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# Restoring Fairness to Dukego

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Abstract. In this paper we correct an analysis of the two-player perfect-information game Dukego given in Chapter 19 of *Winning Ways*. In particular, we characterize the board dimensions that are fair, i.e., those for which the first player to move has a winning strategy.

## 1. Introduction

The game of Quadraphage, invented by R. Epstein (see [3] and [4]), pits two players against each other on a (generalized) $m \times n$ chess board. The Chess player possesses a single chess piece such as a King or a Knight, which starts the game on the center square of the board (or as near as possible if $mn$ is even); his object is to move his piece to any square on the edge of the board. The Go player possesses a large number of black stones, which she can play one per turn on any empty square to prevent the chess piece from moving there; her object is to block the chess piece so that it cannot move at all. (Thus the phrase "a large number" of black stones can be interpreted concretely as $mn-1$ stones, enough to cover every square on the board other than the one occupied by the chess piece.) These games can also be called Chessgo, or indeed Kinggo, Knightgo, etc. when referring to the game played with a specific chess piece.

Quadraphage can be played with non-conventional chess pieces as well; in fact, if we choose the chess piece to be an "angel" with the ability to fly to any square within a radius of 1000, we encounter J. Conway's infamous angel-vs.-devil game [2]. In this paper we consider the case where the chess piece is S. Golomb's Duke, a Fairy Chess piece that is more limited than a king, in that it moves one square per turn but only in a vertical or horizontal direction. In this game of Dukego, we will call the Chess player $\mathcal{D}$ and the Go player $\mathcal{G}$.

Berlekamp, Conway, and Guy analyzed the game of Dukego in [1], drawing upon strategies developed by Golomb. As they observe, moving first is never a disadvantage in Dukego; therefore for a given board size $m \times n$, either the first

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player to move has a winning strategy (in which case the $m \times n$ board is said to be *fair*), or else $\mathcal{D}$ has a winning strategy regardless of who moves first, or $\mathcal{G}$ has a winning strategy regardless of who moves first. In [1] it is asserted that all boards of dimensions $8 \times n$ $(n \geq 8)$ are fair, while $\mathcal{D}$ has a winning strategy on boards of dimensions $7 \times n$ even if $\mathcal{G}$ has the first move. This is not quite correct, and the purpose of this paper is to completely characterize the fair boards for Dukego. The result of the analysis is as follows:

> *The only fair boards for Dukego are the $8 \times 8$ board, the $7 \times 8$ board, and the $6 \times n$ boards with $n \geq 9$. On a board smaller than these, $\mathcal{D}$ can win even if $\mathcal{G}$ has the first move, and on a board larger than these, $\mathcal{G}$ can win even if $\mathcal{D}$ has the first move.*

We make the convention throughout this paper that when stating the dimensions $m \times n$ of a board, the smaller dimension is always listed first. With this convention, the winner of a well-played game of Dukego is listed in the table below: the entry $*$ denotes a fair board, on which the first player to move can win, while the entries "D" and "G" denote boards on which the corresponding player always has a winning strategy independent of the player to move first.

| | $n \leq 5$ | $n = 6$ | $n = 7$ | $n = 8$ | $n \geq 9$ |
|---|---|---|---|---|---|
| $m \leq 5$ | D | D | D | D | D |
| $m = 6$ | | D | D | D | $*$ |
| $m = 7$ | | | D | $*$ | G |
| $m = 8$ | | | | $*$ | G |
| $m \geq 9$ | | | | | G |

As a variant of these Quadraphage games, we can allow $\mathcal{G}$ to have both white (wandering) and black (blocking) stones, where the white stones can be moved from one square of the board to another once played. In this variant, $\mathcal{G}$ has the following options on each of her turns: place a stone of either color on an empty square of the board, move a white stone from one square of the board to any empty square, or pass. With a limited number of stones, it might be the case that $\mathcal{G}$ cannot completely immobilize the Duke, yet can play in such a way that the Duke can never reach any of the edge squares. For instance, it is shown in [1] that $\mathcal{G}$ can win (in this sense of forcing an infinite draw) against $\mathcal{D}$ on an $8\times8$ board with only three white stones, or with two white stones and two black stones, or with one white stone and four black stones. In this paper we show the following:

*If $\mathcal{G}$ has at most two white stones and at most one black stone, then $\mathcal{D}$ can win this variant of Dukego on a board of any size, regardless of who has the first move. On the other hand, if $\mathcal{G}$ has at least three white stones, or two white stones and at least two black stones, then the winner of this variant of Dukego is determined by the size of the board and the player with the first move in exactly the same way as in the standard version of Dukego (as listed in the table above).*

In the analysis below, we consider the longer edges of the board to be oriented horizontally, thus defining the north and south edges of the board, so that an $m \times n$ board (where by convention $m \leq n$) has $m$ rows and $n$ columns. Also, if one or both of the dimensions of the board are even, we make the convention that the Duke's starting position is the southernmost and easternmost of the central squares of the board.

## 2. How $\mathcal{D}$ Can Win

In this section we describe all the various situations (depending on the board size, the player to move first, and the selection of stones available to $\mathcal{G}$) in which the chess player $\mathcal{D}$ has a winning strategy.

We begin with the simple observation that if the Duke is almost at the edge of the board—say, one row north of the southernmost row—and $\mathcal{G}$ has (at most) one stone in the southernmost two rows, then $\mathcal{D}$ can win even if it is $\mathcal{G}$'s turn. For after $\mathcal{G}$ plays a second stone, one of the stones must be directly south of the Duke to prevent an immediate win by $\mathcal{D}$. By symmetry, we can assume that the second stone is to the west of the Duke, whereupon $\mathcal{D}$ simply moves east every turn; even if $\mathcal{G}$ continues to block the southern edge by placing stones directly south of the Duke on each turn, $\mathcal{D}$ will eventually win by reaching the east edge. If $\mathcal{D}$ is in this situation, we say that $\mathcal{D}$ has an Imminent Win on the south edge of the board (and similarly for the other edges).

It is now easy to see that $\mathcal{D}$ can always win on a $5 \times n$ board even if $\mathcal{G}$ has the first move. The Duke will be able to move either north or south (towards one the long edges) on his first turn, since $\mathcal{G}$ cannot block both of these squares on her first turn; and then $\mathcal{D}$ will have an Imminent Win on the north or south edge of the board, correspondingly. Of course, this implies that $\mathcal{D}$ can win on any $3 \times n$ or $4 \times n$ board as well, regardless of who moves first (Dukego on $1 \times n$ and $2 \times n$ boards being less interesting still). Similarly, $\mathcal{D}$ can win on a $6 \times n$ board if he has the first move, since he can move immediately into an Imminent Win situation along the south edge of the board (recalling our convention that the Duke starts in the southernmost and/or easternmost central square).

We can also see now that $\mathcal{D}$ can win on any size board if $\mathcal{G}$ is armed with only two white (wandering) stones. $\mathcal{D}$ selects his favorite of the four compass directions, for instance south, and pretends that the row directly adjacent to the Duke in that direction is the edge of the board. By adopting an Imminent Win strategy for this fantasy edge row, $\mathcal{D}$ will be able either to reach the east or west edge of the board for a true win, or else to move one row to the south for a fantasy win. But of course, repeating this Fantastic Imminent Win strategy will get the Duke closer and closer to the south edge of the board, until his last fantasy win is indeed a win in reality. Similarly, $\mathcal{D}$ can always win if $\mathcal{G}$ adds a single black (blocking) stone to her two white stones. $\mathcal{D}$ plays as above until $\mathcal{G}$ is forced to play her black stone (if she never does, then we have just seen that $\mathcal{D}$ will win); once the black stone is played, $\mathcal{D}$ rotates the board so that the black stone is farther north than the Duke, and then uses this Fantastic Imminent Win strategy towards the south edge.

We now describe a slightly more complicated situation in which $\mathcal{D}$ has a winning strategy. Suppose that the Duke is two rows north of the southernmost row and three rows west of the easternmost row, and that there are no stones in the two southernmost rows or in the two easternmost rows of the board, nor is there a stone directly east of the Duke (see the left diagram of Figure 1). Then we claim that $\mathcal{D}$ can win even if it is $\mathcal{G}$'s turn to move. For (referring to the second diagram of Figure 1) $\mathcal{G}$ must put a stone in square **A** to block $\mathcal{D}$ from moving into an Imminent Win along the south edge. $\mathcal{D}$ moves east to square **1**, whereupon $\mathcal{G}$ must put a stone in square **B** to block an Imminent Win along the east edge. But this is to no avail, as $\mathcal{D}$ then moves to square **2** to earn an Imminent Win along the south edge anyway. If $\mathcal{D}$ is in this situation described in Figure 1, we say that $\mathcal{D}$ has a Corner Win in the southeast corner of the board (and similarly for the other corners).

Notice that it is necessary for the shaded area to be empty for the Corner Win to be in force. If $\mathcal{G}$ has a stone in the second-southernmost row somewhere to the west of the Duke, then she can place a stone on square **2** to safely guard the south edge of the board. Similarly, if $\mathcal{G}$ has a stone in the second-easternmost row somewhere to the north of the Duke, then she can defend both edges of the board by playing a stone at square **A** on her first turn and one at square **3** on her second turn.

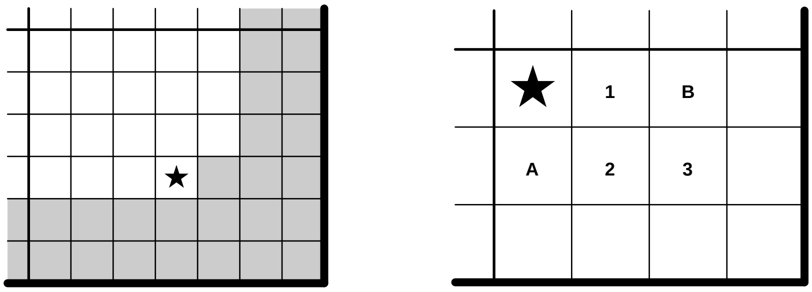


**Figure 1.** The Corner Win: If $\mathcal{G}$ has no stones in the shaded region (left), then $\mathcal{D}$ can win from position $\star$ (right).

We can now see that $\mathcal{D}$ can win on a $6\times 8$ board regardless of who has the first move, since the Duke starts the game in a Corner Win position. This implies that $\mathcal{D}$ can win on $6\times 6$ and $6\times 7$ boards regardless of who has the first move. Also, we can argue that $\mathcal{D}$ can win on a $7\times 7$ board even if $\mathcal{G}$ has the first move. By symmetry, we can assume that $\mathcal{G}$'s first stone is placed to the northwest of the Duke or else directly north of the Duke, whereupon $\mathcal{D}$ can move to the south and execute a Corner Win in the southeast corner of the board. Similarly, $\mathcal{D}$ can win on an $8\times 8$ board (and thus on a $7\times 8$ board as well) if he has the first move, since he can again gain a Corner Win situation by moving south on his first turn.

## 3. How $\mathcal{G}$ Can Win

We have now shown all we claimed about $\mathcal{D}$'s ability to win; it's time to give $\mathcal{G}$ her turn. We start by exhibiting strategies for $\mathcal{G}$ to win on a $7\times 8$ board and on a $6\times 9$ board with the first move. To begin, we assume that $\mathcal{G}$ possesses only three white (wandering) stones; since having extra stones on the board is never disadvantageous to $\mathcal{G}$, the strategy will also show that $\mathcal{G}$ can win with a large number of black stones and no white stones.

The squares labeled with capital letters in Figure 2 are the strategic squares for $\mathcal{G}$'s strategies on these boards. The key to reading the strategies from Figure 2 is as follows: whenever the Duke moves on a square labeled with some lowercase letters, $\mathcal{G}$ must choose her move to ensure that the squares with the corresponding capital letters are all covered. Other squares may be covered as well, as this is never a liability for $\mathcal{G}$. If the Duke is on a square with a + sign as well as some lowercase letters, $\mathcal{G}$ must position a tactical stone on the edge square adjacent to the Duke (blocking an immediate win) as well as having strategic stones on the corresponding uppercase letters. All that is required to check that these are indeed winning strategies is to verify that every square marked with lowercase

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | | | | |
| | A | ab+ | ab+ | ab+ | ab+ | B | |
| | ae+ | abe | ab | ab | abc | bc+ | |
| | ae+ | ae | be | b | bc | bc+ | |
| | ae+ | ade | de | bd | bcd | bc+ | |
| | E | de+ | de+ | D | cd+ | C | |
| | | | | | | | |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |
| | A | ab+ | B | bc+ | C | cd+ | D | |
| | ag+ | abg | bg | bc | ce | cde | de+ | |
| | ag+ | afg | fg | cf | ef | def | de+ | |
| | G | fg+ | fg+ | F | ef+ | ef+ | E | |
| | | | | | | | | |

**Figure 2.** $\mathcal{G}$'s strategies on a $7\times 8$ board (left) and on a $6\times 9$ board (right) with three white stones.

letters (counting + as a lowercase letter for this purpose) contains all of the letters, save at most one, of each of its neighbors, so that $\mathcal{G}$ can correctly change configurations by moving at most one white stone.

In the case of the 7×8 board, the Duke begins on the square marked "b" in the left-hand diagram of Figure 2 (recalling our convention about the precise beginning square for $\mathcal{D}$ on boards with one or both dimensions even), so $\mathcal{G}$'s first move will be to place a stone on the square marked **B**. Notice how $\mathcal{G}$ counters the instant threat of a Corner Win by $\mathcal{D}$ in the southeast corner, by playing her first stone on the second-easternmost column of the board at **B**, a move which also begins the defense of the north and east edges of the board against direct charges by the Duke.

In the case of the 6×9 board, we need to make another convention about the beginning of the game. The Duke begins on the more southern of the two central squares (the square marked "cf" in the right-hand diagram of Figure 2), and we stipulate that $\mathcal{G}$'s first stone be played at square **F**. We may also assume that $\mathcal{D}$'s first move is not to the north, for in this case $\mathcal{G}$ may rotate the board 180 degrees and pretend that $\mathcal{D}$ is back in the starting position, moving her white stone from the old square **F** to the new square **F**. Eventually, $\mathcal{D}$ will move east or west, to a square marked either "ef" or "fg", and at this point $\mathcal{G}$ begins to consult the right-hand diagram for her strategy, playing her second stone at square **E** or **G**, respectively.

$\mathcal{G}$'s strategy on the 7×8 board can be converted into a strategy using two white stones and two black stones without too much difficulty. As before, $\mathcal{G}$ begins by placing a white stone on the square **B**. The goal of $\mathcal{G}$ is to establish her two black stones on squares **A** and **C** (or on **B** and **E**), and then use her white stones both on the strategic squares **B** and **E** (or **A** and **C**, respectively) and as tactical stones blocking immediate wins. The strategic square **D** is only used to keep $\mathcal{D}$ in check until the two black stones can be established. The conversion is straightforward and we omit the details.

Less straightforward, however, is showing that $\mathcal{G}$ also has a strategy for winning on the 6×9 board with the first move, if she has two white stones and two black stones. We include in Figure 3 a full strategy for $\mathcal{G}$ in this situation. $\mathcal{G}$'s goal is to coerce $\mathcal{D}$ into committing to one of the two sides of the board, corresponding to diagram 2 or diagram 3 in Figure 3. Then she will be able to establish her black stones on squares **A** and **F** (or **D** and **F**), and use her white stones both as strategic stones on squares **G** and **H** (or **E** and **H**, respectively) and as tactical stones blocking immediate wins for $\mathcal{D}$.

$\mathcal{G}$ begins by reading the topmost diagram (labeled 1). Since the Duke starts on the square marked "f", $\mathcal{G}$ places a stone on the square **F**; since square **F** is shaded black in the diagram, this stone must be a black stone. When the Duke moves to a square marked with a number, $\mathcal{G}$ immediately switches to the corresponding diagram in Figure 3 and moves according to the Duke's current position. For example, suppose that the Duke's first move is to the east, onto the

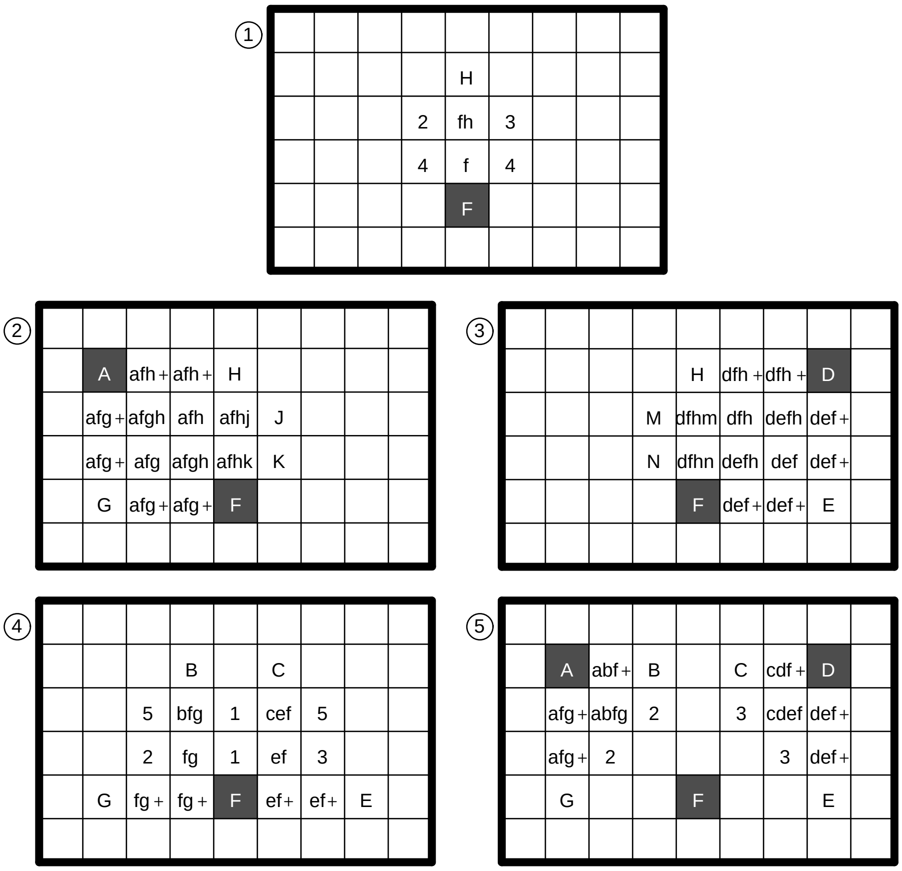


**Figure 3.** $\mathcal{G}$'s strategy on a $6\times 9$ board with two white stones and two black stones.

square marked **4**; in this case $\mathcal{G}$ switches to diagram 4, where the Duke's square is marked "ef", indicating that $\mathcal{G}$ must add a stone to square **E** (a white stone, since square **E** is not shaded black) in addition to her existing black stone on square **F**. We remark, to ameliorate one potential source of confusion, that diagram 5 is really a combination of two smaller diagrams, one for each side of the board; in particular, there will never be a need to have black stones simultaneously at squares **A**, **D**, and **F**.

Of course, any opening move for $\mathcal{G}$ other than placing a stone on square **F** would lead to a quick Imminent Win for $\mathcal{D}$ along the south edge. It turns out, though, that even if $\mathcal{G}$'s first move is to play a *white* stone at **F**, a winning strategy exists for $\mathcal{D}$ (assuming still that $\mathcal{G}$ has exactly two stones of each color at her disposal). A demonstration of this would be somewhat laborious, and so we leave the details for the reader's playtime.

To conclude this section, we are now in a position to argue that $\mathcal{G}$, armed with a large number of black stones or with three white stones or with two stones of each color, can win on a 7×9 board (and thus on any larger board) even if $\mathcal{D}$ has the first move. By symmetry we may assume that $\mathcal{D}$'s first move is either to the south or to the east. If $\mathcal{D}$ moves east on his first move, then $\mathcal{G}$ ignores the westernmost column of the board and adopts the above 7×8 strategy on the remainder of the board; alternatively, if $\mathcal{D}$ moves south on his first turn, then $\mathcal{G}$ ignores the northernmost row of the board and adopts the above 6×9 strategy on the rest of the board.

## 4. Afterthoughts

It is not quite true that we have left no stone unturned (pun unintended) in our analysis of Dukego. For instance, it is unclear exactly how many black stones $\mathcal{G}$ needs to win the original version of Dukego (no white stones) on the various board sizes; determining these numbers would most likely involve a fair amount of computation to cover $\mathcal{D}$'s initial strategies.

Somewhat more tractable, however, would be to determine how many black stones $\mathcal{G}$ needs to win when she also possesses a single white stone. The strategy given in [1] for $\mathcal{G}$ on an 8×8 board works perfectly well when $\mathcal{G}$ has one white stone (to be used tactically) and four black stones (to be placed strategically), and this is the best that $\mathcal{G}$ could hope for. On the other hand, $\mathcal{G}$'s strategies on the 7×8 and 6×9 boards as described above require five and seven black stones, respectively, to go along with the single white stone (and it requires some care to show that seven black stones suffice for the 6×9 board, since we need to account for $\mathcal{D}$ moving north on his first move).

We believe that $\mathcal{G}$ *cannot* win on either a 7×8 board or a 6×9 board with a single white stone and only four black stones. If this is the case, then the least number of black stones that $\mathcal{G}$ can win with, when supplemented by a white stone, would be five on a 7×8 board; but we don't know whether the analogous number on a 6×9 board is five, six, or seven.

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