Psi_x^* \delta_\varnothing,\ \delta_\varnothing \right).$$

Applying Wick's theorem to this equality, or simply unraveling the definitions on its right-hand side, we obtain:

THEOREM 3.1.

$$\rho(X) = \det \left[K(x_i, x_j)\right]_{1 \le i,j \le s},$$

where the kernel K is defined by

$$K(i,j) = \left(\Psi_i \Psi_j^* \delta_\varnothing,\, \delta_\varnothing\right).$$

Observe that

$$(\psi_l \psi_m^* \delta_\varnothing, \delta_\varnothing) = \begin{cases} 1 & \text{if } l = m < 0, \\ 0 & \text{otherwise .} \end{cases}$$

Therefore, applying formulas (3–7) we obtain:

THEOREM 3.2.

$$K(i,j) = \sum_{m=-1/2,-3/2,\dots} [i \to m]\, [j \to m]^* , \tag{3–9}$$

with the convention (3–8) *about matrix elements without parameters.*

Formula (3–9) is the analog of the [Borodin et al. 2000, Proposition 2.9] for the discrete Bessel kernel.

We conclude this section with a formula which, upon substitution of formulas (3–2) and (3–3) for matrix elements, becomes the formula of [Borodin and Olshanski 2000].

THEOREM 3.3.

$$K(i,j) = \frac{z'\sqrt{\xi}\,\left[i \to \tfrac{1}{2}\right]\,\left[j \to -\tfrac{1}{2}\right]^* - z\dfrac{\sqrt{\xi}}{(\xi-1)^2}\,\left[i \to -\tfrac{1}{2}\right]\,\left[j \to \tfrac{1}{2}\right]^*}{i-j}, \tag{3–10}$$

where for $i = j$ the right-hand side is defined by continuity.

More generally, set

$$K(i,j)_{\alpha,\beta} = \left(\Psi_i(\alpha,\beta)\,\Psi_j^*(\alpha,\beta)\,\delta_\varnothing,\, \delta_\varnothing\right),$$

where

$$\Psi_k = e^{\alpha U} e^{\beta D}\, \psi_k\, e^{-\beta D} e^{-\alpha U} = \sum_i\, [k \to i]_{\alpha,\beta,z,z'}\, \psi_i$$

$$\Psi_k^* = e^{\alpha U} e^{\beta D}\, \psi_k^*\, e^{-\beta D} e^{-\alpha U} = \sum_i\, [k \to i]^*_{\alpha,\beta,z,z'}\, \psi_i^* .$$

We will prove that

$$K(i,j)_{\alpha,\beta} = \Big(\beta z' \left[i \to \tfrac{1}{2}\right]_{\alpha,\beta,z,z'} \left[j \to -\tfrac{1}{2}\right]^*_{\alpha,\beta,z,z'} - \alpha(\alpha\beta-1)z \left[i \to -\tfrac{1}{2}\right]_{\alpha,\beta,z,z'} \left[j \to \tfrac{1}{2}\right]^*_{\alpha,\beta,z,z'}\Big)\Big/(i-j). \tag{3–11}$$

First we treat the case $i \neq j$, in which we can clear denominators in (3–11). Computing

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-2\alpha\beta & 2\alpha(\alpha\beta-1) \\ -2\beta & 2\alpha\beta-1 \end{pmatrix}$$

we conclude that

$$e^{\alpha U} e^{\beta D} L e^{-\beta D} e^{-\alpha U} = L + T,$$

where

$$T = -2\alpha\beta L + 2\beta D + 2\alpha(\alpha\beta - 1)U.$$

This can be rewritten as

$$\begin{aligned} [L, e^{\alpha U} e^{\beta D}] &= -T e^{\alpha U} e^{\beta D}, \\ [L, e^{-\alpha U} e^{-\beta D}] &= e^{-\alpha U} e^{-\beta D} T. \end{aligned} \tag{3–12}$$

From (3–12) and the equality $[L, \psi_i \psi_j^*] = 2(i-j)\psi_i \psi_j^*$ we get

$$\left[L,\ \Psi_i(\alpha,\beta)\,\Psi_j^*(\alpha,\beta)\right] = -\left[T,\ \Psi_i(\alpha,\beta)\,\Psi_j^*(\alpha,\beta)\right] + 2(i-j)\,\Psi_i(\alpha,\beta)\,\Psi_j^*(\alpha,\beta). \tag{3–13}$$

Since $\delta_\varnothing$ is an eigenvector of L we have

$$\left([L,\ \Psi_i(\alpha,\beta)\,\Psi_j^*(\alpha,\beta)]\,\delta_\varnothing,\ \delta_\varnothing\right) = 0.$$

Expand this equality using (3–13) and the relations

$$\begin{aligned} T\delta_\varnothing &= -2\alpha\beta z z' \delta_\varnothing + 2\alpha(\alpha\beta - 1) z \delta_\Box, \\ T^*\delta_\varnothing &= -2\alpha\beta z z' \delta_\varnothing + 2\beta z' \delta_\Box, \end{aligned}$$

where T^* is the operator adjoint to T and $\delta_\Box$ is the vector corresponding to the partition $(1, 0, 0, \ldots)$. We obtain

$$\begin{aligned} &(i-j)K(i,j)_{\alpha,\beta} \\ &\quad = \beta z' \left(\Psi_i(\alpha,\beta)\,\Psi_j^*(\alpha,\beta)\,\delta_\varnothing, \delta_\Box\right) - \alpha(\alpha\beta - 1) z \left(\Psi_i(\alpha,\beta)\,\Psi_j^*(\alpha,\beta)\,\delta_\Box, \delta_\varnothing\right). \end{aligned}$$

In order to obtain (3–11) for $i \neq j$, it now remains to observe that

$$(\psi_l \psi_m^* \delta_\varnothing, \delta_\Box) = \begin{cases} 1 & \text{if } l = \frac{1}{2} \text{ and } m = -\frac{1}{2}, \\ 0 & \text{otherwise}, \end{cases}$$

$$(\psi_l \psi_m^* \delta_\Box, \delta_\varnothing) = \begin{cases} 1 & \text{if } l = -\frac{1}{2} \text{ and } m = \frac{1}{2}, \\ 0 & \text{otherwise}. \end{cases}$$

In the case $i = j$ we argue by continuity. It is clear from (3–9) that $K(i,j)$ is an analytic function of i and j and so is the right-hand side of (3–10). The passage from (3–9) to (3–10) is based on the fact that the product i times $[i \to m]_{\alpha,\beta,z,z'}$ is a linear combination of $[i \to m]_{\alpha,\beta,z,z'}$ and $[i \to m \pm 1]_{\alpha,\beta,z,z'}$ with coefficients that are linear functions of m. Since the matrix coefficients are, essentially, the hypergeometric function, such a relation must hold for any i, not just half-integers. Hence, (3–9) and (3–10) are equal for any $i \neq j$, not necessarily half-integers. Therefore, they are equal for $i = j$.

3.5. Rim-Hook Analogs. The same principles apply to rim-hook analogs of the z-measures, which were also considered by S. Kerov (private communication).

Recall that a rim hook of a diagram λ is, by definition, a skew diagram λ/μ that is connected and lies on the rim of λ. Here connected means that the squares have to be connected by common edges, not just common vertices. Rim hooks of a diagram λ are in the following 1-1 correspondence with the squares of λ: given a square $\square \in \lambda$, the corresponding rim hook consists of all squares on the rim of λ which are (weakly) to the right of and below $\square$. The length of this rim hook is equal to the hook-length of $\square$.

The entire discussion of the previous section applies to the more general operators

$$\begin{aligned} U_r v_k &= \left(z + \tfrac{k}{r} + \tfrac{1}{2}\right) v_{k+r}, \\ L_r v_k &= \left(\tfrac{2k}{r} + z + z'\right) v_k, \\ D_r v_k &= \left(z' + \tfrac{k}{r} - \tfrac{1}{2}\right) v_{k-r}, \end{aligned}$$

which satisfy the same $\mathfrak{sl}(2)$ commutation relations. The easiest way to check the commutation relations is to consider $\frac{k}{r}$ rather than k as the index of v_k; the above formulas then become precisely the formulas (3–1). The operator U_r acts on the basis $\{\delta_\lambda\}$ as follows

$$U_r \delta_\lambda = \sum_{\mu=\lambda+\text{rim hook}} (-1)^{\text{height}+1} \left(z + \frac{1}{r^2} \sum_{\square\in\text{rim hook}} c(\square) \right) \delta_\mu,$$

where the summation is over all partitions μ which can be obtained from λ by adding a rim hook of length r, height is the number of horizontal rows occupied by this rim hook and $c(\square)$ stands, as usual for the content of the square $\square$. Similarly, the operator D_r removes rim hooks of length r. These operators were considered by Kerov (private commutation).

It is clear that the action of the operators $e^{\alpha U_r}$ and $e^{\beta D_r}$ on a half-infinite wedge product like

$$v_{s_1} \wedge v_{s_2} \wedge v_{s_3} \wedge \cdots,$$

essentially (up to a sign which disappears in formulas like (3–5)) factors into the tensor product of r separate actions on

$$\bigwedge_{s_i \equiv k+\frac{1}{2} \bmod r} v_{s_i}, \quad k = 0, \dots, r-1.$$

Consequently, the analogs of the correlation functions (3–5) have again a determinantal form with a certain kernel $K_r(i,j)$ which has the following structure. If $i \equiv j \mod r$ then $K_r(i,j)$ is essentially the kernel $K(i,j)$ with rescaled arguments. Otherwise, $K_r(i,j) = 0$.

This factorization of the action on $\Lambda^{\frac{\infty}{2}} V$ is just one more way to understand the following well-known phenomenon. Let $\mathbb{Y}_r$ be the partial ordered set formed by partitions with respect to the following ordering: $\mu \leq_r \lambda$ if μ can be obtained from λ by removing a number of rim hooks with r squares. The minimal elements of $\mathbb{Y}_r$ are called the r-cores. The r-cores are precisely those partitions which do not have any hooks of length r. We have

$$\mathbb{Y}_r \cong \bigsqcup_{r\text{-cores}} (\mathbb{Y}_1)^r \tag{3–14}$$

as partially ordered sets. Here the Cartesian product $(\mathbb{Y}_1)^r$ is ordered as follows:

$$(\mu_1, \dots, \mu_r) \leq (\lambda_1, \dots, \lambda_r) \quad \Longleftrightarrow \quad \mu_i \leq_1 \lambda_i \ \text{ for } i = 1, \dots, r,$$

and the partitions corresponding to different r-cores are incomparable in the $\leq_r$-order. Combinatorial algorithms which materialize the isomorphism (3–14) are discussed in [James and Kerber 1981, Section 2.7]. The r-core and the r-tuple of partitions which the isomorphism (3–14) associates to a partition λ are called the r-core of λ and the r-quotient of λ. Among more recent papers dealing with r-quotients let us mention [Fomin and Stanton 1997] where an approach similar to the use of $\Lambda^{\frac{\infty}{2}} V$ is employed, an analog of the Robinson–Schensted algorithm for $\mathbb{Y}_r$ is discussed, and further references are given.

Factorization (3–14) and the corresponding analog of the Robinson–Schensted algorithm play the central role in the recent paper [Borodin 1999]; see also [Rains 1998].

Acknowledgements

I want to thank A. Borodin, S. Kerov, G. Olshanski, and A. Vershik for numerous discussions of the z-measures. I also want to thank the organizers of the Random Matrices program at MSRI, especially P. Bleher, P. Deift, and A. Its. My research was supported by NSF under grant DMS-9801466.

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