cover of M_2, so that $\pi_1(M_4)$ is a normal subgroup of $\pi_1(M_2)$. Consider the action of $\pi_1(M_2)$ on $\pi_1(M_4)$ by conjugation. $\pi_1(M_4)$ has a trivial center, so in other words the action of $\pi_1(M_4)$ on itself is effective. Then for every $\alpha \in \pi_1(M_2)$, since some power of α^k is in $\pi_1(M_4)$, α must conjugate $\pi_1(M_4)$ non-trivially. Thus $\pi_1(M_2)$ is isomorphic to a group of automorphisms of $\pi_1(M_4)$, so by Mostow's theorem it is a discrete group of isometries of H^n. □

In the three-dimensional case, it seems likely that M_1 would actually be hyperbolic. Waldhausen proved that two Haken manifolds which are homotopy equivalent are homeomorphic, so this would follow whenever M_1 is Haken. There are some sorts of properties of three-manifolds which do not change under passage to a finite-sheeted cover. For this reason (and for its own sake) it would be interesting to have a better understanding of the commensurability relation among three-manifolds. This is difficult to approach from a purely topological point of view, but there is a great deal of information about commensurability given by a hyperbolic structure. For instance, in the case of a complete non-compact 6.31

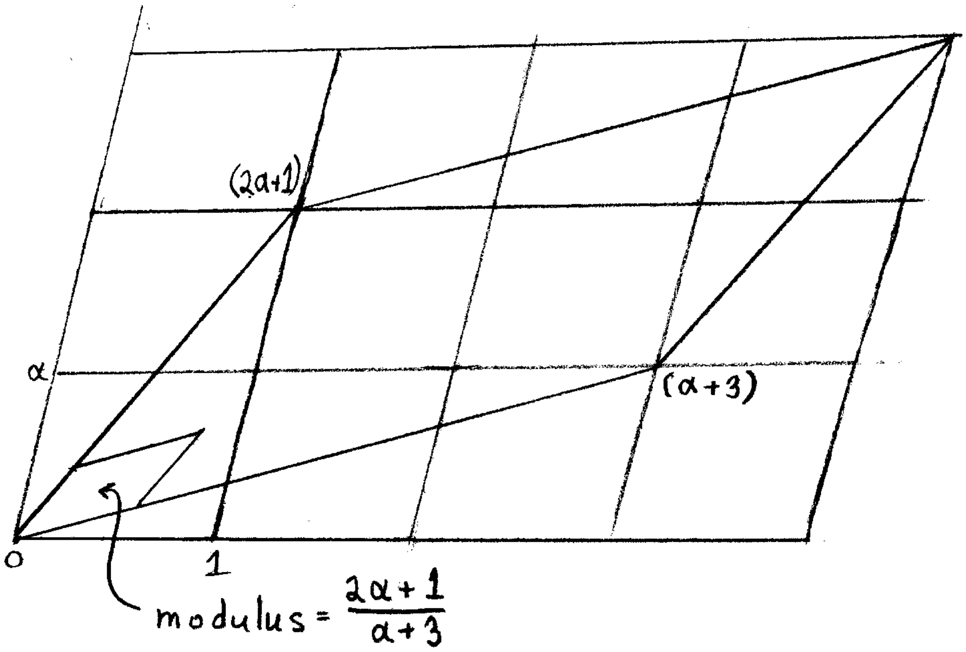
hyperbolic three-manifold M of finite volume, each cusp gives a canonical Euclidean structure on a torus, well-defined up to similarity. A convenient invariant for this structure is obtained by arranging M so that the cusp is the point at ∞ in the upper half space model and one generator of the fundamental group of the cusp is a translation $z \mapsto z+1$. A second generator is then $z \mapsto z+\alpha$. The set of complex numbers $\alpha_1 \dots \alpha_k$ corresponding to various cusps is an invariant of the commensurability class of M well-defined up to the equivalence relation

$$\alpha_i \sim \frac{n\alpha_i + m}{p\alpha_i + q},$$

where

$$n, m, pq \in \mathbb{Z}, \qquad \begin{vmatrix} n & m \\ p & q \end{vmatrix} \neq 0.$$

$(n, m, p$ and q depend on $i)$.



In particular, if $\alpha \sim \beta$, then they generate the same fields $\mathbb{Q}(\alpha) = \mathbb{Q}(\beta)$. 6.32

Note that these invariants α_i are always algebraic numbers, in view of

PROPOSITION 6.7.4. *If* Γ *is a discrete subgroup of* $\mathrm{PSL}(2, \mathbb{C})$ *such that* H^3/Γ *has finite volume, then* Γ *is conjugate to a group of matrices whose entries are algebraic.*

PROOF. This is another easy consequence of Mostow's theorem. Conjugate Γ so that some arbitrary element is a diagonal matrix

$$\begin{bmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{bmatrix}$$

and some other element is upper triangular,

$$\begin{bmatrix} \lambda & x \\ 0 & \lambda^{-1} \end{bmatrix}.$$

The component of Γ in the algebraic variety of representations of Γ having this form is 0-dimensional, by Mostow's theorem, so all entries are algebraic numbers. $\square$

One can ask the more subtle question, whether all entries can be made algebraic integers. Hyman Bass has proved the following remarkable result regarding this question:

THEOREM 6.7.5 (Bass). *Let* M *be a complete hyperbolic three-manifold of finite volume. Then either* $\pi_1(M)$ *is conjugate to a subgroup of* $\mathrm{PSL}(2, \mathcal{O})$, *where* $\mathcal{O}$ *is the ring of algebraic integers, or* M *contains a closed incompressible surface (not homotopic to a cusp).*

The proof is out of place here, so we omit it. See Bass. As an example, very few knot complements seem to contain non-trivial closed incompressible surfaces. The property that a finitely generated group Γ is conjugate to a subgroup of $\mathrm{PSL}(2, \mathcal{O})$
is equivalent to the property that the *additive* group of matrices generated by Γ 6.33
is finitely generated. It is also equivalent to the property that the trace of every element of Γ is an algebraic integer. It is easy to see from this that every group commensurable with a subgroup of $\mathrm{PSL}(2, \mathcal{O})$ is itself conjugate to a subgroup of $\mathrm{PSL}(2, \mathcal{O})$. (If $\mathrm{Tr}\,\gamma^n = a$ is an algebraic integer, then an eigenvalue λ of γ satisfies $\lambda^{2n} - a\lambda^n + 1 = 0$. Hence λ, λ^{-1} and $\mathrm{Tr}\,\gamma = \lambda + \lambda^{-1}$ are algebraic integers).

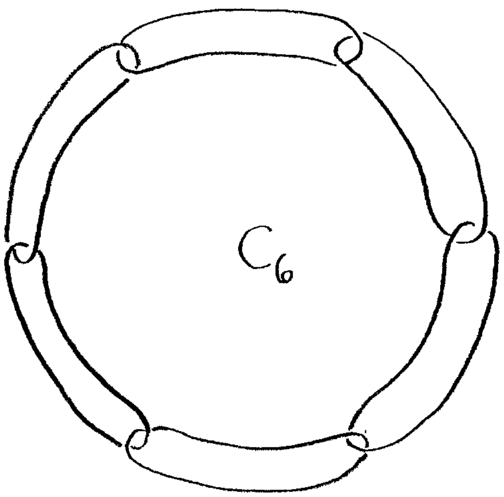
If two manifolds are commensurable, then their volumes have a rational ratio. We shall see examples in the next section of incommensurable manifolds with equal volume.

Questions 6.7.6. Does every commensurability class of discrete subgroups of $\mathrm{PSL}(2, \mathbb{C})$ have a finite collection of maximal groups (up to isomorphism)?

Is the set of volumes of three-manifolds in a given commensurability class a discrete set, consisting of multiples of some number V_0?

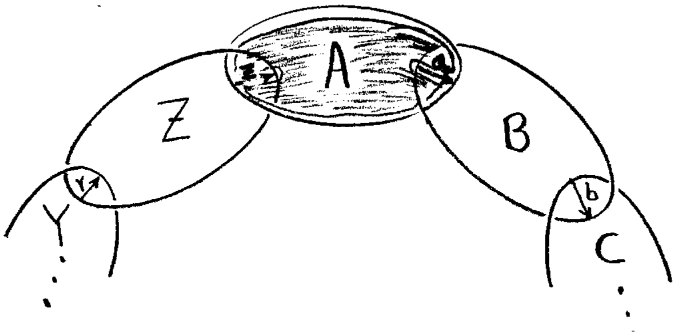
6.8. Some Examples

Example 6.8.1. Consider the k-link chain C_k pictured below:

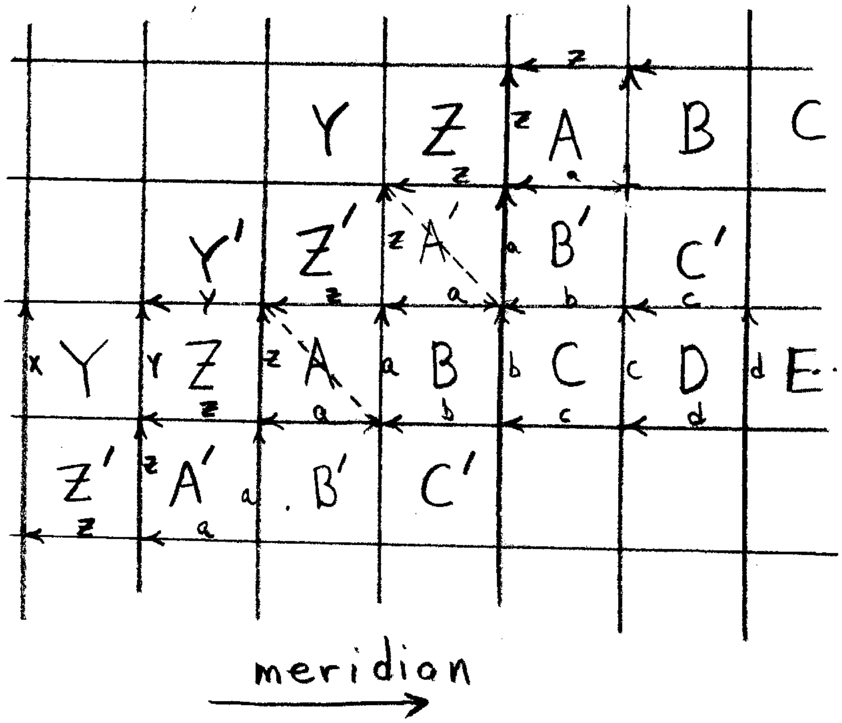


6.34

If each link of the chain is spanned by a disk in the simplest way, the complement of the resulting complex is an open solid torus.

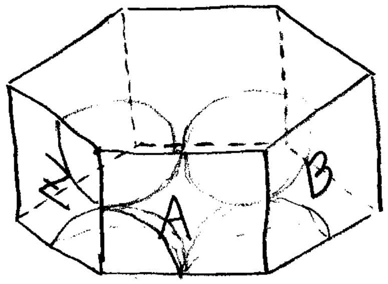


$S^3 - C_k$ is obtained from a solid torus, with the cell division below on its boundary, by deleting the vertices and identifying.

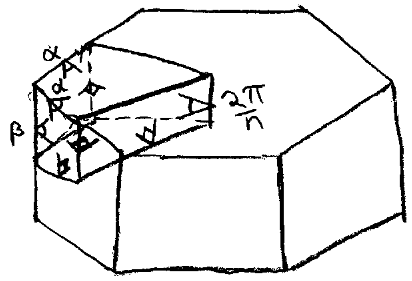


6.35

To construct a hyperbolic structure for $S^3 - C_k$, cut the solid torus into two drums.

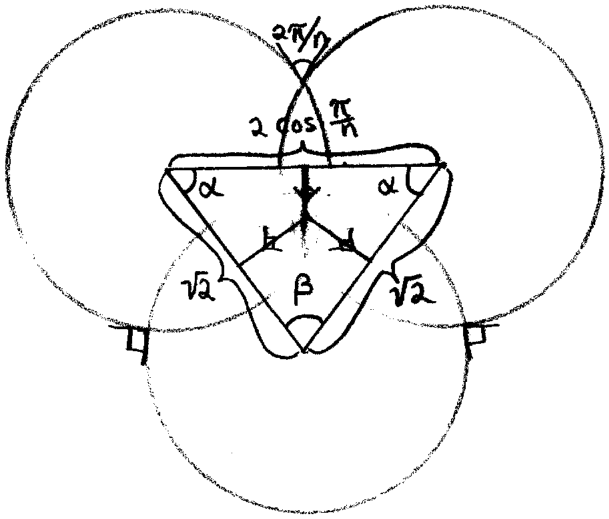


Let P be a regular k-gon in H^3 with all vertices on S^2_∞. If P' is a copy of P obtained by displacing P along the perpendicular to P through its center, then P' and P can be joined to obtain a regular hyperbolic drum. The height of P' must be adjusted so that the reflection through the diagonal of a rectangular side of the drum is an isometry of the drum. If we subdivide the drum into $2k$ pieces as shown,



6.36

the condition is that there are horospheres about the ideal vertices tangent to three faces. Placing the ideal vertex at ∞ in upper half-space, we have a figure bounded by three vertical Euclidean planes and three Euclidean hemispheres of equal radius r. Here is a view from above:



From this figure, we can compute the dihedral angles α and β of the drum to be

$$\alpha = \operatorname{arc\,cos}\left(\frac{\cos \pi/k}{\sqrt{2}}\right), \quad \beta = \pi - 2\alpha.$$

Two copies of the drum with these angles can now be glued together to give a hyperbolic structure on $S^3 - C_k$. (Note that the total angle around an edge is $4\alpha + 2\beta = 2\pi$. Since the horospheres about vertices are matched up by the gluing maps, we obtain a complete hyperbolic manifold).

From Milnor's formula (6), p. 7.15, for the volume, we can compute some values.

6.37

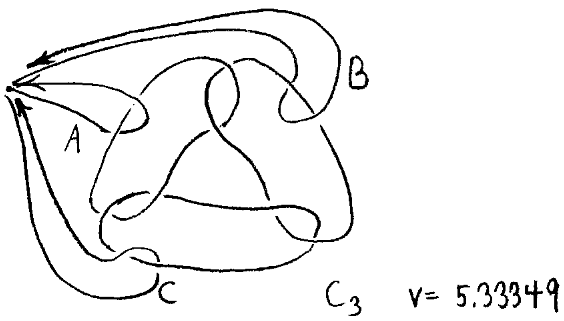
k	$v(S^3 - C_k)$	$v(S^3 - C_k)/k$	
2	0	0	(Seifert fiber space)
3	5.33349	1.77782	$\sim \mathrm{PSL}(2, \mathcal{O}_7)$
4	10.14942	2.53735	$\sim \mathrm{PSL}(2, \mathcal{O}_3)$
5	14.60306	2.92061	
6	18.83169	3.13861	
7	22.91609	3.27373	
10	34.691601	3.4691601	
50	182.579859	3.65159719	
200	732.673784	3.66336892	
1000	3663.84264	3.66384264	
8000	29310.8990	3.66386238	
∞	∞	3.66386238	Whitehead link

Note that the quotient space $(S^3 - C_k)/\mathbb{Z}_k$ by the rotational symmetry of C_k is obtained by generalized Dehn surgery on the White head link W, so the limit of 6.38
$v(C_k)/k$ as $k \to \infty$ is the volume of $S^3 - W$.

Note also that whenever k divides l, then there is a degree $\frac{l}{k}$ map from $S^3 - C_l$ to $S^3 - C_k$. This implies that $v(S^3 - C_l)/l > v(S^3 - C_k)/k$. In fact, from the table it is clear that these numbers are strictly increasing with k.

The cases $k = 3$ and 4 have particular interest.

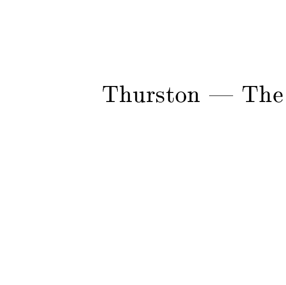
EXAMPLE 6.8.2. The volume of $S^3 - C_3$ per cusp has a particularly low value (1.7778). The holonomy of the hyperbolic structure can be described by



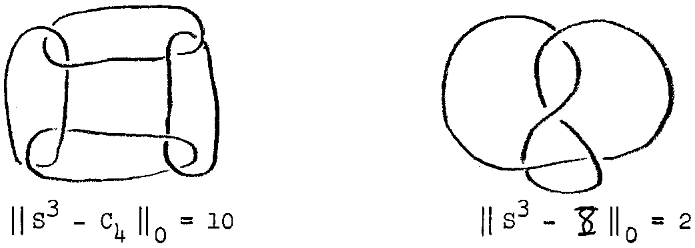
$$H(A) = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$$
$$H(B) = \begin{bmatrix} 1+\alpha & \alpha \\ -\alpha & 1-\alpha \end{bmatrix}$$
$$H(C) = \begin{bmatrix} 1 & 0 \\ -\alpha & 1 \end{bmatrix}$$

where $\alpha = \frac{-1+\sqrt{-7}}{2}$. Thus $\pi_1(X^3 - C_3)$ is a subgroup of $\mathrm{PSL}(2, \mathcal{O}_7)$ where $\mathcal{O}_d$ is the ring of integers in $\mathbb{Q}\sqrt{-d}$. See §7.4. Referring to Humbert's formula 7.4.1, we find $v(H^3/\,\mathrm{PSL}(2, \mathcal{O}_7) = .8889149\ldots$, so $\pi_1(S^3 - C_3)$ has index 6 in this group.

EXAMPLE 6.8.3. When $k = 4$, the rectangular-sided drum becomes a cube with all dihedral angles 60°. This cube may be subdivided into five regular ideal tetrahedra: 6.39



Thus $S^3 - C_4$ is commensurable with S^3- figure eight knot, since $\pi_1(S^3 - C_4)$ preserves a tiling of H^3 by regular ideal tetrahedra.

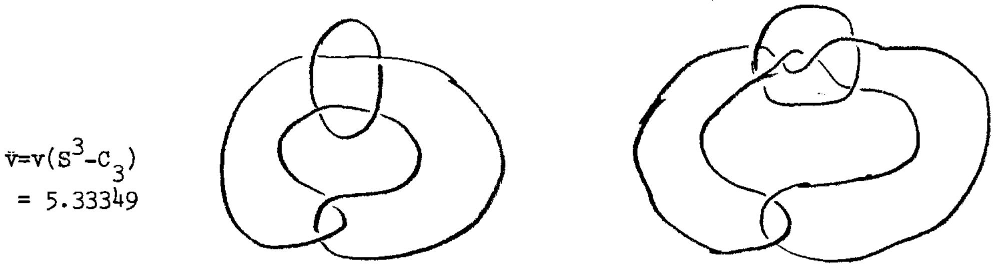


commensurable with $\mathrm{PSL}(2, \mathcal{O}_3)$

$S^3 - C_k$ is homeomorphic to many other link complements, since we can cut along any disk spanning a component of C_k, twist some integer number of times and glue back to obtain a link with a complement homeomorphic to that of C_k. Furthermore, if we glue back with a half-integer twist, we obtain a link whose complement is hyperbolic with the same volume as $S^3 - C_k$. This follows since twice-punctured spanning disks are totally geodesic thrice-punctured spheres in the hyperbolic structure of $S^3 - C_k$. The thrice-punctured sphere has a unique hyperbolic structure, and all six isotopy classes of diffeomorphisms are represented by isometries. 6.40

Using such operations, we obtain these examples for instance:

Example 6.8.4.



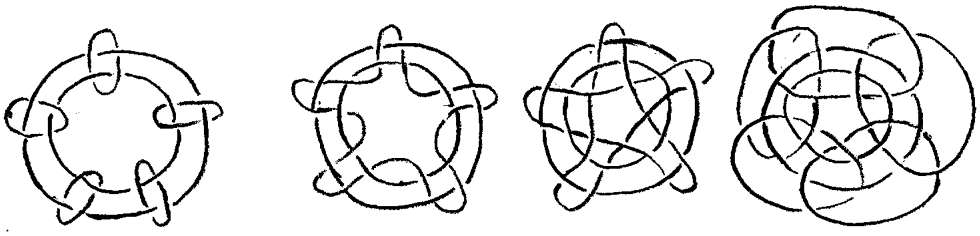
commensurable with C_3

The second link has a map to the figure-eight knot obtained by erasing a component of the link. Thus, by 6.5.6, we have

$$v(S^3 - C_3) = 5.33340\ldots > 2.02988 = v(S^3 - \text{figure eight knot}).$$

These links are commensurable with C_3, since they give rise to identical tilings of H^3 by drums. As another example, the links below are commensurable with C_{10}:

Example 6.8.5.



$k = 5$ Commensurable with C_{10} $\quad v = 34.69616$

6.41

The last three links are obtained from the first by cutting along 5-times punctured disks, twisting, and gluing back. Since this gluing map is a diffeomorphism of the surface which extends to the three-manifold, it must come from an isometry of a 6-punctured sphere in the hyperbolic structure. (In fact, this surface comes from the top of a 10-sided drum).

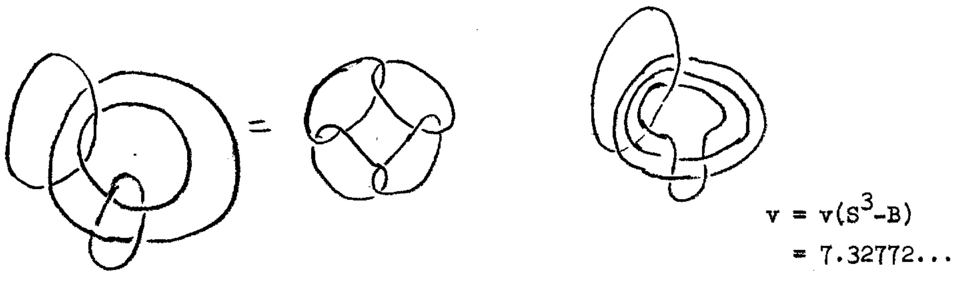
The compex modulus associated with a cusp of C_n is

$$\frac{1}{2}\quad\left(1+\sqrt{\frac{1+\sin^2\frac{\pi}{n}}{\cos^2\frac{\pi}{n}}}\;i\right).$$

Clearly we have an infinite family of incommensurable examples.

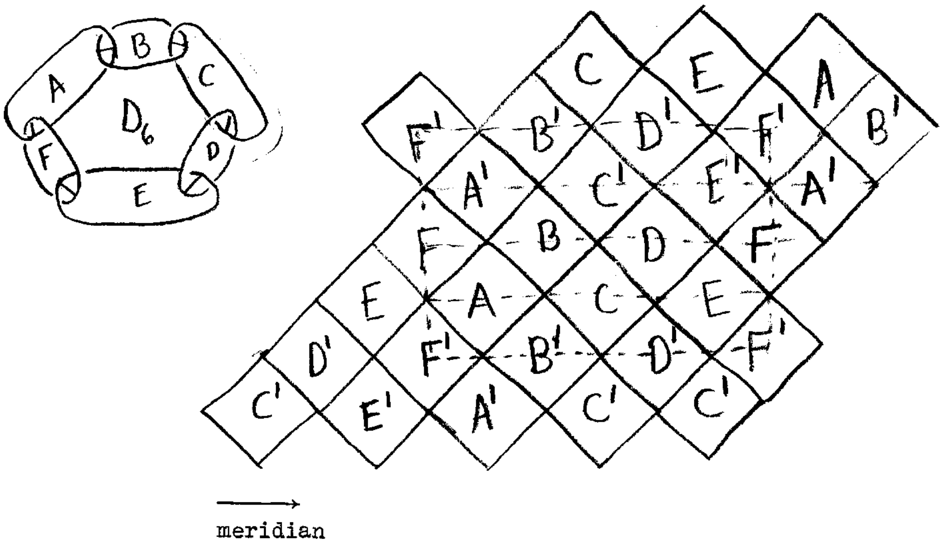
By passing to the limit $k \to \infty$ and dividing by $\mathbb{Z}$, we get these links commensurable with $S^3 - W$ and $S^3 - B$, for instance:

EXAMPLE 6.8.6.

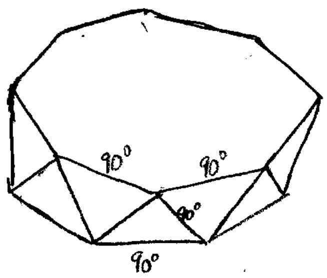


Many other chains, with different amounts of twist, also have hyperbolic structures. They all are obtained, topologically, by identifying faces of a tiling of the boundary of a solid torus by rectangles. Here is another infinite family $D_{2k}(\geq 3)$ which is easy to compute: 6.42

EXAMPLE 6.8.7.



Hyperbolic structures can be realized by subdividing the solid torus into 4 drums with triangular sides:



6.43

Regular drums with all dihedral angles 90° can be glued together to give $S^3 - D_k$. By methods similar to Milnor's in 7.3, the formula for the volume is computed to be

$$v(S^3 - D_{2k}) = 8k\big(\mu(\tfrac{\pi}{4} + \tfrac{\pi}{2k}) + \mu(\tfrac{\pi}{4} - \tfrac{\pi}{2k})\big).$$

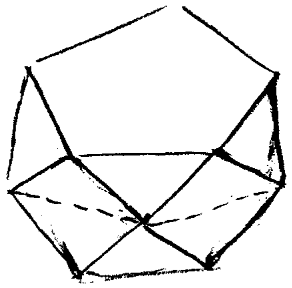
Thus we have the values

k	$v(S^3 - D_{2k})$	$v(S^3 - D_{2k})/(2k)$
3	14.655495	2.44257
4	24.09218	3.01152
5	32.55154	3.25515
6	40.59766	3.38314
100	732.750	3.66288
1000	7327.705	3.66386
∞	∞	3.66386

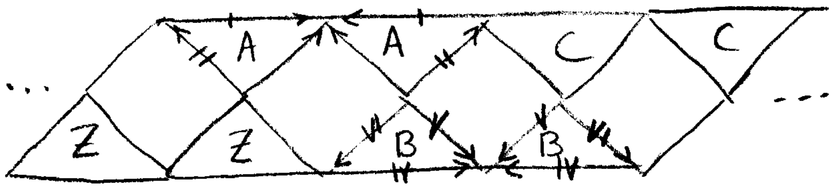
The cases $k = 3$ and $k = 4$ have algebraic significance. They are commensurable with $\mathrm{PSL}(2, \mathcal{O}_1)$ nad $\mathrm{PSL}(2, \mathcal{O}_2)$, respectively. When $k = 3$, the drum is an octahedron and $v(S^3 - D_{2k}) = 4v(S^3 - W)$.

Note that the volume of $(S^3 - D_{12})$ is 20 times the volume of the figure-eight knot complement. 6.44

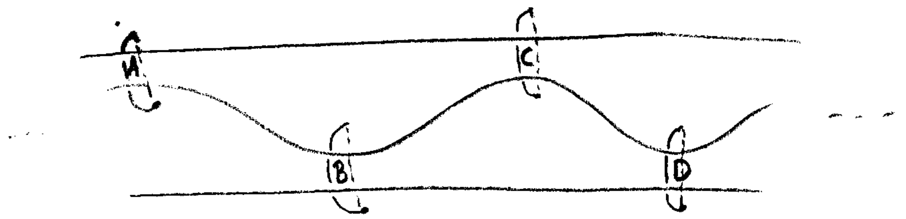
Two copies of the triangular-sided drum form this figure:



The faces may be glued in other patterns to obtain link complements. For instance, if k is even we can first identify



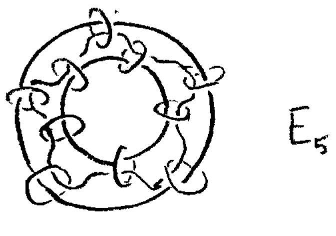
the triangular faces, to obtain a ball minus certain arcs and curves on the boundary.



6.45

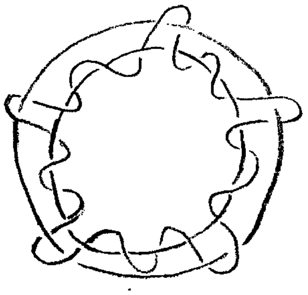
If we double this figure, we obtain a complete hyperbolic structure for the complement of this link, E_l:

Example 6.8.8.

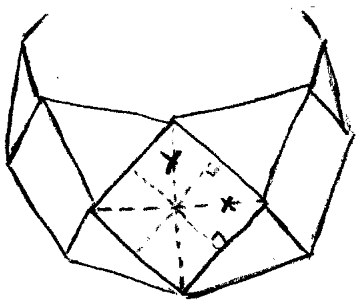


Alternatively, we can identify the boundary of the ball to obtain

Example 6.8.9.



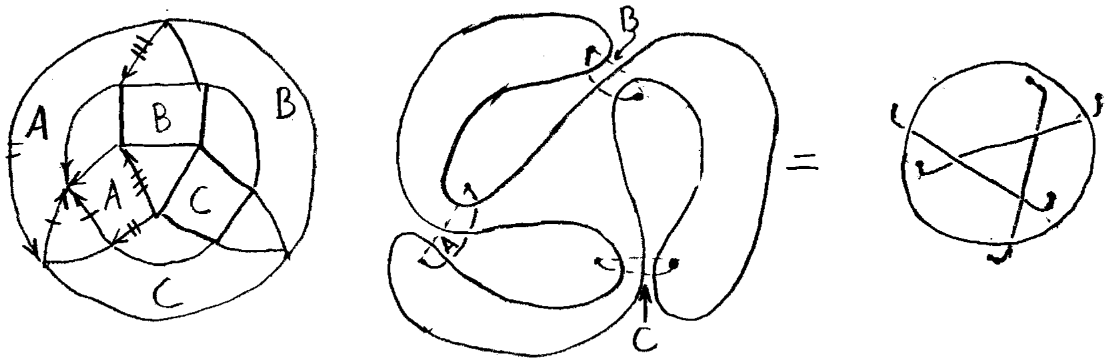
In these examples, note that the rectangular faces of the doubled drums



6.46

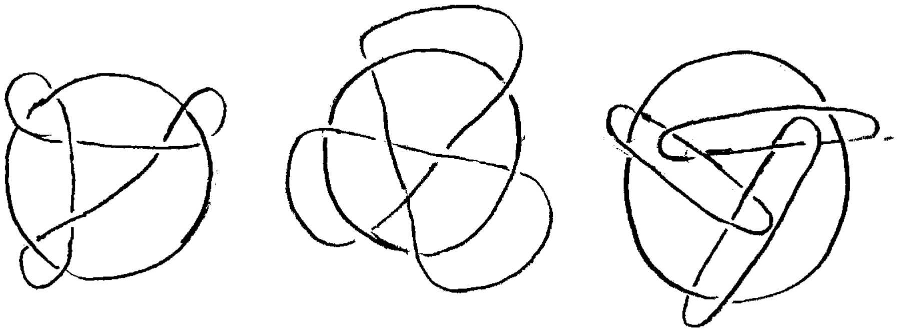
have complete symmetry, and some of the link complements are obtained by gluing maps which interchange the diagonals, while others preserve them. These links are generally commensurable even when they have the same volume; this can be proven by computing the moduli of the cusps.

There are many variations. Two copies of the drum with 8 triangular faces, glued together, give a cube with its corners chopped off. The 4-sided faces can be glued, to obtain the ball minus these arcs and curves:



The two faces of the ball may be glued together (isometrically) to give any of these link complements: 6.47

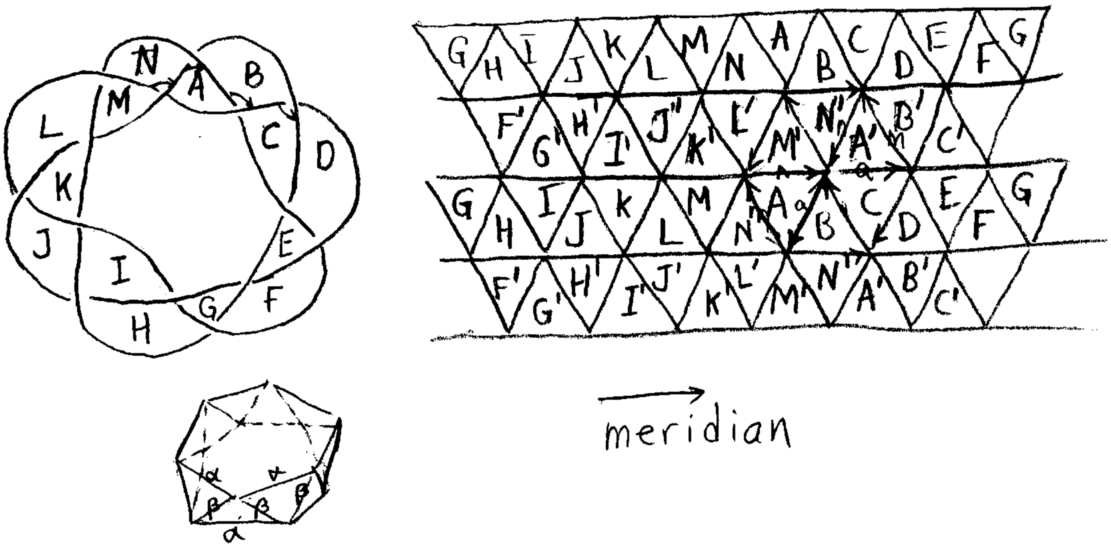
Example 6.8.10.



$$v = 12.04692 = \tfrac{1}{2}v(S^3 - D_8) > v(C^3) \text{ (commensurable with } \mathrm{PSL}(2, \mathbb{Z}\sqrt{-2}))$$

The sequence of link complements, F_n below can also be given hyperbolic structures obtained from a third kind of drum:

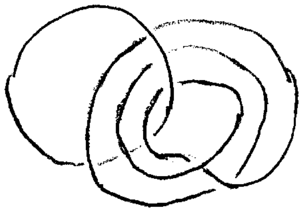
Example 6.8.11.



6.48

The regular drum is determined by its angles α and $\beta = \pi - \alpha$. Any pair of angles works to give a hyperbolic structure; one verifies that when the angle $\alpha = \arccos(\cos\frac{\pi}{2n} - \frac{1}{2})$, the hyperbolic structure is complete. The case $n = 1$ gives a trivial knot. In the case $n = 2$, the drums degenerate into simplices with 60° angles, and we obtain once more the hyperbolic structure on F_2 = figure eight knot. When $n = 3$, the angles are 90°, the drums become octahedra and we obtain $F_3 = B$. Passing to the limit $n = \infty$, and dividing by $\mathbb{Z}$, we obtain the following link, whose complement is commensurable with S^3 − figure eight knot:

Example 6.8.12.



$$v = 4 - 05977\ldots$$

With these examples, many maps between link complements may be constructed. The reader should experiment for himself. One gets a feeling that volume is a very good measure of the complexity of a link *complement*, and that the ordinal structure is really inherent in three-manifolds.