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Hypercube Tic-Tac-Toe

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ABSTRACT. We study the analogue of tic-tac-toe played on a k-dimensional hypercube of side n. The game is either a first-player win or a draw. We are primarily concerned with the relationships between n and k (regions in n-k space) that correspond to wins or draws of certain types. For example, for each given value of k, we believe there is a critical value n_d of n below which the first player can force a win, while at or above this critical value, the second player can obtain a draw. The larger the value of n for a given k, the easier it becomes for the second player to draw. We also consider other "critical values" of n for each given k separating distinct behaviors. Finally, we discuss and prove results about the misère form of the game.

1. Introduction

Hypercube tic-tac-toe is a two-person game played on an n^k "board" (i.e. a k-dimensional hypercube of side n). (The familiar 3×3 game has $k = 2$ and $n = 3$. Several editions of the 4^3 game, $k = 3$ and $n = 4$, are commercially available.) In all these games the two players take turns. Each player claims a single one of the n^k cells with his/her *symbol* (traditionally O's and X's, or "noughts and crosses", as the game is known in the UK), and the first player to complete a "path" of length n (in any straight line, including any type of diagonal) is the winner. If all n^k cells are filled (with the two kinds of symbols) but no solid-symbol path has been completed, the game is declared a draw.

Since the first move cannot be a disadvantage, with best play the first player should never lose. Hence, in the ideal world, the first player seeks a win, while the second player tries to draw. For each given value of k, we believe there is a critical value n_d of n below which the first player can force a win, while at or above this critical value, the second player can obtain a draw. This exact value of n is exceedingly difficult to determine as a function of k. (The larger the value of n for a given k, the easier it becomes for the second player to draw.)

There are several other "critical values" of n for each given k. The smallest of these is the value of n below which the first player *must* win, no matter how well or poorly the two players play. Thus, for all $k > 1$, the 2^k board is a win for the first player on his/her second move, independent of the actual sequence of moves. Another critical value, $n_p \geq n_d$, is the value at or above which the second player can force a draw by a "pairing strategy".

There are exactly $((n+2)^k - n^k)/2$ possible winning paths on the n^k board. If it is possible to "dedicate" two cells of the hypercube exclusively to each path, the second player can draw by occupying the second dedicated cell whenever the first player occupies the first dedicated cell on a single path. A *necessary* condition for a pairing strategy to exist is that the number of cells must be at least twice the number of paths, i.e.,

$$n^k \geq (n+2)^k - n^k,$$

which is easily seen to be equivalent to

$$n \geq \frac{2}{2^{1/k}-1}.$$

The *Hales–Jewett Conjecture* is that for every k, when $n \geq \frac{2}{2^{1/k}-1}$, i.e. for all $n \geq \left\lceil \frac{2}{2^{1/k}-1} \right\rceil$, the second player can force a draw. A stronger conjecture would be that for each k, for all $n \geq n_k = \left\lceil \frac{2}{2^{1/k}-1} \right\rceil$, a draw for the second player by "pairing strategy" can be found.

It is very tempting to conjecture that

$$n_k = \left\lceil \frac{2}{2^{1/k}-1} \right\rceil = \left\lfloor \frac{2k}{\ln 2} \right\rfloor$$

for all integers $k \geq 1$. Somewhat surprisingly, this conjecture is false. Even more surprisingly, the first failure of this conjecture occurs at $k = 6,847,196,937$ dimensions, where a "board" of side $\lfloor \frac{2k}{\ln 2} \rfloor = 19{,}756{,}834{,}129$ is too small (by just a little) to allow a pairing strategy. The next failure occurs at $k = 27,637,329,632$ dimensions, where a "board" of side $n = \lfloor \frac{2k}{\ln 2} \rfloor = 79{,}744{,}476{,}806$ is too small (by just a little) to allow a pairing strategy.

Are there infinitely many (albeit incredibly sparse) exceptional values of k? Can an explicit pairing strategy be exhibited for specific pairs or classes of pairs, of k and n_k? Are there values of k such that no pairing strategy exists when $n = n_k$? These questions and others will be explored.

In the (n,k)-plane, "phase changes" occur from "forced win for first player", to "win by strategy for first player", to "draw by strategy" for second player, to "draw by pairing strategy" for second player. It should be easier to describe these *regions* in (n,k) "phase space" than to calculate the locations of their precise boundaries.

What we have just discussed is the normal form of the game. In the *misère* form, the first player to form a straight path of length n is the *loser*. We will consider the misère form later on, but unless otherwise specified we will always be talking about the normal form.

2. An Elementary Result

Theorem 1. *The number of winning paths on the n^k hypercube is*

$$\tfrac{1}{2}\big((n+2)^k - n^k\big).$$

Proof A (Geometric/Intuitive): Embed the n^k hypercube in an $(n+2)^k$ hypercube which extends one unit farther in each direction in each of the k dimensions than the original hypercube (see Figure 1). Then each winning path in the n^k hypercube terminates in exactly two "border cells" of the enlarged hypercube, and these two border cells are unique to that path. Moreover, every border cell is at the end of a path, so that the $(n+2)^k - n^k$ border cells are in two-to-one correspondence with the winning paths. □

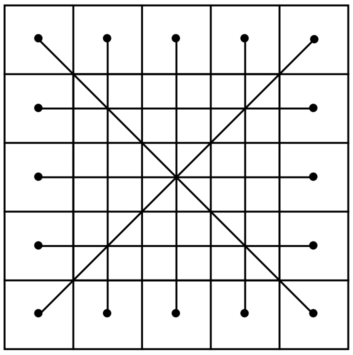

Figure 1. The familiar $n = 3, k = 2$ tic-tac-toe board is embedded in an $n = 5, k = 2$ board. Each of the 8 winning paths terminates in exactly 2 border cells of the 5×5 board: $\frac{1}{2}(5^2 - 3^2) = 8$.

Proof B (Algebraic/Rigorous): Represent each cell of the n^k hypercube by its coordinate k-vector $\alpha = (a_1, a_2, \ldots, a_k)$, where $1 \le a_i \le n$ for each i, $1 \le i \le k$. A winning path P is an ordered sequence of n such vectors, $P = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, in which the i^{th} component, for each i, either runs from 1 up to n, or from n down to 1, or remains constant at any one of the n values, except that we do not allow all k components to remain constant (since all n vectors in P would then degenerate to the same cell, and we would not have a path). Thus the number of allowed sequences which represent paths is $(n+2)^k - n^k$. However, the path $P = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is the same as the path $P' = \{\alpha_n, \alpha_{n-1}, \ldots, \alpha_1\}$, so there are only $\frac{1}{2}((n+2)^k - n^k)$ *unoriented* paths. □

3. Regions in "$n - k$ Space"

We first observe that having the first move cannot be a disadvantage; so the first player looks for a winning strategy, while the second player looks for a drawing strategy. (This assumes intelligent players of comparable skill.)

If $k \geq 2$ and $n = 2$, the first player must complete a winning path on her second move, independently of how well or poorly she plays. More generally, when n is "small" compared to k, it is impossible to assign all n^k cells to two players without at least one winning path having been created for one of the two players. For $k = 1$, we see that $n = 1$ is "small enough" but $n = 2$ is not. For $k = 2$, we have $n = 2$ is "small enough" (as mentioned more generally) but $n = 3$ is not (i.e. *ordinary* tic-tac-toe *can* result in a draw, and in fact always will with best play by both players).

It is also known that for $k = 3$, no draw is possible on the 3^3 "board", but that draws *are* possible on the 4^3 "board" (which has been available commercially from several manufacturers). The "obvious" conjecture that the critical value for all k is $n = k$ (since it is true for $k = 1, 2, 3$) was first disproved some forty years ago (by A. W. Hales), as follows:

Form the 4^4 hypercube as the tensor product of the following two 4^2 "boards", where we represent the cells of the two players by $+$ (for $+1$) and $-$ (for -1).

$$\begin{array}{|c|c|c|c|}\hline + & + & - & - \\ \hline + & - & + & - \\ \hline - & + & - & + \\ \hline - & - & + & + \\ \hline \end{array} \quad \ast \quad \begin{array}{|c|c|c|c|}\hline + & - & + & + \\ \hline + & + & + & - \\ \hline + & + & - & + \\ \hline - & + & + & + \\ \hline \end{array}$$

Note that the *left* factor has 2 plusses and 2 minuses on each tic-tac-toe path (an even number, but not 0 or 4, of each; while the *right* factor has 3 plusses and 1 minus on each tic-tac-toe path. Each "winning path" on the resulting 4^4 hypercube is either a constant from one factor times a path from the other factor (and therefore not "four identical symbols"), or the term-by-term products of a path from the first factor and a path from the second factor; but clearly such a term-by-term product path will have an *odd* number of minuses, and therefore cannot have all four cells the same.

Since the left factor has equally many $+$'s and $-$'s, this will also be true of the tensor product. Thus, the 4^4 draw could occur in an actual game, especially if the two players cooperated to achieve it. $\square$

In general, this suggests that the critical n for each k (for "no draw is possible on the n^k board) satisfies $n \leq k$, and this n is monotonically non-decreasing as k increases. (The monotonic property is easily proved. The precise expression of this critical n as a function of k is not known.)

The principal regions in $n - k$ space are the following:

1. The first player *must* win (as when $n = 2$ for $k \geq 2$).

2. Since no draw is possible, the first player should have a relatively easy win (as when $n = k = 3$, where playing in the center on the first move is already devastating).
3. Although draws are possible, there is a win for the first player with best play. (It is known [1] that exhaustive computer searching has shown that the 4^3 "board" is in this category.)
4. Although there is no trivial drawing strategy for the second player (as in region 5, below), the second player can always draw with best play. (This is the case for the familiar 3^2 board. While mathematicians will consider the drawing strategy "trivial" because it is so easily learned, it does not meet our definition of "trivial" given in Region 5; nor does it meet the layman's notion of "trivial" since this game is still widely played.)
5. The second player has a "trivial" draw by a *pairing strategy*. In a pairing strategy, two of the n^k cells are explicitly dedicated to each of the $\frac{1}{2}((n+2)^k - n^k)$ winning paths. (There may be some undedicated cells left over.) Whenever the first player claims one dedicated cell, the second player then immediately claims the other cell dedicated to the same path, if he hasn't already claimed it. (If he already has, he is free to claim *any* unclaimed cell.) Clearly, the first player can never complete a winning path if the second player is able to follow this strategy.

   When $k = 1$, the line of length $n = 2$ *forces* the second player to draw by an automatic pairing of the only two "cells" of the "board".

   When $k = 2$, the smallest board with a pairing strategy has $n = 5$, as shown in Figure 2.

   The second player can even give the first player a "handicap" of the center square, as well as the first move, and still draw by the pairing shown in Figure 2.

v	i	a	a	f
j	b	h	u	b
c	i		g	c
d	u	h	d	f
j	e	e	g	v

Figure 2. A pairing strategy for the 5×5 board. Two each of a through e are dedicated to the rows, two each of f through j are dedicated to the columns, and two each of u and v to the diagonals. The center cell is left undesignated.

A more suggestive way to indicate the same pairing strategy is shown in Figure 3.

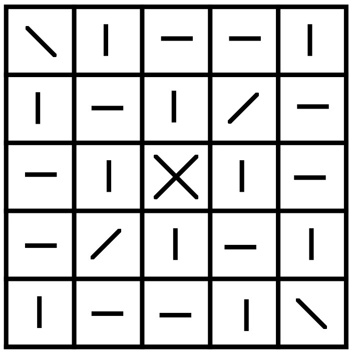

Figure 3. The pairing strategy for the 5×5 board, shown with horizontal, vertical, and diagonal strokes to indicate the type of "winning paths" to which the cells are dedicated. Note the symmetry of this pattern relative to each of the two diagonals, as well as under $180°$ rotation.

On the 6×6 board, there are $6^2 = 36$ cells and $\frac{1}{2}(8^2 - 6^2) = 14$ paths. If we dedicate *all six* cells on each diagonal to that diagonal, we have $36 - 12 = 24$ remaining cells, to assign to $14 - 2 = 12$ remaining paths. This can be done as shown in Figure 4, which has the full D_4 symmetry of the square board.

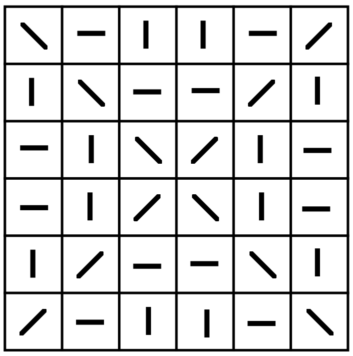

Figure 4. Representation of a pairing strategy on the 6×6 board. The horizontal and vertical midlines can be regarded as "reflectors" for this pattern. (So too can the diagonals.)

For $k = 3$, the smallest "board" with a pairing strategy is the 8^3, which has 512 cells and $\frac{1}{2}(10^3 - 8^3) = 244$ paths. However, the four "body diagonals" have 8 cells each, and if we dedicate *all* of these to their respective (non-overlapping) body diagonals, we are left with $512 - 32 = 480$ *cells*, and $244 - 4 = 240$ *paths*, i.e., exactly two cells available per path. If we divide the 8^3 "board" into octants, each 4^3, by the three mid-planes, we can assign "strokes" to the sixty available cells in the first octant (the other 4 cells were on a body diagonal), and then use the three mid-planes as mirrors to assign "strokes" (as in Figures 5 and 6) to all the remaining cells in the other octants, to end up with two dedicated cells per winning path (having treated the body-diagonal paths separately.)

To show the formation of the assignment, for pairing strategy purpose, to the $4 \times 4 \times 4$ "first octant" of the $8 \times 8 \times 8$ board, we first show, in Figure 5, the dedication of cells to ranks, files, and (vertical) columns.

X	=	O	X
O	=	‖	‖
‖	‖	=	O
X	O	=	X

=	O	‖	‖
=	X	X	O
O	X	X	=
‖	‖	O	=

O	‖	=	O
‖	X	X	=
=	X	X	‖
O	=	‖	O

X	=	‖	X
=	O	O	‖
‖	O	O	=
X	‖	=	X

$\times$	unassigned	‖	file cell
$=$	rank cell	$\circ$	(vertical) column cell

Figure 5. Partial assignment of cells to paths, in the "first octant" of the $8\times8\times8$ game.

We complete the assignment with three orientations of diagonals (including body diagonals, to which *all* their cells are dedicated) in Figure 6.

▲	=	O	▼
O	=	‖	‖
‖	‖	=	O
▶	O	=	╲

=	O	‖	‖
=	▲	▼	O
O	▶	╲	=
‖	‖	O	=

O	‖	=	O
‖	╲	▶	=
=	▼	▲	‖
O	=	‖	O

╲	=	‖	▶
=	O	O	‖
‖	O	O	=
▼	‖	=	▲

Figure 6. Assignment of cells to "winning paths", in the "first octant" of the $8 \times 8 \times 8$ game. The arrows indicate the type of diagonals to which these cells are dedicated. These four 4×4 patterns are to be stacked with the left-most on top. Then the upper left corner of that top layer is at a corner of the $8 \times 8 \times 8$ board, and that cell is dedicated to the corresponding body diagonal.

Several of these examples of pairing strategies were shown in [4].

In general, a *necessary* condition for a pairing strategy to exist is that the number of cells must at least be equal to twice the number of winning paths, that is,

$$n^k \geq (n+2)^k - n^k, \quad \text{or, equivalently,} \quad n \geq \frac{2}{2^{1/k}-1}.$$

Accordingly, let us *define* $n_k = \left\lceil \frac{2}{2^{1/k}-1} \right\rceil$.

Thus, $n_1 = 2$, $n_2 = 5$, $n_3 = 8$, and we have seen that pairing strategies really do exist for these values of n_k. At the present time (July, 2000), successful pairing strategies have also been reported for the next two cases: $n_4 = 11$ and $n_5 = 14$. Whether a pairing strategy exists for every $(n_k)^k$ hypercube is *not* known, but it *has* been shown [3] that a somewhat more elaborate drawing strategy exists for the second player for these cases, at least for all large k (specifically, for $k \geq 100$), proving a conjecture in [2] for these values.

In the next section, we will examine the validity of replacing $n_k = \left\lceil \frac{2}{2^{1/k}-1} \right\rceil$, the round-up of an *exponential* expression in k, by the much simpler $n_k = \left\lfloor \frac{2k}{\ln 2} \right\rfloor$, the round-down of a *linear* expression in k.

4. The Linearized Approximation to n_k

We have $n_k = \left\lceil \frac{2}{2^{1/k}-1} \right\rceil$, where, letting $a = 2^{1/k}$, we have

$$\frac{1}{2^{1/k}-1} = \frac{a^k - 1}{a - 1} = 1 + a + a^2 + \cdots + a^{k-1} \approx \int_0^k a^t \, dt = \int_0^k e^{t \ln a} dt$$
$$= \frac{1}{\ln a} \cdot a^t \Big|_{t=0}^{k} = \frac{a^k - 1}{\ln a} = \frac{k}{\ln 2}(2-1) = \frac{k}{\ln 2},$$

and since taking the upper limit of integration to be k (rather than, say, $k-1$), this suggests that $\frac{1}{2^{1/k}-1}$ has been rounded upward to $\frac{k}{\ln 2}$, giving some heuristic motivation to believing the "identity" $n_k = \left\lceil \frac{2}{2^{1/k}-1} \right\rceil \stackrel{?}{=} \left\lfloor \frac{2k}{\ln 2} \right\rfloor$.

Alternatively,

$$2 > \frac{(n+2)^k}{n^k} = \left(1 + \frac{2}{n}\right)^k \approx e^{2k/n},$$

from which $n_k \approx \frac{2k}{\ln 2}$.

This belief is easily strengthened by routine computer verification for the first 10^j values of k, for each $j = 1, 2, 3, 4, 5, 6, 7, 8, 9$. (Multiple precision is certainly required long before reaching $k = 10^9$.) Surely this constitutes "proof beyond a reasonable doubt", and would almost certainly convince not only a jury, but an engineer, a statistician, even a physicist, but (we hope) not a true mathematician. Because this purported "identity" is not always true!

The first failure occurs at $k = 6{,}847{,}196{,}937$. That is, if one is playing hypercube tic-tac-toe in $k = 6{,}847{,}196{,}937$ dimensions on a board which is $n = \left\lfloor \frac{2k}{\ln 2} \right\rfloor = 19{,}756{,}834{,}129$ on a side, the number of cells is slightly less than twice the number of winning paths, so no true pairing strategy can possibly exist! (If the two players each make 10^9 moves per second, how many eons will it take to claim all n^k cells?)

And, *horribile dictu*, this first failure is not the last! It is, to be sure, rather isolated, but the *second* failure occurs at $k = 27{,}637{,}329{,}632$, where the value $n = \left\lfloor \frac{2k}{\ln 2} \right\rfloor = 79{,}744{,}476{,}806$ again fails to allow twice as many cells as paths on the n^k "board". More careful power series analysis shows that the difference $\frac{2k}{\ln 2} - \frac{2}{2^{1/k}-1}$ equals $1 - \varepsilon$, where $0 < \varepsilon < \frac{\ln 2}{6k}$. Using results from the theory of diophantine approximation, "failure" can only occur when k is the denominator of a continued fraction convergent for $\frac{2}{\ln 2}$, which greatly facilitates computation.

Worst of all, we *believe* (though it is not yet proved) that the set of k's for which

$$\left\lfloor \frac{2k}{\ln 2} \right\rfloor \neq \left\lceil \frac{2}{2^{1/k} - 1} \right\rceil$$

is an *infinite* subsequence of the positive integers! This is related to the "Markov constant" $M\left(\frac{2}{\ln 2}\right)$. (Fortunately the chance of landing on one of these deadly values of k "at random" is not very great.)

A conjecture about all positive integers k that fails for the first time at $k =$ 6,847,196,937 is impressive, but not record-setting. However, the number of cells in the n^k hypercube for this value of k and $n = \left\lfloor \frac{2k}{\ln 2} \right\rfloor$ may be one of the larger integers that has occurred "naturally".

5. Hypercube Tic-Tac-Toe and Combinatorial Phase Space

The five regions described in Section 2 above in $n-k$ space partition the lattice points in one quadrant of the Euclidean plane into five "connected" regions. (If we use cells of quadrille paper rather than points for each pair (k, n), these regions are more likely to be connected and simply connected.) The hard problem is to find the precise boundaries of these regions — i.e. to locate exactly where the "phase transitions" occur, between the different "states" in game space. What is undoubtedly easier, and probably more "useful", is to obtain qualitative results on the shapes of these regions and their boundaries, and to get fairly good inequalities of the sort: "if $c_1 k < n < c_2 k$, then (k, n) is in region j", for each j from 1 to 5.

The connectedness of Region 5 in n-k phase space is actually provable. The first part is that a pairing pattern on n^{k+1} obviously imposes a pairing pattern on n^k. It is also true that a pairing pattern on n^k extends to a pairing pattern on $(n+1)^k$, but here one must be careful. Instead of extending at the edges, it is easier to extend from the middle.

Assume that the successful pairing designations have already been made on the n^k "board". We now insert k mid-hyperplanes into the n^k configuration. If n is odd, all split cells are "replicated" (i.e. their designations as rank cells, body diagonal cells, etc. are inherited by each offspring cell). If n is even, use the mid-hyperplanes as one-way mirrors to generate a duplication of one of the adjacent layers. The crucial point is that by adjoining the new layers "centrally", all diagonals remain diagonals.

By centrally enlarging the "board", all new paths are blocked if all old paths were blocked; all new paths have at least two dedicated cells if all old paths had at least two dedicated cells.

In Figure 7, we see this "central enlargement" illustrated to go from 3×3 to 4×4, and from 4×4 to 5×5.

DIMENSION

SIDE $n \backslash k$	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2	5	1	1	1	1	1	1	1
3	5	4	2	2	2	2	2	2
4	5	4	3	3	2	2	2	2
5	5	5	4	3	3	3	2	2
6	5	5	4	3	3	3	3	3
7	5	5	4	3	3	3	3	3
8	5	5	5	4	3	3	3	3
9	5	5	5	4	3	3	3	3
10	5	5	5	4	4	3	3	3
11	5	5	5	5	4	3	3	3
12	5	5	5	5	4	4	3	3

Table 1. Regions in $n-k$ phase space. (Dotted boundaries and circled numbers are uncertain.)

1	2	3
4	5	6
7	8	9

$\rightarrow$

1	2	2	3
4	5	5	6
4	5	5	6
7	8	8	9

A	B	C	D
E	F	G	H
I	J	K	L
M	N	O	P

$\rightarrow$

A	B	B	C	D
E	F	F	G	H
E	F	F	G	H
I	J	J	K	L
M	N	N	O	P

Figure 7. Central enlargements of even and odd boards.

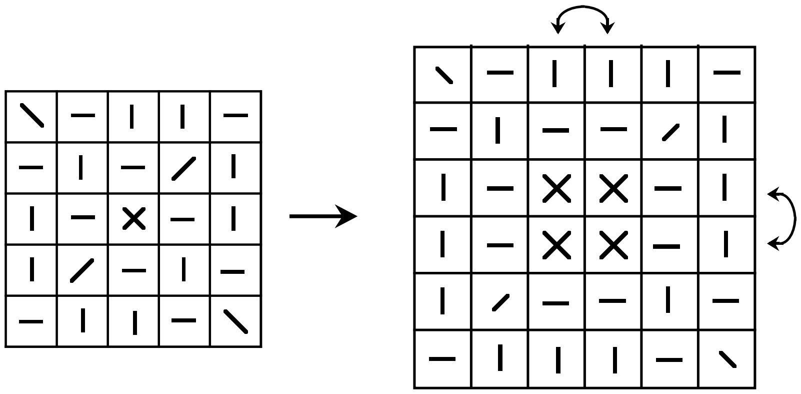

Figure 8. Extending a "pairing strategy" from 5^2 to 6^2.

In Figure 8, an actual pairing strategy is extended from the 5×5 board to the 6×6 board by this method.

Note that we have shown more than just connectedness. We have shown that Region 5 is "row-column convex" — i.e. that any horizontal or vertical line joining two points of the region lies entirely in the region. This implies that the region is simply connected.

A similar argument shows that Region 2 is also "row-column convex".

6. Misère Hypercube Tic-Tac-Toe

There is one sweepingly general result.

Theorem 2. *The first player can achieve at least a draw on the n^k board whenever $n > 1$ is odd (in the misère case).*

Proof. When n is odd, the n^k "board" has a central cell. The first player should start by claiming this central cell, and thereafter playing diametrically opposite every subsequent move of her opponent. It is clear that the first player will never be completing a path that includes the central cell; and any other path completed by the first player will be a mirror image of a path already completed by the second player. □

Corollary. *For (n, k) in "Region 2", i.e. where the n^k hypercube is too small to fail to have a path when all filled in, if $n > 1$ is odd, the first player wins the misère game using the strategy in the proof of Theorem 2.*

Example. On the $3 \times 3 \times 3$ "board", one would naively expect that the *worst* move for the first player (under misère rules) would be to claim the central cube, since this is on the most paths (thirteen paths, versus seven paths for a corner cell, five paths for a mid-face cell, and only four paths for an edge cell). Yet, by the Corollary, this is a winning move for the first player. (A computer program

e	f	g
l	m	h
k	j	i

a	b	c
d	x	d
c	b	a

i	j	k
h	m	l
g	f	e

Figure 9. First player winning strategy for $3 \times 3 \times 3$ Misère Tic-Tac-Toe.

could determine whether or not it is the only winning first move; but in any case, it is the first move with the simplest winning strategy.)

7. Tic-Tac-Toe on Projective Planes

The n^2 board resembles the structure of a finite affine geometry. It can be extended to the finite projective plane of order n, with a total of n^2+n+1 points on n^2+n+1 lines, each line containing $n+1$ points. Tic-Tac-Toe generalizes to these projective "boards" in the obvious way: the two players take turns claiming *points*, and the first player to complete a *line* is the winner. This game is an easy win (for the first player) on the 7-point plane, and a fairly easy draw (for the second player) on the 13-point plane. (On the 7-point plane, no draw is possible.) Note that in projective Tic-Tac-Toe, if one player completes a path, the other cannot possibly, even if given all the remaining "points".

8. On the Boundary Between Regions 3 and 4

We call the n^k-game a *win* if the first player wins, given best play by both sides; otherwise, we call it a *draw*.

We offer the following three hypotheses, all assuming $n_1 \geq 2$ and $k_1 \geq 2$.

Hypothesis 1. If the $n_1^{k_1}$ game is a draw, then the $n_1^{k_1-1}$ game is a draw. ("row convexity")

Hypothesis 2. If the $n_1^{k_1}$ game is a draw, then the $(n_1+1)^{k_1}$ game is a draw ("column convexity").

Hypothesis 3. If the $n_1^{k_1}$ game is a draw, then the n^k game is a draw for all $k \leq k_1$ and all $n \geq n_1$.

Clearly, Hypothesis 3 is true if and only if Hypotheses 1 and 2 are both true. In this case, the union of regions 1, 2 and 3, and the union of regions 4 and 5, are both connected and simply connected.

The next result is not quite so obvious.

Theorem 3. *If, for specific k_1 and even n_1, Hypothesis* 1 *is true, then Hypothesis* 2 *is true.*

Proof. Interpreting geometrically the binomial expansion

$$(n_1+1)^{k_1} = n_1^{k_1} + k_1 n_1^{k_1-1} + \binom{k_1}{2} n_1^{k_1-2} + \cdots + k_1 n_1 + 1,$$

we see that the $(n_1+1)^{k_1}$ hypercube can be decomposed, relative to its k_1 mid-hyperplanes (which are unique because n_1+1 is odd) into $2^{k_1}-1$ hypercubes of side n_1 and dimension j, $1 \le j \le k_1$, plus the unique central cell. For example, the single $n_1^{k_1}$ hypercube results from squeezing back together the 2^{k_1} pieces which are left after all the mid-hyperplanes have been removed; the k_1 hypercubes of size $n_1^{k_1-1}$ result from squeezing back together the cells that are, respectively, in each of the k_1 hyperplanes but not in two or more; etc. Such a decomposition of the 5^3 hypercube ($n_1 = 4$, $k_1 = 3$) is shown in Figure 10. (This figure is used only to indicate how the decomposition works, and *not* to suggest that the 4^3-game is a draw, which in fact it is not.)

A	A	C	A	A
A	A	C	A	A
B	B	E	B	B
A	A	C	A	A
A	A	C	A	A

A	A	C	A	A
A	A	C	A	A
B	B	E	B	B
A	A	C	A	A
A	A	C	A	A

D	D	G	D	D
D	D	G	D	D
F	F	X	F	F
D	D	G	D	D
D	D	G	D	D

A	A	C	A	A
A	A	C	A	A
B	B	E	B	B
A	A	C	A	A
A	A	C	A	A

A	A	C	A	A
A	A	C	A	A
B	B	E	B	B
A	A	C	A	A
A	A	C	A	A

Figure 10. The 5^3 (hyper)cube decomposed into: one 4^3 (hyper)cube (the A-cells); three 4^2 hypercubes (i.e., squares), indicated by the letters B,C, and D, respectively; three 4^1 hypercubes (i.e., lines), indicated by the letters E, F, and G, respectively; and the central cell, X.

Using Hypothesis 1, the second player draws by always replying to the first player in the same sub-hypercube where the first player has just moved, and using the drawing strategy for that sub-hypercube. (If the first player ever occupies the central cell, the second player then gets a "free move", which may lead to additional free moves later on, as in the proof that Region 5 is "row-column convex".) Because n_1+1 is odd, every path in the $(n_1+1)^{k_1}$ hypercube requires a winning path in one of the $2^k - 1$ sub-hypercubes, and by the strategy just described, no such path will ever be completed by the first player. □

Notes. 1. It appears that Hypothesis 1 may be easier to prove than the n_1-is-odd case of Hypothesis 2.

2. Analogous to the theorem just proved, it is possible to show that "$n_1^{k_1}$ is a draw" implies "$(n_1+2)^{k_1}$ is a draw" whether n_1 is even or odd. (This greatly reduces the uncertainty in the shape of the boundary between Region 3 and Region 4.)

3. We can prove the assertion in Note 2 as follows: 1) We embed the $n_1^{k_1}$ hypercube in an $(n_1+2)^{k_1}$ hypercube, as in the geometric approach to counting winning paths. We then deal with the $2k_1$ added hypersurfaces in much the same

way as with the k_1 mid-hyperplanes in the previous theorem. (This time, the sub-hypercubes of dimensions $j = 1, 2, \ldots, k_1$, are still geometrically connected, and don't need to be squeezed back together.) There are now 2^{k_1} zero-dimensional vertices, so if the first player ever claims one of them, the second player can claim the diametrically opposite one (just to give a rule for dealing with free moves).

4. In Table 1, Region 5 is "row-column convex" and progagates mostly vertically. In a strongly analogous sense, Region 2 is also "row-column convex" and progagates mostly horizontally.

9. Additional Notes

(i) On the 3^3 board, the central cell is so powerful that if the first player is forbidden to occupy it on his initial move, the second player wins by occupying it in reply (in the normal game). The entire game tree is easily searched "by hand".

(ii) A further generalization of the n^k game with n in a row is to the n^k game with r in a row. That is, the first player to claim r consecutive cells along any straight path is the winner (or, in the misère version, the loser). Games that are dull draws for given n and k may become interesting when it is merely required to get r-in-a-row (for some $r < n$). This generalization provides a common framework for Tic-Tac-Toe and Go Moku.

References

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[2] Hales, A. W., and Jewett, R. I., "Regularity and Positional Games," *Transactions of the American Mathematical Society*, vol. 106, no. 2, Feb. 1963, 222–229.

[3] Beck, J., "On Positional Games", *Journal of Combinatorial Theory (Series A)*, vol. 30, no. 2, March, 1981, 117–133.

[4] Berlekamp, E. R., Conway, J. H., and Guy, R. K., *Winning Ways for Your Mathematical Plays*, Academic Press, 1982; see especially Vol. II, pp. 667–679.

[5] Beck, J., "Games, Randomness and Algorithms," in *The Mathematics of Paul Erdös, I*, R. L. Graham and J. Nešetřil, eds., Springer-Verlag, 1997.

Supplemental Annotated Bibliography

1. Reference [2] first appeared, with the same title and authors, as Jet Propulsion Laboratory Report no. 32–134, January 31, 1962.

2. Several results presented at the January, 1972, MAA meeting in Las Vegas, Nevada (including the pairing pattern on the 8^3, and how to enlarge a pairing from n^k to $(n+1)^k$) were contained in a letter dated April 6, 1970, from S. W. Golomb to J. L. Selfridge. Some of this material was included in [4].
3. Three of Martin Gardner's *Mathematical Games* columns in *Scientific American* dealing with aspects of generalized Tic-Tac-Toe were:
   a. March, 1957, included in *The Scientific American Book of Mathematical Puzzles and Diversions* (Simon and Schuster, 1959) is rather elementary, but does describe winning paths on the 4^4 game (without the proof that draws are possible).
   b. August, 1971, included in *Wheels, Life, and Other Mathematical Amusements* (Freeman, 1983), Chapter 9, describes pairing strategies on n^2 boards for all $n \geq 5$.
   c. April, 1979, included in *Fractals, Hypercards, and More* (Freeman, 1992) deals with the generalization of Tic-Tac-Toe to polyominoes.
4. Reference [5], József Beck's chapter "Games, Randomness and Algorithms", in *The Mathematics of Paul Erdös, I*, edited by R. L. Graham and J. Nešetřil, Springer-Verlag, 1997, has what is probably the most up-to-date published results on n^k-hypercube Tic-Tac-Toe. One example is that the second player can draw (not necessarily by a pairing strategy) provided that $n > (\log_2 3+\varepsilon)k$ and $n > n_0(\varepsilon)$, which is asymptotically better than the $n > \frac{2k}{\ln 2}$ conjecture first proposed in Reference [2]. (The underlying method for this improved result is attributed to Erdös and Selfridge.)

   In recent, as yet unpublished work, Beck has shown that the second player can force a draw if $k = \mathrm{O}(n^2/\log n)$, improving his previous result of $k = \mathrm{O}(n^{3/2})$. This is close to best possible, in a sense, since it is known that if $k = \Omega(n^2)$ then the first player has a "weak win", i.e. can occupy n-in-a-line, though not necessarily first.

   See, also, P. Erdös and J. L. Selfridge, "On a combinatorial game", *Journal of Combinatorial Theory, Series A*, vol. 14, no. 3, May 1973, 298–301.
5. Another reference is: J. L. Paul, *Tic-Tac-Toe in n-dimensions, Mathematics Magazine*, vol. 51, no. 1, Jan. 1978, 45–49.
6. A more elementary reference is: Mercer, G. B., and Kolb, J. R., "Three-dimensional ticktacktoe", *Mathematics Teacher*, vol. 64, no. 2, February, 1971, 119–122. (They show that the number of winning paths on the $n \times n \times n$ cube is $3n^2+6n+4$, without any of the more general insights contained in Theorem 1.)

Historical Note: Except for the "conjecture" on n_k with its counterexamples (from 1999) in Section 4, the even more recent work described in Section 8, and the results cited in references [1], [3], [4], and [5], the work in this paper predates 1972. Some results already appeared in [2]. Other results were presented at the January, 1972, meeting of the MAA in Las Vegas, Nevada (in a session titled

"Players, Probabilities, and Profits", on the morning of 21 January) by S. W. Golomb, in a talk titled "Games Mathematicians Play".

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