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On the Lattice Structure of Finite Games

DAN CALISTRATE, MARC PAULHUS, AND DAVID WOLFE

ABSTRACT. We prove that games born by day n form a distributive lattice, but that the collection of all finite games does not form a lattice.

Introduction

A great deal is known about the partial order structure of large subsets of games. See, for instance, [BCG82] [Con76] for a complete characterization of games generated by numbers, and infinitesimals such as $\uparrow$ and $*n$. Linear operators applied to these games of temperature zero can often leverage this characterization to apply to hot games, such as positions occurring in Go [BW94] and Domineering [Ber88] [Wol93]. Some general results are known about the group structure of games, including a complete characterization of the group generated by games born by day 3 [Moe91], but surprisingly little has been written about the overall structure of the partial-ordering of games. Here we prove that the games born by day n form a distributive lattice, but that the collection of all finite games do not form a lattice.

We assume the reader is already familiar with combinatorial game theory definitions as in [BCG82] or [Con76]. In particular, we assume knowledge of the definitions of a game [BCG82, p. 21], sums and negatives of games [BCG82, p. 33], and the standard partial ordering on games [BCG82, p. 34].

The Lattices

Define the games born by day n, which we'll denote by $\mathcal{G}[n]$, recursively:

$$\mathcal{G}[0] \stackrel{\text{def}}{=} \{0\}$$
$$\mathcal{G}[n] \stackrel{\text{def}}{=} \{\{G^L \mid G^R\} :\ G^L, G^R \subseteq \mathcal{G}[n-1]\}$$

A *lattice*, $(S, \geq)$, is a partial order with the additional property that any pair of elements, $x, y \in S$ has a least upper bound or *join* denoted by $\vee$, and a greatest lower bound or *meet* denoted by $\wedge$. I.e., $x \geq x \vee y$ and $y \geq x \vee y$, and for any $z \in S$, if $z \geq x$ and $z \geq y$ then $z \geq x \vee y$. (Reverse all inequalities for $x \wedge y$.) In a *distributive lattice*, meet distributes over join (or, equivalently, join distributes over meet.) I.e, for all $x, y, z \in S$, $x \wedge (y \vee z) = (x \wedge y) \vee (y \wedge z)$.

We'll give a constructive proof that the games born by day n form a lattice by explicit construction of the join and meet operations. First, define

$$\lceil G \rceil \stackrel{\text{def}}{=} \{H \in \mathcal{G}[n-1] :\ H \not\leq G\}, \text{ and}$$
$$\lfloor G \rfloor \stackrel{\text{def}}{=} \{H \in \mathcal{G}[n-1] :\ H \not\geq G\}.$$

Then define the join and meet operations (over games born by day n) by

$$G_1 \vee G_2 \stackrel{\text{def}}{=} \left\{ G_1^L, G_2^L \ \mid \ \lceil G_1 \rceil \cap \lceil G_2 \rceil \right\}, \text{ and}$$
$$G_1 \wedge G_2 \stackrel{\text{def}}{=} \left\{ \lfloor G_1 \rfloor \cap \lfloor G_2 \rfloor \ \mid \ G_1^R, G_2^R \right\}.$$

Observe that $G_1 \vee G_2$ and $G_1 \wedge G_2$ are in $\mathcal{G}[n]$ since their left and right options are chosen from $\mathcal{G}[n-1]$.

Theorem 1. *The games born by day n form a lattice, with the join and meet operations given above.*

Proof. To verify these operations define a lattice, it suffices to show that

$$G_1 \vee G_2 \geq G_i \text{ (for } i = 1, 2\text{), and} \tag{0–1}$$
$$\text{if } G \geq G_1 \text{ and } G \geq G_2 \text{ then } G \geq G_1 \vee G_2. \tag{0–2}$$

($G_1 \wedge G_2$ can be verified symmetrically.)

To see (0–1), we'll show the difference game $(G_1 \vee G_2) - G_i$ (for $i = 1$ and $i = 2$) is greater or equal to 0, i.e., that Left wins moving second on this difference game. Left can respond to a Right move to $(G_1 \vee G_2) - G_i^L$ by moving to $G_i^L - G_i^L$. If, on the other hand, Right moves to $H - G_i$ where $H \in \lceil G_1 \rceil \cap \lceil G_2 \rceil$, then by definition of $\lceil G_i \rceil$, $H \not\leq G_i$, and hence Left wins moving first on $H - G_i$.

To see (0–2), suppose $G \geq G_1$ and $G \geq G_2$, and we'll show Left wins moving second on the difference game $G - (G_1 \vee G_2)$. Observe that any right option G^R of G is greater or incomparable to G, and hence is greater or incomparable to both G_1 and G_2. Therefore, $G^R \in \lceil G_1 \rceil \cap \lceil G_2 \rceil$. Thus, Left can respond to a Right move to $G^R - (G_1 \vee G_2)$ by moving to $G^R - G^R$. If, on the other hand, Right moves on the second component to some $G - G_i^L$ (for $i = 1$ or $i = 2$), Left has a winning response since $G \geq G_i$. ☐

Theorem 2. *The lattice of games born by day n is distributive.*

Proof. First, observe the following identities:

$$\lfloor G_1 \vee G_2 \rfloor = \lfloor G_1 \rfloor \cup \lfloor G_2 \rfloor, \text{ and} \tag{0–3}$$
$$\lceil G_1 \wedge G_2 \rceil = \lceil G_1 \rceil \cup \lceil G_2 \rceil. \tag{0–4}$$

(To see the first, $\lfloor G_1 \vee G_2 \rfloor = \{X :\ X \not\geq G_1 \text{ or } X \not\geq G_2\} = \lfloor G_1 \rfloor \cup \lfloor G_2 \rfloor$.)

We wish to show $H \wedge (G_1 \vee G_2) = (H \wedge G_1) \vee (H \wedge G_2)$. Expanding both sides, call them S_1 and S_2, and rewriting S_2 using (0–3) and (0–4),

$$\begin{aligned} S_1 &= H \wedge (G_1 \vee G_2) = \{\ \lfloor H \rfloor \cap \lfloor G_1 \vee G_2 \rfloor \ \mid\ H^R,\ \lceil G_1 \rceil \cap \lceil G_2 \rceil\ \} \\ S_2 &= (H \wedge G_1) \vee (H \wedge G_2) = \{\ \lfloor H \rfloor \cap \lfloor G_1 \rfloor, \lfloor H \rfloor \cap \lfloor G_2 \rfloor \mid \lceil H \wedge G_1 \rceil \cap \lceil H \wedge G_2 \rceil\ \} \\ &= \{\ \lfloor H \rfloor \cap \lfloor G_1 \vee G_2 \rfloor \ \mid\ \lceil H \rceil,\ \lceil G_1 \rceil \cap \lceil G_2 \rceil\ \} \end{aligned}$$

Clearly, $S_1 \geq S_2$, since S_2 has additional right options. To see that $S_2 \geq S_1$, we'll confirm Left wins second on the difference game $S_2 - S_1$. All right options match up except those moving S_2 to $X \in \lceil H \rceil$. By definition of $\lceil H \rceil$, $X \not\leq H$. Also, $H \geq S_1$, since S_1 is formed by the meet $H \wedge (G_1 \vee G_2)$. Hence $X \not\leq S_1$, and Left can win moving first on $X - S_1$. □

Theorem 3. *The collection of finite games,* $\mathcal{G} = \bigcup_{n \geq 0} \mathcal{G}[n]$, *is not a lattice.*

Proof. We'll prove the stronger statement that no two incomparable games, G_1 and G_2, have a join in $\mathcal{G}$. We'll do this by arguing that if $G > G_1$ and $G > G_2$, then $G >= H_n$ for some n, where

$$H_n \stackrel{\text{def}}{=} \{G_1, G_2 \ \|\ G_1, G_2 \ \mid\ -n\}$$

Since $H_0 > H_1 > H_2 > \cdots$, the theorem follows.

Suppose $G > G_1$ and $G > G_2$, and denote G's birthday by n. Note that all followers G' of G satisfy $-n < G' < n$. We'll confirm that Left wins moving second on the difference game $G - H_n$. Right cannot win by moving H_n to G_i (for $i = 1$ or $i = 2$), since $G > G_i$. When Right's initial move is on G, Left replies on the second component, $-H_n$, leaving $G^R - \{G_1, G_2 \mid -n\}$. Either Right plays on the first component, and Left wins by moving on the second component leaving $G^{RR} + n > 0$. Or Right moves the second component to some G_i and Left has a winning move since $G > G_i$. □

Lattices up to Day 3

The specific structure of the distributive lattice of games born by day n remains elusive. We show the day 1 and day 2 lattices here; the day 2 lattice corrects errors found in [Guy96, p. 55] [Guy91, p. 15]. The lattice has 22 games divided among 9 levels. (Lattice edges need only be drawn between adjacent levels.)

By extending the software package, *The Gamesman's Toolkit* [Wol96] [Wol], we find the lattice born by day 3 has 1474 games and can be drawn on 45 levels, with the number of games on successive levels being 1, 2, 3, 5, 8, 9, 12, 14, 17, 20, 24, 26, 30, 34, 39, 45, 52, 58, 65, 72, 77, 81, 86, 81, …, 3, 2, 1. As with the games born by day 2, the partial ordering appears to be composed of many sub-lattices which are hypercubes. In addition, the day-3 lattice has 44 *join-irreducible* elements whose partial order completely determines the lattice.

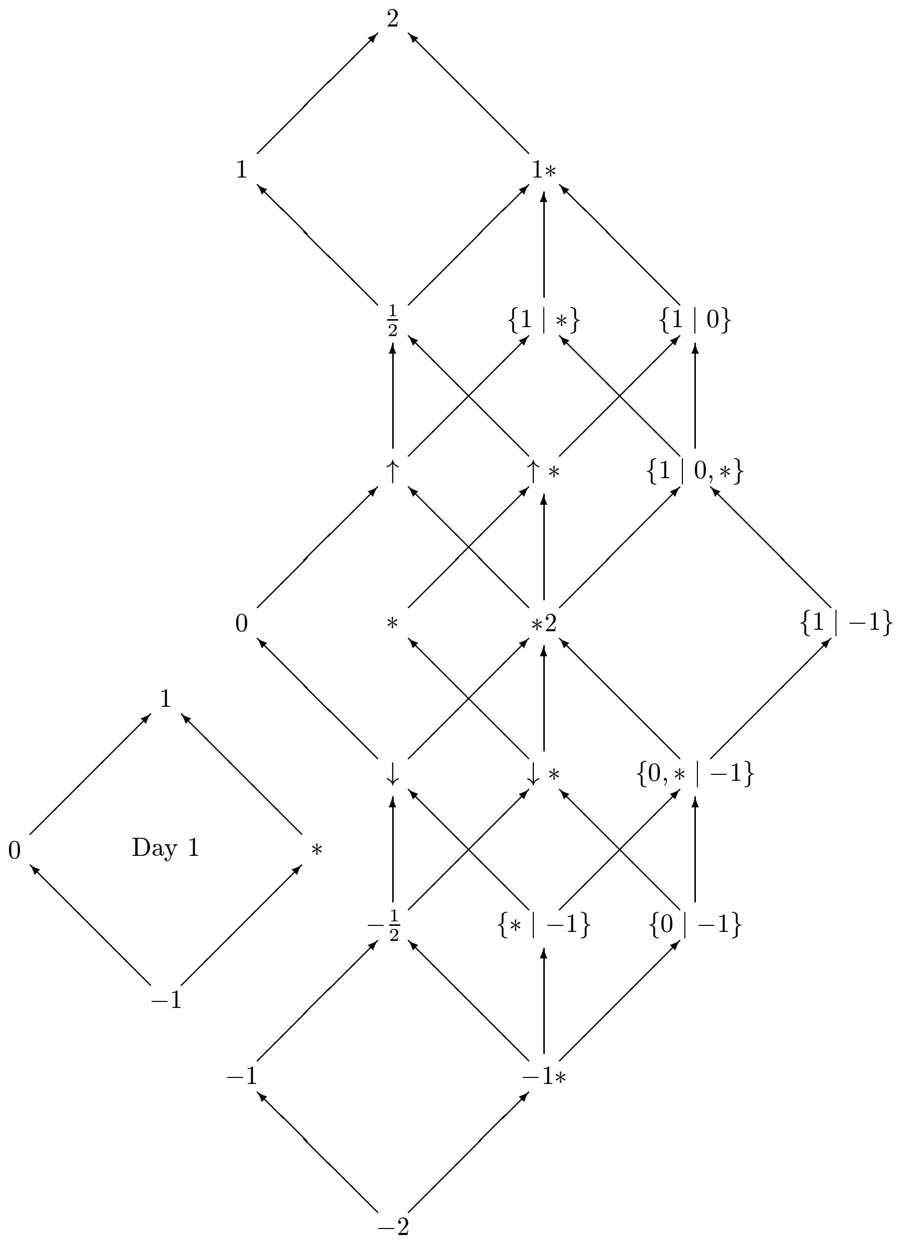

Figure 1. Games born by days 1 and 2.

These 44 elements are of the form g and $\{g \mid -2\}$, where g is one of the 22 games born by day 2. (Refer to a book on lattice theory such as [Bir67] or [DP90] for appropriate definitions and theorems.)

It would be interesting to describe the exact structure of the day 3 lattice, and (if possible) subsequent lattices.

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Dan Calistrate
calistrate@shaw.ca

Marc Paulhus
paulhusm@math.ucalgary.ca

David Wolfe
Department of Mathematics and Computer Science
Gustavus Adolphus College
800 West College Avenue
Saint Peter, MN 56082
United States
wolfe@gustavus.edu