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The Abstract Structure of the Group of Games

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ABSTRACT. We compute the abstract group structure of the group **Ug** of partizan games and the group **ShUg** of short partizan games. We also determine which partially ordered cyclic groups are subgroups of **Ug** and **ShUg**.

As in [2], let **Ug** be the group of all partizan combinatorial games, let **No** be the field of surreal numbers, and for G in **Ug**, let $L(G)$ and $R(G)$ be the Left and Right sections of G, respectively. If $L(G)$ is the section just to the left or right of some number z, we say that z is the *Left stop* of G, and similarly for $R(G)$ and the *Right stop*. Let **ShUg** be the group of all short games in **Ug**; that is, **ShUg** is the set of all games born before day ω, or of all games which can be expressed in a form with only finitely many positions. For games U and integers n, we write

$$nU = n.U = \begin{cases} 0, & & \text{if } n=0; \\ U+\cdots+U & (n \text{ summands}), & \text{if } n \text{ is a positive integer;} \\ (-U)+\cdots+(-U) & (-n \text{ summands}), & \text{if } n \text{ is a negative integer.} \end{cases}$$

Also, recall from [1, Chapter 8] the definition of Norton multiplication, for a game G and a game $U > 0$:

$$G.U = \begin{cases} \text{as above}, & \text{if } G \text{ equals an integer;} \\ \{G^L.U+U^L, G^L.U+(U+U-U^R) \mid & \\ \quad G^R.U-U^L, G^R.U-(U+U-U^R)\}, & \text{otherwise.} \end{cases} \tag{0–1}$$

Here, G^L, G^R, U^L, and U^R range independently over the left options of G, right options of G, left options of U, and right options of U, respectively. To define $G.U$, we must fix a form of G and sets of Left and Right options for U.

We will say that a subgroup $\mathfrak{X}$ of **Ug** has the *integer translation property* if it contains the integers and, whenever either Left or Right has a winning move in a sum $A_1+\cdots+A_n$ of games from $\mathfrak{X}$, not all integers, he also has a winning move in an A_j which is not equal to an integer.

Lemma 1. *The real numbers have the integer translation property.*

Proof. Let x be an integer and G be a nonintegral real number, and set $G' = \{G^L{+}x | G^R{+}x\}$. It will do to show that $G' = G{+}x$. Let $L = \sup_{G^L} R(G^L)$ and $R = \inf_{G^R} L(G^R)$. Then since G is a number, $L < R$, and G is the simplest number satisfying $L < G < R$ [2, Theorem 56]. But $\sup_{G^L} R(G^L{+}x)$ equals $L{+}x$, $\inf_{G^R} L(G^R{+}x)$ equals $R{+}x$, and $L{+}x < R{+}x$, so G' will be the simplest number satisfying $L{+}x < G' < R{+}x$. Since $L{+}x < G{+}x < R{+}x$, to prove $G' = G{+}x$, we only need to show that no simpler number than $G{+}x$ satisfies $L{+}x < G{+}x < R{+}x$. Suppose S is born before $G{+}x$ and satisfies $L{+}x < S < R{+}x$. Since $G{+}x$ is real, and hence born on or before day ω, S must be a dyadic rational, and obviously $L < S{-}x < R$; but also, since G is the simplest number between L and R, G must be born before or at the same time as $S{-}x$, so G is a dyadic rational, $G = (2m{+}1)/2^n$, say, for integers $n > 0$ and m. Then for G to be the simplest number between L and R, we must have $m/2^{n-1} \leq L < (2m{+}1)/2^n$ and $(2m{+}1)/2^n < R \leq (m{+}1)/2^{n-1}$. Therefore,

$$(m{+}2^{n-1}x)/2^{n-1} \leq L{+}x < (2m{+}1{+}2^n x)/2^n$$

and

$$(2m{+}1{+}2^n x)/2^n < R{+}x \leq (m{+}1{+}2^{n-1}x)/2^{n-1},$$

so $(2m{+}1{+}2^n x)/2^n = G{+}x$ is in fact the simplest number between $L{+}x$ and $R{+}x$. □

Theorem 2. *Suppose we have a subgroup $\mathcal{X}$ of* $\mathbf{Ug}$ *with the integer translation property, and such that every $H \in \mathcal{X}$ can be written in a form $\hat{H}$, where all positions of $\hat{H}$ are in $\mathcal{X}$. Fix a game $U > 0$ and sets of Left and Right options for U, and define $G.U$ for each G in $\mathcal{X}$ by using* (0–1) *with the form $\hat{G}$ for G and the given sets of options for U. Then, for all G and H in $\mathcal{X}$, $(G{+}H).U = G.U{+}H.U$, and if $G \geq H$, then $G.U \geq H.U$.*

Proof. [1, Chapter 8]. □

Let $\mathcal{X}$ be the subgroup of real numbers. We fix forms for each real number by letting each dyadic rational have its canonical form; that is,

$$0 = \{|\},$$

$$n = \{n{-}1|\} \qquad \text{and} \qquad -n = \{|-(n{-}1)\}$$

for integers $n > 0$, and

$$(2m{+}1)/2^n = \{m/2^{n-1}|(m{+}1)/2^{n-1}\}$$

for integers $n > 0$ and m. We let each real r that is not a dyadic rational have form

$$r = \{\lfloor r\rfloor, \lfloor 2r\rfloor/2, \lfloor 4r\rfloor/4, \ldots | \ldots, \lceil 4r\rceil/4, \lceil 2r\rceil/2, \lceil r\rceil\}.$$

By Lemma 1, the real numbers have the integer translation property, so we can now apply Theorem 2 to define $r.U$, where r is a real number and $U > 0$ is a game with specified sets of options.

Corollary 3. *For all real numbers r and s, and all games $U > 0$ with specified sets of options, $(r+s).U = r.U+s.U$, and if $r \geq s$, then $r.U \geq s.U$.*

Proof. Immediate. □

Lemma 4. *If $n \geq 2$ is an integer and $x \in \mathbf{No}$ is positive, then $G_{nx} = (2/n).\{2x|x\}-3x/n$ has order n. The nonzero multiples of G_{nx} all have Left stops of x/n or larger.*

Proof. By Corollary 3, G_{nx} has order dividing n. Let $U = \{2x|x\}$; then $U^L = U+U-U^R = 2x$. Observe that $0.U$ has Right stop 0 and $1.U$ has Left stop $2x$. It follows by induction that for all dyadic rationals d in $(0, 1)$, $d.U$ has Left stop $2x$ and Right stop 0, and then, for r real in $(0, 1)$, $r.U$ also has stops $2x$ and 0. Similarly, since $1.U$ has Right stop x and $2.U = 3x$ has Left stop $3x$, $r.U$ has stops $3x$ and x for all r real in $(1, 2)$, and $1.U = U$ clearly has stops $2x$ and x. This implies that $r.U$ is not a number for real r in $(0, 2)$, so $m.G_{nx} \neq 0$ for $m = 1, \ldots, n-1$. Our claim on the Left stop of the multiples of G_{nx} follows from the computation of the stops of $r.U$. □

No is the unique, up to isomorphism, universally embedding totally ordered field [2, Theorems 28 and 29]. We will prove a similar result about **Ug**.

An abelian group $\mathcal{X}$ is *universally embedding* if, given any abelian group $\mathcal{G}$ whose members form a set, and an embedding of a subgroup $\mathcal{H}$ of $\mathcal{G}$ in $\mathcal{X}$, the embedding can be extended to an embedding of $\mathcal{G}$ in $\mathcal{X}$. The members of such a group necessarily form a proper class.

Theorem 5. **Ug** *is a universally embedding abelian group.*

Proof. By Zorn's Lemma, it will do to show that if an abelian group $\mathcal{G}$ is generated by its subgroup $\mathcal{H}$ and its member $x \notin \mathcal{H}$, and there is an embedding j of $\mathcal{H}$ in **Ug**, then there is an embedding of $\mathcal{G}$ in **Ug** extending j. Let M be the set of integers m with $mx \in \mathcal{H}$. M is a subgroup of the integers. If $M = 0$, pick a large ordinal α, exceeding every element of $j(\mathcal{H})$, and embed $\mathcal{G}$ in **Ug** by sending x to α. Otherwise, M is cyclic, generated by $m > 1$, say. If $G_0 = j(mx)$, pick an ordinal $\beta > -G_0$ and sets of options for $G_0+\beta$, and set $G_1 = (1/m).(G_0+\beta)-\beta/m$. Obviously, $m.G_1 = G_0$. Let $\mathcal{X}$ be the subgroup of **Ug** generated by $j(\mathcal{H})$ and G_1, and let α be an ordinal such that $\alpha/2m$ exceeds every element of $\mathcal{X}$. Now we can map $\mathcal{G}$ to **Ug** by sending x to $G_1+G_{m\alpha}$, and this will be an embedding if $q.(G_1+G_{m\alpha}) \neq j(h)$ for all $h \in \mathcal{H}$ and $q \in \{1, \ldots, m-1\}$. But if $q.(G_1+G_{m\alpha}) = j(h)$, then $q.G_{m\alpha} \in \mathcal{X}$, and since $q.G_{m\alpha}$ has Left stop at least α/m, $\alpha/2m \not\geq q.G_{m\alpha}$. This contradicts our choice of α. Hence we have embedded $\mathcal{G}$ into **Ug**. □

Theorem 6. *Any universally embedding abelian group is isomorphic to* **Ug**.

Proof. Transfinite induction and a back-and-forth argument suffice to construct an isomorphism between any two universally embedding abelian groups. □

Call a subgroup $\mathcal{G}$ of **ShUg** *odd-closed* if whenever G is a short game, n is an odd integer, and $n.G \in \mathcal{G}$, then $G \in \mathcal{G}$. Call it *position-closed* if whenever H is a position of the canonical form of $G \in \mathcal{G}$, then $H \in \mathcal{G}$.

Theorem 7. *Position-closed subgroups of* **ShUg** *are odd-closed.*

Proof. By a remark in [2], if G is short and n is odd, G is an integral linear combination of positions of (any form of) $n.G$. ☐

Theorem 8. [2, Theorem 92] *All short games have infinite order or order a power of* 2.

We now determine the abstract group structure of **ShUg**. Let $\mathcal{D}$ be the additive group of dyadic rationals.

Theorem 9. **ShUg** *is isomorphic to the direct sum of countably many* $\mathcal{D}$*s and countably many* $\mathcal{D}/\mathbb{Z}$*s.*

Proof. We will find subgroups $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \ldots$ and $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \ldots$ of **ShUg** such that:

(i) Each $\mathcal{G}_l$ is a direct sum of $\mathcal{S}_0, \ldots, \mathcal{S}_l$.
(ii) $\bigcup_{l \geq 0} \mathcal{G}_l =$ **ShUg**.
(iii) Each $\mathcal{G}_l$ is position-closed (and hence odd-closed.)
(iv) Each $\mathcal{S}_l$ is isomorphic to either $\mathcal{D}$ or $\mathcal{D}/\mathbb{Z}$.

This will prove that **ShUg** is a countable direct sum of $\mathcal{D}$s and $\mathcal{D}/\mathbb{Z}$s; this proves the theorem, unless possibly only finitely many $\mathcal{D}$s or $\mathcal{D}/\mathbb{Z}$s appear in the sum. If there were only finitely many $\mathcal{D}/\mathbb{Z}$s, k, say, then the subgroup of **ShUg** of games of order 2 would be $(\mathbb{Z}/2\mathbb{Z})^k$, which contradicts the existence of infinitely many games $(*, *2, *3, *4, \ldots)$ of order 2. Also, the tinies $+_1, +_2, +_3, \ldots$, generate a subgroup of **ShUg** isomorphic to the direct sum of countably many $\mathbb{Z}$s. Since this subgroup is torsion-free, it will map to an isomorphic subgroup of the quotient of **ShUg** by its torsion subgroup. If there are only finitely many $\mathcal{D}$s in **ShUg**, k say, $\mathcal{D}^k$ will then have a subgroup isomorphic to $\mathbb{Z}^{k+1}$, which is impossible. Therefore, the direct sum must be as claimed.

We now proceed to the proof of 1–4. Well-order **ShUg** so that all options of H always precede H. (In this proof, by options and positions of a short game, we will always mean the options and positions of its canonical form.) We induce on l. Let $\mathcal{G}_0 = \mathcal{S}_0 = \mathcal{D}$. Clearly, 1, 3, and 4 are then true for $l = 0$. Otherwise, assume 1, 3, and 4 for $l = 0, \ldots, i$. Let q_i be the first short game not in $\mathcal{G}_i$, according to our order (so all options of q_i are in $\mathcal{G}_i$), and let r_i be an element of $q_i + \mathcal{G}_i$ with minimal order. Suppose that $2^b t r_i$ is in $\mathcal{G}_i$, where t is odd and $b \geq 0$. By odd-closure, $2^b r_i = z$, say, is in $\mathcal{G}_i$. Since $\mathcal{G}_i$ is 2-divisible, we see that there is y in $\mathcal{G}_i$ with $2^b y = z$. Then $2^b(r_i - y) = 0$, so $r_i - y$ has order dividing 2^b; by minimality of order, r_i also has order dividing 2^b, so $2^b r_i = 0$ and therefore $2^b t r_i = 0$. Hence $\mathcal{G}_i + \mathbb{Z} r_i$ is a direct sum. In fact, it is also position-closed; to see this, it will do to show that all positions of r_i are in $\mathcal{G}_i+\mathbb{Z}r_i$. Let $r_i = q_i+x$, $x \in \mathcal{G}_i$; all positions of r_i will equal $q'+x'$, where q' is a position of q_i and x' is a position of x. If q' isn't equal to q_i, then $q'+x'$ is already in $\mathcal{G}_i$; otherwise, $q_i+x' = r_i+(x'-x)$ is in $\mathcal{G}_i+\mathbb{Z}r_i$. This proves position-closure. Now for short games H, define

$$\phi(H) = \tfrac{1}{2}.(H+2N_H)-N_H$$

where N_H is the minimal nonnegative integer such that $H+2N_H > 0$. By our earlier remarks, $2\phi(H) = H$ for all H. Define

$$r_{ij} = \begin{cases} r_i, & j = 0, \\ \phi(r_{i(j-1)}), & j > 0. \end{cases}$$

Let $\mathcal{S}_{i+1} = \bigcup_{j\geq 0} \mathbb{Z}r_{ij}$. Evidently, $\mathcal{S}_{i+1}$ is isomorphic to $\mathcal{D}$ (if r_i has infinite order) or $\mathcal{D}/\mathbb{Z}$ (if r_i has order a power of 2.) Let $\mathcal{G}_{i+1} = \mathcal{G}_i+\mathcal{S}_{i+1}$. 4 is then certainly true. 1 will be true if the sum is direct. Let $2^k t r_{ij}$ be in $\mathcal{G}_i$, t odd, $k \geq 0$. By odd-closure, $2^k r_{ij}$ is in $\mathcal{G}_i$; if $k \leq j$, then $2^{j-k}2^k r_{ij} = 2^j r_{ij} = r_i$ is in $\mathcal{G}_i$, which is impossible. If $k > j$, then $2^k r_{ij} = 2^{k-j} r_i$ is in $\mathcal{G}_i$, and thus equals zero, since $\mathcal{G}_i+\mathbb{Z}r_i$ was direct. Hence $\mathcal{G}_i+\mathcal{S}_{i+1}$ is direct. For 3 to be true, we need $\mathcal{G}_{i+1}$ position-closed. It will do to show that for all j, all positions of r_{ij} are in $\mathcal{G}_{i+1}$. We induce on j. If $j = 0$, we have proved this above. Otherwise, we observe that any position of $\frac{1}{2}.K$, except $\frac{1}{2}.K$, is an integral linear combination of positions of K; therefore, any position of $r_{ij} = \phi(r_{i(j-1)})$ is either an integer translate of r_{ij} or an integer translate of an integral linear combination of positions of $r_{i(j-1)}$. The result then follows from the induction hypothesis.

This concludes the induction, proving that 1, 3, and 4 are true for all i. For 2, if some short game is not in $\bigcup_{l\geq 0} \mathcal{G}_l$, let K be the first such game, in our order. K will then eventually be chosen as some q_i; but $q_i \in \mathcal{G}_{i+1}$, which is a contradiction. This concludes the proof. $\square$

We would like to determine the abstract structure of **Ug** and **ShUg** as abstract partially ordered abelian groups. We have not done this, but we can approach the problem by first looking at cyclic subgroups of both groups. Any finite cyclic subgroup of **Ug** or **ShUg** must have all nonzero members incomparable with 0; so look at an infinite cyclic subgroup of either one, generated by G, say. We can't have $n.G > 0$ and $m.G < 0$ for positive m and n, since then $mn.G$ would have to be both positive and negative. Therefore either all positive multiples of G are positive or incomparable with 0, or all positive multiples of G are negative or incomparable with 0. By replacing G by $-G$ if necessary, we can assume that all positive multiples of G are positive or incomparable with 0. In this case, the set $\mathcal{S}$ of nonnegative integers n such that $n.G \geq 0$ must obviously be a submonoid of $\mathbb{Z}_{\geq 0}$. We will show that for $G \in$ **ShUg**, and hence also for $G \in$ **Ug**, all such submonoids can occur.

Lemma 10. *$F = \{2|-1, \{0|-4\}\}$ has $n.F$ incomparable with 0, for all nonzero integers n.*

Proof. First, we induce on n to show that $2+n.F \geq 0$ for all $n \geq 0$. If $n = 0$, this is clear. Otherwise, look at Right's first move. It can be to $1+(n-1).F$. Left has then won if $n = 1$; otherwise, he can respond on F to to $3+(n-2).F$, which is positive or zero by the induction hypothesis. Right's other first move is to $2+\{0|-4\}+(n-1).F$. In this case, Left should respond on $\{0|-4\}$, leaving $2+(n-1).F$, which is positive or zero by the induction hypothesis.

Now, it will do to show that both players have a winning first move from $n.F$ for all positive integers n. If $n > 0$, Left can move from $n.F$ to $2+(n-1).F$, and this is positive or zero by the above remarks. To show that Right has a winning first move, we induce on n. If $n = 1$, Right can move from F to -1 and win. If $n \geq 2$, Right's first move should be to $\{0|-4\}+(n-1).F$. If Left responds to $(n-1).F$, we have a good move by the induction hypothesis. Otherwise, Left must respond to $2+\{0|-4\}+(n-2).F$. If $n = 2$, Right can move to -2 and win. If $n = 3$, Right can move to $2+\{0|-4\}+\{0|-4\} = -2$ and win. Finally, if $n \geq 4$, Right can move to $-2+(n-2).F$. Left's only response is then to $(n-3).F$, and we can win this by the induction hypothesis. □

F has temperature 2 and mean value 0, so for all numbers $\varepsilon > 0$ and integers n, we have $-2-\varepsilon < n.F < 2+\varepsilon$.

Lemma 11. *All submonoids of $\mathbb{Z}_{\geq 0}$ are finitely generated.*

Proof. Let $\mathcal{S}$ be a submonoid of $\mathbb{Z}_{\geq 0}$. If it has no nonzero members, the result is obvious. Otherwise, let $n > 0$ be in $\mathcal{S}$, and for each $i > 0$, let $\mathcal{S}_i = \{j \in \{0, \ldots, n-1\} | j+ni \in \mathcal{S}\}$. Then $\mathcal{S}_1$, $\mathcal{S}_2$, … is a nondecreasing sequence of subsets of $\{0, \ldots, n-1\}$, so there must be some i_0 for which $\mathcal{S}_i = \mathcal{S}_{i_0}$ for all $i \geq i_0$. Then $\mathcal{S}$ is generated by $\mathcal{S} \cap \{1, 2, \ldots, n(i_0+1)-1\}$. □

Theorem 12. *If $\mathcal{S}$ is a submonoid of $\mathbb{Z}_{\geq 0}$, generated by positive integers a_1, …, a_n, then for all integers $m > 0$ and $M > 6$,*

$$G = \{M, M+a_1.F, M+a_2.F, \ldots, M+a_n.F | -M-F\}$$

will have $2m.G > 0$ if m is in $\mathcal{S}$, and $2m.G || 0$ otherwise.

Proof. Let $a_0 = 0$, and let $T = \{a_0, a_1, \ldots, a_n\}$. We make the following claims.

Claim 1. *For all integers b and nonnegative integers c, d, e, and q where $c+d+e \geq 2$, $V_{bcdeq} = (c+d+e).M+b.F-2c-d+e.\{0|-4\}+q.G$ is positive or zero.*

Proof of Claim 1. We induce on q. Let e' be the remainder when e is divided by 2. If $q = 0$, $V_{bcdeq} \geq b.F+(c+d+e).(M-2)+e'.\{2|-2\}$. But since $M > 6$, $(c+d+e).(M-2) > 8$, so this is positive. If $q > 0$, look at Right's first move in V_{bcdeq}. If it is in $\{0|-4\}$, we reply from G to M; we are then in a position equal to $V_{bcd(e+1)(q-1)}$, which is positive or zero by the induction hypothesis. If it is in F or $-F$, we reply from G to M; we are then in a position $V_{(b+\beta)(c+\gamma)(d+\delta)(e+\varepsilon)(q-1)}$, where β is 1 or -1, γ, δ, and ε are each 0 or 1, and $\gamma+\delta+\varepsilon = 1$. In any case, this is positive or zero by the induction hypothesis. The only other possibility for Right's first move is that it is in G. If $q \geq 2$, we reply from G to M. We are then at $V_{(b-1)cde(q-2)}$, which is positive or zero by the induction hypothesis. If $q = 1$, Right's move was to

$$\begin{aligned}&(c{+}d{+}e{-}1).M{+}(b{-}1).F{-}2c{-}d{+}e.\{0|{-}4\}\\ &\quad\geq (c{+}d{+}e{-}1).(M{-}2){+}(b{-}1).F{-}2{+}e'.\{2|{-}2\},\end{aligned}$$

and since $M > 6$, $(c{+}d{+}e{-}1).(M{-}2) > 4$, so

$$(c{+}d{+}e{-}1).(M{-}2){-}2{+}e'.\{2|{-}2\} > 0.$$

Since $(b{-}1).F$ is not negative, we have a position which is positive or incomparable with zero, which we can win. □

Claim 2. *For all nonnegative integers m and n, not both zero, there is a winning strategy for Left playing first in $2m.G{-}n.F$.*

Proof of Claim 2. We induce on m. We know the claim already if $m = 0$. Otherwise, Left should open to $M{+}(2m{-}1).G{-}n.F$. Right may respond on G, to $(2m{-}2).G{-}(n{+}1).F$; we have a good move from this by the induction hypothesis. If $n > 0$, Right may also respond on $-F$, to $M{+}(2m{-}1).G{-}(n{-}1).F{-}2$. In this case, we should respond on G to $2M{+}(2m{-}2).G{-}(n{-}1).F{-}2$, which is positive or zero by Claim 1. □

Claim 3. *For all integers b and nonnegative integers c, d, e, and q where $c{+}d{+}e \geq 1$, $W_{bcdeq} = -(1{+}c{+}d{+}e).M{+}b.F{+}2c{+}d{+}e.\{4|0\}{+}q.G$ is negative or zero.*

Proof of Claim 3. We induce on q. Let e' be the remainder when e is divided by 2. If $q = 0$, $W_{bcdeq} \leq b.F{-}M{+}(c{+}d{+}e).(2{-}M){+}e'.\{2|{-}2\}$. Since $M > 6$, $-M{+}(c{+}d{+}e).(2{-}M) < -10$, so this is negative. If $q > 0$, look at Left's first move in W_{bcdeq}. If it is in $\{4|0\}$, we reply from G to $-M{-}F$; we are then in a position equal to $W_{(b-1)cd(e+1)(q-1)}$, which is negative or zero by the induction hypothesis. If it is in F or $-F$, we also reply in G; we are then in a position $W_{(b+\beta)(c+\gamma)(d+\delta)(e+\varepsilon)(q-1)}$, where β is 0 or -2, γ, δ, and ε are each 0 or 1, and $\gamma{+}\delta{+}\varepsilon = 1$. This is negative or zero by the induction hypothesis. The only other possibility is that it is in G. If $q \geq 2$, we reply from G to $-M{-}F$, leaving a position of $W_{b'cde(q-2)}$, for some integer b'. This is negative or zero by the induction hypothesis. If $q = 1$, Left's move was to

$$\begin{aligned}&-(c{+}d{+}e).M{+}b'.F{+}2c{+}d{+}e.\{4|0\} \qquad \text{(for some integer } b'\text{)}\\ &\quad\leq (c{+}d{+}e).(2{-}M){+}b'.F{+}e'.\{2|{-}2\},\end{aligned}$$

and since $M > 6$, $(c{+}d{+}e).(2{-}M) < -4$, so this is negative. □

Claim 4. *For all nonnegative integers m not in $\mathcal{S}$, there is a winning strategy for Right playing first in $2m.G$.*

Proof of Claim 4. We open by moving from G to $-M-F$, and we continue doing this as long as Left's reply to our play is also in G. If this goes on for $2m$ moves, we will end up moving from some position $(b_1+\cdots+b_m-m).F$, where $b_1, \ldots, b_m \in T$. This cannot be zero as $m \notin \mathcal{S}$, so, by Lemma 10, we are moving from a game incomparable with 0 and will hence win. If this does not go on for $2m$ moves, Left responds in F or $-F$ at some point, leaving a position of the form $M+W_{bcdeq}$, where c, d, and e are nonnegative, $q \geq 1$, and $c+d+e=1$. We should respond by moving from G to $-M-F$. This leaves the position $W_{(b-1)cde(q-1)}$, which is negative or zero by Claim 3. $\square$

Claim 5. *For $m \in \mathcal{S}$, there is a winning strategy for Left playing second in $2m.G$.*

Proof of Claim 5. Since m is in $\mathcal{S}$, we can express m as a sum of the positive a_i's; pad this with zeroes to make a sum of exactly m terms, so that

$$0=(b_1-1)+(b_2-1)+\cdots+(b_m-1), \qquad b_1, \ldots, b_m \in T.$$

We may arrange these terms so that all initial partial sums are nonpositive. Then when Right opens, by moving from G to $-M-F$, our first response is on another copy of G, to $M+b_1.F$; if he moves on G again, our second response is from G to $M+b_2.F$, and so on. If this goes on for $2m$ moves, we will win, by moving to 0. Otherwise, Right responds on $-F$ at some point, leaving a position

$$(a+1).F+2q.G-2, \text{where } 1 \leq q < m \text{ and } a=b_1-1+\cdots+b_{m-q}-1<0. \quad (0\text{–}2)$$

We claim that we have a winning strategy from all positions (0–2). To prove this, induce on q. We should always respond to

$$M+(a'+2).F+(2q-1).G-2, \text{where } a'=b_1-1+\cdots+b_{m-q}-1+b_{m+1-q}-1.$$

Right must move from this position. If he moves on F or $-F$, respond from G to M; then the position is of the form $V_{b(c+1)de(2q-2)}$, where c, d, and e are nonnegative and $c+d+e=1$. This is positive or zero by Claim 1. If he moves on G and $q>1$, then his move is to a position (0–2) with q decreased by one, which we can win by the induction hypothesis. Finally, if he moves on G and $q=1$, then $a'=0$, so the position is now $F-2$, from which we move immediately to 0. $\square$

The theorem now follows immediately from Claims 2, 4, and 5. $\square$

References

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