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The Geometry and Topology of Three-Manifolds

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This is an electronic edition of the 1980 notes distributed by Princeton University. The text was typed in TEX by Sheila Newbery, who also scanned the figures. Typos have been corrected (and probably others introduced), but otherwise no attempt has been made to update the contents. Genevieve Walsh compiled the index.

Numbers on the right margin correspond to the original edition's page numbers.

Thurston's *Three-Dimensional Geometry and Topology*, Vol. 1 (Princeton University Press, 1997) is a considerable expansion of the first few chapters of these notes. Later chapters have not yet appeared in book form.

Please send corrections to Silvio Levy at levy@msri.org.

Introduction

These notes (through p. 9.80) are based on my course at Princeton in 1978–79. Large portions were written by Bill Floyd and Steve Kerckhoff. Chapter 7, by John Milnor, is based on a lecture he gave in my course; the ghostwriter was Steve Kerckhoff. The notes are projected to continue at least through the next academic year. The intent is to describe the very strong connection between geometry and low-dimensional topology in a way which will be useful and accessible (with some effort) to graduate students and mathematicians working in related fields, particularly 3-manifolds and Kleinian groups.

Much of the material or technique is new, and more of it was new to me. As a consequence, I did not always know where I was going, and the discussion often tends to wanter. The countryside is scenic, however, and it is fun to tramp around if you keep your eyes alert and don't get lost. The tendency to meander rather than to follow the quickest linear route is especially pronounced in chapters 8 and 9, where I only gradually saw the usefulness of "train tracks" and the value of mapping out some global information about the structure of the set of simple geodesic on surfaces.

I would be grateful to hear any suggestions or corrections from readers, since changes are fairly easy to make at this stage. In particular, bibliographical information is missing in many places, and I would like to solicit references (perhaps in the form of preprints) and historical information.

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CHAPTER 1

Geometry and three-manifolds

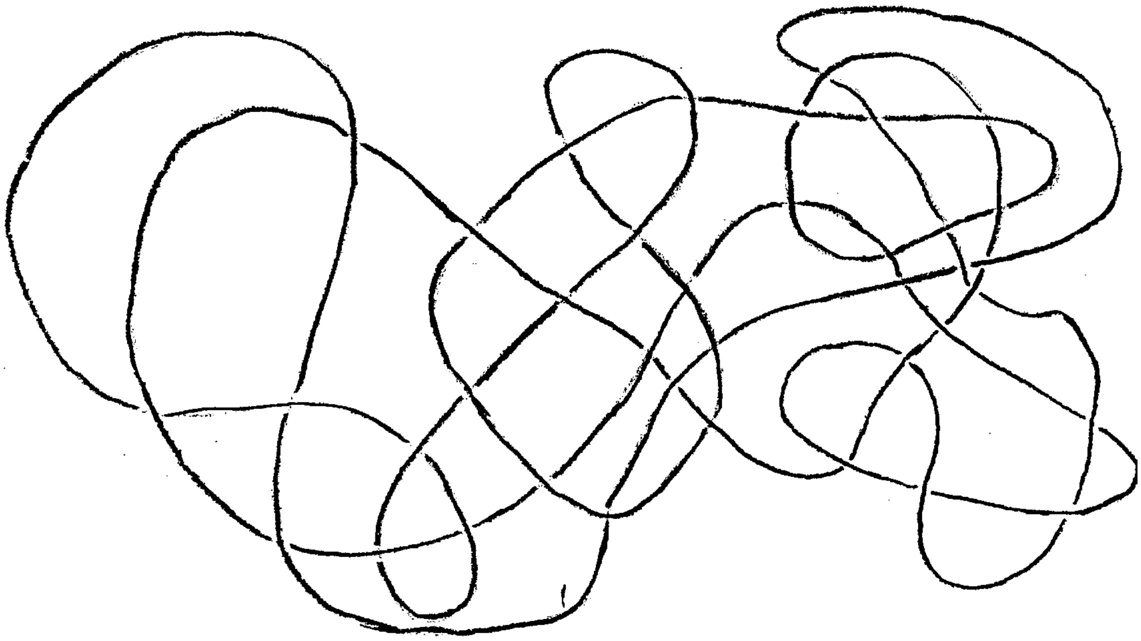
1.1

The theme I intend to develop is that topology and geometry, in dimensions up through 3, are very intricately related. Because of this relation, many questions which seem utterly hopeless from a purely topological point of view can be fruitfully studied. It is not totally unreasonable to hope that eventually all three-manifolds will be understood in a systematic way. In any case, the theory of geometry in three-manifolds promises to be very rich, bringing together many threads.

Before discussing geometry, I will indicate some topological constructions yielding diverse three-manifolds, which appear to be very tangled.

0. Start with the three sphere S^3, which may be easily visualized as $\mathbb{R}^3$, together with one point at infinity.

1. Any knot (closed simple curve) or link (union of disjoint closed simple curves) may be removed. These examples can be made compact by removing the interior of a tubular neighborhood of the knot or link.



1.2

The complement of a knot can be very enigmatic, if you try to think about it from an intrinsic point of view. Papakyriakopoulos proved that a knot complement has fundamental group $\mathbb{Z}$ if and only if the knot is trivial. This may seem intuitively clear, but justification for this intuition is difficult. It is not known whether knots with homeomorphic complements are the same.

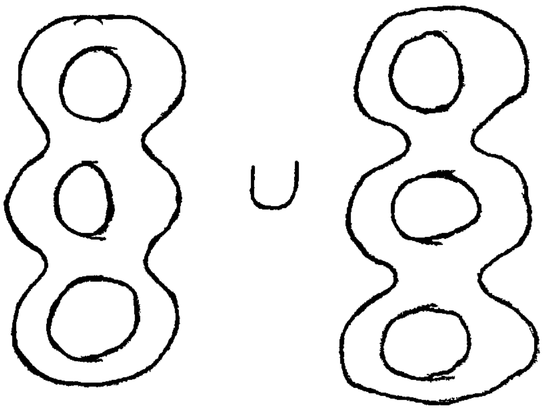
2. Cut out a tubular neighborhood of a knot or link, and glue it back in by a different identification. This is called *Dehn surgery.* There are many ways to do this, because the torus has many diffeomorphisms. The generator of the kernel of the inclusion map $\pi_1(T^2) \to \pi_1$ (solid torus) in the resulting three-manifold determines the three-manifold. The diffeomorphism can be chosen to make this generator an arbitrary primitive (indivisible non-zero) element of $\mathbb{Z} \oplus \mathbb{Z}$. It is well defined up to change in sign.

Every oriented three-manifold can be obtained by this construction (Lickorish) . It is difficult, in general, to tell much about the three-manifold resulting from this construction. When, for instance, is it simply connected? When is it irreducible? (*Irreducible* means every embedded two sphere bounds a ball).

Note that the homology of the three-manifold is a very insensitive invariant. The homology of a knot complement is the same as the homology of a circle, so when Dehn surgery is performed, the resulting manifold always has a cyclic first homology group. If generators for $\mathbb{Z} \oplus \mathbb{Z} = \pi_1(T^2)$ are chosen so that $(1,0)$ generates the homology of the complement and $(0,1)$ is trivial then any Dehn surgery with invariant $(1,n)$ yields a homology sphere. 3. *Branched coverings.* If L is a link, then any finite-sheeted covering space of $S^3 - L$ can be compactified in a canonical way by adding circles which cover L to give a closed manifold, M. M is called a 1.3
branched covering of S^3 over L. There is a canonical projection $p : M \to S^3$, which is a local diffeomorphism away from $p^{-1}(L)$. If $K \subset S^3$ is a knot, the simplest branched coverings of S^3 over K are then n-fold cyclic branched covers, which come from the covering spaces of $S^3 - K$ whose fundamental group is the kernel of the composition $\pi_1(S^3 - K) \to H_1(S^3 - K) = \mathbb{Z} \to \mathbb{Z}_n$. In other words, they are unwrapping S^3 from K n times. If K is the trivial knot the cyclic branched covers are S^3. It seems intuitively obvious (but it is not *known*) that this is the only way S^3 can be obtained as a cyclic branched covering of itself over a knot. Montesinos and Hilden (independently) showed that every oriented three-manifold is a branched cover of S^3 with 3 sheets, branched over some knot. These branched coverings are not in general regular: there are no covering transformations.

The formation of irregular branched coverings is somehow a much more flexible construction than the formation of regular branched coverings. For instance, it is not hard to find many different ways in which S^3 is an irregular branched cover of itself.

5. *Heegaard decompositions.* Every three-manifold can be obtained from two handlebodies (of some genus) by gluing their boundaries together.



1.4

The set of possible gluing maps is large and complicated. It is hard to tell, given two gluing maps, whether or not they represent the same three-manifold (except when there are homological invariants to distinguish them).

6. *Identifying faces of polyhedra.* Suppose $P_1, \ldots, P_k$ are polyhedra such that the number of faces with K sides is even, for each K.

Choose an arbitrary pattern of orientation-reversing identifications of pairs of two-faces. This yields a three-complex, which is an oriented manifold except near the vertices. (Around an edge, the link is automatically a circle.)

There is a classical criterion which says that such a complex is a manifold if and only if its Euler characteristic is zero. We leave this as an exercise.

In any case, however, we may simply remove a neighborhood of each bad vertex, to obtain a three-manifold with boundary.

The number of (at least not obviously homeomorphic) three-manifolds grows very quickly with the complexity of the description. Consider, for instance, different ways to obtain a three-manifold by gluing the faces of an octahedron. There are

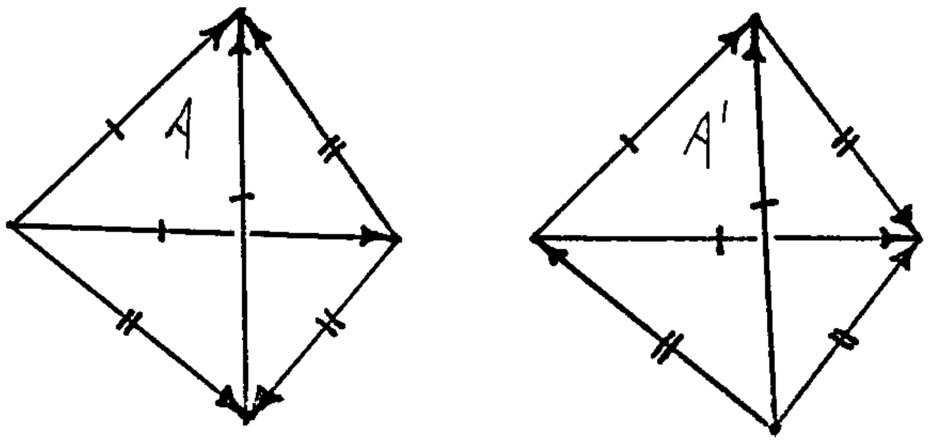
$$\frac{8!}{2^4 \cdot 4!} \cdot 3^4 = 8{,}505$$

possibilities. For an icosahedron, the figure is 38,661 billion. Because these polyhedra are symmetric, many gluing diagrams obviously yield homeomorphic results—but this reduces the figure by a factor of less than 120 for the icosahedron, for instance.

In two dimensions, the number of possible ways to glue sides of $2n$-gon to obtain an oriented surface also grows rapidly with n: it is $(2n)!/(2^n n!)$. In view of the amazing fact that the Euler characteristic is a complete invariant of a closed oriented surface, huge numbers of these gluing patterns give identical surfaces. It seems unlikely that 1.5

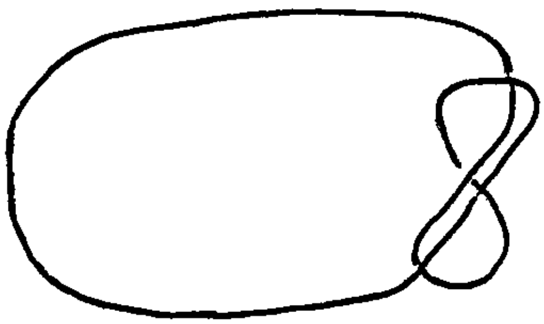
such a phenomenon takes place among three-manifolds; but how can we tell?

Example. Here is one of the simplest possible gluing diagrams for a three-manifold. Begin with two tetrahedra with edges labeled:



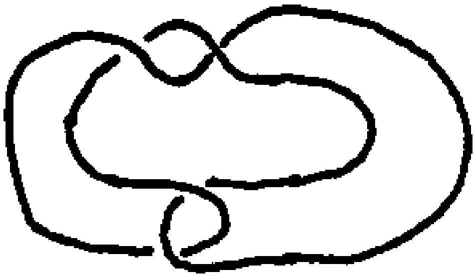
There is a unique way to glue the faces of one tetrahedron to the other so that arrows are matched. For instance, A is matched with A'. All the $/\!\!\!\longrightarrow$ arrows are identified and all the $/\!\!/\!\!\!\longrightarrow$ arrows are identified, so the resulting complex has 2 tetrahedra, 4 triangles, 2 edges and 1 vertex. Its Euler characteristic is $+1$, and (it follows that) a neighborhood of the vertex is the cone on a torus. Let M be the manifold obtained by removing the vertex.

It turns out that this manifold is homeomorphic with the complement of a figure-eight knot.



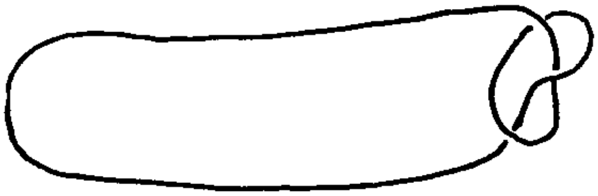
"Figure eight knot."

1.6

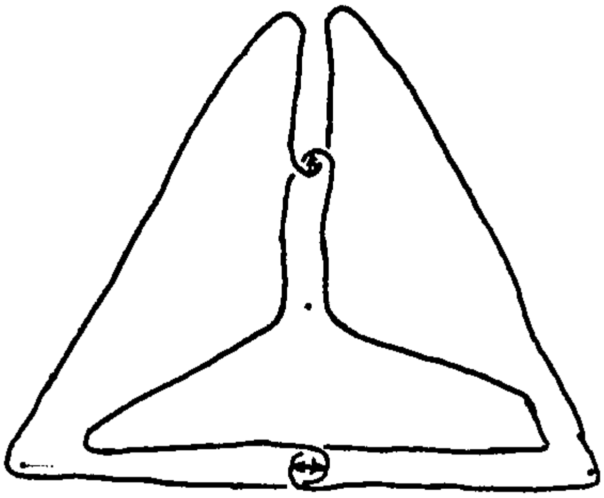


Another view of the figure-eight knot

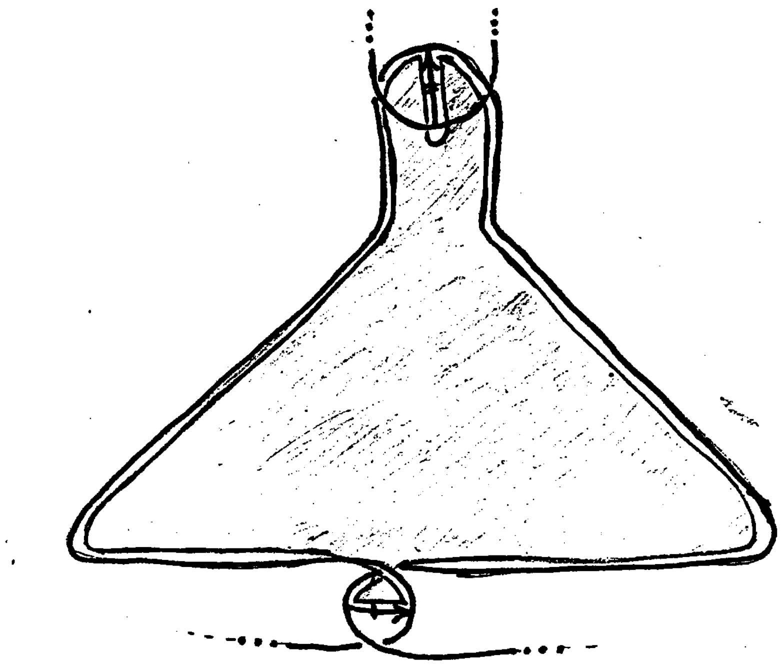
This knot is familiar from extension cords, as the most commonly occurring knot, after the trefoil knot



In order to see this homeomorphism we can draw a more suggestive picture of the figure-eight knot, arranged along the one-skeleton of a tetrahedron. The knot can be



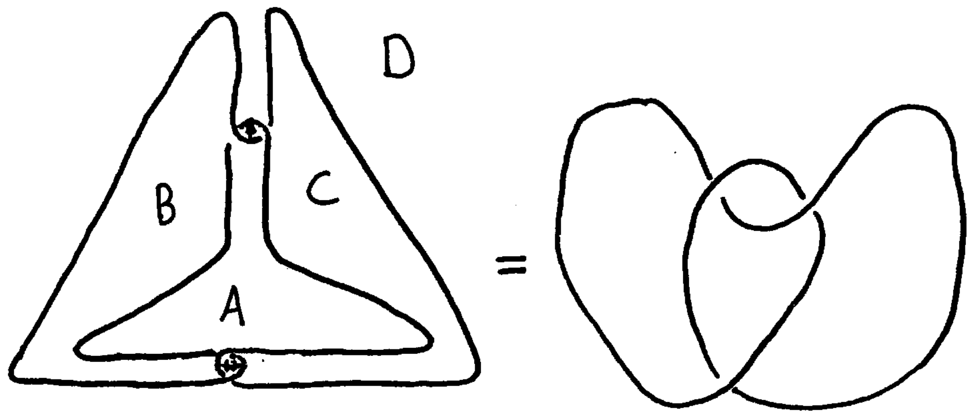
Tetrahedron with figure-eight knot, viewed from above spanned by a two-complex, with two edges, shown as arrows, and four two-cells, one for each face of the tetrahedron, in a more-or-less obvious way: 1.7



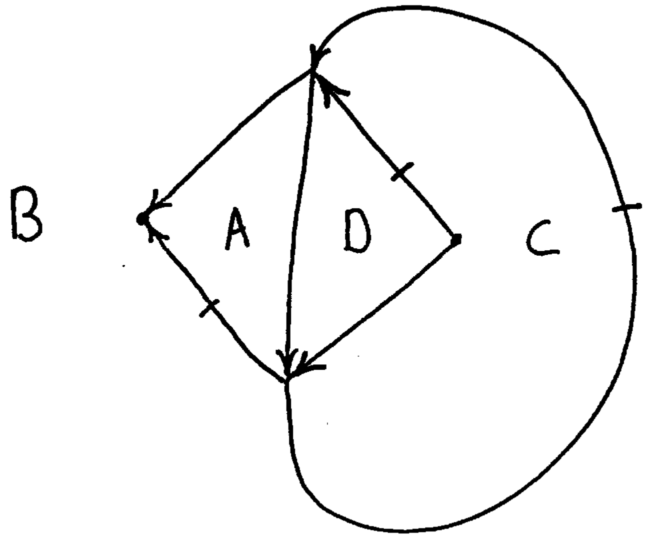
This pictures illustrates the typical way in which a two-cell is attached. Keeping in mind that the knot is not there, the cells are triangles with deleted vertices. The two complementary regions of the two-complex are the tetrahedra, with deleted vertices.

We will return to this example later. For now, it serves to illustrate the need for a systematic way to compare and to recognize manifolds.

Note. Suggestive pictures can also be deceptive. A trefoil knot can similarly be arranged along the one-skeleton of a tetrahedron: 1.8



From the picture, a cell-division of the complement is produced. In this case, however, the three-cells are not tetrahedra.



The boundary of a three-cell, flattened out on the plane.