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Transfinite Chomp

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ABSTRACT. Chomp is a Nim-like combinatorial game played in $\mathbb{N}^d$ or some finite subset. This paper generalizes Chomp to transfinite ordinal space Ω^d. Transfinite Chomp exhibits regularities and closure properties not present in the smaller game. A fundamental property of transfinite Chomp is the existence of certain initial winning positions, including rectangular positions $2\times\omega$, $3\times\omega^\omega$, $2\times 2\times\omega^3$, and $2\times 2\times\omega\times\omega$. Many open questions remain for both transfinite and finite Chomp.

Introduction and Notation

In the game of Chomp, cookies are laid out at the lattice points $\mathbb{N}^d$ where $\mathbb{N}$ denotes the natural numbers and play is in $d \in \mathbb{Z}^+$ dimensions. The cookie at the origin is poisonous. Two players alternate biting into the configuration, each bite eating the cookies in an infinite box from some lattice point outward in all directions, until one player loses by eating the poison cookie. The game can start from a position with finitely many bites already taken from $\mathbb{N}^d$ rather than from all of $\mathbb{N}^d$. Chomp was invented by David Gale in [Ga74] and christened by Martin Gardner. When Chomp begins from a finite rectangle it is isomophic to an earlier game, Divisors, due to Schuh [Sch52]. See also [BCG82b], pp.598–606.

This paper considers Chomp on Ω^d where Ω denotes the ordinals, a subject the first author began studying in the early 1990s. This transfinite version of Chomp has been mentioned in *Mathematical Intelligencer* columns [Ga93; Ga96]; these are anthologized in [Ga98].

Identifying each ordinal a with the set $\Omega_a = \{x \in \Omega : x < a\}$, the sets

$$\begin{array}{lll} a, & \overline{a} = \{x \in \Omega : x \geq a\} & \text{for } a \in \Omega, \\ a \times b, & \overline{(a,b)} = \overline{a} \times \overline{b} & \text{for } (a,b) \in \Omega^2 \end{array}$$

are the boxes at the origin and the Chomp bites in one and two dimensions. Similarly in d dimensions, the Chomp boxes and bites are the corresponding

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$v_1 \times \cdots \times v_d$ and $\overline{v} = \overline{v}_1 \times \cdots \times \overline{v}_d$ for $v \in \Omega^d$. Every Chomp position X is a finite union of boxes, and conversely. As we will see, every Chomp game must terminate after finitely many bites.

Working in the ordinals gives a more satisfyingly complete picture of Chomp. For example, a position of two equal-height columns, $X = 2 \times h$, is an N-position (next player wins) for all $h \in \mathbb{Z}^+$, but it is a P-position (previous player wins) for $h = \omega$, and then it is an N-position again for all $h > \omega$. For three equal-height columns, $3 \times h$ is a P-position if and only if $h = \omega^\omega$. Another result is that the six-dimensional position $\omega \times \omega \times \omega \times 2 \times 2 \times 2$—i.e., the cartesian product of $\mathbb{N}^3$ with a $2 \times 2 \times 2$ cube—is a P-position.

The join operator on Chomp positions is denoted "$+$," the difference operator "$-$," and "$\subset$" denotes proper subposition. Every Chomp bite is a difference operation

$$X \mapsto X - \overline{v} \qquad \text{for some } v \in \Omega^d,$$

valid only for positions X such that $X - \overline{v} \subset X$. Since parts of the analysis use the last direction for special purposes, vectors $v \in \Omega^k$ are often written $v = (u, k)$ with $u \in \Omega^{d-1}$ and $k \in \Omega$.

The minimal excluded element operator is denoted mex. Thus for any proper subset $S \subset \Omega$,

$$\operatorname{mex}(S) = a, \text{ where } a \notin S \text{ and } x \in S \text{ for all } x < a.$$

1. The Ordinals, Very Briefly

This section gives the bare basics on ordinals needed to read the paper.

The ordinal numbers Ω with ordinal addition $\uplus$ and ordinal multiplication $\star$ extend the natural numbers $(\mathbb{N}, +, \cdot)$ to the infinite. The operations are associative and satisfy the distributive property, but they are not commutative. Every ordinal x has an ordinal successor $x \uplus 1$ and the supremum of every set of ordinals is an ordinal. The finite ordinals are just $\mathbb{N}$. The first infinite ordinal is $\omega = \sup(\mathbb{N})$. Infinite ordinals include ω, $\omega \uplus 1$, $\omega \uplus 2$, $\dots$, $\omega \star 2$, $\omega \star 2 \uplus 1$, $\dots$, $\omega \star 3$, $\dots$, ω^2, $\omega^2 \uplus 1$, $\dots$, $\omega^2 \star 2$, $\dots$, $\omega^2 \star 3 \uplus \omega \star 5 \uplus 19$, $\dots$, ω^3, $\dots$, ω^4, $\dots$, ω^ω, $\dots$

The ordinals are totally ordered and they are well founded, meaning every nonempty subset contains a minimal element and so the mex operator makes sense.

Every nonzero ordinal x can be written uniquely as

$$x = \hat{c} \uplus \omega^{e_0} \star c_0$$

with e_0 an ordinal, $0 < c_0 < \omega$, and $\hat{c}$ an ordinal multiple (possibly 0) of $\omega^{e_0 \uplus 1}$. Recursively expanding $\hat{c}$ as long as it is nonzero gives a unique expression

$$x = \omega^{e_k} \star c_k \uplus \omega^{e_{k-1}} \star c_{k-1} \uplus \cdots \uplus \omega^{e_1} \star c_1 \uplus \omega^{e_0} \star c_0$$

for some finite k, with $e_k > e_{k-1} > \cdots > e_1 > e_0$ a descending chain of ordinals, and $0 < c_i < w$ for $i \in \{0, \ldots, k\}$. This form for an ordinal number is its base ω expansion, also known as its Cantor normal form.

Instead of ordinal addition and multiplication, we nearly always use commutative operators called *natural* addition and multiplication, denoted by ordinary "+" and "·". Natural addition of two ordinals written in base ω simply adds coefficients of equal powers of ω, where missing terms are taken to have coefficient 0. Ordinal addition satisfies

$$\omega^e \uplus y = \begin{cases} \omega^e + y & \text{if } y < \omega^{e+1} \\ y & \text{if } y \geq \omega^{e+1}. \end{cases}$$

This property and associativity completely define ordinal addition. Though we seldom use ordinal addition, explicitly noting when we do so, we do make a point of arranging natural operations to agree with the ordinal operations, e.g., writing $\omega + 1$ (which equals $\omega \uplus 1$) rather than $1 + \omega$ (which does not equal $1 \uplus \omega = \omega$), and writing $\omega \cdot 2$ (which equals $\omega \star 2$) rather than $2 \cdot \omega$ (which does not equal $2 \star \omega = \omega$).

2. Size

Every Chomp position X has an ordinal size, denoted size(X).

To compute size, start by expressing X as a finite overlapping sum of boxes at the origin. Each side of each box is uniquely expressible as a finite sum of powers of ω (including $1 = \omega^0$), e.g., $\omega \cdot 2 + 3 = \omega + \omega + 1 + 1 + 1$. (Here and elsewhere we use so-called "natural addition" or "polynomial addition" of ordinals, as compared to concatenating order types, or "ordinal addition.") This decomposition of the sides induces a unique decomposition of each box as a finite disjoint sum of translated boxes all of whose sides are powers of ω. Construct the finite set S of translated boxes that decompose the boxes of X, and then remove any box contained in some other box of S, creating a new set S'. Then size(X) is just the sum of the sizes of the elements of S', with the size of each box being the product of the lengths of its sides.

For example, when X is finite, size(X) is just the number of points in X. In this case, S consists of distinct unit boxes, eliminating repeats since it is a set rather than a multiset. And S' is simply S again since there are no inclusions of distinct unit boxes.

As another example, consider the position $X = (\omega + 1) \times 2 + (\omega \cdot 2) \times 1$. Decomposing the summands yields (using "@" to specify translation)

$$S_{(\omega+1)\times 2} = \{(\omega \times 1)@(0,0), (\omega \times 1)@(0,1), (1 \times 1)@(\omega, 0), (1 \times 1)@(\omega, 1)\},$$
$$S_{(w\cdot 2)\times 1} = \{(\omega \times 1)@(0,0), (\omega \times 1)@(\omega, 0)\},$$

and

$$S' = S_{(\omega+1)\times 2} \cup S_{(w\cdot 2)\times 1} - \{(1\times 1)@(\omega,0)\},$$

because the subtrahend is a subset of $(\omega\times 1)@(\omega,0)$ in $S_{(w\cdot 2)\times 1}$. Thus $\mathrm{size}(X) = \mathrm{size}(S') = \omega\cdot 3+1$. (See Figure 1, where the including box is shaded.)

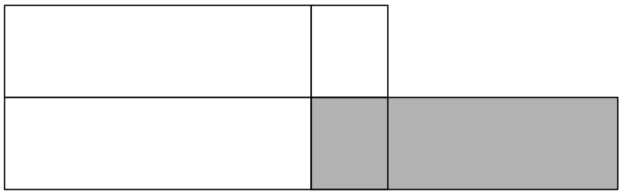

Figure 1. A Chomp position of size $\omega\cdot 3+1$.

As a third example, let $X = 1\times\omega\times\omega^2 + \omega\times\omega^2\times 1 + \omega^2\times 1\times\omega$. (See Figure 2.) Then $\mathrm{size}(X) = \omega^3\cdot 3$. Here S' contains just three terms, the summands of X.

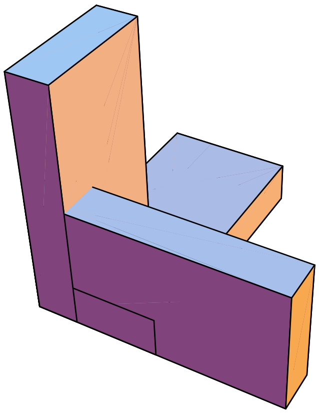

Figure 2. A Chomp position of size $\omega^3\cdot 3$.

This example illustrates the difficulty of trying to compute size from a fully disjoint decomposition: there are finite fully disjoint decompositions of X, but they all have component sums exceeding $\mathrm{size}(X)$; and there are fully disjoint decompositions whose component sums come to $\mathrm{size}(X)$, but they are all infinite. In our construction, the finite decomposition used to compute $\mathrm{size}(X)$ is semi-disjoint, meaning whenever two elements of S' have nonempty intersection, then the size of the intersection is at least a factor of ω less than the size of either intersectand.

If Y is reachable from X by one bite, then $\mathrm{size}(Y) < \mathrm{size}(X)$. To see this, note that S'_Y changes one or more components in S'_X, either removing them or replacing them by components with smaller total sum. It follows that Chomp terminates in finitely many moves.

3. Grundy Values

The Grundy value function on Chomp positions,

$$G : \{\text{Chomp positions in } \Omega^d\} \longrightarrow \Omega,$$

is

$$G(X) = \operatorname{mex}\{G(Y) : Y \text{ can be reached from } X \text{ in one bite}\}.$$

In particular, the poison cookie has Grundy value 1, and a column of single cookies has Grundy value equal to its height. Only the empty position has Grundy value 0 since it is reachable from any other position.

A Chomp position X is a P-position if and only if $G(X) = 1$. The case $X = \varnothing$ is clear. As for nonempty X, observe that if $G(X) = 1$ then every nonempty Y left by a bite into X satisfies $G(Y) > 1$, so some second bite into Y gives a nonempty position X' with $G(X') = 1$ again. (In general, any Grundy value-increasing Chomp bite is *reversible* in this fashion—a fact we will exploit several times.) So if $G(X) = 1$, the previous player wins by reversing bites until the next player is left with the poison cookie; and if $G(X) > 1$, the next player wins by biting X down to Y with $G(Y) = 1$.

A simple upper bound on Grundy values is clear: since Chomp bites decrease position size, it follows by induction that $G(X) \le \operatorname{size}(X)$ for all positions X.

Though it only seems to matter whether or not Grundy values are 1, knowing them in general will let us construct P-positions and execute winning strategies.

4. Other Termination Criteria

The astute reader may reasonably object that P-positions should have Grundy value 0.

In fact, Chomp can be defined with different termination conditions, leading to definitions of Grundy value different from the unrestricted definition used so far here. (We will use a certain restricted Grundy value later in this paper.)

The poison cookie description of Chomp can be viewed in two ways. First, the poison cookie specifies 1-restricted Chomp, i.e., the bite $\overline{(0, \dots, 0)}$ is forbidden and the last bite wins. The restricted Grundy values for this game are smaller by 1 than unrestricted Grundy values in the finite case (so now a P-position is specified by Grundy value 0 as it should be), but they catch up at $G = \omega$ and are equal from then on. In a larger theory of Grundy values, the poison cookie bite has the special value "loony" as defined in [BCG82a], Chapter 12.

Second, the poison cookie describes unrestricted misere Chomp, i.e., all bites are allowed but the last bite loses. In this context, an unrestricted Grundy value of 1 is the natural P-position criterion. Unrestricted Chomp Chomp is "tame" in the sense of [BCG82a], Chapter 13, so its misere analysis is tractable.

One can restrict Chomp more generally, disallowing a set of moves, and one can play a misere version of restricted Chomp. Even 1-restricted Chomp is not tame, however (e.g., the 1-restricted misere position $(3 \times 1) + (1 \times 3)$, like the misere Nim sum $2 + 2$, does not reduce to a Nim heap), and its misere analysis already requires general misere game theory.

For an isolated game, 1-restricted Chomp and unrestricted misere Chomp are the same. But a sum of Chomp positions played unrestricted misere is not equivalent to the same sum played 1-restricted: in the misere sum, only the last poison cookie is fatal. The 1-restricted sum is a P-position if and only if the 1-restricted Grundy values of the components have Nim sum 0. The unrestricted misere sum is equivalent to misere Nim, where the sum is a P-position if and only if the Nim sum of the unrestricted Grundy values equals

$$\begin{cases} 1 & \text{if every component has Grundy value 0 or 1,} \\ 0 & \text{if any component has Grundy value 2 or more.} \end{cases}$$

5. The Fundamental Theorem

The fundamental theorem of Chomp requires a construction. Let $d > 1$ be a finite ordinal, let A be a d-dimensional Chomp position, and let B (standing for "base") be a nonnull $(d-1)$-dimensional Chomp position. For any ordinal h, let $E(A, B, h)$ denote the d-dimensional Chomp position consisting of A and an infinite column over B in the last direction, the whole thing then truncated in the last direction at height h. That is,

$$E(A, B, h) = A + (B \times \Omega) - \overline{(0, \ldots, 0, h)}.$$

Call this the *extension of A by B to height h.* (See Figure 3 for two examples with the same A and B; in the second example the truncation eats into A.)

Theorem 5.1 (Fundamental Theorem of Chomp). *Suppose d, A, and B are given, where $d > 1$ is a finite ordinal, A is a d-dimensional Chomp position, and B is a nonnull $(d-1)$-dimensional Chomp position. Then there is a unique ordinal h such that $E(A, B, h)$, the extension of A by B to height h, is a P-position, meaning the second player wins.*

Once one realizes this, the proof is almost self-evident: extend A by B to the first—and only—height that doesn't give an N-position. Here is the formal argument:

Uniqueness is easy, granting existence. Let h be the least ordinal such that $E(A, B, h)$ is a P-position, and consider any greater ordinal, $k > h$. Since $E(A, B, k)$ reaches $E(A, B, h)$ in one bite, it is an N-position.

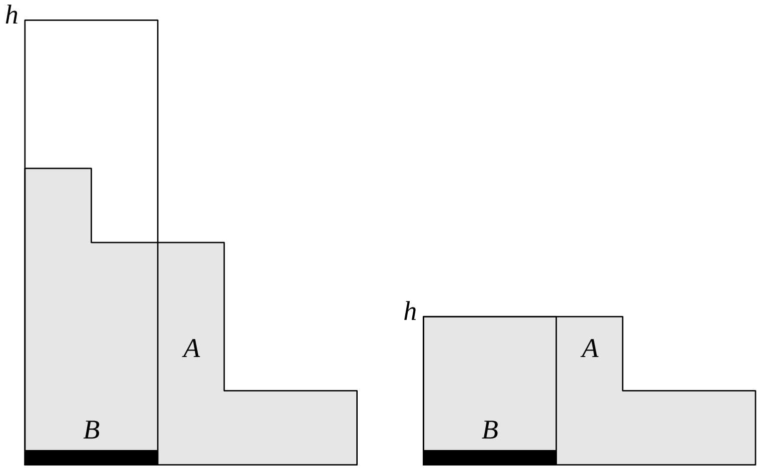

Figure 3. Extensions of A by B to height h.

In showing that h exists we further show how (in principle) to find it. We will construct a height function H on suitable pairs A, B of Chomp positions, such that

$$H(A,B) = h \qquad \text{if} \qquad E(A,B,h) \text{ is a P-position.}$$

The construction is by an outer induction on B and an inner induction on A.

The outer basis case is the poison cookie $B = 1^{d-1} \subset \Omega^{d-1}$, denoted B_0.

The inner basis case is $A = \varnothing$. The extension of $\varnothing$ by B_0 to height 1, $E(\varnothing, B_0, 1)$, gives the d-dimensional poison cookie, a P-position. Thus we have $H(\varnothing, B_0) = 1$.

For the inner induction step, take nonnull A and assume that for each $A' \subset A$, some extension $E(A', B_0, H(A', B_0))$ is a P-position. Let h be the minimal nonzero excluded member from the set of such prior P-position heights $H(A', B)$ as A' ranges over the positions reachable from A by one bite, said bite not eating into the column over B_0. Thus $h = \operatorname{mex}(M_0)$ where

$$M_0 = \{0\} \cup \bigl\{h : h = H(A - \overline{v}, B_0),\ A - \overline{v} \subset A,\ v = (u,k),\ B_0 - \overline{u} = B_0,\ h > k\bigr\}.$$

Then $E(A, B_0, h)$ is a P-position. For if it were an N-position, then some bite would take it to a P-position, $E(A, B_0, h) - \overline{v}$ for $v = (u,k)$ with $k < h$. There are two cases:

(i) If $B_0 - \overline{u} = B_0$ then the bite eats into $E(A, B_0, h)$ without truncating the column over B_0, leaving $E(A - \overline{v}, B_0, h)$ with the bite $\overline{v}$ as in the definition of M_0. This is an N-position since $h \notin M_0$ means $h \neq H(A - \overline{v}, B_0)$.

(ii) If $B_0 - \overline{u} = \varnothing$ then the bite truncates $E(A, B_0, h)$ in the last direction, leaving $E(A, B_0, k)$ with $k < h$; we may assume $k > 0$ else the game was just lost. This is an N-position since $k = H(A - \overline{v'}, B_0)$ for some bite $\overline{v'}$

as in the definition of M_0, and said bite takes $E(A, B_0, k)$ to the P-position $E(A - \overline{v'}, B_0, k)$.

So $H(A, B_0)$ is defined by for all A. Note how the first case relies on h being excluded from M_0, while the second relies on all $k < h$ belonging to M_0.

Returning to the outer induction, suppose that for some B, the height function $H(A', B')$ is defined for all pairs A', B' satisfying either of (i) $\varnothing \subset B' \subset B$, or (ii) $B' = B$ and $A' \subset A$. To define $H(A, B)$, construct a sequence of sets M_i, none of whose elements can serve as $H(A, B)$, and show that $\operatorname{mex}(M_\omega)$ does so. So, starting from an M_0 as above,

$$M_0 = \{0\} \cup \big\{h : h = H(A - \overline{v}, B),\ A - \overline{v} \subset A,\ v = (u, k),\ B - \overline{u} = B,\ h > k\big\},$$

adjoin for each succeeding ordinal $i + 1$ the heights of certain P-positions,

$$M_{i+1} = \left\{ \begin{array}{l} h : h = H(A + E(A, B, k) - \overline{v}, B - \overline{u}), \\ \qquad v = (u, k),\ \varnothing \subset B - \overline{u} \subset B,\ k \in M_i,\ h > k \end{array} \right\},$$

and let

$$M_\omega = \bigcup_{i<\omega} M_i.$$

Let $h = \operatorname{mex}(M_\omega)$. Then $E(A, B, h)$ is a P-position. For as before, if it were an N-position, then some bite would leave a P-position, $E(A, B, h) - \overline{v}$ for $v = (u, k)$ with $k < h$. This time there are three cases:

(i) If $B - \overline{u} = B$ then the bite eats into $E(A, B, h)$ without truncating the column over B, leaving $E(A - \overline{v}, B, h)$ with the bite $\overline{v}$ as in the definition of M_0. This is an N-position as before.

(ii) If $\varnothing \subset B - \overline{u} \subset B$ then the bite eats into part of the column over B, leaving $A + E(A, B, k) - \overline{v}$. Since $k < h$, we have $k \in M_i$ for some i. Let $h' = H(A + E(A, B, k) - \overline{v}, B - \overline{u})$; either $h' > k$, so $h' \in M_{i+1}$ and thus $h \neq h'$, or $h' \leq k < h$ and again $h \neq h'$; in either case, the bite has left an N-position.

(iii) If $B - \overline{u} = \varnothing$ then the bite truncates $E(A, B, h)$ in the last direction, leaving $E(A, B, k)$ for some $k < h$; we may assume $k > 0$ else N has just lost. If $k \in M_0$ then a bite as in the definition of M_0 leaves a P-position; if $k \in M_{i+1}$ for some i then a bite as in the definition of M_{i+1} leaves a P-position.

This completes the proof. This outer inductive step actually covers the basis case as well: when $B = B_0$, the construction of M_ω simply gives M_0 since all other M_i are empty.

Corollary 5.2 (Size Lemma). *For any nonempty d-dimensional Chomp position A, let $A - (B_0 \times \Omega)$ denote all of A except its intersection with the tower over the $(d-1)$-dimensional poison cookie B_0. Then*

$$H(A, B_0) \leq 1 + \operatorname{size}(A - (B_0 \times \Omega)).$$

In particular, if $A - (B_0 \times \Omega)$ is finite then so is $H(A, B_0)$.

This follows from the proof of the Fundamental Theorem since $H(A, B_0)$ is the minimal excluded element from the set M_0, and every element of M_0 is less than size$(A - (B_0 \times \Omega))$ by induction.

6. Two Constructions

Along with extending Chomp positions to P-positions, The Fundamental Theorem can be used to find Grundy values and to construct extensions with arbitrary Grundy values.

To find the Grundy value of an arbitrary position, let X be d-dimensional and raise it to height 1 in the $(d+1)$st dimension, creating the position $A = X \times 1$; note that A is essentially the same thing as X, i.e., the bites out of A and X correspond perfectly and so $G(A) = G(X)$. Apply the Fundamental Theorem to A and $B = B_0$, the poison cookie in $d+1$ dimensions; this creates a P-position Y from the original X by adding an orthogonal column of single cookies of some height h,

$$Y = (X \times 1) + (1^d \times h).$$

(See Figure 4.) In fact, the column has height $h = G(X)$, because in that case the previous player wins by a pairing strategy: if the next bite is from $X \times 1$ leaving a Grundy value smaller than $G(X)$, bite the orthogonal column down to the same value; if the next bite is from $X \times 1$ and reversible, i.e., it leaves a larger Grundy value, then bite into what's left of $X \times 1$ restoring the Grundy value to $G(X)$; if the next bite is from the orthogonal column, reducing its Grundy value below $G(X)$, then bite into $X \times 1$ reducing its Grundy value by the same amount. So we have constructed a column of the desired height $G(X)$. This sort of pairing strategy will be used frequently throughout the paper.

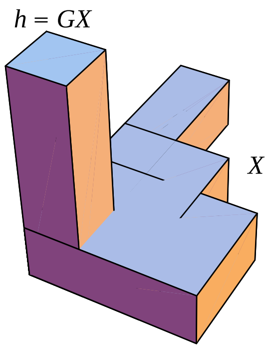

Figure 4. Finding a Grundy value.

To extend a position and get an arbitary Grundy value g, start with X as before and carry out a similar construction, only the column of single cookies extends to height g in a prepended zeroth dimension:

$$A = (1 \times X) + (g \times 1^d).$$

Again apply the Fundamental Theorem to A and any nonempty $1 \times B$ in dimensions $0, \dots, d$, extending A in the last direction to get a P-position,

$$Y = E((1 \times X) + (g \times 1^d), 1 \times B, h).$$

(See Figure 5, where B is taken to be B_0, the poison cookie.) Again the win is by a pairing strategy, showing that the d-dimensional extension of X,

$$E(X, B, h),$$

has the same Grundy value as the orthogonal column, i.e., the desired value g.

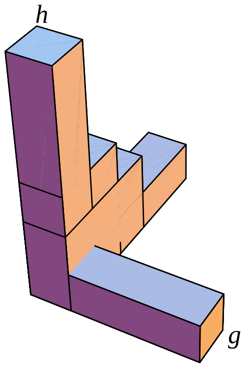

Figure 5. Constructing a position with Grundy value g.

Proposition 6.1 (Beanstalk Lemma). *Let A be any finite d-dimensional Chomp position, and let h be any infinite ordinal. Then the Chomp position*

$$A + (1^{d-1} \times h),$$

obtained by adding a tower of height h to A, has infinite Grundy value.

To see this, note that for every finite ordinal g, the second construction just given—adding a tower to A over B_0 to obtain a position with Grundy value g—adds a tower of finite height by the Size Lemma. Adding an infinite tower to A thus gives an infinite Grundy value.

A more general statement of the Beanstalk Lemma is that for any ordinal h, $G(A + (1^{d-1} \times h))$, where A is finite, has the same highest term as h. That is, if $h = \omega^i \cdot a_i + \cdots$ with $0 < a_i < \omega$ then also $G(A + (1^{d-1} \times h)) = \omega^i \cdot a_i + \cdots$. This phenomenon of the dominant term of size determining the dominant term of Grundy value is called *stratification*, and it is ubiquitous in transfinite Chomp analysis. By contrast, the dominant term of size actually equalling the dominant term of Grundy value is particular to this case, an artifact of a single column's Grundy value being its height.

7. P-Ordered Positions

Define a Chomp position to be *P-ordered* if its P-subpositions are totally ordered by inclusion. Every non-P-ordered position contains a minimal non-P-ordered position $(3\times1)+(2\times2)+(1\times3)$ or $(2\times1\times1)+(1\times2\times1)+(1\times1\times2)$, so every P-position must be, up to congruence, a subposition of $2\times\Omega$ or $(1\times\Omega)+(\Omega\times1)$. Thus the complete list of P-ordered P-positions, up to rotation and inclusion, is

$$2\times\omega, \{(1\times(i+1))+(2\times i) : 0\le i<\omega\}, \{(1\times a)+(a\times1) : 0<a\}.$$

Theorem 7.1. *If X is any Chomp position and P is any P-ordered P-position, then $G(X\times P)=G(X)$. In particular, if X is a P-position, then so is $X\times P$.*

To prove this, let $g=G(X)$, construct the product $X\times P$, and prepend to this main body a column of g single cookies in the zeroth direction, constructing the position

$$Y=(1\times X\times P)+(g\times1\times1).$$

(These triple products and others to follow refer to $\Omega\times\Omega^d\times\Omega^e$ where $X\subset\Omega^d$ and $P\subset\Omega^e$; thus "1" often will mean 1^d or 1^e.) The idea is to show that Y is a P-position by exhibiting a winning strategy.

Let $Y_0=Y$. For $i\ge0$, let Y_{2i+1} be the result of an arbitrary bite applied to Y_{2i} and let Y_{2i+2} be the result of a to-be-specified bite applied to Y_{2i+1}. The specified bite will maintain two invariants of the even positions Y_{2i}:

(I1) Vertical sections of the main body are P-positions, i.e., for all $z\in\Omega$, $Y_{2i}\cap(1\times\{z\}\times P)$ either is empty or is a P-position. (Strictly speaking, it needs to be translated to the origin — we're being a bit casual to avoid even more notation.)

(I2) What's left of the prepended column retains the same Grundy value as what's left of X, i.e., $G(Y_{2i}\cap(g\times1\times1))=G(Y_{2i}\cap(1\times X\times1))$.

The invariants clearly hold for Y_0.

Bites into Y_{2i} fall under three cases:

(i) $\overline{(0,x,p)}$ with $p>0$. This bite preserves invariant (I2). Answer it with $\overline{(0,x,p')}$, where $\overline{p'}$ is the bite into P that restores the section $Y_{2i+1}\cap(1\times\{x\}\times P)$ to a P-position $Y_{2i+2}\cap(1\times\{x\}\times P)$. The bite $\overline{(0,x,p)}$ may have left other sections $Y_{2i+1}\cap(1\times\{\xi\}\times P)$ not P-positions for $\xi\in\overline{x}$; but the answering bite restores all such sections in Y_{2i+2} to P-positions, thus restoring invariant (I1) while preserving (I2). This property depends on P being P-ordered.

(ii) $\overline{(0,x,0)}$ with $x \neq 0$, i.e., a truncation in the X-direction. This bite preserves invariant (I1). Let $g' = G(Y_{2i} \cap (1 \times X \times 1))$, and $g'' = G(Y_{2i+1} \cap (1 \times X \times 1))$; note $g' \neq g''$. Also note that $G(Y_{2i} \cap (g \times 1 \times 1)) = g'$ by invariant (I2). If $g'' < g'$, answer the bite by truncating the orthogonal column with $\overline{(g'',0,0)}$, restoring (I2). If $g'' > g'$, answer the bite with $\overline{(0,x',0)}$, reversing the bite into X to make $G(Y_{2i+2} \cap (1 \times X \times 1) = g'$, also restoring (I2). Either of these answering bites also preserves (I1).

(iii) $\overline{(h,0,0)}$, i.e., a bite into the orthogonal column. This bite preserves (I1). We had $G(Y_{2i} \cap (1 \times X \times 1) > h$ before this bite, so answer it with any bite $\overline{(0,x,0)}$ that makes $G(Y_{2i+2}) \cap (1 \times X \times 1) = h$, restoring (I2) while preserving (I1).

This completes the proof.

Applying the theorem twice, if X is any P-position and P_1, P_2 are P-ordered P-positions, then $X \times P_1 \times P_2$ is a P-position. In particular, letting $X = P_1 = P_2 = \omega \times 2$ and then permuting axes, $\omega \times \omega \times \omega \times 2 \times 2 \times 2$ is a P-position. The winning strategy is

- Answer $\overline{(r,t,i+1,s,u,0)}$ with $\overline{(r,t,i,s,u,1)}$ and vice versa.
- Answer $\overline{(r,i+1,0,s,0,0)}$ with $\overline{(r,i,0,s,1,0)}$ and vice versa.
- Answer $\overline{(i+1,0,0,0,0,0)}$ with $\overline{(i,0,0,1,0,0)}$ and vice versa.

8. Side-Top Positions

Consider two 2-dimensional Chomp positions S and T (for "side" and "top"), with S finite and T at most two cookies wide. Let $S@(2,0)$ denote the translation of S rightward by 2 and let $T@(0,\omega)$ denote the translation of T upward by ω. Construct the Chomp position

$$U = (2 \times \omega) + S@(2,0) + T@(0,\omega),$$

i.e., a 2-wide, ω-high column with finitely much added to its side and any amount to its top. (See Figure 6.) This section gives the criterion for U to be a P-position.

Doing so requires a certain notion of restricted Grundy value, cf. the section earlier in this paper of other termination criteria. Let $\mathcal{X}$ denote a collection of Chomp positions. The $\mathcal{X}$-restricted Grundy value function

$$G_{\mathcal{X}} : \mathcal{X} \longrightarrow \Omega$$

is defined as

$$G_{\mathcal{X}}(X) = \text{mex}\{G_{\mathcal{X}}(Y) : Y \in \mathcal{X}, Y \text{ can be reached from } X \text{ in one bite}\}.$$

Let $\square S$ denote $S@(2,0)$ left-extended two units to the axis, i.e.,

$$\square S = (2 \times s) + S@(2,0),$$

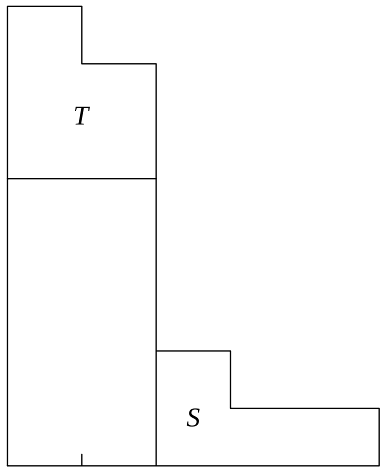

Figure 6. A side-top position.

where s is the height of S. Let $\mathcal{S}$ denote the set of all side positions $\tilde{S}$ such that $H(\Box\tilde{S}, 2)$ is infinite.

Theorem 8.1 (Side-Top Theorem). *Let a side-top position $U = (2 \times \omega) + S@(2,0) + T@(0,\omega)$ be given. If $H(\Box S, 2)$ is finite then U is an N-position. If $H(\Box S, 2)$ is infinite then*

$$U \text{ is a P-position} \quad \Longleftrightarrow \quad G_{\mathcal{S}}(S) = G(T).$$

The first statement is clear, for if $H(\Box S, 2)$ is finite then the bite truncating U to that height in the second direction wins. For the rest of the proof, $H(\Box S, 2)$ is infinite.

To show "$\Longleftarrow$" of the second statement, assume $G_{\mathcal{S}}(S) = G(T)$. Take any bite $\overline{(a,b)}$, other than the complete bite $a = b = 0$ of course.

If b is infinite then the bite takes $T@(0,\omega)$ down to some $\tilde{T}@(0,\omega)$ and leaves the rest of U intact. Let $g = G(\tilde{T}) \neq G(T)$. If $g > G(T)$ then a further bite into $\tilde{T}@(0,\omega)$ reverses the first one, while if $g < G(T) = G_{\mathcal{S}}(S)$ then a bite into $S@(2,0)$ takes it down to some $\tilde{S}@(2,0)$ with $G(\tilde{S}) = g$ and $H(\Box\tilde{S}, 2)$ infinite.

Now we consider b finite. If $a = 0$ then the bite leaves a finite position, thus leaving a losing position since $H(\Box S, 2)$ is infinite. If $a = 1$ then the bite leaves a position with one infinite column plus a finite part. The Beanstalk Lemma from earlier shows that this has infinite Grundy value, meaning its Grundy value isn't 1, so it is a losing position.

Finally, if $a \geq 2$ then player N has bitten into $S@(2,0)$, leaving position $(2 \times \omega) + \tilde{S}@(2,0) + T@(0,\omega)$ for some finite $\tilde{S}$. Either $H(\Box\tilde{S}, 2)$ is finite or it is infinite. If it is finite then the bite truncating the second direction to that height leaves a P-position. If it is infinite then $\tilde{S} \in \mathcal{S}$. Let $g = G(\tilde{S})$, so that $g \neq G_{\mathcal{S}}(S) = G(T)$; if $g < G(T)$ then bite into $T@(0,\omega)$, getting $\tilde{T}@(0,\omega)$ with

$G(\tilde{T}) = g$; if $g > G_{\mathcal{S}}(S)$ then bite into $\tilde{S}@(2,0)$ to restore Grundy value $G_{\mathcal{S}}(S)$. This covers all cases.

For "$\Longrightarrow$" of the second statement, let $G_{\mathcal{S}}(S) \neq G(T)$. If $G_{\mathcal{S}}(S)$ is larger, bite into $S@(2,0)$ to produce a P-position as just shown; if $G(T)$ is larger, bite into $T@(0,\omega)$ similarly.

9. Two-wide Chomp

A two-wide Chomp position takes the form $X = (1 \times h) + (2 \times k)$ with $h \geq k$. Thus

$$h = \omega^i \cdot u + a, \quad k = \omega^i \cdot v + b,$$

where $u > 0$, $u \geq v$, $a \geq b$ when $u = v$, $a < \omega^i$, $b < \omega^i$, and when $i < \omega$ then also $u < \omega^i$. This section gives the Grundy values $G(X)$, renotated for convenience

$$G(\omega^i \cdot u + a, \omega^i \cdot v + b).$$

When $i = 0$, the position is just two columns of finite heights $u \geq v$. Computing from first principles gives the following Grundy values shown in Table 1. Reading down the diagonals shows what's going on. For $u - v$ even, the Grundy

$G(u,v)$	$v=0$	1	2	3	4	5	6	$\cdots$
$u=0$	0							
1	1	2						
2	2	1	3					
3	3	4	1	5				
4	4	3	5	1	6			
5	5	6	4	7	1	8		
6	6	5	7	4	8	1	9	
7	7	8	6	9	4	10	1	$\cdots$
8	8	7	9	6	10	4	11	$\cdots$
9	9	10	8	11	7	12	4	$\cdots$
10	10	9	11	8	12	7	13	$\cdots$
11	11	12	10	13	9	14	7	$\cdots$

Table 1. Two-wide Grundy values

values skip every third value; for $u - v$ odd, they iterate at every second step for a while and then stabilize. The formula is

$$G(u,v) = \begin{cases} u - v + \lfloor \frac{3v+1}{2} \rfloor & \text{if } u - v \text{ is even,} \\ \min\{u - v + \lfloor \frac{v}{2} \rfloor, \frac{3(u-v)-1}{2}\} & \text{if } u - v \text{ is odd.} \end{cases} \tag{9–1}$$

The next case is $i = 1$, i.e., the column heights are $\omega \cdot u + a$, $\omega \cdot v + b$, with all of u, v, a, b finite and $u > 0$, $u \geq v$, $a \geq b$ when $u = v$.

Consider the subcase $u = v$, $a = b = 0$. Biting only into the right column gives a Chomp position whose left column has greater order of magnitiude than its right; by an extension of the Beanstalk Lemma such a position has infinite Grundy value. On the other hand, biting into both columns and thus truncating the position at a lower height gives (by induction) a finite Grundy value. Thus, $G(\omega^i \cdot u, \omega^i \cdot u)$ is the mex of the set of Grundy values of smaller positions with two columns of equal height h. These run through

$$\begin{aligned}
\mathbb{N} \setminus \{3k+1\} &\quad \text{as } 0 \le h < \omega, \\
\{3k+1\} \setminus \{9k+4\} &\quad \text{as } \omega \le h < \omega \cdot 2, \\
\{9k+4\} \setminus \{27k+13\} &\quad \text{as } \omega \cdot 2 \le h < \omega \cdot 3, \\
&\ \ \vdots \\
\left\{3^{u-1}k + \frac{3^{u-1}-1}{2}\right\} \setminus \left\{3^u k + \frac{3^u - 1}{2}\right\} &\quad \text{as } \omega \cdot (u-1) \le h < \omega \cdot u.
\end{aligned}$$

And thus $G(\omega, \omega) = 1$, $G(\omega \cdot 2, \omega \cdot 2) = 4$, $G(\omega \cdot 3, \omega \cdot 3) = 13$, and in general $G(\omega \cdot u, \omega \cdot u) = (3^u - 1)/2$. Each time the column heights reach a new multiple of ω, the Grundy values have filled up the naturals minus an arithmetic progression, with the complementary progression—an iterate of the initial $\{3k+1\}$—becoming sparser each time.

Continuing to assume $i = 1$ and $u = v$ but now allowing a and b nonzero gives the position $X = (2 \times \omega^i \cdot u) + ((1 \times a) + (2 \times b))@(0, \omega^i \cdot u)$. By the Beanstalk Lemma, the only new bites giving positions of small enough Grundy value to worry about are the bites into the finite top part. These bites give Grundy values from the beginning of the omitted arithmetic progression $\{3^u k + (3^u - 1)/2\}$, thus

$$G(\omega \cdot u + a, \omega \cdot u + b) = 3^u G(a, b) + \frac{3^u - 1}{2}. \tag{9–2}$$

We already know the right side here, thanks to (1).

Still keeping $i = 1$ but now taking $u > v$, the formula is

$$G(\omega \cdot u + a, \omega \cdot v + b) = \omega \cdot (u - v) + (a \oplus b) \quad \text{when } u > v, \tag{9–3}$$

where "$\oplus$" denotes Nim sum. This is easy to see when $b = 0$ and the position is $X = (2 \times \omega \cdot v) + (1 \times \omega \cdot (u - v) + a)@(0, \omega \cdot v)$: bites leaving two equal height columns give all finite Grundy values except an arithmetic progression; bites into the top part give $\omega \cdot (u - v) + a$ more Grundy values, filling in the missing progression and then all other values less than $\omega \cdot (u - v) + a$; and bites into the right column give larger Grundy values.

To prove (3) when $b > 0$, first note the recursive formula

$$G(\omega \cdot u + a, \omega \cdot v + b) = G(\omega \cdot (u - v) + a, b),$$

which follows from observing that taking base $B = 2 \times \omega \cdot v$ and top $T = (1\times\omega\cdot(u-v)+a)+(1\times b)@(1,0)$, the orthogonal sum of positions $B+T@(0,\omega\cdot v)$ and T (cf. various prior constructions) is a P-position. The pairing strategy is clear: match bites into either top part, and if $B+T@(0,\omega\cdot v)$ is bitten down to two columns of equal height less than $\omega \cdot v$, giving a position of finite Grundy value, the Beanstalk Lemma says that a matching bite into T gives the same Grundy value.

With the recursive formula established it suffices to prove (3) when the position is T. To do so, take the orthogonal sum of T with a single column of height $\omega \cdot (u-v) + (a \oplus b)$. This time the pairing strategy to win is: the Nim sum allows matching bites into the finite parts; the simple upper bound $G \leq$ size from Section 2 shows that other bites into T leave positions of Grundy value less than $\omega \cdot (u-v)$, matched by bites into the column; and the Beanstalk Lemma shows that $G(T) \geq \omega \cdot (u-v)$, so bites deeper than $a \oplus b$ into the column can be matched by bites into T.

The analysis for $2 < i < \omega$ is similar to $i = 1$. When $u = v$ the formula is

$$G(\omega^i \cdot u + a, \omega^i \cdot u + b) = \omega^{i-1} \cdot u + G(a,b). \tag{9–4}$$

This is easy to establish first when $a = b = 0$, and then in general. When $u > v$, the formula is

$$G(\omega^i \cdot u + a, \omega^i \cdot v + b) = \omega^i \cdot (u-v) + (a \oplus b). \tag{9–5}$$

Again this follows from the recursive formula

$$G(\omega^i \cdot u + a, \omega^i \cdot v + b) = G(\omega^i \cdot (u-v) + a, b),$$

which is established as above.

Finally, for $i \geq \omega$, the formula is

$$G((\omega^\omega)^j \cdot u + a, (\omega^\omega)^j \cdot v + b) = (\omega^\omega)^j \cdot G(u,v) + \begin{cases} G(a,b) & \text{if } u = v, \\ a \oplus b & \text{if } u > v. \end{cases} \tag{9–6}$$

Here $j > 0$, $0 < u < \omega^\omega$, $u \geq v$, $a < (\omega^\omega)^j$, $b < (\omega^\omega)^j$.

To see this, first let $j = 1$ and $a = b = 0$. Build up the formulas $G(\omega^\omega, \omega^\omega) = \omega^\omega \cdot 2 = \omega^\omega \cdot G(1,1)$, $G(\omega^\omega \cdot 2, \omega^\omega) = \omega^\omega \cdot 1 = \omega^\omega \cdot G(2,1)$, $G(\omega^\omega \cdot 2, \omega^\omega \cdot 2) = \omega^\omega \cdot 3 = \omega^\omega \cdot G(2,2)$, ..., and in general,

$$G(\omega^\omega \cdot u, \omega^\omega \cdot v) = \omega^\omega \cdot G(u,v) \text{ for finite } u \geq v$$

by considering the bites into these configurations.

Similarly, $G(\omega^\omega \cdot \omega, \omega^\omega \cdot \omega) = \omega^\omega$: bites truncating both columns give positions with Grundy values $\omega^\omega \cdot (\mathbb{N} \setminus \{3k+1\})$, and bites truncating the second column give positions with Grundy value on the order of $\omega^\omega \cdot \omega$. This continues on to $G(\omega^\omega \cdot \omega \cdot 2, \omega^\omega \cdot \omega \cdot 2) = \omega^\omega \cdot 4$, $G(\omega^\omega \cdot \omega \cdot 3, \omega^\omega \cdot \omega \cdot 3) = \omega^\omega \cdot 13$, ..., until the gaps in $\mathbb{N}$ are filled in and $G(\omega^\omega \cdot \omega^2, \omega^\omega \cdot \omega^2) = \omega^\omega \cdot \omega$.

Essentially the same argument now gives $G(\omega^\omega \cdot \omega^3, \omega^\omega \cdot \omega^3) = \omega^\omega \cdot \omega^2$, $G(\omega^\omega \cdot \omega^4, \omega^\omega \cdot \omega^4) = \omega^\omega \cdot \omega^3$, ..., until catching up at $G((\omega^\omega)^2, (\omega^\omega)^2) = (\omega^\omega)^2$. From here the whole argument repeats to establish the formula for $j = 2$, $a = b = 0$, and similarly for all ordinals j.

The term in (9–6) when $(a, b) \neq (0, 0)$ is argued exactly as in earlier cases.

Formulas (9–1)–(9–6) cover all cases.

10. Three-Wide Chomp

Let A be a three-wide Chomp position,

$$A = (1 \times u) + (2 \times v) + (3 \times x), \qquad u \geq v \geq x.$$

This section gives the conditions for A to be a P-position.

First we dispense with some small positions and some large ones: when the right column has finite height, i.e., $x < \omega$, the Beanstalk Lemma reduces the case $v < \omega$ to finite calculation, and the Side-Top Theorem covers the case $v \geq \omega$. On the other hand, the discussion below will show that the tall box $A = 3 \times \omega^\omega$ is a P-position, so any superposition of this box, i.e., any other position with $x \geq \omega^\omega$, is an N-position.

This leaves the case $\omega \leq x < \omega^\omega$, where the analysis is detailed. The first step is to decompose A into components, two bottoms (denoted with a subscript "b") and three tops (subscript "t"). For some unique i with $1 \leq i < \omega$, we have $\omega^i \leq x < \omega^{i+1}$, so write

$$\begin{aligned}
u &= \omega^{i+1} \cdot u_{i+1} + \omega^i \cdot u_i + a && (u_i < \omega, a < \omega^i),\\
v &= \omega^{i+1} \cdot v_{i+1} + \omega^i \cdot v_i + b && (v_i < \omega, b < \omega^i),\\
x &= \omega^i \cdot x_i + c && (0 < x_i < \omega, c < \omega^i).
\end{aligned}$$

The precise decomposition of A depends on the nature of u, v, and x. Specifically,

(i) If $u_{i+1} = v_{i+1} = 0$ and $u_i = v_i = x_i$ then let

$$U_b = 2 \times \omega^i \cdot u_i, \quad X_b = (1 \times \omega^i \cdot x_i)@(2, 0),$$
$$U_t = ((1 \times a) + (2 \times b) + (3 \times c))@(0, \omega^i \cdot u_i), \quad V_t = \varnothing, \quad X_t = \varnothing.$$

(ii) If $v_{i+1} = 0$ and $v_i = x_i$ and either $u_{i+1} > 0$ or $u_i > v_i$ then let

$$U_b = (1 \times (\omega^{i+1} \cdot u_{i+1} + \omega^i \cdot u_i)) + (2 \times \omega^i \cdot v_i),$$
$$X_b = (1 \times \omega^i \cdot x_i)@(2, 0), \quad U_t = (1 \times a)@(0, \omega^{i+1} \cdot u_{i+1} + \omega^i \cdot u_i),$$
$$V_t = ((1 \times b) + (2 \times c))@(1, \omega^i \cdot v_i), \quad X_t = \varnothing.$$

(iii) If $u_{i+1} = v_{i+1} > 0$ and $u_i = v_i$ then let

$$U_b = 2 \times \omega^{i+1} \cdot u_{i+1}, \quad X_b = (1 \times \omega^i \cdot x_i)@(2,0),$$
$$U_t = ((1 \times (\omega^i \cdot u_i + a)) + (2 \times (\omega^i \cdot v_i + b)))@(0, \omega^{i+1} \cdot u_{i+1}),$$
$$V_t = \varnothing, \quad X_t = (1 \times c)@(2, \omega^i \cdot x_i).$$

(iv) Otherwise let

$$U_b = (1 \times (\omega^{i+1} \cdot u_{i+1} + \omega^i \cdot u_i)) + (2 \times (\omega^{i+1} \cdot v_{i+1} + \omega^i \cdot v_i)),$$
$$X_b = (1 \times \omega^i \cdot x_i)@(2,0), \quad U_t = (1 \times a)@(0, \omega^{i+1} \cdot u_{i+1} + \omega^i \cdot u_i),$$
$$V_t = (1 \times b)@(1, \omega^{i+1} \cdot v_{i+1} + \omega^i \cdot v_i), \quad X_t = (1 \times c)@(2, \omega^i \cdot x_i).$$

(See Figures 7a and 7b. Note that 7b only illustrates one instance of the fourth, general case.)

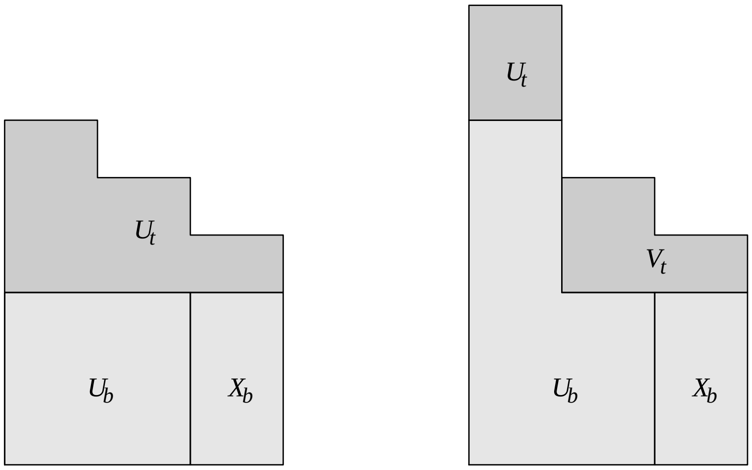

Figure 7a. Three-wide positions: first and second cases.

The criterion to be established is that in these cases the three-wide Chomp position A is a P-position if and only if

$$G(U_b) = G(X_b) \quad \text{and} \quad G(U_t) \oplus G(V_t) \oplus G(X_t) = 0. \tag{10–1}$$

This reduces the three-wide question to questions about one-wide and two-wide subpositions: even in the case with a three-wide top subposition, the bottom subpositions already fail the test.

To show the criterion it suffices to show that if A is a position that satisfies (7) then every bite into A gives a position that doesn't; and if A is a position not satisfying (7) then some bite into A gives a position that does.

So assume first that A satisfies (7). Bite into A, obtaining a new position with components U_b', X_b', etc. If the bite doesn't touch any bottom component then it changes exactly one top component, violating $G(U_t') \oplus G(V_t') \oplus G(X_t') = 0$ in (7).

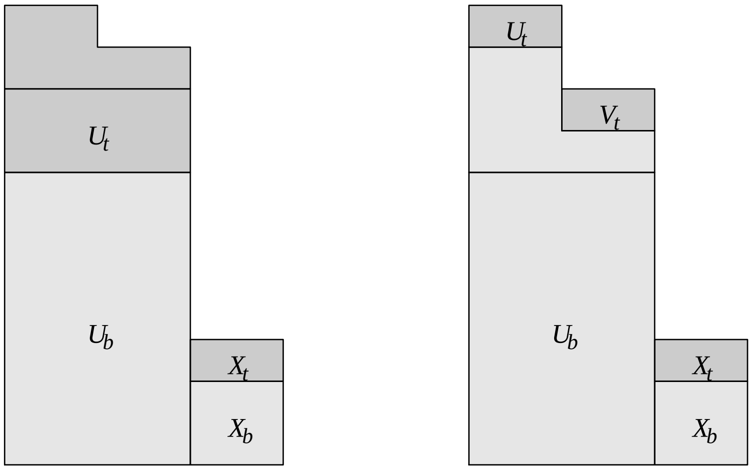

Figure 7b. Three-wide positions: third and fourth cases.

If the bite reduces U_b to U_b' but doesn't touch X_b then $G(U_b') \neq G(U_b) = G(X_b) = G(X_b')$, again violating (7). If the bite reduces X_b to X_b' but doesn't touch U_b then possibly $U_b' \supset U_b$ and $G(U_b') > G(U_b)$ in the new decomposition, but in any case $G(U_b') \geq G(U_b) = G(X_b) > G(X_b')$. Finally, if the bite touches both bottom components then it is either $\overline{(1,z)}$ for some z, in which case $G(U_b') > G(U_b)$ by two-wide results and $G(X_b) > G(X_b')$; or the bite is $\overline{(0,z)}$ for some z, in which case $U_b' = 2 \times h$ and $X_b' = (1 \times h)@(2,0)$ for some h, so that $G(U_b') \neq G(X_b')$.

For the other half of the argument, assume that position A, with components U_b, X_b, etc., doesn't satisfy (7). We seek a bite into A, giving a new position with components U_b', X_b', etc., that does.

If $G(U_b) = G(X_b)$ then $G(U_t) \oplus G(V_t) \oplus G(X_t) \neq 0$ and the disjoint grouping strategy (cf. the pairing strategy used earlier) provides a bite into exactly one top component restoring the second condition in (7).

If $G(U_b) < G(X_b)$ then the answering bite to restore (7) is some $\overline{(2,z)}$ that touches X_b. All such bites satisfy $U_b' \supseteq U_b$ (proper containment only when the decomposition index i is altered by the bite), $G(U_b') \geq G(U_b)$, and $G(X_b') < G(X_b)$. The needed bite has $z = \omega^j \cdot z_j + c'$ such that $\omega^j \cdot z_j = G(U_b')$ and $c' = G(U_t') \oplus G(V_t')$.

If $G(U_b) > G(X_b)$, then the answering bite is some $\overline{(0,z)}$ touching U_b but not X_b. All such bites give $G(U_b') < G(U_b)$ in this decomposition. To make the top Grundy values sum to zero, use the fact that for any component U_b arising in our decomposition, any bite $\overline{(1,y)}$ in U_b increases the Grundy value, i.e., $G(U_b - \overline{(1,y)}) > G(U_b)$; it follows by the definition of Grundy value as a mex that $G(X_b)$ takes the form $G(U_b - \overline{(0,z')})$ for some z'. A larger bite height $z > z'$ will leave bottom piece $U_b' = U_b - \overline{(0,z')}$ with the right Grundy value and also leave a two-wide top piece U_t' with $G(U_t') = G(X_t')$.

To complete the discussion of three-wide Chomp, we show that the tall box $3 \times \omega^\omega$ is a P-position by itemizing how every bite leaves an N-position. That is, every bite into $3 \times \omega^\omega$ can be answered with another bite leaving a smaller P-position described above. Bites take the form $\overline{(e, z)}$ with $0 \le e < 3$ and $0 \le z < \omega^\omega$. One can check that the P-positions enumerated don't include any position $3 \times \omega^\omega - \overline{(e, z)}$. Responses to the bite are analyzed by the cases $e = 2, 1, 0$.

If $e = 2$, the bite truncates the third column. Let $u = H(3 \times z, 2)$. If $u < \omega^\omega$ then by definition the answering bite is $\overline{(0, u)}$. To show $u < \omega^\omega$ we may assume $u \ge \omega$, else there is nothing to prove. The argument has two cases, depending on z.

When $z < \omega$, the position $3 \times \omega^\omega - \overline{(2, z)}$ is a Side-Top position and the Side-Top Theorem tells us how to proceed: recalling the notion of restricted Grundy value $G_\mathcal{S}$, bite the top part of the position down to $2 \times a$ such that $G(2 \times a) = G_\mathcal{S}(1 \times z) \le G(1 \times z) = z < \omega$. From the results on two-wide Chomp we know that $G(2 \times a) < \omega$ if and only if $a < \omega^2$; since we are biting into $2 \times \omega^\omega$, there is plenty of room to bite the top part down to $2 \times a$.

When $z \ge \omega$, write, using the terminology of the three-wide algorithm, $z = x = \omega^i \cdot x_i + c$ with $0 < x_i < \omega$ and $c < \omega^i$. Now the answering bite to $\overline{(2, x)}$ is $\overline{(0, u)}$, where $u = \omega^{i+1} \cdot x_i + a$ is found from the algorithm with $G(2 \times a) = c$. From the two-wide results, a is unique and satisfies $a < \omega^{i+1}$. This completes the case $e = 2$.

If $e = 1$, the bite is $\overline{(1, z)}$, truncating the last two columns at the same height. Let $u = H(3 \times z, 1)$. Again the answering bite is clearly $\overline{(0, u)}$ once we know that $u < \omega^\omega$, and again showing this breaks into two cases depending on z. When $z < \omega$, the Beanstalk Lemma says that $u < \omega$ as well. When $z \ge \omega$, write $z = x = \omega^i \cdot x_i + c$ with $0 < x_i < \omega$ and $c < \omega^i$, and u works out to $u = \omega^i \cdot 2x_i + G(2 \times c)$ — to see this, check that the position $1 \times u + 3 \times x$ is identified by the three-wide algorithm as a P-position. Note $G(2 \times c) < \omega^i$ by two-wide results. This completes the case $e = 1$.

If $e = 0$, the bite is $\overline{(0, z)}$, leaving a rectangle $3 \times z$ with $z < \omega^\omega$. When $z < \omega$, the rectangle is finite. A result on finite Chomp says that the rectangle must be an N-position, so some finite bite into it leaves a P-position. The bite takes the form $\overline{(1, v)}$ or $\overline{(2, x)}$ and is believed to be unique.

When $\omega \le z < \omega^2$, write $z = \omega \cdot (n + 1) + c$ with $0 \le n < \omega$ and $c < \omega$. Answer the bite $\overline{(0, z)}$ with $\overline{(2, x)}$ for the unique $x < \omega$ that satisfies $G_\mathcal{S}(x) = G(2 \times (\omega \cdot n + c))$, where again $G_\mathcal{S}$ is restricted Grundy value as in the Side-Top Theorem and x is found from that theorem.

When $z \ge \omega^2$, write $z = \omega^{i+1} \cdot z_{i+1} + c$ with $0 < z_{i+1} < \omega$ and $c < \omega^{i+1}$. Answer $\overline{(0, z)}$ with $\overline{(2, x)}$ where $x = \omega^i \cdot z_{i+1} + G(2 \times c)$ is found from the three-wide P-position algorithm; again $G(2 \times c) < w^i$ by two-wide results. In this case, when z_{i+1} is even there is sometimes also a winning bite $\overline{(1, v)}$; finding it is an exercise for the interested reader. This completes the case $e = 0$.

11. A Three-Dimensional Example

This section will present the three-dimensional Chomp P-positions with 2-by-2 base, including $2 \times 2 \times \omega^3$.

We need a variant of the Side-Top Theorem. Consider a pair of 3-dimensional Chomp positions F and T (for "front" and "top"), with F a finite subset of $2 \times \Omega \times \Omega$, and T a subset of $1 \times 2 \times \Omega$. Let $B = 1 \times 2 \times \omega$ be the one-deep, two-wide infinite brick of height ω. Let $F@(1,0,0)$ denote the translation of F one unit forward, meaning perpendicular to the wide dimension of the brick, and let $T@(0,0,\omega)$ denote the translation of T upward by ω. Construct the Chomp position

$$U = B + F@(1,0,0) + T@(0,0,\omega),$$

the brick with finitely much added to its front and any amount to its top. (See Figure 8.)

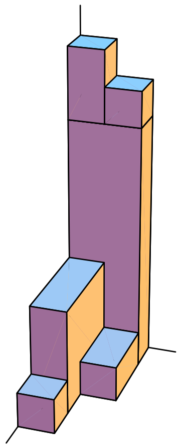

Figure 9. A front-top position.

Let $\Box F$ denote $F@(1,0,0)$ back-extended one unit to the wall, i.e., $\Box F = F \cap (1 \times 2 \times \Omega) + F@(1,0,0)$. Let $\mathcal{F}$ denote the set of all front positions $\tilde{F}$ such that $H(\Box\tilde{F}, 1 \times 2)$ is infinite.

Theorem 11.1 (Front-Top Theorem)**.** *Let a front-top position*

$$U = B + F@(1,0,0) + T@(0,0,\omega)$$

be given. If $H(\Box F, 1 \times 2)$ is finite then U is an N-position. If $H(\Box F, 1 \times 2)$ is infinite then

$$U \text{ is a P-position} \iff G_{\mathcal{F}}(F) = G(T).$$

(Here, as in the Side-Top Theorem earlier, $G_{\mathcal{F}}$ denotes restricted Grundy value.)

The proof is virtually identical to the Side-Top case. A more complicated Base-Top Theorem holds, where finite base material may be added in all directions around the brick, but this is all we need.

Defining an ordinal subtraction operation helps with the decompositions to be used in this section. For ordinals $b \le z$, let $b\backslash z$ be the unique ordinal t such that $b \uplus t = z$, where "$\uplus$" denotes ordinal addition. For instance, $\omega\backslash(\omega \cdot (c+1)+d) = \omega \cdot c + d$ while $\omega\backslash z = z$ if $z \ge \omega^2$. In Chomp terms, $b\backslash z$ is the amount of a z-high tower that extends above base height b.

Returning to subpositions of $2 \times 2 \times \Omega$, introduce the bird's-eye view notation

$$\begin{bmatrix} vz \\ uy \end{bmatrix} = (1 \times 1 \times u) + (1 \times 2 \times v) + (2 \times 1 \times y) + (2 \times 2 \times z)$$

where $u \ge v \ge y \ge z$ without loss of generality. Thus we are looking at the position down the third axis — the origin is at the lower left corner, the first axis goes right, the second up the page. Since this section involves many decompositions, extend the bird's-eye notation also to

$$\begin{bmatrix} v \\ u \end{bmatrix} = 1 \times 1 \times u + 1 \times 2 \times v$$

so that

$$\begin{bmatrix} vz \\ uy \end{bmatrix} = \begin{bmatrix} v \\ u \end{bmatrix} + \begin{bmatrix} z \\ y \end{bmatrix} @(1,0,0).$$

The finite P-positions in $2 \times 2 \times \Omega$ have a nice closed form, unlike the three-wide case. They are found by repeated application of the Fundamental Theorem, giving

$$\begin{array}{lll} \begin{bmatrix} v & 0 \\ v+10 \end{bmatrix} & \text{for } 0 \le v < \omega, & \begin{bmatrix} 10 \\ 11 \end{bmatrix}, \\ \begin{bmatrix} v & 1 \\ v+21 \end{bmatrix} & \text{for } 1 \le v < \omega, & \begin{bmatrix} 21 \\ 22 \end{bmatrix}, \\ \begin{bmatrix} v & 2 \\ v+32 \end{bmatrix} & \text{for } 2 \le v < \omega, & \begin{bmatrix} 32 \\ 33 \end{bmatrix}, \\ \text{etc.} & & \end{array}$$

We now classify the subpositions of $2 \times 2 \times \Omega$ into six types and characterize the P-positions for each type. For the classification, write uniquely $u = \hat{u} + \omega \cdot u_1 + u_0$ with $\hat{u}$ a multiple of ω^2 and u_1, u_0 finite; similarly for v, y, z.

Type A: $z < \omega$, $z = y$.

Type B: $z < \omega$, $z < y$.

Type C: $\omega \le z < \omega^3$, $\hat{z} = \hat{v}$, $y < \omega^\omega$ (the third condition actually follows from the first two).

Type D: $\omega \le z < \omega^3$, $\hat{z} < \hat{v}$, $y < \omega^\omega$.

Type E: $\omega \le z < \omega^3$, $y \ge \omega^\omega$.

Type F: $z \ge \omega^3$.

Further, a type C, D, or E position is *short* if $z < \omega^2$ and *tall* if $z \geq \omega^2$. A type D position is *thick* if $z\backslash u < \omega^2 \cdot 2$ and *thin* if $z\backslash u \geq \omega^2 \cdot 2$.

The following properties characterize the P-subpositions of $2 \times 2 \times \Omega$.

PA1: A type A position with $v < \omega$ is a P-position if and only if $u = v+z+1$.

PA2: A type A position with $v \geq \omega$ is a P-position if and only if

$$G(\omega\backslash u, \omega\backslash v) = z.$$

(This is the notation introduced earlier for Grundy values of two-wide positions.)

PB1: The P-positions of type B are $\left[\begin{smallmatrix} z+1 & z \\ z+1 & z+1 \end{smallmatrix}\right]$ for $z < \omega$.

PB2: If $z \geq \omega$ and h is finite then both $\left[\begin{smallmatrix} v & z \\ u & y \end{smallmatrix}\right] - \overline{(0,0,h+1)}$ and $\left[\begin{smallmatrix} v & z \\ u & y \end{smallmatrix}\right] - \overline{(1,1,h)}$ are N-positions.

Properties PA1 and PB1 follow from finite iteration of the Fundamental Theorem, as already observed. PB2 follows immediately from PB1 by noting that answering bite $\overline{(0,0,h+1)}$ with bite $\overline{(1,1,h)}$ and vice versa leaves a P-position.

PA2 follows from the Front-Top Theorem. The restricted Grundy value $G_{\mathcal{F}}(z,z) = z$ for the front portion $(1 \times 2 \times z)@(1,0,0)$ of a PA2 position is a consequence of the finite P-positions enumerated by PA1 and PB1.

The following property PA3 combines with PB2 to analyze all bites of finite height in superpositions of $2 \times 2 \times \omega$ (types C through F). With the one exception noted in PA3, all such bites leave N-positions.

PA3: If $z \geq \omega$ and h is finite then $\left[\begin{smallmatrix} v & z \\ u & y \end{smallmatrix}\right] - \overline{(0,1,h)}$ and $\left[\begin{smallmatrix} v & z \\ u & y \end{smallmatrix}\right] - \overline{(1,0,h)}$ are N-positions unless $\left[\begin{smallmatrix} v & z \\ u & y \end{smallmatrix}\right]$ is a short type C position with $u_1 = v_1$.

To see this, note that PA3 is a statement about bites that leave two columns of finite height. (In the case of bite $\overline{(0,1,h)}$, follow it by a mirror-reflection to restore our symmetry-breaking assumption $v \geq y$ before continuing the analysis.) The P-positions of this form are described by PA2: they satisfy $G(\omega\backslash u, \omega\backslash v) = h < \omega$. By two-wide Grundy value results, $G_2(\omega\backslash u, \omega\backslash v) < \omega$ if and only if $u < \omega^2$ and $u_1 = v_1$. This is precisely the exception in PA3; under any other conditions the bite leaves an N-position.

We next characterize P-positions of types C and D. The argument here is long and intricate, so the reader may just want to read through the results PC1 and PD1 and then skip onward to the discussion of type E.

The characterizations of type C and type D P-positions are proved by transfinite induction on Chomp positions, noting that all properties of a position are proved from properties of strictly smaller positions reached by biting into it. The induction hypothesis is the conjunction of all propositions PCn and PDn, including pending auxiliary ones. The basis of the induction is provided by propositions PAn and PBn.

Every type C position with $v \geq y$ can be written uniquely as

$$\begin{bmatrix} v\,z \\ u\,y \end{bmatrix} = B + T@(0,0,h),$$

where

$$h = \begin{cases} \omega & \text{if } \hat{z} = 0, \\ \omega^2 \cdot k & \text{if } \hat{z} = \omega^2 \cdot k,\ 0 < k < \omega, \end{cases}$$

$$B = \begin{bmatrix} hh \\ hh \end{bmatrix}, \quad T \in \left\{ \begin{bmatrix} vz \\ uy \end{bmatrix} : \hat{v} = 0 \right\}.$$

The B and T in this decomposition are called the base and top pieces.

Proposition PC1 now characterizes type C P-positions up to a restricted Grundy value calculation on the comparatively small top piece. After establishing PC1 and PD1, we will briefly consider some specifics of the calculation.

PC1: A type C Position as just decomposed is a P-position if and only if $T \in \mathcal{T}$ and $G_{\mathcal{T}}(T) = 0$, where

$$\mathcal{T} = \left\{ \begin{bmatrix} vz \\ uy \end{bmatrix} : u_1 > \max(v_1, y_1) \text{ and } \hat{u} = 0 \right\}.$$

Characterizing a type D P-position is accomplished by decomposing it into a base B, and two-wide front and top F and T. Specifically,

$$B = \begin{bmatrix} \omega^2\omega \\ \omega^2\omega \end{bmatrix}, \quad F = \begin{bmatrix} \omega\backslash z \\ \omega\backslash y \end{bmatrix}, \quad T = \begin{bmatrix} \omega^2\backslash v \\ \omega^2\backslash u \end{bmatrix}.$$

Thus $\left[\begin{smallmatrix} v & z \\ u & y \end{smallmatrix}\right] = B + F@(1,0,\omega) + T@(0,0,\omega^2)$.

PD1: A type D position as just decomposed is a P-position if and only if $G(F) = G(T)$.

Proving PC1 and PD1 requires several auxiliary propositions.

PC2: A type C position with $\hat{u} = \hat{v}$ and $u_1 = v_1$ is an N-position.

PC3: In a type C position with either $\hat{u} > \hat{v}$ or both $\hat{u} = \hat{v}$ and $u_1 > v_1$, every bite that intersects the base B leaves an N-position.

The proofs of PC2 and PC3 are briefly deferred.

PC4: If a short type C position is a P-position, then $u < \omega^2$.

For PC4, the Size Lemma gives $u \leq 1 + v + y + z$. In a short type C position, $\hat{v} = \hat{y} = \hat{z} = 0$ and the result follows.

PD2: If a type D position has $\hat{u} = \hat{v} = \hat{y}$, then it is an N-position.

To see PD2 note that the givens imply $G(T) < \omega^2$ in the type D decomposition, while $G(F) \geq \omega^2$. Thus PD2 follows from the inductive hypothesis on PD1.

PD3: If a type D position $\left[\begin{smallmatrix} v & z \\ u & y \end{smallmatrix}\right]$ satisfies $\hat{z} < \hat{y}$ and $\hat{v} < \hat{u}$ then

$$G(F) = G(T) \iff G(\tilde{F}) = G(\tilde{T}),$$

where (using row vectors to indicate decomposition in the other direction)

$$\tilde{F} = \left[\omega\backslash v \omega\backslash z\right], \quad \tilde{T} = \left[\omega^2\backslash u \omega^2\backslash y\right].$$

That is, for a type D position where neither the shortest nor the tallest column has the same ω^2-coefficient as either intermediate column, the decomposition to determine whether it is a P-position can be made in either direction. PD3 follows from 2-wide Grundy value results.

We now prove PC1 from these auxiliary propositions and then resume the deferred proofs of PC2 and PC3. The PC1 proof uses the decomposition of a type C position into top and base pieces T and B.

Consider the set of type C positions with base B of a particular height h, $B = 2 \times 2 \times h$. Let $\mathcal{T}(h)$ denote the set of top pieces of such positions for which every bite intersecting B leaves an N-position. Then it follows easily from the definition of restricted Grundy value that the P-positions of type C are precisely those of the form $B + T@(0,0,h)$, where $T \in \mathcal{T}(h)$ and $G_{\mathcal{T}(h)}(T) = 0$.

Now it follows from PC2 and PC3 that

$$\mathcal{T}(h) = \left\{ \begin{bmatrix} vz \\ uy \end{bmatrix} : \hat{v} = 0 \text{ and either } \hat{u} > 0 \text{ or } u_1 > \max(v_1, y_1) \right\}$$

for all values of h that occur. Thus $\mathcal{T}(h)$ is independent of h, and whether a type C position is a P-position depends solely on its top piece. Notice that $\mathcal{T}(h)$ (for any h) differs from the set $\mathcal{T}$ in PC1 only in allowing $\hat{u} > \hat{v}$ in the top piece. This is because the restricted Grundy value calculation does not in itself rule out the possibility $\hat{u} > \hat{v}$ in $\mathcal{T}(h)$. But PC4 excludes $\hat{u} > \hat{v}$ directly for short type C P-positions, and $\mathcal{T}(h)$ being independent of h excludes $\hat{u} > \hat{v}$ for tall type C P-positions. Thus for each h, $\mathcal{T}(h) \cap \{[\begin{smallmatrix} v & z \\ u & y \end{smallmatrix}] : \hat{u} = 0\} = \mathcal{T}$, completing the proof of PC1.

To prove PC2, let $k = G(h\backslash u, h\backslash v)$, where h is the height of base B in the type C decomposition; k is finite by two-wide results since $u_1 = v_1$ and $h\backslash u < \omega^2$. We show that PC2 describes an N-position by finding a bite that leaves a P-position. For a short type C position, the bite $\overline{(1,0,k)}$ leaves a P-position of type A by PA2. For a tall type C position, let $h = \omega^2 \cdot m$, $h' = \omega^2 \cdot (m-1)$, and $h'' = w \uplus h'$, where $\uplus$ denotes ordinal addition. Then bite $\overline{(1,0,h''+c)}$ leaves a P-position of type D by PD1, where c is the unique value satisfying $G(c,c) = k$. In the decomposition of this type D P-position, $F = \left[\begin{smallmatrix} h'+c \\ h'+c \end{smallmatrix}\right]$, $T = \left[\begin{smallmatrix} h'+(h\backslash v) \\ h'+(h\backslash u) \end{smallmatrix}\right] = \left[\begin{smallmatrix} \omega^2\backslash v \\ \omega^2\backslash u \end{smallmatrix}\right]$, and $G(F) = G(T) = \omega \cdot (m-1) + k$.

To prove PC3, we sketch the cases showing that every bite $\overline{(i,j,k)}$ intersecting the base in a type PC3 position leaves an N-position. PB2 and PA3 show this for all finite k, covering all bites in short type C positions, so consider only $k \geq \omega$ and tall type C positions; also, $k < h$ since the bite intersects the base. If $(i,j) = (0,0)$, we get a type C N-position by PC2. If $(i,j) = (1,1)$, we get a type D N-position by PD2.

If $(i,j) = (1,0)$, we get a type D position with a decomposition such that $G(T) > G(F)$, an N-position by PD1. To see why $G(T) > G(F)$ here, note that for some $m < h$ and $f < \omega^2$,

$$T = \begin{bmatrix} \omega^2 \cdot (h-1) + (h\backslash v) \\ \omega^2 \cdot (h-1) + (h\backslash u) \end{bmatrix} = \begin{bmatrix} \omega^2\backslash v \\ \omega^2\backslash u \end{bmatrix}, \quad F = \begin{bmatrix} \omega^2 \cdot m + f \\ \omega^2 \cdot m + f \end{bmatrix}.$$

We have $G(h\backslash u, h\backslash v) \geq \omega$ since $u_1 > v_1$ or $\hat{u} > \hat{v}$, and $G(f, f) < \omega$ since $f < \omega^2$, hence $G(T) = \omega\cdot(h-1)+G(h\backslash u, h\backslash v) \geq \omega\cdot h$, and $G(F) = \omega\cdot m+G(f, f) < \omega\cdot h$. Finally, the result for $(i, j) = (0, 1)$ follows symmetrically from $(i, j) = (1, 0)$ by interchanging v and y.

We now prove PD1, that a type D position is a P-position if and only if $G(F) = G(T)$ in its decomposition. We first show that every type D position with $G(F) \neq G(T)$ can be bitten to a position with $G(F') = G(T')$, where the primes mean "after the bite." If $G(F) > G(T)$, find a bite (known to exist by 2-wide Grundy value results) that bites F without touching T, leaving a type D position with $G(F') = G(T)$ and $T' = T$. If $G(T) > G(F)$, find a bite that bites T without touching F, leaving a type D position with $G(T') = G(F)$ and $F' = F$. Case analysis (details omitted) confirms this can be done.

To finish PD1 we sketch the cases showing that every bite $\overline{(i, j, h)}$ from a type D position with $G(F) = G(T)$ leaves an N-position. PB2 and PA3 show this for all finite h, so take $h \geq \omega$. If $(i, j) = (0, 0)$, any h leaving a type C position (reducing two, three, or four columns) leaves an N-position by PC2. Otherwise a type D position is left (by reducing one, two, or three columns). Reducing three columns to type D leaves an N-position by PD2. Reducing one or two columns to type D leaves an N-position by PD1, since $G(T') \neq G(T)$ and $G(F') = G(F)$. If $(i, j) = (1, 1)$ or $(i, j) = (1, 0)$, we get a type D N-position by PD1 since $G(F') \neq G(F)$ but $G(T') = G(T)$. This leaves the final case $(i, j) = (0, 1)$. If $\hat{h} = \hat{z} = \hat{y}$, we get a type C N-position by PC4. If $\hat{z} < \hat{h}$ and $h \geq y$, we get $G(T') \neq G(T)$ but $G(F') = G(F)$, an N-position by PD1. If $\hat{z} < \hat{h} = \hat{v}$ and $h < y$, we get $G(T') = G(\tilde{T})$ but $G(F') \neq G(\tilde{F})$, an N-position by PD3 and PD1 (recall that the "$\tilde{\ }$" notation arises from cleaving in the other direction). If $\hat{z} < \hat{h} < \hat{v}$ and $h < y$, we get $G(F') < G(F) \uplus \omega^2$ (since $h < y$ and $\hat{h} \leq \hat{y}$) and $G(T') \geq G(T) \uplus \omega^2$ (since $\hat{y} < \hat{v}$, by PD3), where $\uplus$ denotes ordinal addition and $F = \left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right]$, $T = \left[\begin{smallmatrix} v \\ u \end{smallmatrix}\right]$, $F' = \left[hz\right]$, $T' = \left[uy\right]$. Since $G(F) \uplus \omega^2 = G(T) \uplus \omega^2$ follows from $G(F) = G(T)$, this gives $G(F') < G(T')$, an N-position by PD1. Finally, if $\hat{h} < \hat{z}$ or if $\hat{h} = \hat{z} < \hat{y}$, we get $G(T') >= \omega^2$ (since $\hat{y} < \hat{u}$) and $G(F') < \omega^2$, an N-position by PD1.

We now briefly consider specific examples of type C P-positions, which so far have only been characterized up to a restricted Grundy value calculation on positions with 2×2 cross section and height less than ω^2.

Type C P-positions $\left[\begin{smallmatrix} v & z \\ u & y \end{smallmatrix}\right]$ with $z < \omega \cdot 2$ (i.e., with $z_1 = 1$) are well behaved. They are just those positions with $u_1 + z_1 = v_1 + y_1 + 1$ whose finite top pieces have Nim sum 0. Top pieces at the same level (coefficient of ω) must have their num sum component computed together, so they can be 2-wide Grundy values, or even a 3-column piece with L-shaped cross section (which is computed by the suitable restricted Grundy value function).

Type C P-positions with $z_1 = 2$ start to get interesting. For example, P-positions $\left[\begin{smallmatrix} \omega\cdot 2+b & \omega\cdot 2+d \\ \omega\cdot 3+a & \omega\cdot 2+c \end{smallmatrix}\right]$ are given by precisely those $Q = \left[\begin{smallmatrix} b & d \\ a & c \end{smallmatrix}\right] \in \mathcal{Q}$ that satisfy

$G_{\mathcal{Q}}(Q) = 0$, where

$$\mathcal{Q} = \left\{ \begin{bmatrix} bd \\ ac \end{bmatrix} : a \oplus b \equiv a \oplus c \equiv 1 \,(\text{mod } 3), d \le \min(b,c) \right\}.$$

This fact combines with 2-wide Grundy results and number theory to show that given b, c, and $d \le \min(b,c)$, then there is a P-position

$$\begin{bmatrix} \omega \cdot 2 + b\omega \cdot 2 + d \\ \omega \cdot 3 + a\omega \cdot 2 + c \end{bmatrix} \quad \text{for some } a$$

if and only if $b \oplus c$ is not a power of 2. When $b \oplus c$ is a power of 2, the Fundamental Theorem finds a similar P-position but with taller highest column,

$$\begin{bmatrix} \omega \cdot 2 + b\omega \cdot 2 + d \\ \omega \cdot 5 + a\omega \cdot 2 + c \end{bmatrix} \quad \text{for some } a.$$

Type C P-positions with $z_1 = 2$ satisfy the conditions

$$\begin{aligned} u_1 + z_1 &= v_1 + y_1 + 1 && \text{if } v_1 \text{ or } y_1 \text{ is odd or } v_1 = y_1 = 2, \\ u_1 + z_1 &= v_1 + y_1 + 3 && \text{if } v_1 \text{ and } y_1 \text{ are even.} \end{aligned}$$

Type C P-positions with $z_1 > 2$ become increasingly more complex. They can be found by finite calculation, but we don't prove this here. The principles justifying this claim are the subject of ongoing research.

For a type E positions $\left[\begin{smallmatrix} v & z \\ u & y \end{smallmatrix}\right]$, define the base and top to be

$$B = \begin{bmatrix} \omega^\omega \; \omega \\ \omega^\omega \omega^\omega \end{bmatrix}, \quad T = \begin{bmatrix} \omega^\omega \backslash v \quad 0 \\ \omega^\omega \backslash u \omega^\omega \backslash y \end{bmatrix},$$

so that $\left[\begin{smallmatrix} v & z \\ u & y \end{smallmatrix}\right] = B + T@(0,0,\omega^\omega) + (1 \times 1 \times (\omega \backslash z))@(1,1,\omega)$. Then

PE1: A type E position is a P-position if and only if $G(T) = \omega \backslash z$.

This is a variant of the Front-Top Theorem, with a higher, differently shaped top and an elevated "front." The winning strategy is: anwer any bite into the front or the top with a pairing strategy; the rest of the analysis in this section shows that any other bite leaves an N-position, so answer the bite appropriately.

Property PE1 isn't entirely satisfactory. It refers to the Grundy value of the top, which has an L-shaped cross-section; we haven't discussed such positions in general yet, though the section on P-ordered positions gives the special case result $G(\left[\begin{smallmatrix} a & 0 \\ a+b & a \end{smallmatrix}\right]) = a + b$.

Finally, type F is the simplest to characterize.

PF1: A type F position is a P-position if and only if it is $2 \times 2 \times \omega^3$.

This is shown by answering any bite $\overline{(i,j,h)}$ into $2 \times 2 \times \omega^3$ with another bite that gives a P-position. By properites PB2 and PA3, we may assume $h \ge \omega$.

If the bite is $\overline{(0,0,h)}$ with $\omega \le h < \omega^2$, then the answering bite is

$$\overline{(1, 0, G(2 \times (\omega \backslash h)))},$$

giving a front-top position of type A. If the bite is $\overline{(0,0,h)}$ with $\omega^2 \le h < \omega^2 \cdot 2$, then the answering bite is $\overline{(1,0,\omega + (\omega \backslash h))}$, giving a short thick position of type D with $F = T$ in the decomposition. If the bite is $\overline{(0,0,h)}$ with $\omega^2 \cdot 2 \le h$, then the answering bite is $\overline{(1,0,\omega \backslash h)}$, giving a tall thick position of type D with $F = T$. If the bite is $\overline{(1,1,h)}$, then the answering bite is $\overline{(1,0,\omega^2 + h)}$, giving a thin position of type D with $T = \begin{bmatrix} \omega^3 \\ \omega^3 \end{bmatrix}$, $F = \begin{bmatrix} a \\ \omega^2 + a \end{bmatrix}$ for some $a < \omega^3$, and $G(T) = G(F) = \omega^2$. If the bite is $\overline{(1,0,h)}$, then the answering bite is $\overline{(0,0,\omega^2 + h)}$, giving a thick position of type D. The final case of bite $\overline{(0,1,h)}$ is symmetric.

We conclude the section with some examples of P-positions of the different types.

Finite type A: $\begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & 2 \\ 5 & 2 \end{bmatrix}$, $\begin{bmatrix} 3 & 2 \\ 6 & 2 \end{bmatrix}$.

Type B: $\begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$, $\begin{bmatrix} 3 & 2 \\ 3 & 3 \end{bmatrix}$.

Front-top type A: $\begin{bmatrix} \omega & 0 \\ \omega & 0 \end{bmatrix}$, $\begin{bmatrix} \omega & 1 \\ \omega+1 & 1 \end{bmatrix}$. The latter has top $T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and front $F = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, with $G(T) = G_{\mathcal{F}}(F) = 1$.

Short type C: $\begin{bmatrix} \omega & \omega \\ \omega \cdot 2 & \omega \end{bmatrix}$, $\begin{bmatrix} \omega+1 & \omega \\ \omega \cdot 2+1 & \omega \end{bmatrix}$.

Tall type C: $\begin{bmatrix} \omega^2 & \omega^2 \\ \omega^2+\omega & \omega^2 \end{bmatrix}$, $\begin{bmatrix} \omega^2+1 & \omega^2 \\ \omega^2+\omega+1 & \omega^2 \end{bmatrix}$.

Thick type D: $\begin{bmatrix} \omega^2 & \omega \\ \omega^2 & \omega \end{bmatrix}$, $\begin{bmatrix} \omega^2 & \omega \\ \omega^2+1 & \omega+1 \end{bmatrix}$, $\begin{bmatrix} \omega^2 \cdot 2 & \omega^2 \\ \omega^2 \cdot 2 & \omega^2 \end{bmatrix}$ with $T = F = \begin{bmatrix} \omega^2 \\ \omega^2 \end{bmatrix}$.

Thin type D: $\begin{bmatrix} \omega^2 & \omega \\ \omega^2 \cdot 2 & \omega^2 \end{bmatrix}$, $\begin{bmatrix} \omega^2 \cdot 2 & \omega^2 \\ \omega^2 \cdot 3 & \omega^2 \cdot 2 \end{bmatrix}$, $\begin{bmatrix} \omega^2 \cdot 2 & \omega \\ \omega^2 \cdot 2 & \omega \cdot 2 \end{bmatrix}$ with $T = \begin{bmatrix} \omega^2 \\ \omega^2 \end{bmatrix}$ and $F = \begin{bmatrix} 0 \\ \omega \end{bmatrix}$, $\begin{bmatrix} \omega^3 & \omega \\ \omega^3 & \omega^2 \end{bmatrix}$ with $T = \begin{bmatrix} \omega^3 \\ \omega^3 \end{bmatrix}$ and $F = \begin{bmatrix} 0 \\ \omega^2 \end{bmatrix}$, $\begin{bmatrix} \omega^3 & \omega^2 \\ \omega^3 & \omega^2 \cdot 2 \end{bmatrix}$ with $T = \begin{bmatrix} \omega^3 \\ \omega^3 \end{bmatrix}$ and $F = \begin{bmatrix} \omega^2 \\ \omega^2 \cdot 2 \end{bmatrix}$.

Type E: $\begin{bmatrix} \omega^\omega & \omega \\ \omega^\omega & \omega^\omega \end{bmatrix}$, $\begin{bmatrix} \omega^\omega+\omega & \omega+1 \\ \omega^\omega+\omega & \omega^\omega \end{bmatrix}$, $\begin{bmatrix} \omega^\omega & \omega^2 \\ \omega^\omega+\omega^2 & \omega^\omega \end{bmatrix}$, $\begin{bmatrix} \omega^\omega+\omega^3 & \omega^2 \\ \omega^\omega+\omega^3 & \omega^\omega \end{bmatrix}$.

Type F: $\begin{bmatrix} \omega^3 & \omega^3 \\ \omega^3 & \omega^3 \end{bmatrix}$.

12. Open Questions

In all cases we have examined to date, any two Chomp positions whose sizes agree past some power of ω (i.e., both sizes are $\sum_{i \ge i_0} \omega^i \cdot a_i$ plus possibly different lower order terms) also have Grundy values agreeing past the same power of ω (i.e., both values are $\sum_{i \ge i_0} \omega^i \cdot b_i$ plus possibly different lower order terms). We conjecture that this property always holds. Specifically, for any Chomp position X and ordinal i, define the stratification $\operatorname{strat}(X, \omega^i)$ to be the Chomp position obtained from X by deleting all rectangles of size less than ω^i in the construction of size(X). For ordinal j, define $\operatorname{strat}(j, \omega^i) = \operatorname{strat}(1 \times j, \omega^i)$.

Conjecture 12.1 (Stratification Conjecture)**.** *For all Chomp positions X and Y and ordinals i, if strat(X, w^i) = strat(Y, w^i) then*

$$\mathit{strat}(G(X), w^i) = \mathit{strat}(G(Y), w^i).$$

We would like to know which sets of Chomp positions have computable subsets of P-positions.

A winning strategy for a set of Chomp positions can be viewed as a pair of functions, one which identifies P-positions and N-positions, and another which identifies winning moves by mapping each N-position to one or more P-positions reachable from it in one bite. A complete analysis of a set of P-positions would give a winning strategy for each subposition of any member of the set. For instance, the discussion of two-wide Chomp gave a complete analysis of $(1 \times 2 \times \Omega) + (\Omega \times 1 \times 1)$. The discussion of P-ordered positions such as $\omega \times \omega \times 2 \times 2$ gave a winning strategy, but not a complete analysis.

It is not difficult to show that the set of P-positions contained in any Chomp position with a finite part and either two one-wide transfinite stalks or one two-wide transfinite stalk is recursive.

We are confident that the set of P-positions in $3 \times \Omega$ is recursive (though this has not been fully proved) and consider it very likely that the set of P-positions in $(1 \times 3 \times \Omega) + (\Omega \times 1 \times 1)$ is recursive.

However we don't know the Grundy value $G(4 \times \omega)$, or even whether it is computable. Put another way, we don't know if $(1 \times 4 \times \omega) + (\omega^2 \times 1 \times 1)$ has a recursive set of P-positions. The sets

$$\{(a, b, g) : G((4 \times a) + (3 \times b) + (2 \times \omega)) = g\},$$
$$\{(a, b, g) : G((4 \times a) + (3 \times b)) = g\}$$

are recursive, but the set

$$L = \{g : G((4 \times a) + (2 \times \omega)) = g \text{ or } G(4 \times a) = g \text{ for some } a < \omega\}$$

is recursively enumerable and not known to be recursive. This is of interest because $G(4 \times \omega) = \operatorname{mex}(L)$. We know L contains all $g < 46$, we don't know if $46 \in L$, but if $46 \in L$ then $46 = G((4 \times a) + (2 \times \omega))$ for some $a > 480$. If $\operatorname{mex}(L)$ is infinite then we believe $G(4 \times \omega) = \omega \cdot 2$.

13. Conclusion

Extending Chomp from the naturals to the ordinals gives it a pleasing structure.

The main tools used here are size, the Fundamental Theorem, pairing strategies, "change of venue" arguments, and stratification. The Fundamental Theorem extends any position in one direction by any nonempty base to produce a P-position, leading to constructions that find Grundy values and create certain extensions with any given Grundy value. Pairing orthogonal summands with the same Grundy value creates a P-position, as does taking the cartesian product of an arbitrary P-position and any product of P-ordered P-positions. Stratification estimates a Grundy value by looking at the dominant piece of a position, while change of venue arguments switch strategies in response to bites that alter a position's large structure.

Results include the Grundy values of all two-wide positions, a list of all three-wide P-positions, and a list of some three-dimensional P-positions with a 2-by-2 base. In particular, the boxes $2 \times \omega$, $3 \times \omega^\omega$, $2 \times 2 \times \omega^3$, and $\omega \times \omega \times \omega \times 2 \times 2 \times 2$ are all P-positions.

We briefly touched on the computability of sets of P-positions, giving one example at the boundary of our current knowledge, a candidate for a small uncomputable set.

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