William P. Thurston

The Geometry and Topology of Three-Manifolds

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This is an electronic edition of the 1980 notes distributed by Princeton University. The text was typed in TEX by Sheila Newbery, who also scanned the figures. Typos have been corrected (and probably others introduced), but otherwise no attempt has been made to update the contents. Genevieve Walsh compiled the index.

Numbers on the right margin correspond to the original edition's page numbers.

Thurston's *Three-Dimensional Geometry and Topology*, Vol. 1 (Princeton University Press, 1997) is a considerable expansion of the first few chapters of these notes. Later chapters have not yet appeared in book form.

Please send corrections to Silvio Levy at levy@msri.org.

CHAPTER 4

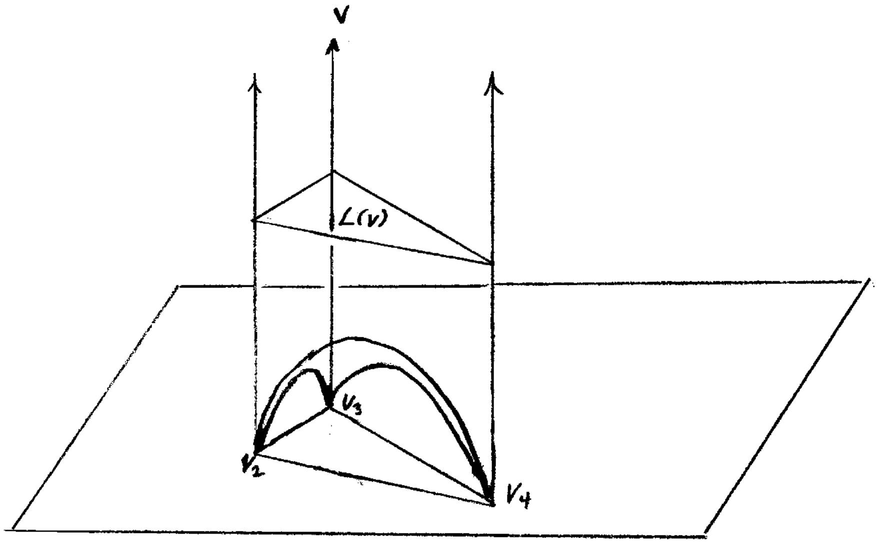
Hyperbolic Dehn surgery

A hyperbolic structure for the complement of the figure-eight knot was constructed in 3.1. This structure was in fact chosen to be complete. The reader may wish to verify this by constructing a horospherical realization of the torus which is the link of the ideal vertex. Similarly, the hyperbolic structure for the Whitehead link complement and the Borromean rings complement constructed in 3.3 and 3.4 are complete.

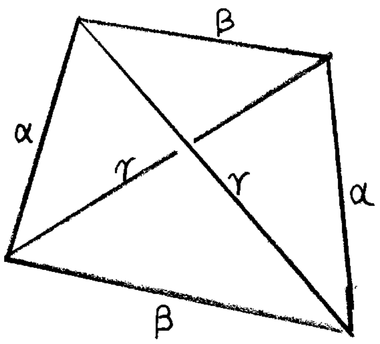
There is actually a good deal of freedom in the construction of hyperbolic structures for such manifolds, although most of the resulting structures are not complete. We shall first analyze the figure-eight knot complement. To do this, we need an understanding of the possible shapes of ideal tetrahedra.

4.1. Ideal tetrahedra in H^3.

The link $L(v)$ of an ideal vertex v of an oriented ideal tetrahedron T (which by definition is the set of rays in the tetrahedron through that vertex) is a Euclidean triangle, well-defined up to orientation-preserving similarity. It is concretely realized as the intersection with T of a horosphere about v. The triangle $L(v)$ actually determines T up to congruence. To see this, picture T in the upper half-space model and arrange it so that v is the point at infinity. The other three vertices of T form a triangle in E^2 which is in the same similarity class as $L(v)$. Consequently, if two 4.2
tetrahedra T and T' have vertices v and v' with $L(v)$ similar to $L(v')$, then T' can be transformed to T by a Euclidean similarity which preserves the plane bounding upper half-space. Such a similarity is a hyperbolic isometry.

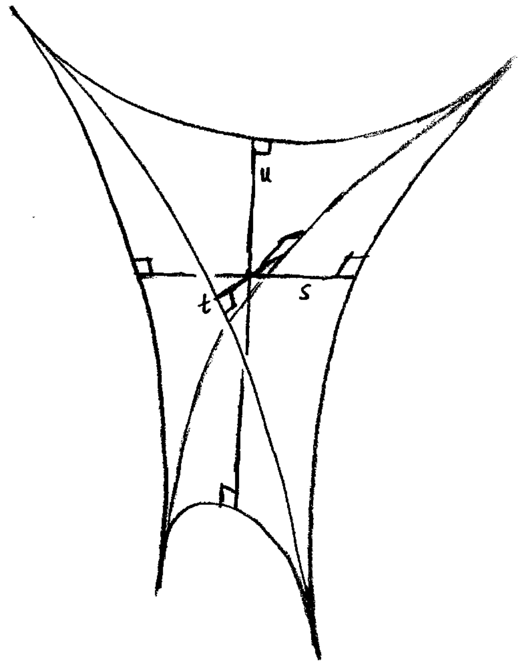


It follows that T is determined by the three dihedral angles α, β and γ of edges incident to the ideal vertex v, and that $\alpha + \beta + \gamma = \pi$. Using similar relations among angles coming from the other three vertices, we can determine the other three dihedral angles:



4.3

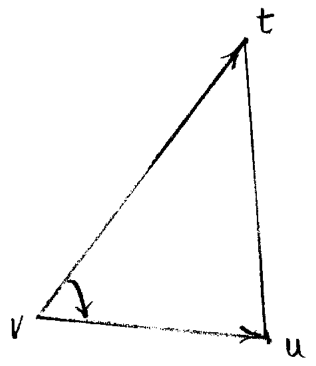
Thus, dihedral angles of opposite edges are equal, and the oriented similarity class of $L(v)$ does not depend on the choice of a vertex v! A geometric explanation of this phenomenon can be given as follows. Any two non-intersecting and non-parallel lines in H^3 admit a unique common perpendicular. Construct the three common perpendiculars s, t and u to pairs of opposite edges of T. Rotation of π about s, for instance, preserves the edges orthogonal to s, hence preserves the four ideal vertices of T, so it preserves the entire figure. It follows that s, t and u meet in a point and that they are pairwise orthogonal. The rotations of π about these three axes are the three non-trivial elements of $z_2 \oplus z_2$ acting as a group of symmetries of T.

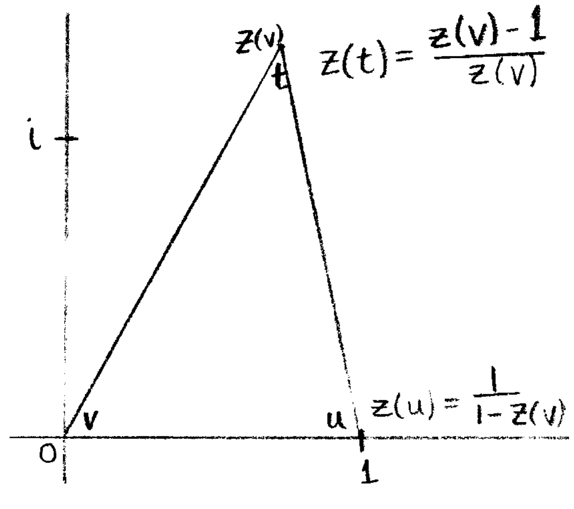


In order to parametrize Euclidean triangles up to similarity, it is convenient to regard E^2 as $\mathbb{C}$. To each vertex v of a triangle $\Delta(t, u, v)$ we associate the ratio 4.4

$$\frac{(t-v)}{(u-v)} = z(v)$$

of the sides adjacent to v. The vertices must be labelled in a clockwise order, so that





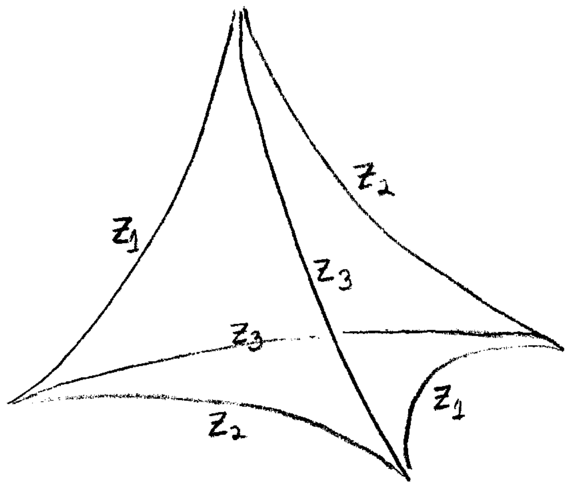
$\operatorname{Im}\big(z(v)\big) > 0$. Alternatively, arrange the triangle by a similarity so that v is at 0 and u at 1; then t is at $z(v)$. The other two vertices have invariants

4.1.1.
$$\begin{aligned} z(t) &= \tfrac{z(v)-1}{z(v)} \\ z(u) &= \tfrac{1}{1-z(v)}. \end{aligned}$$

Denoting the three invariants z_1, z_2, z_3 in clockwise order, with any starting point, we have the identities 4.5

4.1.2.
$$\begin{gathered} z_1\, z_2\, z_3 = -1 \\ 1 - z_1 + z_1 z_2 = 0 \end{gathered}$$

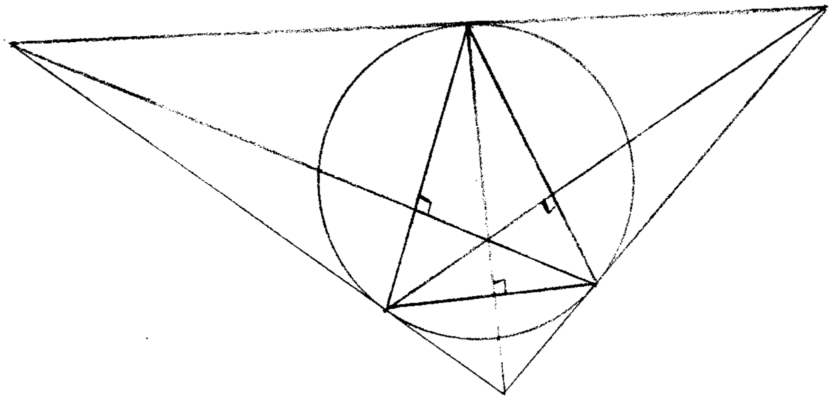
We can now translate this to a parametrization of ideal tetrahedra. Each edge e is labelled with a complex number $z(e)$, opposite edges have the same label, and the three distinct invariants satisfy 4.1.2 (provided the ordering is correct.) Any z_i determines the other two, via 4.1.2.



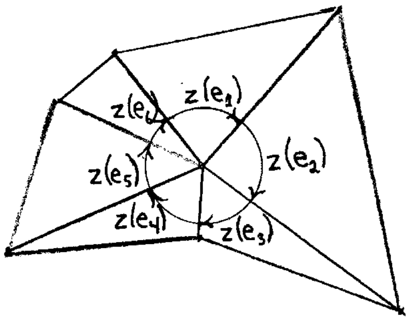
4.2. Gluing consistency conditions.

Suppose that M is a three-manifold obtained by gluing tetrahedra $T_i, \ldots, T_j$ and then deleting the vertices, and let K be the complex which includes the vertices. 4.6

Any realization of $T_1, \ldots, T_j$ as ideal hyperbolic tetrahedra determines a hyperbolic structure on $(M - (1-\text{skeleton}))$, since any two ideal triangles are congruent. Such a congruence is uniquely determined by the correspondence between the vertices. (This fact may be visualized concretely from the subdivision of an ideal triangle by its altitudes.)



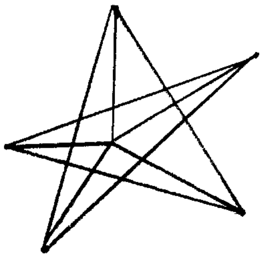
The condition for the hyperbolic structure on $(M - (1 - \text{skeleton}))$ to give a hyperbolic structure on M itself is that its developing map, in a neighborhood of each edge, should come from a local homeomorphism of M itself. In particular, the sum of the dihedral angles of the edges $e_1, \ldots, e_k$ must be 2π. Even when this condition is satisfied, though, the holonomy going around an edge of M might be a non-trivial translation along the edge. To pin down the precise condition, note that for each ideal vertex v, the hyperbolic structure on $M - (1 - \text{skeleton})$ gives a similarity structure to $L(v) - (0 - \text{skeleton})$. The hyperbolic structure extends over an edge e of M if and 4.7
only if the similarity structure extends over the corresponding point in $L(v)$, where v is an endpoint of e. Equivalently, the similarity classes of triangles determined by $z(e_1), \ldots, z(e_k)$ must have representatives which can be arranged neatly around a point in the plane:



The algebraic condition is

4.2.1. $$z(e_1) \cdot z(e_2) \cdot \cdots \cdot z(e_k) = 1.$$

This equation should actually be interpreted to be an equation in the universal cover $\tilde{\mathbb{C}}^*$, so that solutions such as 4.8



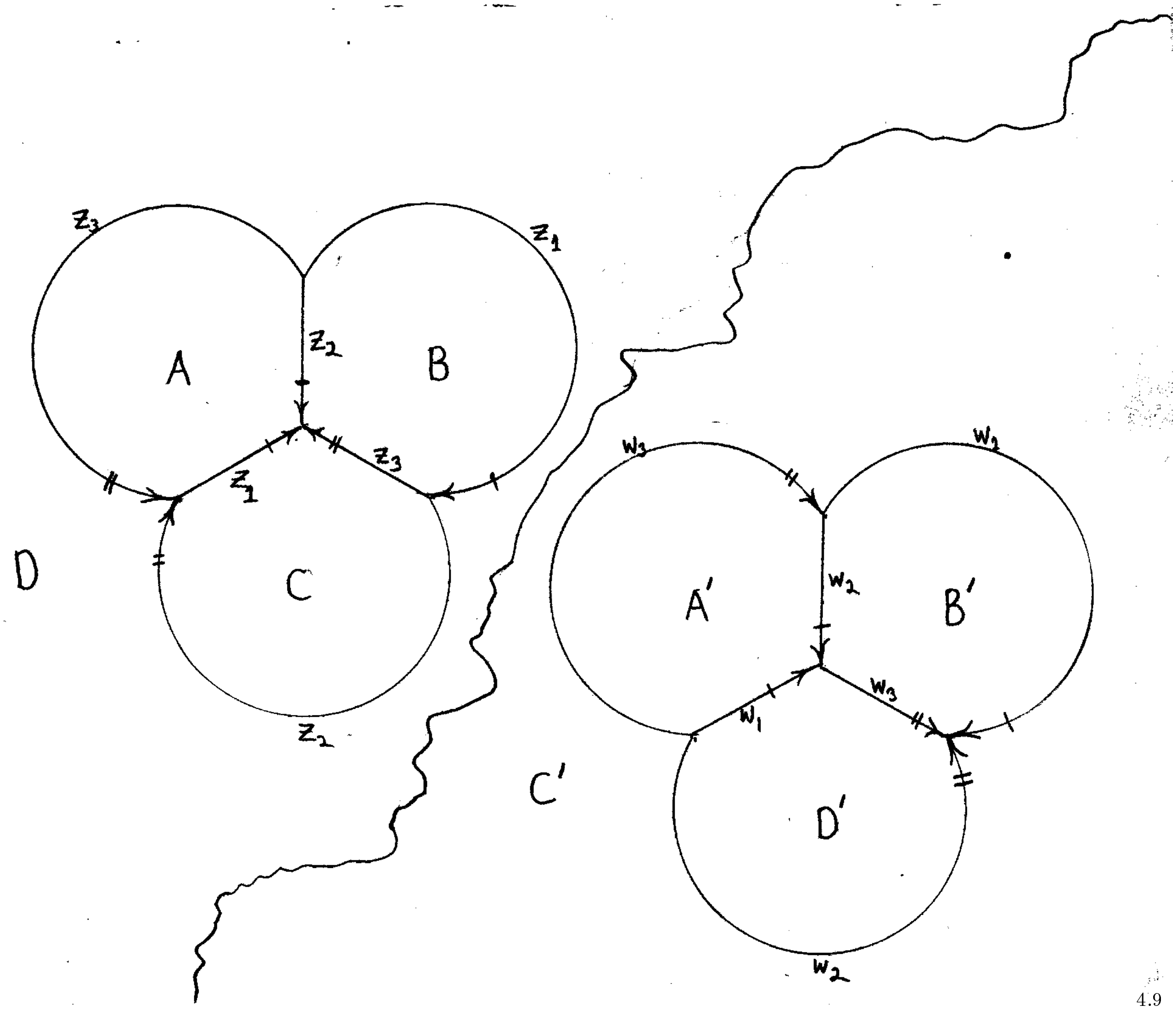
are ruled out. In other words, the auxiliary condition

4.2.2. $$\arg z_1 + \cdots + \arg z_k = 2\pi$$

must also be satisfied, where $0 < \arg z_i \leq \pi$.

4.3. Hyperbolic structure on the figure-eight knot complement.

Consider two hyperbolic tetrahedra to be identified to give the figure eight knot complement:



4.9

We read off the gluing consistency conditions for the two edges:

$$(\not\longrightarrow) z_1^2 z_2 w_1^2 w_2 = 1 \qquad (/\!\!/\!\!\longrightarrow) z_3^2 z_2 w_3^2 w_2 = 1.$$

From 4.1.2, note that the product of these two equations,

$$(z_1 z_2 z_3)^2 (w_1 w_2 w_3)^2 = 1$$

is automatically satisfied. Writing $z = z_1$, and $w = w_1$, and substituting the expressions from 4.1.1 into $(\not\longrightarrow)$, we obtain the equivalent gluing condition,

4.3.1. $$z(z-1)\, w(w-1) = 1.$$

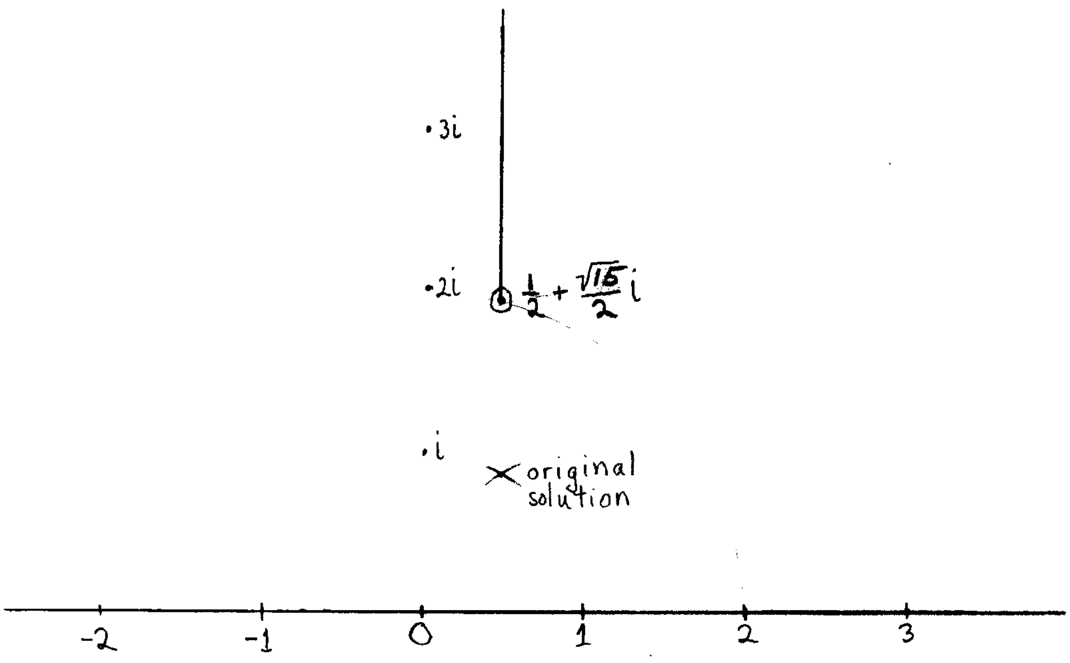
We may solve for z in terms of w by using the quadratic formula.

4.3.2.
$$z = \frac{1 \pm \sqrt{1 + 4/w(w-1)}}{2}$$

We are searching only for geometric solutions

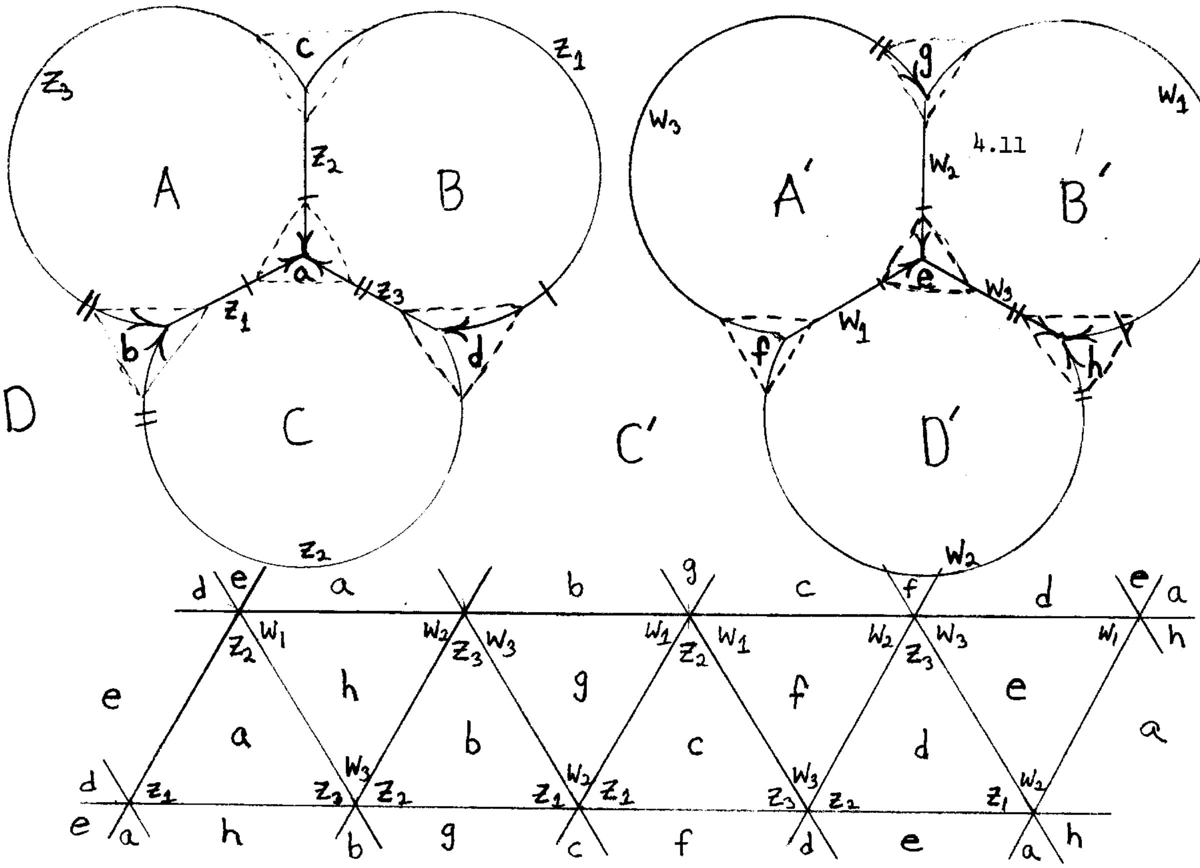
$$\operatorname{Im}(z) > 0 \quad \operatorname{Im}(w) > 0$$

so that the two tetrahedra are non-degenerate and positively oriented. For each value of w, there is at most one solution for z with $\operatorname{Im}(z) > 0$. Such a solution exists provided that the discriminant $1 + 4/w(w-1)$ is not positive real. Solutions are therefore parametrized by w in this region of $\mathbb{C}$: 4.10

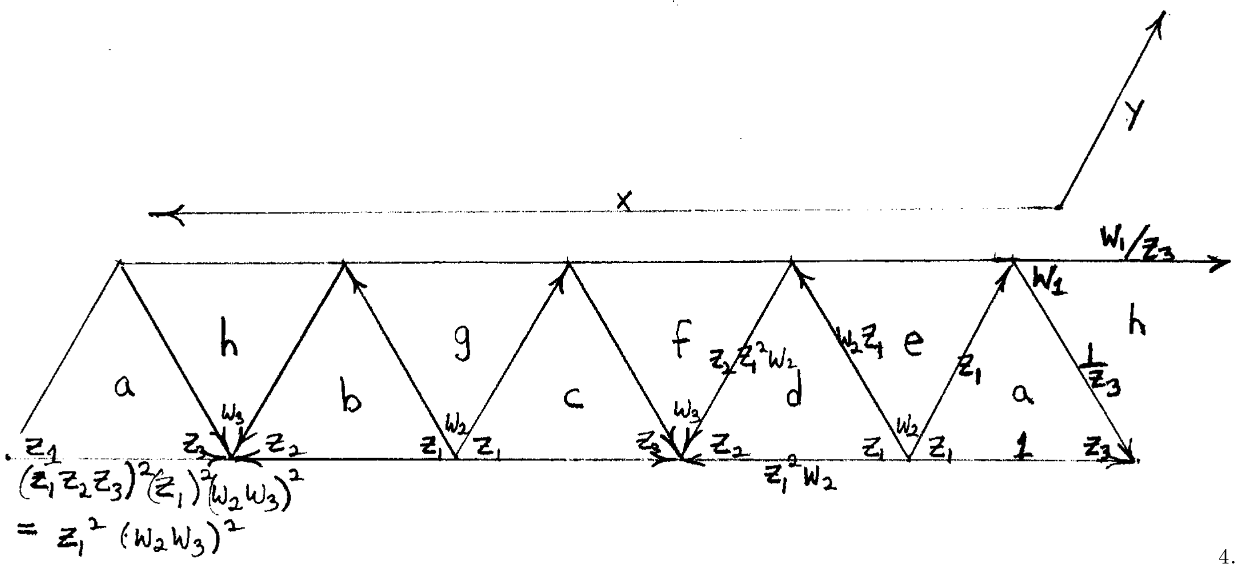


Note that the original solution given in 3.1 corresponds to $w = z = \sqrt[3]{-1} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$.

The link L of the single ideal vertex has a triangulation which can be calculated from the gluing diagram: 4.11



Now let us compute the derivative of the holonomy of the similarity structure on L. To do this, regard directed edges of the triangulation as vectors. The ratio of any two vectors in the same triangle is known in terms of z or w. Multiplying appropriate ratios, we obtain the derivative of the holonomy:



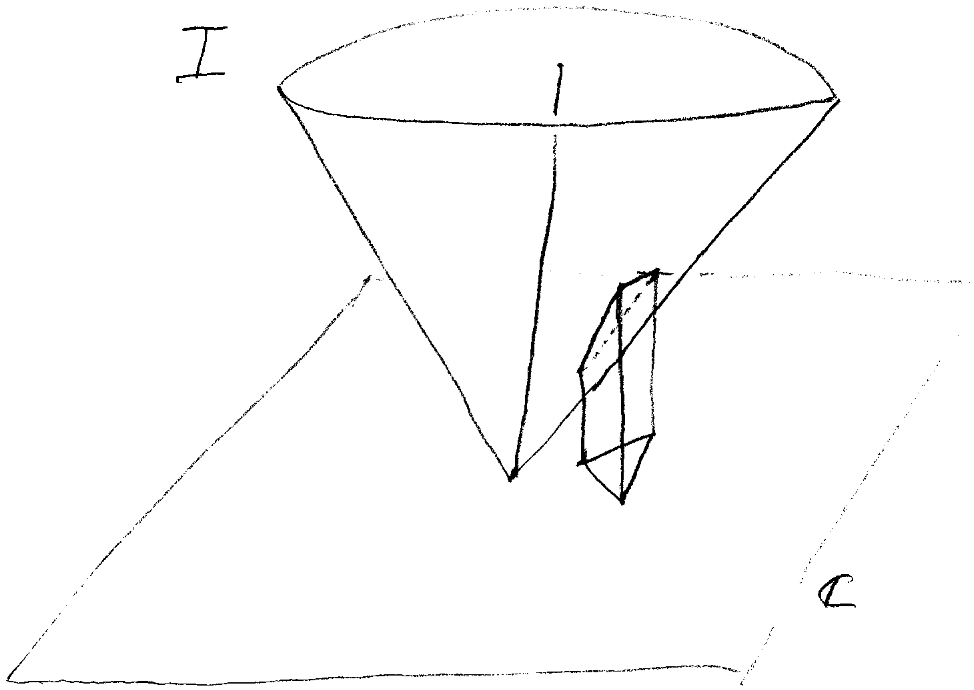
4.12

$$
\begin{aligned}
H'(x) &= z_1^2 (w_2 w_3)^2 = (\tfrac{z}{w})^2 \\
H'(y) &= \tfrac{w_1}{z_3} = w(1-z).
\end{aligned}
$$

Observe that if M is to be complete, then $H'(x) = H'(y) = 1$, so $z = w$. From 4.3.1, $(z(z-1))^2 = 1$. Since $z(z-1) < 0$, this means $z(z-1) = -1$, so that the only possibility is the original solution $w = z = \sqrt[3]{-1}$.

4.4. The completion of hyperbolic three-manifolds obtained from ideal polyhedra.

Let M be any hyperbolic manifold obtained by gluing polyhedra with some vertices at infinity, and let K be the complex obtained by including the ideal vertices. The completion $\bar{M}$ is obtained by completing a deleted neighborhood $\mathcal{N}(v)$ of each ideal vertex v in k, and gluing these completed neighborhoods $\bar{\mathcal{N}}(v)$ to M. The developing map for the hyperbolic structure on $\mathcal{N}(v)$ may be readily understood in terms of the developing map for the similarity structure on $L(v)$. To do this, choose coordinates so that v is the point at infinity in the upper half-space model. The developing images of corners of polyhedra near v are "chimneys" above some polygon in the developing image of $L(v)$ on $\mathbb{C}$ (where $\mathbb{C}$ is regarded as the boundary of upper half-space.) If M is not complete near v, we change coordinates if necessary by a translation of $\mathbb{R}^3$ so that the developing image of $L(v)$ is $\mathbb{C} - 0$. The holonomy for $\mathcal{N}(v)$ now consists of similarities of $\mathbb{R}^3$ which leave invariant the z-axis and the $x - y$ plane ($\mathbb{C}$). Replacing $\mathcal{N}(v)$ by a smaller neighborhood, we may assume that the developing image I of $\mathcal{N}(v)$ is a solid cone, minus the z-axis. 4.13



The completion of I is clearly the solid cone, obtained by adjoining the z-axis to I. It follows easily that the completion of

$$\widetilde{\mathcal{N}(v)} = \tilde{I}$$

is also obtained by adjoining a single copy of the z-axis.

The projection

$$p : \widetilde{\mathcal{N}(v)} \to \mathcal{N}(v)$$

extends to a surjective map $\bar{p}$ between the completions. [$\bar{p}$ exists because p does not increase distances. $\bar{p}$ is surjective because a Cauchy sequence can be replaced by a Cauchy path, which lifts to $\mathcal{N}\tilde{(v)}$.] Every orbit of the holonomy action of $\pi_1\big(\mathcal{N}(v)\big)$ on the z-axis is identified to a single point. This action is given by

$$H(\alpha) : (0, 0, z) \mapsto |H'(\alpha)| \cdot (0, 0, z)$$

where the first $H(\alpha)$ is the hyperbolic holonomy and the second is the holonomy of 4.14
$L(v)$. There are two cases:

Case 1. The group of moduli $\{|H'(\alpha)|\}$ is dense in $\mathbb{R}_+$. Then the completion of $\mathcal{N}(v)$ is the one-point compactification.

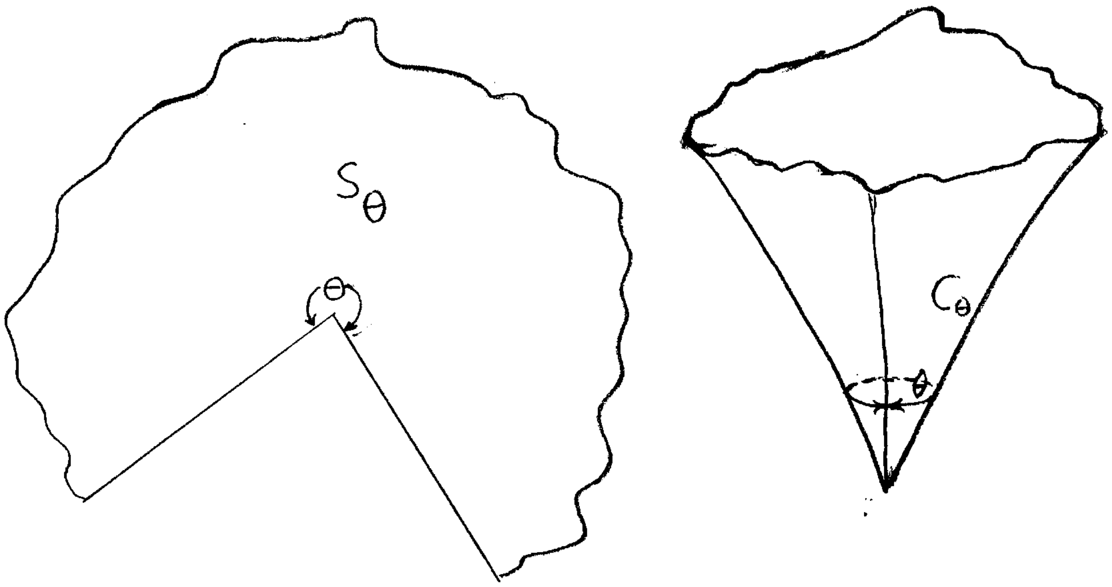
Case 2. The group of moduli $\{|H'(\alpha)|\}$ is a cyclic group. Then the completion

$$\overline{\mathcal{N}(v)}$$

is *topologically* a manifold which is the quotient space $\widetilde{\overline{(\partial \mathcal{N})}}/H$, and it is obtained by adjoining a circle to $\mathcal{N}(v)$. Let $\alpha_1 \in \pi_1(L(v))$ be a generator for the kernel of $\alpha \mapsto |H'(\alpha)|$ and let $1 < |H'(\alpha_2)|$ generate the image, so that α_1 and α_2 generate $\pi_1(L(v)) = \mathbb{Z} \oplus \mathbb{Z}$. Then the added circle in

$$\overline{\mathcal{N}(v)}$$

has length $\log |H'(\alpha_2)|$. A cross-section of $\bar{\mathcal{N}}(v)$ perpendicular to the added circle is a cone C_θ, obtained by taking a two-dimensional hyperbolic sector S_θ of angle θ, $[0 < \theta < \infty]$ and identifying the two bounding rays:



4.15

It is easy to make sense of this even when $\theta > 2\pi$. The cone angle θ is the argument of the element $\tilde{H}'(\alpha_2) \in \tilde{\mathbb{C}}^*$. In the special case $\theta = 2\pi$, C_θ is non-singular, so

$$\overline{\mathcal{N}(v)}$$

is a hyperbolic manifold. $\overline{\mathcal{N}(v)}$ may be seen directly in this special case, as the solid cone $I \cup (z - \text{axis})\,\text{modulo}\, H$.

4.5. The generalized Dehn surgery invariant.

Consider any three-manifold M which is the interior of a compact manifold $\hat{M}$ whose boundary components $P_1, \ldots, P_k$ are tori. For each i, choose generators a_i, b_i for $\pi_1(P_i)$. If M is identified with the complement of an open tubular neighborhood of a link L in S^3, there is a standard way to do this, so that a_i is a *meridian* (it bounds a disk in the solid torus around the corresponding component of L) and b_i is a *longitude* (it is homologous to zero in the complement of this solid torus in S^3). In this case we will call the generators m_i and l_i.

We will use the notation $M_{(\alpha_1,\beta_1),\dots,(\alpha_k,\beta_k)}$ to denote the manifold obtained by gluing solid tori to M so that a meridian in the i-th solid torus goes to $\alpha_i, a_i + \beta_i b_i$. If an ordered pair (α_i, β_i) is replaced by the symbol ∞, this means that nothing is glued in near the i-th torus. Thus, $M = M_{\infty,\dots,\infty}$.

These notions can be refined and extended in the case M has a hyperbolic structure whose completion $\bar{M}$ is of the type described in 4.4. (In other words, if M is not complete near P_i, the developing map for some deleted neighborhood $\mathcal{N}_i$ of P_i should be a covering of the deleted neighborhood I of radius r about a line in H^3.) The developing map D of $\mathcal{N}_i$ can be lifted to $\tilde{I}$, with holonomy $\tilde{H}$. The group of isometries of $\tilde{I}$ is $\mathbb{R} \oplus \mathbb{R}$, parametrized by (translation distance, angle of rotation); this parametrization is well-defined up to sign. 4.16

Definition 4.5.1. The generalized Dehn surgery invariants (α_i, β_i) for $\bar{M}$ are solutions to the equations

$$\alpha_i \tilde{H}(a_i) + \beta_i \tilde{H}(b_i) = (\text{rotation by } \pm 2\pi),$$

(or, $(\alpha_i, \beta_i) = \infty$ if M is complete near P_i).

Note that (α_i, β_i) is unique, up to multiplication by -1, since when M is not complete near P_i, the holonomy $\tilde{H} : \pi_1(\mathcal{N}_i) \to \mathbb{R} \oplus \mathbb{R}$ is injective. We will say that $\bar{M}$ is a hyperbolic structure for

$$M_{(\alpha_1,\beta_1),\dots,(\alpha_k,\beta_k)}.$$

If all (α_i, β_i) happen to be primitive elements of $\mathbb{Z} \oplus \mathbb{Z}$, then $\bar{M}$ is the topological manifold $M_{(\alpha_1,\beta_1),\dots,(\alpha_k,\beta_k)}$ with a non-singular hyperbolic structure, so that our extended definition is compatible with the original. If each ratio α_i/β_i is the rational number p_i/q_i in lowest terms, then $\bar{M}$ is topologically the manifold $M_{(p_1,q_1),\dots,(p_k,q_k)}$. The hyperbolic structure, however, has singularities at the component circles of $\bar{M} - M$ with cone angles of $2\pi(p_i/\alpha_i)$ [since the holonomy $\tilde{H}$ of the primitive element $p_i a_i + q_i b_i$ in $\pi_1(P_i)$ is a pure rotation of angle $2\pi(p_i/\alpha_i)$]. 4.17

There is also a topological interpretation in case the $(\alpha_i, \beta_i) \in \mathbb{Z} \oplus \mathbb{Z}$, although they may not be primitive. In this case, all the cone angles are of the form $2\pi/n_i$, where each n_i is an integer. Any branched cover of $\bar{M}$ which has branching index n_i around the i-th circle of $\bar{M} - M$ has a non-singular hyperbolic structure induced from $\bar{M}$.

4.6. Dehn surgery on the figure-eight knot.

For each value of w in the region R of $\mathbb{C}$ shown on p. 4.10, the associated hyperbolic structure on $S^3 - K$, where K is the figure-eight knot, has some Dehn surgery invariant $d(w) = \pm(\mu(w), \lambda(w))$. The function d is a continuous map from R to the one-point compactification $\mathbb{R}^2 / \pm 1$ of $\mathbb{R}^2$ with vectors v identified to $-v$. Every primitive element (p, q) of $\mathbb{Z} \oplus \mathbb{Z}$ which lies in the image $d(R)$ describes a closed manifold $(S^3 - K)_{(p,q)}$ which possesses a hyperbolic structure.

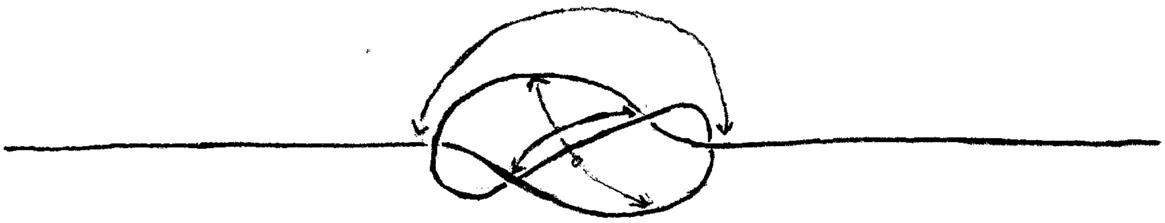
Actually, the map d can be lifted to a map $\tilde{d} : R \to \hat{\mathbb{R}}^2$, by using the fact that the sign of a rotation of

$$(\widetilde{H^3 - z\text{-axis}})$$

is well-defined. (See §4.4. The extra information actually comes from the orientation of the z-axis determined by the direction in which the corners of tetrahedra wrap around it). $\tilde{d}$ is defined by the equation $\tilde{d}(w) = (\mu, \lambda)$ where 4.18

$$\mu \tilde{H}(m) + \lambda \tilde{H}^2(l) = (\text{a rotation by } +2\pi)$$

In order to compute the image $\tilde{d}(R)$, we need first to express the generators l and m for $\pi_1(P)$ in terms of the previous generators x and y on p. 4.11. Referring to page 6, we see that a meridian which only intersects two two-cells can be constructed in a small neighborhood of K. The only generator of $\pi_1(L(v))$ (see p. 4.11) which intersects only two one-cells is $\pm y$, so we may choose $m = y$. Here is a cheap way to see what l is. The figure-eight knot can be arranged (passing through the point at infinity) so that it is invariant by the map $v \mapsto -v$ of $\hat{\mathbb{R}}^3 = S^3$.



This map can be made an isometry of the complete hyperbolic structure constructed for $S^3 - K$. (This can be seen directly; it also follows immediately from Mostow's Theorem, ...). This hyperbolic isometry induces an isometry of the Euclidean structure on $L(v)$ which takes m to m and l to $-l$. Hence, a geodesic representing l must be orthogonal to a geodesic representing m, so from the diagram on the bottom of 4.19
p. 4.11 we deduce that the curve $l = +x + 2y$ is a longitude. (Alternatively, it is not hard to compute m and l directly).

From p. 4.12, we have

4.6.1.
$$\begin{array}{rcl} H(m) &=& w(1-z) \\ H(l) &=& z^2(1-z)^2 \end{array}$$

The behavior of the map $\tilde{d}$ near the boundary of R is not hard to determine. For instance, when w is near the ray $\mathrm{Im}(w) = 0$, $\mathrm{Re}(w) > 1$, then z is near the ray $\mathrm{Im}(z) = 0$, $\mathrm{Re}(z) < 0$. The arguments of $\tilde{H}(m)$ and $\tilde{H}(l)$ are easily computed by analytic continuation from the complete case $w = z = \sqrt[3]{-1}$ (when the arguments are 0) to be

$$\arg \tilde{H}(m) = 0 \quad \arg \tilde{H}(l) \approx \pm 2\pi.$$

Consequently, (μ, λ) is near the line $\lambda = +1$. As $w \to 1$ we see from the equation

$$z(1-z)\, w(1-w) = 1$$

that

$$|z|^2 \cdot |w| \to 1$$

so (μ, λ) must approach the line $\mu + 4\lambda = 0$. Similarly, as $w \to +\infty$, then $|z|\,|w|^2 \to 1$, so (μ, λ) must approach the line $\mu - 4\lambda = 0$. Then the map $\tilde{d}$ extends continuously to send the line segment

$$\overline{1, +\infty}$$

to the line segment 4.20

$$\overline{(-4, +1), (+4, +1)}.$$

There is an involution τ of the region R obtained by interchanging the solutions z and w of the equation $z(l-z)\, w(l-w) = 1$. Note that this involution takes $H(m)$ to $1/H(m) = H(-m)$ and $H(l)$ to $H(-l)$. Therefore $\tilde{d}(\tau w) = -\tilde{d}(w)$. It follows that $\tilde{d}$ extends continuously to send the line segment

$$\overline{-\infty, 0}$$

to the line segment

$$\overline{(+4, -1), (-4, -1)}.$$

When $|w|$ is large and $0 < \arg(w) < \pi/2$, then $|z|$ is small and

$$\arg(z) \approx \pi - 2\arg(w).$$

Thus $\arg \tilde{H}(m) \approx \arg w$, $\arg \tilde{H}(l) \approx 2\pi - 4\arg w$ so $\mu \arg w + \lambda(2\pi - 4\arg w) = 2\pi$. By considering $|H(m)|$ and $|H(l)|$, we have also $\mu - 4\lambda \approx 0$, so $(\mu, \lambda) \approx (4, 1)$.

There is another involution σ of R which takes w to

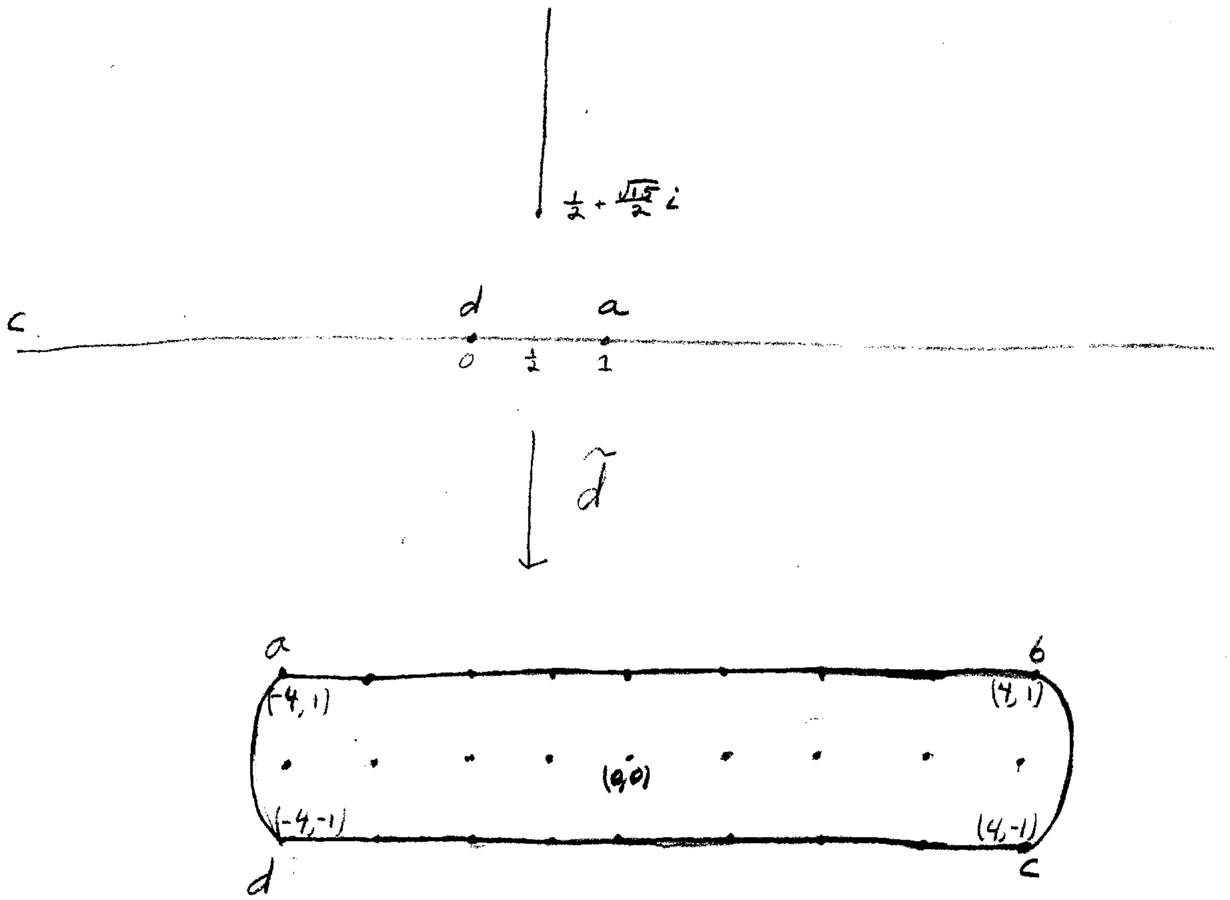
$$\overline{1 - w}$$

(and z to $\overline{1-z}$). From 4.6.1 we conclude that if $\tilde{d}(w) = (\mu, \lambda)$, then $\tilde{d}(\sigma w) = (\mu, -\lambda)$. With this information, we know the boundary behavior of $\tilde{d}$ except when w or τw is near the ray r described by

$$\operatorname{Re}(w) = \tfrac{1}{2}, \quad \operatorname{Im}(w) \geq \tfrac{\sqrt{15}}{2}\, i.$$

The image of the two sides of this ray is not so neatly described, but it does not represent a true boundary for the family of hyperbolic structures on $S^3 - K$, as w crosses r from right to left, for instance, z crosses the real line in the interval $(0, \frac{1}{2})$. For a while, a hyperbolic structure can be constructed from the positively oriented simplex determined by w and the negatively oriented simplex determined by z, by cutting the z-simplex into pieces which are subtracted from the w-simplex to leave a polyhedron P. P is then identified to give a hyperbolic structure for $S^3 - K$. 4.21

For this reason, we give only a rough sketch of the boundary behavior of $\tilde{d}$ near r or $\tau(r)$:



Since the image of $\tilde{d}$ in $\hat{\mathbb{R}}^2$ does not contain the origin, and since $\tilde{d}$ sends a curve winding once around the boundary of R to the curve $abcd$ in $\hat{\mathbb{R}}^2$, it follows that the image of $\tilde{d}(R)$ contains the exterior of this curve. 4.22

In particular

Theorem 4.7. *Every manifold obtained by Dehn surgery along the figure-eight knot K has a hyperbolic structure, except the six manifolds:*

$$(S^3 - K)_{(\mu,\lambda)} = (S^3 - K)_{(\pm\mu,\pm\lambda)}$$

where (μ, λ) is $(1,0)$, $(0,1)$, $(1,1)$, $(2,1)$, $(3,1)$ or $(4,1)$.

The equation

$$(S^3 - K)_{(\alpha,\beta)} = (S^3 - K_{(-\alpha,\beta)}$$

follows from the existence of an orientation reversing homeomorphism of $S^3 - K$.

I first became interested in studying these examples by the beautiful ideas of Jørgensen (compare Jørgensen, "Compact three-manifolds of constant negative curvature fibering over the circle," *Annals* 106 (1977) 61-72). He found the hyperbolic structures corresponding to the ray $\mu = 0$, $\lambda > 1$, and in particular, the integer and half-integer (!) points along this ray, which determine discrete groups.

The statement of the theorem is meant to suggest, but not imply, the true fact that the six exceptions do *not* have hyperbolic structures. Note that at least

$$S^3 = (S^3 - K)_{(1,0)}$$

does not admit a hyperbolic structure (since $\pi_1(S^3)$ is finite). We shall arrive at an understanding of the other five exceptions by studying the way the hyperbolic structures are degenerating as (μ, λ) tends to the line segment

$$\overline{(-4,1),(4,1)}.$$

4.8. Degeneration of hyperbolic structures.

Definition 4.8.1. A codimension-k foliation of an n-manifold M is a $\mathcal{G}$-structure, on M, where $\mathcal{G}$ is the pseudogroup of local homeomorphisms of $\mathbb{R}^{n-k} \times \mathbb{R}^k$ which 4.23
have the local form

$$\phi(x,y) = (f(x,y),\, g(y)).$$

In other words, $\mathcal{G}$ takes horizontal $(n-k)$-planes to horizontal $(n-k)$-planes. These horizontal planes piece together in M as $(n-k)$-submanifolds, called the *leaves* of the foliation. M, like a book without its cover, is a disjoint union of its leaves.

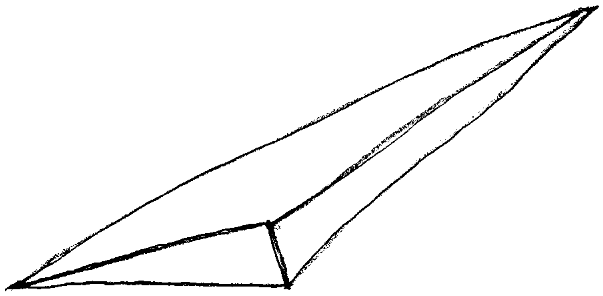
For any pseudogroup $\mathcal{H}$ of local homeomorphisms of some k-manifold N^k, the notion of a codimension-k foliation can be refined:

DEFINITION 4.8.2. An $\mathcal{H}$-*foliation* of a manifold M^n is a $\mathcal{G}$-structure for M^n, where $\mathcal{G}$ is the pseudogroup of local homeomorphisms of $\mathbb{R}^{n-k} \times N^k$ which have the local form

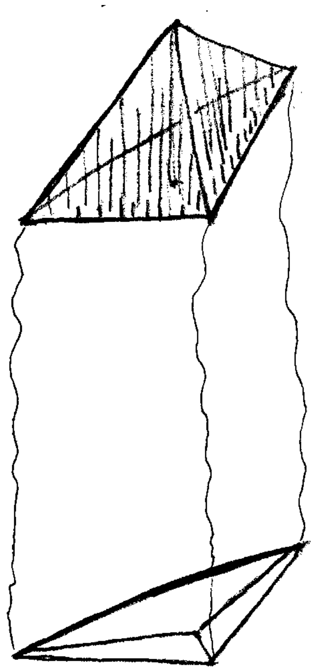
$$\phi(x, y) = (f(x, y),\, g(y))$$

with $g \in \mathcal{H}$. If $\mathcal{H}$ is the pseudogroup of local isometries of hyperbolic k-space, then an $\mathcal{H}$-foliation shall, naturally, be called a codimension-k hyperbolic foliation. A hyperbolic foliation determines a hyperbolic structure for each k-manifold transverse to its leaves.

When w tends in the region $R \subset \mathbb{C}$ to a point $\mathbb{R} - [0, 1]$, the w-simplex and the z-simplex are both flattening out, and in the limit they are flat: 4.24



If we regard these flat simplices as projections of nondegenerate simplices A and B (with vertices deleted), this determines codimension-2 foliations on A and B, whose leaves are preimages of points in the flat simplices:



4.25

A and B glue together (in a unique way, given the combinatorial pattern) to yield a hyperbolic foliation on $S^3 - K$. You should satisfy yourself that the gluing consistency conditions for the hyperbolic foliation near an edge result as the limiting case of the gluing conditions for the family of squashing hyperbolic structures.

The notation of the developing map extends in a straightforward way to the case of an $\mathcal{H}$-foliation on a manifold M, when $\mathcal{H}$ is the set of restrictions of a group J of real analytic diffeomorphisms of N^k; it is a map

$$D : \tilde{M}^n \to N^k.$$

Note that D is not a local homeomorphism, but rather a local projection map, or a submersion. The holonomy

$$H : \pi_1(M) \to J$$

is defined, as before, by the equation

$$D \circ T_\alpha = H(\alpha) \circ D.$$

Here is the generalization of proposition 3.6 to $\mathcal{H}$-foliations. For simplicity, assume that the foliation is differentiable:

Proposition 4.8.1. *If J is transitive and if the isotropy subgroups J_x are compact, then the developing map for any $\mathcal{H}$-foliation $\mathcal{F}$ of a closed manifold M is a fibration* labeled 4.8.1def 4.26

$$D : \tilde{M}^n \to N^k.$$