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One-Dimensional Phutball

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Abstract. We consider the game of one-dimensional phutball. We solve the case of a restricted version called Oddish Phutball by presenting an explicit strategy in terms of a potential function.

1. Introduction

J. H. Conway's *Philosophers Football*, otherwise known as *Phutball* [1], is played by two players, *Left* and *Right*, who move alternately. The game is usually played on a 19 by 19 board, starts with a *ball* on a square, one side is designated the Left goal line and the opposite side the Right goal line. A move consists of either placing a stone on an unoccupied square or jumping the ball over a (horizontal, vertical or diagonal) line of stones one end of which is adjacent to the square containing the ball. The stones are removed immediately after being jumped. Jumps can be chained and a player does not have to jump when one is available. A player wins by getting the ball on or over the opponent's goal line. Demaine, Demaine and Eppstein [2] have shown that deciding whether or not a player has a winning jump is NP complete.

In this paper we consider the 1-dimensional version of the game. For example:

$$\text{Left} \quad \cdot\ \cdot\ \diamond\ \diamond\ \bullet\ \diamond\ \diamond\ \cdot\ \diamond\ \cdot\ \cdot\ \cdot \quad \text{Right}$$

is a position on a finite linear strip of squares. Initially, there is a black stone, $\bullet$, which represents the ball. A player on his turn can either place a white stone, $\diamond$, on an empty square, $\cdot$, or jump the ball over a string of contiguous stones one end of which is adjacent to the ball; the ball ends on the next empty square. The stones are removed immediately upon jumping. A jump can be continued if there is another group adjacent to the ball's new position. Left wins by jumping the ball onto or over the rightmost square — Right's goalline; Right wins by jumping on over the leftmost square — Left's goalline.

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It would seem clear that

- Your position can only improve by having a stone placed between the ball and the opponent's goalline; and
- Your position can only improve if an empty square between the ball and the opponent's goalline is deleted.

However, both of these assertions are false as can be seen from the following examples. In all cases it is Left to move first and since Right can win by jumping if Left does not move the ball, Left's move must be a jump.

$$
\begin{array}{ll}
\text{(1) Left wins:} & \diamond\ \bullet\ \diamond \cdot \diamond \cdot \diamond\ \cdot \cdot \cdot \cdot \\
\text{(2) Right wins:} & \diamond\ \bullet\ \diamond \cdot \diamond\ \diamond\ \diamond\ \cdot \cdot \cdot \cdot \\
\text{(3) Right wins:} & \diamond\ \bullet\ \diamond \cdot \diamond\ \diamond\ \cdot \cdot \cdot \cdot
\end{array}
$$

The reader may wish to check that in (1) Left loses if he jumps all three stones. The main continuation for (1) and (2) are given next, the reader may wish to try other variations. An arrow indicates the move was a jump and the numbers give the order of the placement of the stones.

$$
\begin{array}{lll}
(1): & \diamond\ \bullet\ \diamond \cdot \diamond \cdot \diamond\ \cdot \cdot \cdot \cdot & \rightarrow \diamond 3 \cdot 1 \bullet \diamond\ \cdot 2 \cdot \cdot \\
 & & \rightarrow \diamond\ \diamond \cdot \diamond \cdot \diamond\ \cdot 2 \bullet 1 \\
 & \text{Left jumps and wins} & \\
(2): & \diamond\ \bullet\ \diamond \cdot \diamond\ \diamond\ \diamond\ \cdot \cdot \cdot \cdot & \rightarrow \diamond \cdot \cdot \cdot 3 \cdot 1 \bullet 2 \cdot 4 \\
 & & \rightarrow \diamond \cdot 2 \bullet 1 \cdot 3 \cdot \diamond \cdot \diamond \\
 & \text{Right jumps and wins} &
\end{array}
$$

Also the obvious *Never jump backwards* heuristic does not always work.

In $\cdot \cdot \cdot \cdot \cdot \cdot \diamond\ \diamond\ \diamond\ \diamond \cdot \bullet\ \diamond\ \diamond \cdot \diamond\ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$, Right's *only* winning move is to jump backwards to

$$
\cdot \cdot \cdot \cdot \cdot \cdot \diamond\ \diamond\ \diamond\ \diamond \cdot \cdot \cdot 4 \cdot 2 \bullet 1 \cdot 3 \cdot \cdot \cdot \cdot
$$

This splits into two options, again the final plays are left to the reader:

$$
\begin{array}{rl}
\text{Option 1} \rightarrow & \cdot \cdot \cdot \cdot \cdot \cdot \diamond\ \diamond\ \diamond\ \diamond \cdot \cdot \cdot \diamond \cdot \diamond \cdot 3 \cdot 1 \bullet 2 \cdot 4 \\
\rightarrow & \cdot \cdot \cdot \cdot \cdot \cdot \diamond\ \diamond\ \diamond\ \diamond \cdot 2 \bullet 1 \cdot 3 \cdot \cdot \cdot \cdot \cdot \diamond \cdot \diamond \\
\rightarrow & \cdot \cdot 4 \cdot 2 \bullet 1 \cdot 3 \cdot \cdot \cdot \cdot \diamond \cdot \diamond \cdot \cdot \cdot \cdot \cdot \cdot \diamond \cdot \diamond \\
\rightarrow & \cdot \cdot \diamond \cdot \diamond \cdot 3 \cdot 1 \bullet 2 \cdot 4 \diamond \cdot \diamond \cdot \cdot \cdot \cdot \cdot \cdot \diamond \cdot \diamond \\
\text{Option 2} \rightarrow & \cdot \cdot \cdot \cdot \cdot \cdot \diamond\ \diamond\ \diamond\ \diamond \cdot \cdot \cdot 4 \cdot 2 \bullet 1 \cdot 3 \cdot 5 \cdot \cdot \\
\rightarrow & \cdot \cdot \cdot \cdot \cdot \cdot \diamond\ \diamond\ \diamond\ \diamond \cdot 2 \bullet 1 \cdot 3 \cdot \diamond \cdot \diamond \cdot \diamond \cdot \cdot \\
\rightarrow & \cdot \cdot 4 \cdot 2 \bullet 1 \cdot 3 \cdot \cdot \cdot \cdot \diamond \cdot \diamond \cdot \diamond \cdot \diamond \cdot \diamond \cdot \cdot \\
\rightarrow & \cdot \cdot \diamond \cdot \diamond \cdot 3 \cdot 1 \bullet 2 \cdot 4 \diamond \cdot \diamond \cdot \diamond \cdot \diamond \cdot \diamond \cdot \cdot
\end{array}
$$

The other main Right option is:

$$
\begin{array}{ll}
 & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \diamond\ \diamond\ \diamond\ \ \diamond\ 1\,\bullet\ \diamond\ \diamond\ \cdot\ \diamond\ \cdot 2\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\to & \cdot\ \cdot\ 4\ \cdot\ 2\ \bullet 1\ \cdot\ 3\ \cdot\ \cdot\ \cdot\ \diamond\ \diamond\ \cdot\ \diamond\ \cdot 2\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\to & \cdot\ \cdot\ \diamond\ \cdot\ \diamond\ \cdot\ 3\ \cdot\ 1\,\bullet\ 2\ \cdot\ \diamond\ \diamond\ \cdot\ \diamond\ \cdot\ \diamond\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\to & \cdot\ \cdot\ \diamond\ \cdot\ \diamond\ \cdot\ \diamond\ \cdot\ \diamond\ \ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \bullet\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot
\end{array}
$$

Note that on Right's second move it does her no good to jump backwards. Left responds by placing a stone to the right.

2. Oddish Phutball

We do not have a complete strategy for 1-dimensional Phutball and as the preceding examples show, it is a surprisingly difficult and often counter-intuitive game to analyze in full generality. However, if we place a small restriction on the moves allowed (one that naturally arises in actual play between two intelligent beings), then we do have a complete analysis.

Oddish Phutball is 1-dimensional Phutball in which the players will only place stones on the squares at an odd distance from the ball. We therefore eliminate all the squares at an even distance from the ball and we represent the ball by an arrow between two squares. This we call the *Oddish* board. For example, the 1-dimensional Phutball position

$$\cdot\ \cdot\ \diamond\ \cdot\ \cdot\ \cdot\ \diamond\ \cdot\ \ \diamond\,\bullet\,\diamond\ \cdot\ \diamond\ \ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

on the Oddish board becomes

$$\cdot\ \diamond\ \cdot\ \diamond\ \diamond{}^{\downarrow}\diamond\ \diamond\ \ \cdot\ \cdot\ \cdot$$

On the Oddish board the arrow can only move past a set of consecutive stones, stopping anywhere between the stones. The stones passed over by the arrow are removed. The arrow cannot move past an empty square. The winning condition is to move the arrow past the last square on the Oddish board. In this simplified game, one might hypothesize that an optimal player will only jump when he can 'jump to a winning position', that is, jump far enough towards the opponent's goal line that the arrow will never again return to its position before this jump. It turns out that this hypothesis is correct. It follows that one can define a potential function for each player which is the number of stones they must place before they can jump to a winning position. Our analysis will consist of showing how to calculate this potential function and demonstrating that the player with the lower potential always has a winning strategy.

The Left *potential function*, p_L, takes into account only the squares of the Oddish board to the right of the ball. The following are the rules for the calculation.

(i) Summing: Compute the *left sum* by starting with a sum of 0 at the rightmost edge and, moving left, up to the square before the arrow:
add 1 if the next square is empty; and
subtract 1 if the next square is a stone, *except if the sum is* 0*, in which case the sum remains* 0;

(ii) The sum up to but not including the square immediately to the right of the arrow we denote by *psum* and the total sum we denote by *bsum*. The Left potential is then

$$p_L = \lfloor bsum/2 \rfloor + (1 - \delta_p^b)$$

where δ_p^b is the Kronecker delta function which is 1 if $bsum = psum$ (and this can only happen if they both equal 0) and it equals 0 otherwise.

The Right potential, p_R, is calculated similarly.
In calculating Right's potential for the Oddish position

$$\cdot\ \cdot \diamond \diamond \diamond^{\downarrow} \diamond \diamond \diamond\ \cdot\ \cdot\ \cdot\ \cdot \diamond\ \cdot$$

$psum = bsum = 0$ therefore $p_R = 0$. For Left, $psum = 2$ and $bsum = 1$ therefore $p_L = 1$.

Theorem 1. *In a game of Oddish Phutball, Left can win if* $p_L < p_R$*; Right can win if* $p_L > p_R$*; it is a first player win if* $p_L = p_R$.

To prove this result we need several observations.

Lemma 2. *In an Oddish Phutball position:*

(i) *If* $p_L = 0$ *then Left can jump to a position such that* $p_R - p_L \geq 1$.
(ii) *If* $p_L > 0$ *then Left can decrease* p_L *by 1 by placing a stone.*
(iii) *If* $p_L > 0$ *any Left jump increases* $p_R - p_L$ *by at most 1.*

Proof. If there are only stones to the right of the arrow then $p_L = 0$ and Left can jump and win immediately. Therefore, suppose that there are k stones immediately to the right of the arrow before the first empty square. Let m be the sum before these k stones are included in the count.

1: If $p_L = 0$ then $0 < m < k$ and specifically, $k > 1$. Choose k' even such that $k - 1 \leq k' \leq k$. Therefore $k' > 0$ and $\lfloor (k'-1)/2 \rfloor + 1 = \lfloor k'/2 \rfloor$. Then Left jumping over k' stones changes p_L from 0 to at most $\lfloor (k'-1)/2 \rfloor + 1$, i.e. at most $\lfloor k'/2 \rfloor$. Now p_R has changed either from $\lfloor m'/2 \rfloor + 1$ to $\lfloor (m'+k')/2 \rfloor + 1$ for some $m' > 0$ or from 0 to $\lfloor k'/2 \rfloor + 1$. In both cases, the new position has $p_L < p_R$.

2: If $p_L > 0$ then $m \geq k$. Placing a stone on the empty square immediately to the right of this group changes p_L from 1 to 0 if $m = k$ or $m = k + 1$; and if $m \geq k + 2$ then p_L changes from $\lfloor \frac{m-k}{2} \rfloor + 1$ to $\lfloor \frac{m-k}{2} \rfloor$.

3: Suppose that $p_L = \lfloor i/2 \rfloor + 1 > 0$ for some $i \geq 0$. Let Left jump forward over j stones. The Left potential changes from $\lfloor i/2 \rfloor + 1$ to $\lfloor (i+j)/2 \rfloor + 1$. The Right potential changes from either $\lfloor l/2 \rfloor + 1$ to $\lfloor (l+j)/2 \rfloor + 1$ for some $l \geq 0$

or from 0 to $\lfloor j/2 \rfloor + 1$. In all cases, the change in the potential is between $\lfloor j/2 \rfloor$ and $\lfloor j/2 \rfloor + 1$. Therefore $p_R - p_L$ changes by at most 1.

Note that if Left jumps backwards over j stones and $p_R > 0$ the same analysis shows $p_R - p_L$ changes by at most 1. However, if $p_R = 0$ with the Right sum being zero for the last q stones then the Right potential does not change if $j < q - 1$ and changes to $\lfloor (j-q)/2 \rfloor + 1$ if $q - 1 \leq j$ whereas the Left potential has changed from $\lfloor i/2 \rfloor + 1$ to $\lfloor (i+j)/2 \rfloor + 1$ for some $i \geq 0$. Hence the change in the Right potential is not greater than $\lfloor j/2 \rfloor + 1$ and that of the Left is at least $\lfloor j/2 \rfloor$, so again the change in $p_R - p_L$ is less than or equal to 1. □

Proof of Theorem 1. We show that Left cannot lose by giving an explicit strategy and then show that the game is finite and hence Left must win.

We assume first that $p_L < p_R$ and it is Right to move. Note that since $p_R > 0$, Right cannot win in one move by jumping on or over the goal line.

If Right places a stone to the left of the arrow then she decreases her potential by at most 1 and does not affect Left's potential. If she places a stone to the right of the arrow then she does not affect p_R but may decrease p_L. If Right jumps forward or backwards then $p_L - p_R$ increases by at most 1. In all cases, it is then Left's turn with $p_L \leq p_R$.

Suppose that $p_L = 0$ and that there are k stones in the string adjacent to the arrow. Left jumps over k' stones where $k' = k$ or $k' = k - 1$ and k' is even. With this k' we have $\lfloor (k'-1)/2 \rfloor + 1 = \lfloor k'/2 \rfloor$. Then, by Lemma 2.1, $p_L < p_R$ and it is now Right's turn.

If $p_L > 0$ then Left places a stone at the end of the adjacent string of stones and p_L is reduced by at least 1 and therefore by Lemma 2.2, $p_L < p_R$ and it is now Right's turn.

Now suppose that it is Left's move with $p_L \leq p_R$. If $p_L = 0$ then by Lemma 2.1 Left can jump to a position with $p_L - p_R \geq 1$. If $p_L > 0$ then by Lemma 2.2 Left can place a stone to decrease p_L by 1 without changing p_R. In both cases, it is Right's turn with $p_L < p_R$. It follows that if it is Left's move and $p_R \geq p_L$ or Right's move with $p_R > p_L$ then Right cannot win — it is never her move in a position with $p_R = 0$ so she never has a move that allows her to jump to the goal line. We now have to prove that the game finishes after a finite number of moves. Suppose that it does not. Let A be the rightmost square, on the Oddish board, that is changed infinitely often and suppose that the game has progressed far enough so that in the continuation no moves to the right of A are played.

The next time that the arrow passes over A to the right, it cannot be that Right has just jumped backwards past A since then Left's strategy has him placing a stone to the right of A. If Left has just jumped (immediately to the right of A) then Right must place a stone to the left and then Left would place a stone to the right of A, again a contradiction.

Therefore the game is finite and therefore there is a winner. Since the winner is not Right, it must be Left. □

3. 1-dimensional Phutball Revisited

When we attempted to extend our analysis to unrestricted 1-dimensional phutball, we were able to develop a generalized potential function which *almost* works. Instead of the full solution we had hoped for, a small hole in the proof led us to discover the position shown earlier in which the only winning move is a backwards jump. The very existence of such a position implies that a fundamentally different approach is required to fully analyze the general game. Nonetheless, the generalized potential function does provide some insight into the game, so we include it here for completeness.

The Left *potential function*, P_L, takes into account only the ball and all the squares to the right of the ball. The following are the rules for the calculation.

1. Transformation: Replace the ball with the symbol immediately to the right, and for each odd group of contiguous empty squares replace the rightmost with a stone.
2. Summing: In this new configuration, compute the left sum by starting with a sum of 0 at the rightmost edge and, moving left, up to and including the square with the ball:
   add 1 if the next square is empty; and
   subtract 1 if the next square is a stone, *except if the sum is* 0 *in which case the sum remains* 0;
3. Calculation: The sum up to but not including the ball's square we denote by $psum$ and the total sum we denote by $bsum$. So the Left potential can be calculated by
$$P_L = \lfloor bsum/4 \rfloor + (1 - \delta_p^b)$$
   where δ_p^b is the Kronecker delta function which is 1 if $bsum = psum$ (and this can only happen if they both equal 0) and it equals 0 otherwise. The Right potential, P_R, is calculated similarly.

The analysis of when this potential function works and when it breaks down is similar to the analysis of the oddish potential function and is left as an exercise for the reader.

References

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