William P. Thurston

The Geometry and Topology of Three-Manifolds

Electronic version 1.1 - March 2002

http://www.msri.org/publications/books/gt3m/

This is an electronic edition of the 1980 notes distributed by Princeton University. The text was typed in TEX by Sheila Newbery, who also scanned the figures. Typos have been corrected (and probably others introduced), but otherwise no attempt has been made to update the contents. Genevieve Walsh compiled the index.

Numbers on the right margin correspond to the original edition's page numbers.

Thurston's *Three-Dimensional Geometry and Topology*, Vol. 1 (Princeton University Press, 1997) is a considerable expansion of the first few chapters of these notes. Later chapters have not yet appeared in book form.

Please send corrections to Silvio Levy at levy@msri.org.

CHAPTER 7

Computation of volume

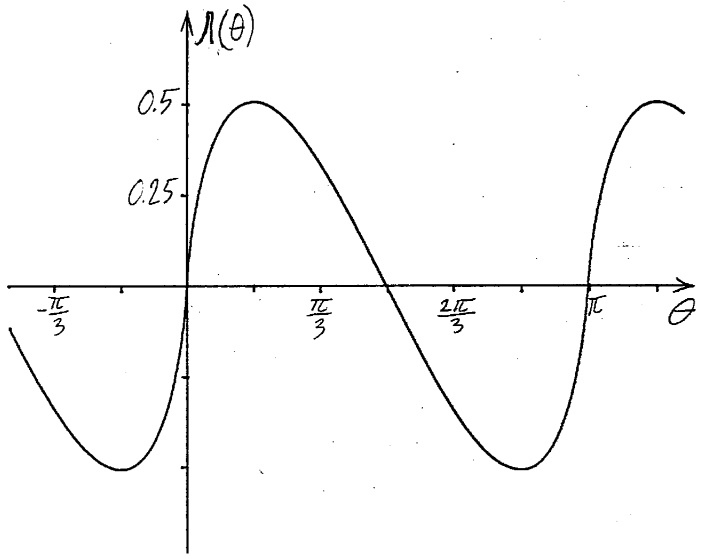
by J. W. Milnor

7.1. The Lobachevsky function $Л(\theta)$.

This preliminary section will decribe analytic properties, and conjecture number theoretic properties, for the function

$$Л(\theta) = -\int_0^\theta \log|2\sin u|\,du.$$

Here is the graph of this function:



Thus the first derivative $Л'(\theta)$ is equal to $-\log|2\sin\theta|$, and the second derivative $Л''(\theta)$ is equal to $-\cot\theta$. I will call $Л(\theta)$ the *Lobachevsky function.* (This name is not quite accurate historically, since Lobachevsky's formulas for hyperbolic volume were expressed rather in terms of the function

$$\int_0^\theta \log(\sec u)\,du = Л(\theta + \pi/2) + \theta\log 2$$

for $|\theta| \leq \pi/2$. However our function $\pi(\theta)$ is clearly a close relative, and is more convenient to work with in practice. Compare Clausen [3]). references are at end of chapter

Another close relative of $\text{л}(\theta)$ is the *dilogarithm function*

$$\psi(z) = \sum_{n=1}^{\infty} z^n/n^2 \quad \text{for } |z| \leq 1,$$

which has been studied by many authors. (See for example [1], [2], [8], [9], [12], [13].) Writing

$$\psi(z) = -\int_0^z \log(1-w)\, dw/w$$

(where $|w| \leq 1$, the substitution $w = e^{2i\theta}$ yields 7.2

$$\log(1-w)\, dw/w = \big(\pi - 2\theta + 2i \log(2\sin\theta)\big)\, d\theta$$

for $0 < \theta < \pi$, hence

$$\psi(e^{2i\theta}) - \psi(1) = -\theta(\pi - \theta) + 2i\text{л}(\theta)$$

for $0 \leq \theta \leq \pi$. Taking the imaginary part of both sides, this proves the following: 7.3

LEMMA 7.1.2. *The Lobachevsky function has uniformly convergent Fourier series expansion*

$$\text{л}(\theta) = \frac{1}{2}\sum_{n=1}^{\infty} \sin(2n\theta)/n^2.$$

Apparently, we have proved this formula only for the case $0 \leq \theta \leq \pi$. However, this suffices to show that $\text{л}(0) = \text{л}(\pi) = 0$. Since the derivative

$$d\text{л}(\theta)/d\theta = -2\log|\sin 2\theta|$$

is periodic of period π, this proves the following.

LEMMA 7.1.3. *The function $\text{л}(\theta)$ is itself periodic of period π, and is an odd function, that is, $\text{л}(-\theta) = -\text{л}(\theta)$.*

It follows that the equation in 7.1.2 is actually valid for all values of θ.

The equation $z^n - 1 = \prod_{j=0}^{n-1}(z - e^{-2\pi ij/n})$ for $z = e^{2\pi iu}$ leads to the trigonometric identity

$$2\sin nu = \prod_{j=0}^{n-1} 2\sin(u + j\pi/n).$$

Integrating the logarithm of both sides and multiplying by n, this yields the following for $n \geq 1$, and hence for all n.

Lemma 7.1.4. *The identity*

$$\text{л}(n\theta) = \sum_{j \bmod n} n\,\text{л}(\theta + j\pi/n)$$

is valid for any integer $n \neq 0$. (Compare [14].)

Here the sum is to be taken over all residue classes modulo $|n|$. Thus for $n = 2$ we get 7.4

$$\tfrac{1}{2}\text{л}(2\theta) = \text{л}(\theta) + \text{л}(\theta + \pi/2),$$

or equivalently

$$\tfrac{1}{2}\text{л}(2\theta) = \text{л}(\theta) - \text{л}(\pi/2 - \theta).$$

As an example, for $\theta = \pi/6$:

$$\tfrac{3}{2}\text{л}(\pi/3) = \text{л}(\pi/6).$$

(It is interesting to note that the function $\text{л}(\theta)$ attains its maximum,

$$\text{л}(\pi/6) = .5074\ldots,$$

at $\theta = \pi/6$.)

It would abe interesting to know whether there are any other such linear relations between various values of $\text{л}(\theta)$ with rational coefficients. Here is an explicit guess.

Conjecture (A). Restricting attention to angles θ which are rational multiples of π, every rational linear relation between the real numbers $\text{л}(\theta)$ is a consequence of 7.1.3 and 7.1.4.

(If we consider the larger class consisting of all angles θ for which $e^{i\theta}$ is algebraic then it definitely is possible to give other Q-linear relations. Compare [4].)

A different but completely equivalent formulation is the following. 7.5

Conjecture (B). Fixing some denominator $N \geq 3$, the real numbers $\text{л}(\pi j/N)$ with j relatively prime to N and $0 < j < N/2$ are linearly independent over the rationals.

These numbers span a rational vector space v_N, conjectured to have dimension $\phi(N)/2$, where it is easy to check that $v_N \subset v_M$ whenever N divides M. Quite likely the elements $\text{л}(\pi j/N)$ with $1 \leq j \leq \phi(N)/2$ would provide an alternative basis for this vector space.

I have tested these conjectures to the following extent. A brief computer search has failed to discover any other linear relations with small integer coefficients for small values of N.

To conclude this section, here is a remark about computation. The Fourier series 7.1.2 converges rather slowly. In order to get actual numerical values for $\text{Л}(\theta)$, it is much better to work with the series

$$\text{Л}(\theta) = \theta\left(1 - \log|2\theta| + \sum_{n=1}^{\infty} \frac{B_n}{2n}\,\frac{(2\theta)^{2n}}{(2n+1)!}\right),$$

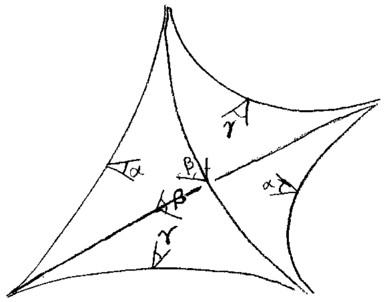
which is obtained by twice integrating the usual Laurent series expansion for the cotangent of θ. Here

$$B_1 = \tfrac{1}{6}, \quad B_2 = \tfrac{1}{30}, \ldots$$

are Bernoulli numbers. This series converges for $|\theta| = \pi$, and hence converges reasonably well for $|\theta| \leq \pi/2$. 7.6

7.2

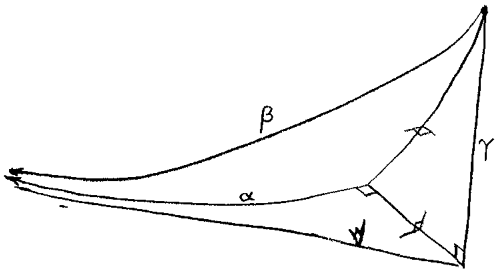
Having discussed the Lobachevsky function, we will see how it arises in the computation of hyperbolic volumes. The first case is the ideal simplex, i.e., a tetrahedron whose vertices are at ∞ and whose edges are geodesics which converge to the vertices at ∞. Such a simplex is determined by the dihedral angles formed between pairs of faces. The simplex intersects any small horosphere based at a vertex in a triangle whose interior angles are precisely the three dihedral angles along the edges meeting at that vertex. Since a horosphere is isometric to a Euclidean plane, the sum of the dihedral angles at an infinite vertex equals 2π. It follows by an easy computation that the dihedral angles of opposite edges are equal.



Call the three dihedral angles determining the simplex α, β, γ and denote the simplex by $\Sigma_{\alpha,\beta,\gamma}$. The main result of this section is:

THEOREM 7.2.1. *The volume of the simplex* $\sum_{\alpha,\beta,\gamma}$ *equals* $\text{Л}(\alpha) + \text{Л}(\beta) + \text{Л}(\gamma)$.

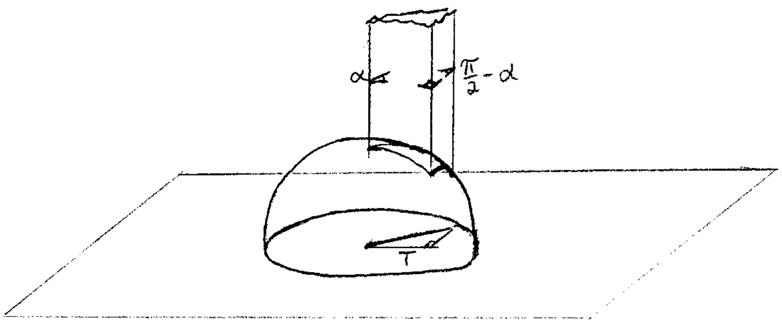
In order to prove this theorem a preliminary computation is necessary. Consider the simplex $S_{\alpha,\beta,\gamma}$ pictured below, with three right dihedral angles and three other 7.7
dihedral angles α, β, γ and suppose that one vertex is at infinity. (Thus $\alpha+\beta = \pi/2$.)



It turns out that any simplex can be divided by barycentric subdivision into simplices with three right angles so this is a natural object to consider. The decomposition of $\sum_{\alpha,\beta,\gamma}$ is demonstrated below, but first a computation, due to Lobachevsky.

LEMMA 7.2.2. *The volume of* $S_{\alpha,\pi/2-\alpha,\gamma}$ *equals* $\frac{1}{4}[\text{л}(\alpha+\gamma)+\text{л}(\alpha-\gamma)+2\text{л}(\pi/2-\alpha)]$.

PROOF. Consider the upper half-space model of H^3, and put the infinite vertex of $S_{\alpha,\pi/2-\alpha,\gamma}$ at ∞. The edges meeting that vertex are just vertical lines. Furthermore, assume that the base triangle lies on the unithemisphere (which is a hyperbolic plane). Recall that the line element for the hyperbolic metric in this model is $ds^2 = \frac{dx^2+dy^2+dz^2}{z^2}$ so that the volume element is $dV = \frac{dz\,dy\,dz}{z^3}$. Projecting the base triangle to the (x, y) plane produces a Euclidean triangle T with angles $\alpha, \pi/2-\alpha, \pi/2$, 7.8
which we may take to be the locus $0 \le x \le \cos\gamma, 0 \le y \le x\tan\alpha$, with γ as above.



REMARK. This projection of the unit hemisphere gives Klein's projective model for H^2. The angles between lines are not their hyperbolic angles; rather, they are the dihedral angles of corresponding planes in H^3.

Now it is necessary to compute

(1).
$$V = \int_{x,y\in T}\int\int_{z\ge\sqrt{1-x^2-y^2}} \frac{dx\,dy\,dz}{z^3}.$$

Integrating with respect to z gives

(2).
$$V = \int_T\int \frac{dx\,dy}{2(1-x^2-y^2)}.$$

Setting $a = \sqrt{1-x^2}$, we have 7.9

$$
(3).\qquad
\begin{aligned}
V &= \int_0^{\cos\gamma} dx \int_0^{x\tan\alpha} \frac{dy}{2(a^2-y^2)} = \int_0^{\cos\gamma} \left(\frac{x}{4a} \log \frac{a + x\tan\alpha}{a - x\tan\alpha} \right) \\
&= \int_0^{\cos\gamma} \frac{dx}{4a} \log \frac{2(a\cos\alpha + x\sin\alpha)}{2(a\cos\alpha - x\sin\alpha)}.
\end{aligned}
$$

If we set $x = \cos\theta$, then $a = \sqrt{1-x^2} = \sin\theta$ and $\frac{dx}{a} = -d\theta$. Then (3) becomes

$$
(4).\qquad
\begin{aligned}
V &= \frac{1}{4} \int_{\pi/2}^{\gamma} -d\theta \log \left(\frac{2\sin(\theta+\alpha)}{2\sin(\theta-\alpha)} \right) \\
&= \frac{1}{4} [\text{л}(\gamma+\alpha) - \text{л}(\gamma-\alpha) - \text{л}(\pi/2+\alpha) + \text{л}(\pi/2-\alpha)].
\end{aligned}
$$

Since $\text{л}(\gamma) - \alpha) = -\text{л}(\alpha-\gamma)$ and $\text{л}(\pi/2+\alpha) = -\text{л}(\pi/2-\alpha)$ by 7.1.3, this is the desired formula. $\square$

Suppose that two vertices are at infinity in $S_{a,\pi/2-\alpha,\gamma}$. Then $\alpha = \gamma$. The lemma above implies that volume

$$
(S_{\alpha,\pi/2-\alpha,a}) = \tfrac{1}{4}[\text{л}(2\alpha) + 2\text{л}(\pi/2-\alpha)].
$$

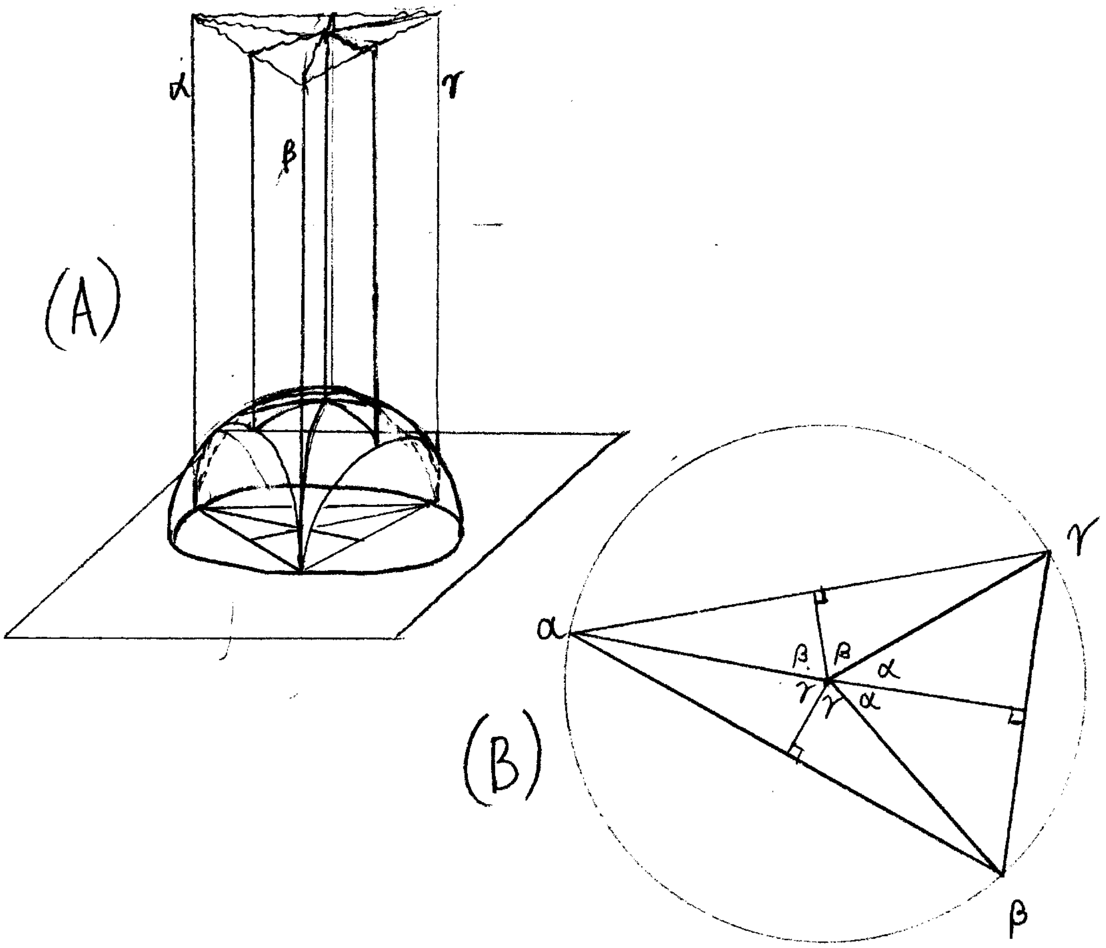
By lemmas 7.1.3 and 7.1.4

$$
\text{л}(\pi/2-\alpha) = -\text{л}(\pi/2+\alpha) \qquad \text{and} \qquad \text{л}(2\alpha) = 2\big(\text{л}(\alpha) + \text{л}(\alpha+\pi/2)\big)
$$

so that

$$
(5).\qquad \text{Volume}\,(S_{\alpha,\pi/2-\alpha,\alpha}) = \tfrac{1}{2}\text{л}(\alpha).
$$

To see how $\sum_{\alpha,\beta,\gamma}$ decomposes into simplices of the above type, consider the upper half-space model of H^3. Put one vertex at the point at infinity and the base 7.10
on the unit sphere. Drop the perpendicular from ∞ to the sphere and draw the perpendiculars from the intersection point x on the base to each of the three edges on the base. Connect x to the remaining three vertices. Taking the infinite cone on the lines in the base gives the decomposition. (See (A) below.) Projecting onto the (x, y) plane gives a triangle inscribed in the unit circle with x projected into its center. Figure (B) describes the case when x is in the interior of the base (which happens when $\alpha, \beta, \gamma < \pi/2$). Not that the pairs of triangles which share a perpendicular are similar triangles. It follows that the angles around x are as described.

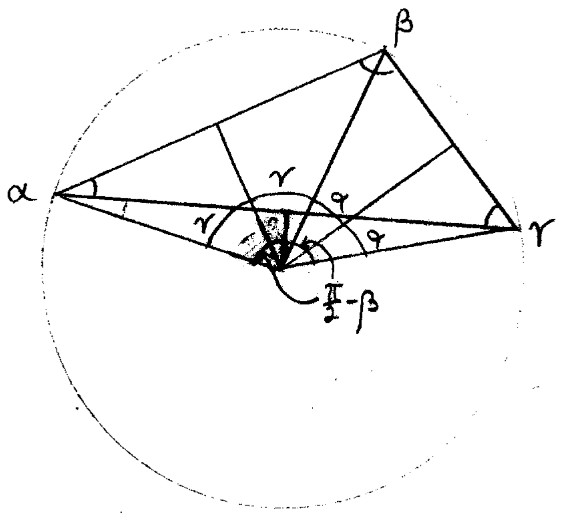


7.11

Each sub-simplex has two infinite vertices and three dihedral angles of $\pi/2$ so that they are of the type considered above. Thus

$$\text{Volume}\Big(\sum_{\alpha,\beta,\gamma}\Big) = 2\big(\tfrac{1}{2}\text{Л}(\gamma) + \tfrac{1}{2}\text{Л}(\beta) + \tfrac{1}{2}\text{Л}(\alpha)\big).$$

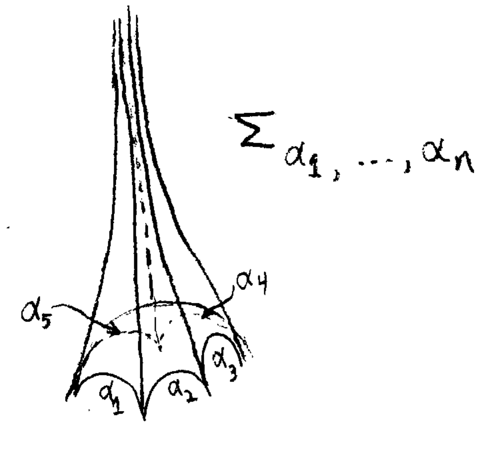
In the case when x is not in the interior of the base triangle, $\Sigma_{\alpha,\beta,\gamma}$ can still be thought of as the sum of six simplices each with three right dihedral angles. However, some of the simplices must be considered to have negative volume. The interested reader may supply the details, using the picture below.



EXAMPLE. The complement of the figure-eight knot was constructed in 3.1 by gluing two copies of $\Sigma_{\pi/3,\pi/3,\pi/3}$. Thus its volume is $6\text{л}(\pi/3) = 2.02988\ldots$.

REMARK. It is not hard to see that the $(\pi/3, \pi/3, \pi/3)$ simplex has volume greater than any other three-dimensional simplex. A simplex with maximal volume must 7.12
have its vertices at infinity since volume can always be increased by pushing a finite vertex out towards infinity. To maximize $V = \text{л}(\alpha) + \text{л}(\beta) + \text{л}(\gamma)$ subject to the restraint $\alpha + \beta + \gamma = 0$ we must have $\text{л}'(\alpha) = \text{л}'(\beta) = \text{л}'(\gamma)$ which implies easily that $\alpha = \beta = \gamma = \pi/3$. (The non-differentiability of $\text{л}(\alpha)$ at $\alpha = 0$ causes no trouble, since V tends to zero when α, β or γ tends to zero.)

Theorem 7.2.1 generalizes to a formula for the volume of a figure which is an infinite cone on a planar n-gon with all vertices at infinity. Let the dihedral angles formed by the triangular faces with the base plane be $(\alpha_1, \ldots, \alpha_n)$ and denote the figure with these angles by $\Sigma_{\alpha_1,\ldots,\alpha_n}$.



THEOREM 7.2.3. (i) $\sum_{i=1}^{n} \alpha_i = \pi$. (ii) Volume $(\Sigma_{\alpha_1,\ldots,\alpha_n}) = \sum_{i=1}^{n} л(\alpha_i)$.

PROOF. The proof is by induction. The case $n = 3$ is Theorem 1. Suppose the theorem to be true for $n = k - 1$. It suffices to prove it for $n = k$. 7.13

Consider the base k-gon for $\Sigma_{\alpha_1,\ldots,\alpha_k}$ and divide it into a $k-1$-gon and a triangle. Take the infinite cone on each of these two figures. If the new dihedral angle on the triangle side is β, the new angle on the $k-1$-gon side in $\pi - \beta$. By the induction hypothesis

$$\left(\sum_{i=1}^{2} \alpha_i\right) + \beta = \pi \quad \text{and} \left(\sum_{i=3}^{n} \alpha_i\right) + \pi - \beta = \pi.$$

Part (i) follows by adding the two equations. Similarly by the induction hypothesis,

$$\mathrm{Vol}(\Sigma_{\alpha_1,\alpha_2,\beta}) = \left(\sum_{i=1}^{2} л(\alpha_i)\right) + л(\beta)$$

and

$$\mathrm{Vol}(\Sigma_{\alpha_3,\ldots,\alpha_n,\pi-\beta}) = \left(\sum_{i=3}^{n} л(\alpha_i)\right) + л(\pi - \beta).$$

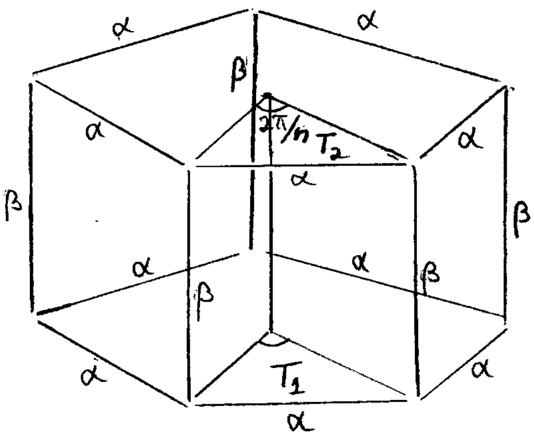
Part (ii) follows easily since $л(\pi - \beta) = -л(\beta)$. □

EXAMPLE. The complement of the Whitehead link was constructed from a regular ideal octahedron which in turn, is formed by gluing two copies of the infinite cone on a regular planar quadrilateral. Thus its volume equals $8л(\pi/4) = 3.66386\ldots$. Similarly, the complement of the Borromean rings has volume $16л(\pi/4) = 7.32772\ldots$ since it is obtained by gluing two ideal octahedra together.

7.3

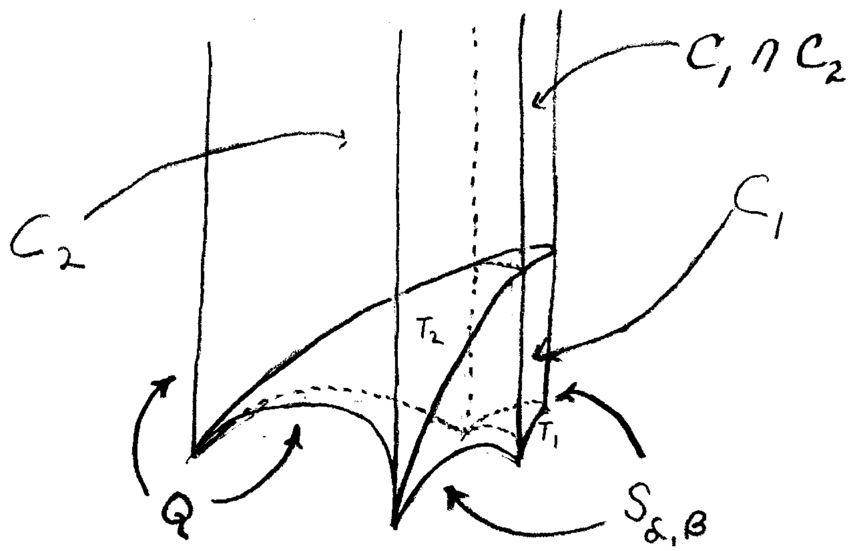
It is difficult to find a general pattern for constructing manifolds by gluing infinite tetrahedra together. A simpler method would be to reflect in the sides of a tetrahedron to form a discrete subgroup of the isometries of H^3. Unfortunately this method yields few examples since the dihedral angles must be of the form π/a, $a \in \mathbb{Z}$ in order that the reflection group be discrete with the tetrahedron as fundamental domain. The only cases when the sum of the angles is π are $\Sigma_{\pi/2,\pi/4,\pi/4}$, $\Sigma_{\pi/3,\pi/3,\pi/3}$ 7.14
and $\Sigma_{\pi/3,\pi/3,\pi/6}$ corresponding to the three Euclidean triangle groups.

Here is a construction for polyhedra in H^3 due to Thurston. Take a planar regular n-gon with vertices at infinity on each of two distinct planes in H^3 and join the corresponding vertices on the two figures by geodesics. If this is done in a symmetric way the sides are planer rectangles meeting each other at angle β and meeting the bases at angle α. Denote the resulting polyhedra by $\mathcal{N}_{\alpha,\beta}$. Note that $2\alpha + \beta = \pi$ since two edges of an n-gon and a vertical edge form a Euclidean triangle at infinity.

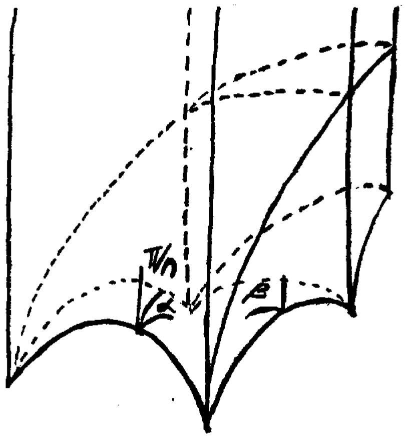


7.15

In order to compute the volume of $\mathcal{N}_{\alpha,\beta}$ consider it in the upper half-space model of H^3. Subdivide $\mathcal{N}_{\alpha,\beta}$ into n congruent sectors $S_{\alpha,\beta}$ by dividing the two n-gons into n congruent triangles and joining them be geodesics. Call the lower and upper triangles of $S_{\alpha,\beta}$, T_1 and T_2 respectively. Consider the infinite cones C_1 and C_2 on T_1 and T_2. They have the same volume since they are isometric by a Euclidean expansion. Hence the volume of $S_{\alpha,\beta}$ is equal to the volume of $Q = (S_{\alpha,\beta} \cup C_2) - C_1$.



Evidently Q is an infinite cone on a quadrilateral. To find its volume it is necessary to compute the dihedral angles at the edges of the base. The angles along the sides are $\frac{\beta}{2}$. The angle at the front face is $\alpha + \gamma$ where γ is the angle between the front face and the top plane of $\mathcal{N}_{\alpha,\beta}$. Consider the infinite cone on the top n-gon of $\mathcal{N}_{\alpha,\beta}$. 7.16
By (1) of Theorem 7.2.3 the angles along its base are π/n. Thus $\gamma = \pi/n$ and the front angle is $\alpha + \pi/n$. Similarly the back angle is $\alpha - \pi/n$.



By (2) of Theorem 7.2.3 we have

$$(6). \qquad \mathrm{Vol}(\mathcal{N}_{\alpha,\beta}) = n\,\mathrm{Vol}(Q) = n\big(2\text{л}(\beta/2) + \text{л}(\alpha+\pi/n) + \text{л}(\alpha-\pi/n)\big).$$

If α and β are of the form π/a, $a \in \mathbb{Z}$ then the group generated by the reflections in the sides of $\mathcal{N}_{\alpha,\beta}$ form a discrete group of isometries of H^3. Take a subgroup Γ which is torsion free and orientation preserving. The quotient space H^3/Γ is an oriented, hyperbolic three-manifold with finite volume.

Since $2\alpha+\beta=\pi$ the only choices for (α,β) are $(\pi/3,\pi/3)$ and $(\pi,4,\pi/2)$. As long as $n > 4$ both of these can be realized since β varies continuously from 0 to $n-2/n$ as the distance between the two base planes of $\mathcal{N}_{\alpha,\beta}$ varies from 0 to ∞. 7.16
Thus we have the following:

Theorem 7.3.1. *There are an infinite number of oriented three-manifolds whose volume is a finite rational sum of* $\text{л}(\theta)$ *for* θ*'s commensurable with* π.

7.4

We will now discuss an arithmetic method for constructing hyperbolic three-manifolds with finite volume. The construction and computation of volume go back to Bianchi and Humbert. (See [5], [7], [10].) The idea is to consider $\mathcal{O}_d$, the ring of integers in an imaginary quadratic field, $\mathbb{Q}(\sqrt{-d})$, where $d \geq 1$ is a square-free integer. Then $\mathrm{PSL}(2,\mathcal{O}_d)$ is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$. Let Γ be a torsion free subgroup of finite index in $\mathrm{PSL}(2,\mathcal{O}_d)$. Since $\mathrm{PSL}(2,\mathbb{C})$ is the group of orientation preserving isometries of H^3, H^3/Γ is an oriented hyperbolic three-manifold. It always has finite volume.

Example. Let $\mathbb{Z}[i]$ be the ring of Gaussian integers. A fundamental domain for the action of $\mathrm{PSL}(2,\mathbb{Z}[i])$ has finite volume. Different choices of Γ give different manifolds; e.g., there is a Γ of index 12 such that H^3/Γ is diffeomorphic to the complement of the Whitehead link; another Γ of index 24 leads to the complement of the Borromean rings. (N. Wielenbert, preprint).

EXAMPLE. In case $d = 3$, $\mathcal{O}_d$ is $\mathbb{Z}[\omega]$ where $\omega = \frac{-1+\sqrt{-3}}{2}$ and there is a subgroup $\Gamma \subset \mathrm{PSL}(2, \mathbb{Z}[\omega])$ of index 12 such that H^3/Γ is diffeomorphic to the complement of the figure-eight knot. (R. Riley, [11]). In order to calculate the volume of $H^3/\,\mathrm{PSL}(2, \mathcal{O}_d)$ in general we recall the following definitions. Define the *discriminant*, 7.18
D, of the extension $Q(\sqrt{-d})$ to be

$$D = \begin{cases} d & \text{if } d \equiv 3 (\operatorname{mod} 4), \\ 4d & \text{otherwise.} \end{cases}$$

If $\mathcal{O}_d$ is considered as a lattice in $\mathbb{T}$, then $\sqrt{D}/2$ is the area of $\mathbb{T}/\mathcal{O}_d$. The *Dedekind ζ-function* for a field K is defined to be

$$\zeta_K(S) = \sum_{\mathfrak{a}} 1/N(\mathfrak{a})^S \quad \text{where}$$

$\mathfrak{a}$ runs through all ideals in $\mathcal{O}$ and $N(\mathfrak{a}) = |\mathcal{O}/\mathfrak{a}|$ denotes the norm of $\zeta(S)$ is also equal to

$$\prod_{\mathfrak{P}} \frac{1}{1 - \dfrac{1}{N(\mathfrak{P})^S}}$$

taking all prime ideals of $\mathfrak{P}$.

THEOREM 7.4.1 (Essentially due to Humbert).

$$\mathrm{Vol}\big(H^3/\,\mathrm{PSL}(2, \mathcal{O}_d)\big) = \tfrac{D^{3/2}}{24}\, \zeta_{\mathbb{Q}(\sqrt{-d})}(2)/\zeta_{\mathbb{Q}}(2).$$

This volume can be expressed in terms of Lobachevsky's function using Hecke's formula

$$\zeta_{\mathbb{Q}(\sqrt{-d})}(S)/\zeta_{\mathbb{Q}}(S) = \sum_{n>0} \frac{\left(\frac{-D}{n}\right)}{n^s}.$$

Here $\left(\frac{-D}{n}\right)$ is the *quadratic symbol* where we use the conventions:

(i) If $n = p_1, \dots, p_t, p_i$ prime then $\left(\frac{-D}{n}\right) = \left(\frac{-D}{p_1}\right)\left(\frac{-D}{p_2}\right) \dots \left(\frac{-D}{p_t}\right)$.
(ii) If $p \mid D$ then $\left(\frac{-D}{p}\right) = 0$; $\left(\frac{-D}{1}\right) = +1$.
(iii) for p an odd prime

$$\left(\frac{-D}{p}\right) = \begin{cases} +1 & \text{if } -D \equiv X^2 (\operatorname{mod} p) \text{ for some } X, \\ -1 & \text{if not.} \end{cases}$$

(iv) For $p = 2$

$$\left(\frac{-D}{p}\right) = \begin{cases} +1 & \text{if } -D \equiv 1 \pmod 8, \\ -1 & \text{if } -D \equiv 5 \pmod 8. \end{cases}$$

(Note that $-D \not\equiv 3 \pmod 4$ by definition.)

The function $n \mapsto \left(\frac{-D}{n}\right)$ is equal to $1/\sqrt{-D}$ times its Fourier transform;* i.e.,

(1).
$$\sum_{k \bmod D} \left(\frac{-D}{k}\right) e^{2\pi i k n/D} = \sqrt{-D}\left(\frac{-D}{n}\right).$$

Multiplying by $1/n^2$ and summing over $n > 0$ we get

(2).
$$\sum_{n>0} 1/n^2 \sum_{k=0}^{n-1} \left(\frac{-D}{k}\right) e^{2\pi i k n/D} = \sqrt{-D} \sum_{n>0} \frac{\left(\frac{-D}{n}\right)}{n^2}.$$

For fixed k the imaginary part of the left side is just the Fourier series for $2\text{Л}(\pi k/D)$. Since the right side is pure imaginary we have:

(3).
$$2 \sum_{k \bmod D} \left(\frac{-D}{k}\right) \text{Л}(\pi k/D) = \sqrt{D} \sum_{n>0} \left(\frac{-D}{n}\right) 1/n^2.$$

Multiplying by $D/24$ and using Hecke's formula leads to

(4).
$$D/12 \sum_{k \bmod D} \left(\frac{-D}{k}\right) \text{Л}(\pi k/D) = \operatorname{Vol}\big(H^3/\operatorname{PSL}(2, \mathcal{O}_d)\big).$$

7.20

Example. In the case $d = 3$, 7.4.4 implies that the volume of $H^3/\big(\operatorname{PSL}(2, \mathbb{Z}[\omega]\big)$ is $\frac{1}{4}\big(\text{Л}(\pi/3) - \text{Л}(2\pi/3)\big) = \frac{1}{2}\text{Л}(\pi/3)$. Recall that the complement of the figure-eight knot $S^3 - K$ is diffeomorphic to H^3/Γ where Γ had index 12 in $\operatorname{PSL}(2, \mathbb{Z}[\omega])$. Thus it has volume $6\text{Л}(\pi/3)$. This agrees with the volume computed by thinking of $S^3 - K$ as two copies of $\Sigma_{\pi/3,\pi/3,\pi/3}$ tetrahedra glued together.

Similarly the volumes for the complements of the Whitehead link and the Borromean rings can be computed using 7.4.4. The answers agree with those computed geometrically in 7.2.

This algebraic construction also furnishes an infinite number of hyperbolic manifolds with volumes equal to rational, finite linear combinations of Л (a rational multiple of π). Note that Conjectures (A) and (B) would imply that any rational relation between the volumes of these manifolds could occur at most as a result of common factors of the integers, d, defining the quadratic fields. In fact, quite likely they would imply that there are no such rational relations.

*Compare Hecke, Vorlesangen über algebr. Zahlen, p. 241. I am grateful to A. Adler for help on this point.

References

[1] L. Euler, *Institutiones calculi integralis*, I, pp. 110-113 (1768).
[2] C. J. Hill, *Journ. reine angew. Math.* (Crelle) 3 (1828), 101-139.
[3] T. Clausen, *Journ. reine angew. Math.* (Crelle) 8 (1932), 298-300.
[4] N. Lobachevsky, *Imaginary Geometry and its Application to Integration* (Russian) Kasan 1836. German translation, Leipzig 1904. (For a modern presentation see B. L. Laptev, Kazan Gos Univ. Uč. Zapiski 114 (1954), 53-77).
[5] L. Bianchi, *Math. Ann.* 38 (1891) 313-333 and 40 (1892), 332-412.
[6] H. Gieseking, Thesis, Münster 1912. (See Magnus, Noneuclidean Tesselations and their Groups, Acad. Press 1974, p. 153.)
[7] G. Humbert, Comptes Rendus 169 (1919) 448-454.
[8] H. S. M. Coxeter, *Quarterly J. Math* 6 (1935), 13-29.
[9] L. Lewin, *Dilogarithms and Associated Functions*, Macdonald (London), 1958.
[10] R. G. Swan, *Bull. AMS* 74 (1968), 576-581.
[11] R. Riley, *Proc. Cambr. Phil. Soc.* 77 (1975) 281-288.
[12] A. M. Gabrielov, I. M. Gel'fand, and M. V. Losik, *Functional Anal. and Appl.* 9 (1975), pp. 49, 193.
[13] S. Bloch, to appear.
[14] D. Kubert and S. Lang, to appear.