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Exhaustive Search in the Game Amazons

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Abstract. Amazons is a young "real world" game that fulfills all defining constraints of combinatorial game theory with its original rule set. We present a program to evaluate small Amazons positions with a given number of amazons on game boards of restricted sizes with canonical combinatorial game theory values. The program does not use an analytical approach, instead it relies on exhaustive search in a bottom-up strategy. Here it is applied on all positions on game boards which fit into an underlying game board of size 11 by 2.

The results show that even under these restrictions Amazons offers a wide spread of game theoretic values, including some very interesting ones. Also the canonical forms of the values can be very complicated.

1. Introduction

"The Game of the Amazons" — or simply "Amazons" — is a relatively new star on the sky of abstract strategic two-person games, invented in 1988 by Argentinian Walter Zamkauskas. Compared with other non-classical games it seems to be well done — easy rules, many interesting choices for each move, a big range of different tactics and strategies to employ, challenging even after many games played. It has yet to prove if it can stand the comparison to classics like chess and go.

Amazons has already built a solid base of players and followers. It has especially produced interest in programmers, leading to several Amazons programming competitions including a yearly world-championship and a tournament at the Mind Sports Olympiad 2000, see for example [Iida and Müller 2000] and [Hensgens and Uiterwijk 2000]. Already there are about a dozen different Amazons programs competing at these tournaments. Some of these programs play strong enough to beat average human players easily.

In contrast to most games of this type combinatorial game theory can be applied directly to Amazons without any changes to the rules. Amazons also often decomposes into at least two independent subgames when the endgame is played out, making the application of combinatorial game theory worthwhile.

The first analysis of Amazons with the means of combinatorial game theory was done by E. Berlekamp, who looked at positions with one amazon per player on boards of size 2 by n [Berlekamp 2000]. Berlekamp calculated only the thermographs for these positions, but even these proved to be quite difficult to analyze.

The idea presented in this article is to produce a database of canonical values in Amazons, in order to find as many different values as possible, to look, how complex they can become and to search for patterns to generalize. A lot of this work still has to be done, but up to now the results show that Amazons is not disappointing our expectations: The canonical forms are complex, offer a wide range of values, and patterns seem to be hard to find.

The approach used to get the canonical data is one of brute computational force, calculating the values of all possible positions from small to big boards. The program searched in 2 by n boards, to walk alongside Berlekamp's thermographic results, checking its own correctness while filling in the large and empty canonical wastelands within the thermographical results.

2. Amazons

2.1. Rules. Amazons is played by two players, White and Black, who interchange moves. White moves first, but according to the standards in combinatorial game theory we assign positive numbers to Black and negative numbers to White.

Amazons is played on a game board of ten times ten squares, with four amazons per player.

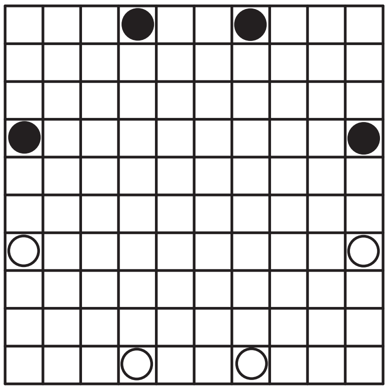

Figure 1. Amazons starting position.

A move consists of two parts, moving an amazon and shooting an arrow with that same amazon. Amazon and arrow both move like a chess queen, diagonally, horizontally or vertically as far as the player wants and no obstacle blocks the way. Neither amazon nor arrow may move onto or over a square which holds another amazon or arrow of either color.

Arrows never move again, working as blocks for the rest of the game, thereby eliminating their square. This limits the game to a maximum of 92 moves. The player who moves last wins.

2.2. Course of a Game. A game of Amazons unfolds in a typical manner. While more and more squares are blocked by arrows, the mobility of the amazons is restricted and the board is divided by firm walls of arrows (marked by crosses in the figures) into several independent rooms. The game practically comes to an end when all these independent, cut-off rooms contain only amazons of one player who thereby normally claims as many more moves as the rooms controlled by his amazons have free squares.

Figure 2 shows an end-position of Amazons. White has no chance to break into the large middle area of the game board if Black plays carefully. Therefore White resigned in this position, because Black will be able to claim larger territories with more remaining moves.

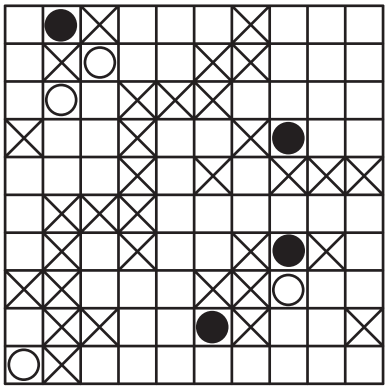

Figure 2. White to move resigns.

In rare cases a room is structured in such a way that the inhabiting amazons cannot move to all of the remaining squares and therefore some moves are lost. The smallest and most typical example of such a "defective" shape is given in Figure 3. Wherever the amazon moves, she has to shoot back on and block

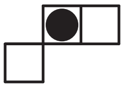

Figure 3. One move left for Black.

the middle square, thereby losing the possibility to move to the third remaining square.

More thorough and extensive information about defective shapes in Amazons can be found in [Müller and Tegos 2002].

2.3. Amazons and Combinatorial Game Theory. It is easy to see that Amazons fulfills all conditions that define a combinatorial game as given in *Winning Ways* [Berlekamp, Conway and Guy 1982]:

(i) There are exactly two players.
(ii) The game has a finite number of distinctive positions.
(iii) From every position each player has a finite set of moves which lead to different other positions.
(iv) Both players take turns moving.
(v) The first player unable to move loses.
(vi) The game ends after a finite number of moves.
(vii) Amazons is a game of complete information.
(viii) There is no element of chance.

Additionally Amazons does the favor to decompose into a sum of several smaller games, making the application of combinatorial game theory not only possible but also very useful. This decomposition happens in practically all Amazons games, although in some cases in a trivial way, where only one of the smaller games contains amazons of both players, therefore yielding a sum of one hot game plus several integers. In other cases the barrier between two parts of the game board is not complete but very difficult to cross, making the application of combinatorial game theory practically possible despite the imperfect boundary.

3. Analyzing Amazons

3.1. General Idea. The approach used here to analyze Amazons is to build a complete database of canonical combinatorial game theory values for Amazons positions within a certain set of small game boards. In order to be able to crosscheck the electronically achieved data with the results from E. Berlekamp it was decided to calculate the game theoretic values of all game boards up to at least size 2 by 11 with one amazon per player.

Because the goal was to get the values for the complete set of game boards, a bottom-up strategy was used, first evaluating all possible Amazons positions for game board size m before proceeding to the positions of game board size $m + 1$. This way the results from smaller game boards were used to determine the values of the larger game boards.

3.2. The Algorithm. The strategy implemented is straightforward. Construct every game board, every position, make every move, look-up the values for the resulting smaller positions, calculate the result for the actual position by eliminating dominated and reversing reversible options. The algorithm in Figure 4 shows this in more detail but may also cloud the essence of the programming strategy.

Obviously this algorithm needs many subroutines, for example to construct the gameboards, set up the positions, make the moves and so on. Additionally it also needs an extensive software package to resolve the calculations of combinatorial game theory and thermography.

4. Technical Aspects

4.1. Programming Language. To choose Java as programming language may be surprising. This was done for personal reasons, but also because the problem is exponential anyway, so even a large speed advantage of another language over Java (such as C) probably would not have allowed to calculate much further. As it turned out the chosen size we wanted to reach — 22 — was just in the range where the choice of the language mattered and the computation with JDK 1.1 took nearly unacceptable 4 months the first time. On a rerun with JDK 1.3 that was necessary because of errors in the code the time dropped to approximately one month.

On the other hand Java lended structure and stability to the program. This greatly eased the programming task itself. Added to this Java is platform independent, which turned out to be very useful several times. Altogether the decision for Java was not regretted.

4.2. Machinery. More of a problem than speed was space. A lot of the programming work went into saving space, as all of the data for smaller game boards and positions had to be available in RAM where it was constantly requested for calculating the game theoretic values of positions on bigger game boards. 500 megabyte of RAM were just enough for the task. Most of the object structure typical for Java of the data had to be destroyed for space saving reasons and was only temporarily rebuilt when the data was needed for calculation.

The first, error prone run of the program was conducted on a Pentium III with 500 megahertz processing speed and 512 megabyte RAM. The rerun was again done on a Pentium III with 500 megahertz, but this time with 1.5 gigabyte of RAM.

4.3. Statistics. Table 1 shows the number of different game boards for the corresponding number of squares on the game board. Mirrored, turned or shifted boards are already excluded. The numbers behave as expected, rising rapidly up to a maximum of 14549 different boards for 14 squares and dropping back to for $m = 1$ to 22 (size of game boards, number of squares)

construct the set B_m of all diagonally connected game boards with m squares that fit into an underlying board of size 2 by 11
eliminate all boards in B_m which are identical except for being mirrored or turned or shifted
for every $b \in B_m$

construct the set P_b of all possible positions on b with one black amazon or one black and one white amazon, excluding positions which are identical except for the colors of the amazons[1]
for every $p \in P_b$

construct the set L_p and the set R_p of all possible moves for Black and White in p

set the cgt-value $\mathrm{v}(p)$ of p to 0

for every $mo \in L_p$

determine the set P_{mo} of all the resulting Amazons positions after move mo[2]

set the cgt-value $\mathrm{v}(mo)$ to 0[3]

for every $r \in P_{mo}$

look-up the cgt-value $\mathrm{v}(r)$ of r[4]

calculate $\mathrm{v}(mo) = \mathrm{v}(mo) + \mathrm{v}(r)$

insert $\mathrm{v}(mo)$ as new left option of $\mathrm{v}(p)$

for every $mo \in R_p$

determine the set P_{mo} of all the resulting Amazons positions after move mo

set the cgt-value $\mathrm{v}(mo)$ to 0

for every $r \in P_{mo}$

look-up the cgt-value $\mathrm{v}(r)$ of r

calculate $\mathrm{v}(mo) = \mathrm{v}(mo) + \mathrm{v}(r)$

insert $\mathrm{v}(mo)$ as new right option of $\mathrm{v}(p)$

compute the canonical form[5] and the thermographic data of $\mathrm{v}(p)$ and save them to disc together with the corresponding moves in L_p and R_p

Figure 4. The basic algorithm. Notes are on the next page.

[1] This step is really processed twice, not once: First for positions with only one black amazon, then for positions with one black and one white amazon.
[2] This may be more than one position because the board may split apart after burning out one square.
[3] $v(mo)$ is the sum of the cgt-values of the resulting positions after move mo. So it is not just the value of one single position.
[4] This value exists because it was already computed before — remember that the size of the board has shrunk. If the size of the board is zero, its cgt-value is 0, too.
[5] Calculating the canonical form especially means eliminating dominated and reversing reversible options.

Figure 4 (notes).

one board with 22 squares as there obviously is just one board with 22 squares that fits into an underlying 11 by 2 board.

size of game boards	number of game boards	number of positions
2	2	2
3	4	12
4	11	66
5	20	200
6	51	765
7	108	2268
8	267	7476
9	609	21924
10	1485	66825
11	3500	192500
12	7429	490314
13	12056	940368
14	14549	1323959
15	12792	1343160
16	8333	999960
17	3961	538696
18	1391	212823
19	340	58140
20	61	11590
21	6	1260
22	1	231
sum	66976	6212539

Table 1. Number of game boards and Amazons positions.

The table also shows the number of different Amazons positions on all of these game boards for every board size for one black and one white amazon. Cases where only the positions of the two amazons are exchanged are excluded, therefore the number of different positions for a board of size m is $\frac{1}{2}m(m-1)$. Not excluded are therefore positions which are symmetrical because of symmetries within the board itself, for instance the two positions of Figure 5.

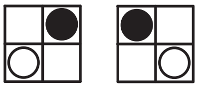

Figure 5. Two "different" positions on a symmetric game board.

The computing time to calculate the game theoretic value for a position also behaved as expected, growing exponentially with the size of the game board, as did the coding space necessary to save the results.

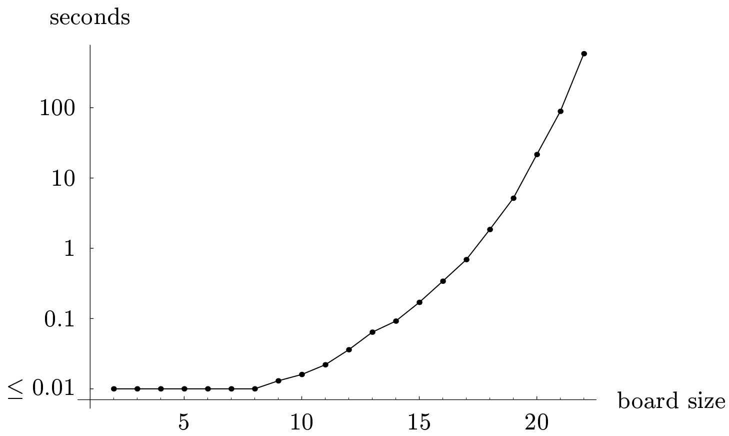

Figure 6. Average computing time per Amazons position, in seconds, as a function of the game-board size.

There are different reasons for the sharp growth in computing time. One is obviously the size of the game board, as larger game boards allow more moves to be made and therefore to produce more complex game theoretic values. These values provide the second reason, as these more complex values for size m, say, now appear in the calculation of the values for size $m+1$, slowing down the process and making the resulting values for size $m+1$ even more complex.

A third reason is the underlying game board of size 11 by 2. The boards of size $m+1$ not only have one more square than the boards of size m, these boards are also more compact. For boards up to size 16 or so it is usual to be stretched thinly, like in Figure 16. Boards with such a shape don't allow many moves by the amazons no matter how big they are.

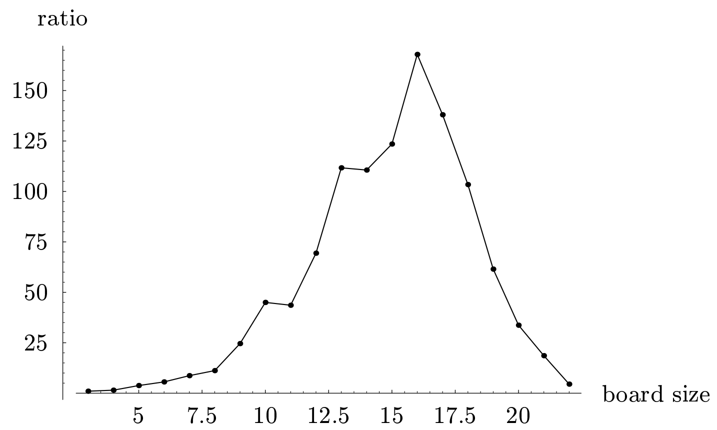

Figure 7. Rising complexity of compact game boards: ratio worst-case/average-case of number of subgames.

The sole game board with 22 squares is an extreme opposite: It is a simple rectangle and every amazon has plenty of different moves available. This behaviour can be clearly seen in Figure 7 which depicts the ratio between the maximum number of subgames for any Amazons position for a given board size with the average number of subgames for all positions on all boards of this size.[1]

What happens in Figure 7 is the disappearance of the average case, which is the thinly stretched game board for smaller sizes. From size 17 onwards all the boards get more compact and therefore the previously worst case now slowly becomes the average case.

5. Game-Theoretic Results

5.1. Amazons is Difficult. E. Berlekamp already proved that the depth of the canonical subgame-tree of an Amazons position can be approximately three quarters of the size of the game board [Berlekamp 2000]. Many of the results this program calculated show that Amazons positions can also have a very broad canonical subgame-tree. One example is the canonical form of one of the positions of Berlekamp's $1000 puzzle (cited in [Berlekamp 2000]) in Figure 8.

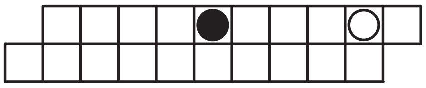

Figure 8. Second game of Berlekamp's G4G3 Problem.

[1]The number of subgames of a position is the number of canonical options in the game theoretic value of a position until a numeric value is reached.

{{14|{11|{8|-8},{{9|{4|{1|-1}},{6|{3|-3},{{2,{2|0}|0,{1|{0|-1},-1}},1.5,{2|0}|-1,{0|-2}}},3},
6|-6,{-3,{{1|-1}|-4}|-7}},{{9,{10|{7|{4|-4},{5,{6|4}|-5,{*,{5|-5}|-6}}},6}|-1},{9,{10|{7|{4|-4
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{0|-2}},*|-1}},{{1|1}|{*,{1,{2|0}|{0|-2},-1}|-1}},{2|-2}}},3}},{9|7},{8,{10|8}|8}|{{3|1},2|{1
|-1},0},{4|-2}},{9,{10|8}|{8|{5|{2|-2},{{1|1}}|{*,{1,{2|0}|{0|-2},-1}|-1}},{{{1|1},1|1}|{*,{
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8.75|4,{5|{4|{1|-1},{{5|{2,{3|{{1|*},0.5|{*|-1},-0.5},*}|{{{1|*},0.5|{*|-1},-0.5},*|-3},-2},{2|-2
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,*|-3},-2},{2|-2},{{{2|0},1|*,0},1|{{1|-1}|-1}}},4,{5|{2|0,{{1|*},0.5|{*|-1},-0.5}},1}|-4,{{{{1
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|{1|-1}},{1|-1}|{{0.5|0},0|-1,{0|-2}},{{1|-1}|-1}},*,{0.75|-1}}|-1,{0|-2}},{{3|1},{2,{4|2}|2},2
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-0.5},0|-2},-1|-5}}}}|{1,{2|0,{{3|{{1|*},0.5|{*|-1},-0.5},*},2|-2,{{{1|*},0.5|{*|-1},-0.5},*|-3}}}|
-1.5,{0,{{2|-0.5},{{1|1},1|0},0.5|-0.5}|-2,{0|-4}},{0|-4}},{{0,*|0}|-1}}|{-1,{0|-2}|{{0|-2},-1|-5,
{-4|-6}},{-3|-5},-4}},{{{7|{4|{4,{5|{2|{{1|*},0.5|{*|-1},-0.5},0},1}|{{{3|{{1|*},0.5|{*|-1},-0.5}
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|{2,{3|{{1|*},0.5|{*|-1},-0.5},*}|{{{1|*},0.5|{*|-1},-0.5},*|-3},-2},{2|-2},{{{2|0},1|*,0},1|{{1
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-3|-7}},{{{2|0},{0|{0|-0.5}}|-3},{0.5,{1,{1|1}|0},{2|-0.5}|-0.5},0,{{1,{2|0}|-0.25},{2|0},0|-1}|
-6,{{{1|-1},{2,{3|{{1|*},0.5|{*|-1},-0.5},*}|{{{1|*},0.5|{*|-1},-0.5},*|-3},-2},0|-4},-3|-7}},-1},
-0.75},{2|0,{1|{0|{{*,{{2|0},1|-1,{0|-2}}|-3},-2|-6},-5,{{{{3|{{1|*},0.5|{*|-1},-0.5},*},2|-2,{{{
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|-0.5},0,{{1,{2|0}|-0.25},{2|0},0|-1}|-6,{{{1|-1},{2,{3|{{1|*},0.5|{*|-1},-0.5},*}|{{{1|*},0.5|{*
|-1},-0.5},*|-3},-2},0|-4},-3|-7}},-1}},1}

Figure 9. Canonical form of the position of Figure 8.

This position has just about 4000 subgames. But it already has a relatively deep subgame tree, which with its longest path needs 15 steps to reach a number. Black in this position has 2 different canonical options, White has 5.

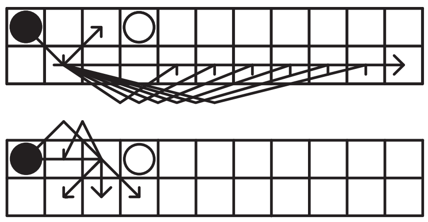

Figure 10. A position with 100833 subgames.

Another position with 100833 subgames is shown in Figure 10. The arrows show all the 13 different canonical moves for Black in this position. The canonical subgame-tree has depth 16, which is only very slightly higher than the lower bound for the maximal depth found by Berlekamp.

In Figure 11 you can see a symmetrical position which gives each amazon 12 different canonical options, together 24.

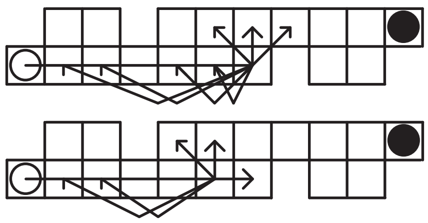

Figure 11. Twelve canonical options for each player.

5.2. Thermographic Results. While the canonical form of Amazons positions on some boards may be nearly impossible to handle, the thermographic data of all computed positions remains simple. Just consider the thermographs for each of the three positions given in Figures 8–11, shown in Figures 12–14.

In fact not one of these more than 6 million Amazons positions has a thermograph where either of its two walls has more than 10 sectors, that is, parts with changing slope, and this includes subzero thermography (see [Berlekamp 1996]), which has no special information value. The average number of sectors is around 2.5 per wall. And while the complexity of the canonical data grows exponentially with the size of the game board, the complexity of the thermographs remains constant from boardsize 15 upward, the worst case growing not at all.

All the thermographs calculated by the program, up to the time of writing, were compared and found to be the same as those found by E. Berlekamp

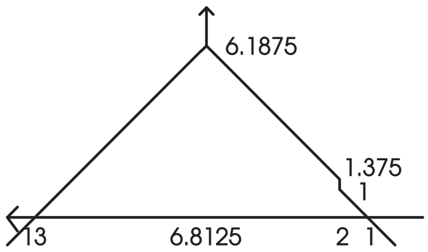

Figure 12. Thermograph of game in Figure 8.

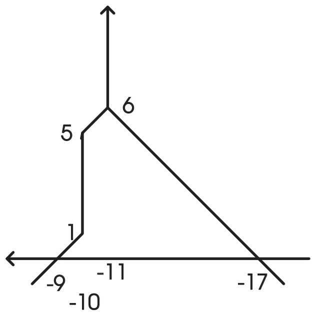

Figure 13. Thermograph of game in Figure 10.

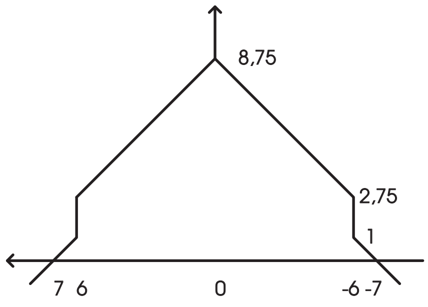

Figure 14. Thermograph of game in Figure 11.

[Berlekamp 2000]. Many canonical forms obviously are out of reach for manual confirmation or testing.

5.3. Special Cases of Interest. This computer aided approach to analyze Amazons also revealed some interesting game theoretic values that were previously unknown in Amazons.

5.3.1. $\frac{7}{8}$ The smallest fraction known before running the program was a quarter, shown in Figure 15.

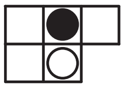

Figure 15. Position with value $\frac{1}{4}$.

The highest denominator in a numeric value found by this program is sixteen. Figure 16 shows the smallest board within this setting that produces a sixteenth.

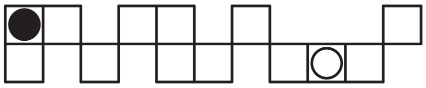

Figure 16. Position with value $-\frac{1}{16}$.

The smallest position which yields an eighth is shown in Figure 17. Interestingly this position does not have value $\frac{1}{8}$ but $\frac{7}{8}$. And the canonical moves of this position are really intriguing.

A game of Amazons normally follows a certain parity pattern: When the first player is to move, there is an even number of squares left. The first player moves and burns out a square, leaving an odd number of squares for the second player, who after his move and shot leaves again the first player with an even number of squares and so on. In this position Black with every move and shot closes off an additional square, thereby changing parity.

This position is also a good example to illuminate the complexity of Amazons. Changing a small detail can alter the value of a position quite a bit. In Figure 17 the dangling leftmost square of the board seems to be irrelevant: It is never moved upon, never shot into, it is just being cutoff somewhere in the course of canonical moves.

Moving that square one space upward basically adds $\frac{1}{8}$ to the game value, making it infinitesimally close to 1. The canonical moves following from this changed position are shown in Figure 18.

In Figure 18, if Black is to move twice in a row, a new option opens up for him. Instead of moving back to where he started, Black can now move even further to the left and shoot back to his starting square in Figure 18, yielding a

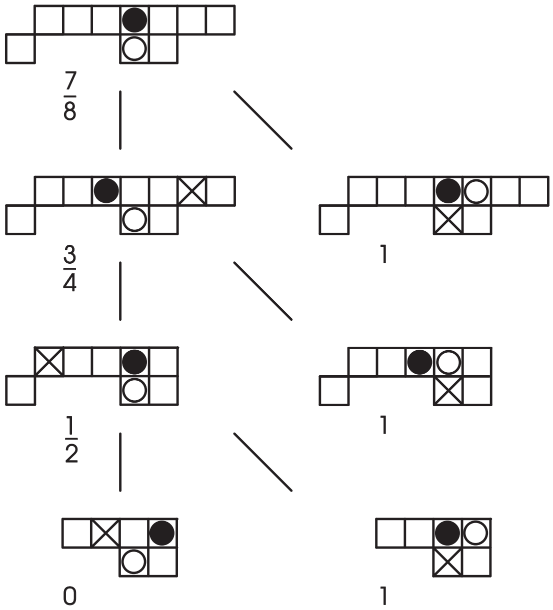

Figure 17. Canonical line of play for $\frac{7}{8}$.

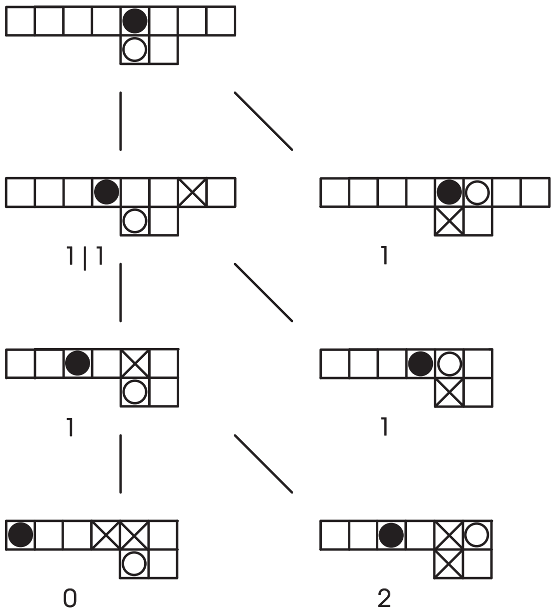

Figure 18. Moving the leftmost square results in $1{\downarrow}$.

solid 1 instead of $\frac{1}{2}$. If Black tries this trick in Figure 17 he ends up in Figure 19, getting only 0, not $\frac{1}{2}$. The reason is that in this position he has no winning move, because Black is left with a defective board like in Figure 3.

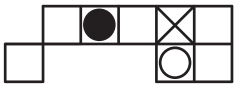

Figure 19. A defective shape.

5.3.2. $*2$ The program also found a position with value $*2$, which is unexpected for games that are not impartial. While $*$ offers just one option to each player and therefore is rather common, any other nimber can only be obtained by allowing more than one infinitesimal option to both players, which must also be exactly the same for both players. $*2 = \{0, *\|0, *\}$ is the smallest of these cases, but even it is hard to imagine in a partisan game like Amazons. Figure 20 shows it.

That this position is a $*2$ becomes clear only after several simplifications. Black and White cannot move to a zero directly. Instead, their first move that will eventually become a zero is reversible, so their opponent will move it instantly to another, hot position. From this position now the first player can move to zero. Figure 21 shows the trick.

From the original position both players can reach a $*$ after just one move, but the following position is a $*$ only because of a reversible move hidden one step further down the tree, see Figure 22.

Because of all these problems it would probably be quite difficult to determine the value of this position without electronic help. To construct a position of value $*2$ by hand seems to be outright impossible.

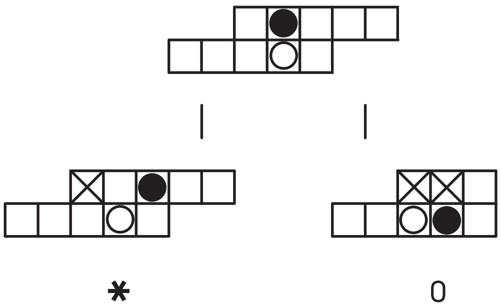

Figure 20. A nimber in Amazons.

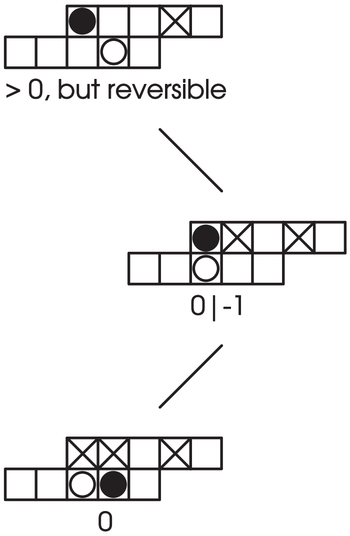

Figure 21. Explaining the 0 in $*2$.

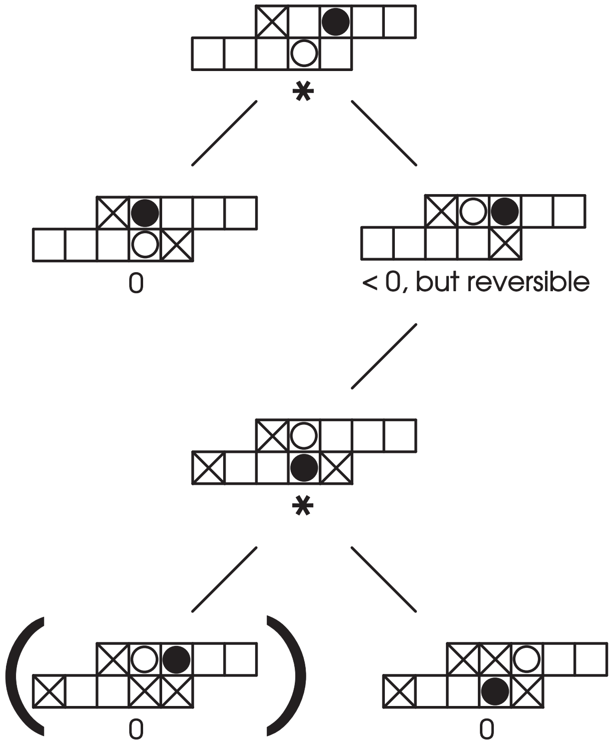

Figure 22. Explaining the $*$ in $*2$.

6. Further Work

Positions like those presented in the preceeding section are some of the bonuses of the brute force approach. Neither the $\frac{7}{8}$ nor the $*2$ have nearly rectangular board shape. That the $*2$ actually is a $*2$ could also not be discerned by looking at the thermograph of the position.

6.1. Other Special Cases. The logical next step is to search the data calculated by the program for other infinitesimals like tinies, minies and other nimbers. $*3$ can now be easily constructed in Amazons, as $*3 = *2 + *$. Nevertheless it would be very interesting to know if a $*3$ can exist on its own on the Amazons board, that is, within one board where all squares are connected. The next theoretical challenge is the $*4$. But every nimber higher than $*2$ would come as a big surprise, because these positions need even more completely symmetrical and infinitesimal options for both players.

6.2. Patterns. More interesting would be to find patterns for constructing positions with certain values. After reaching $\frac{1}{16}$ it seems likely that there is some way to construct all fractions with dyadic denominators. But the structure of the sixteenths found up to now makes it also likely that this construction is very hard to find. Other patterns of interest would be sequences of certain infinitesimals like ups and tinies.

Finally — at least for this article — another way to reach new values or patterns could be to change the shape of the boards or to add more amazons to the fray. The last option obviously is devastating in respect to computing power, but the possibilities abound, too.

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