EXPLICIT FORMULAS FOR THE WALDSPURGER AND BESSEL MODELS

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In this paper we will study certain models of irreducible admissible representations of the split special orthogonal group $SO(2n+1)$ over a nonarchimedean local field. If $n = 1$, these models were considered by Waldspurger [Wa1,Wa2], and arose in his profound studies of the Shimura correspondence. If $n = 2$, they were considered by Novodvorsky and Piatetski-Shapiro [NP], who called them *Bessel models*, and for general n they were studied by Novodvorsky [No]. In the works cited, these authors established the uniqueness of these models; in this paper we establish functional equations and explicit formulas for them. In general, these models arise from a variety of Rankin-Selberg integrals (for example, those of Andrianov [An], Furusawa [Fu1], and Sugano [Su]), and the results of this paper will naturally have applications to the study of L-functions. Moreover, these models arise in the study of the theta correspondence between $SO(2n+1)$ and the double cover of $Sp(2n)$, and they will therefore be of importance in generalizing the work of Waldspurger (see Furusawa [Fu2]).

In the final Section, we present a global application of the explicit formulas: we consider the Eisenstein series (6.1) on $SO(2n+1)$ formed with a cuspidal automorphic representation π on $GL(n)$, and we show that its Bessel period (6.2) is essentially a product of L-series

$$L\big(n(s-1/2)+1/2,\pi\big)\,L\big(n(s-1/2)+1/2,\pi\otimes\eta\big),$$

where η is a quadratic character. This result generalizes work for $n=2$ and base field $\mathbb{Q}$ of Mizumoto [Mi] and Böcherer [Bö], and puts it in a representation-theoretic context.

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This global application is closely related to the results of Bump, Friedberg, and Hoffstein [BFH2]. That paper computes the spherical Whittaker functions on the metaplectic double cover of $Sp(2n)$. (Whittaker models on that group are also unique.) The computation in [BFH2] has the following consequence: if one forms the metaplectic Eisenstein series on the double cover of $GSp(2n)$ with a cuspidal automorphic representation π of $GL(n)$ (which is possible because the cover splits over $GL(n) \subset Sp(2n)$), the Whittaker coefficients of this Eisenstein series are quadratic twists of the standard L-function of π. The close relation between these two computations is a reflection of the following result of Furusawa [Fu2], generalizing the case $n = 2$ in Piatetski-Shapiro and Soudry [PS]: the (special) Bessel coefficient of a cusp form on $SO(2n+1)$ essentially agrees with the Whittaker coefficient of the theta lift on the double cover of $Sp(2n)$. If instead of a cusp form one considers the Eisenstein series (6.1), the theta correspondent on the metaplectic group is the metaplectic Eisenstein series, and our calculation implies that this result of Furusawa for cusp forms is true for these Eisenstein series also. (Our calculation of the Bessel period is in fact direct and independent of [Fu2].)

These results should have an application to the nonvanishing of L-functions under quadratic twists. Namely, there are Rankin-Selberg integrals on the double cover of $GSp(6)$ ([BG]) and on $SO(7)$ ([Gi]) unfolding to Dirichlet series involving the Whittaker (resp. Bessel) periods described above, that is, to Dirichlet series whose individual coefficients are the quadratic twists of a standard $GL(3)$ L-series. (The two constructions give Dirichlet series whose individual coefficients are Euler products which agree at almost all places.) Arguing as in [BFH1], one should be able to show that an infinite number of these quadratic twists are nonzero. In fact, these integrals are the next members of a series beginning with Siegel's calculation of the Mellin transform of a metaplectic $GL(2)$ Eisenstein series and including integrals of Hecke type on the double cover of $GSp(4)$ (due in a nonmetaplectic context to Novodvorsky; see [BFH1]) and on $SO(5)$ (due to Maass). The elucidation of this scenario owes much to discussions with Duke, Ginzburg, Goldfeld, and Hoffstein. In particular, the verification that our results could be applied to the evaluation of (6.2) was first worked out in conversation with Ginzburg.

1. Notations and Statement of Results. Let F be a nonarchimedean local field of characteristic different from 2. Let $\mathcal{O}$ denote the ring of integers of F, ϖ denote a local uniformizer, q denote the cardinality of the residue field $\mathcal{O}/\varpi\mathcal{O}$, and $|\ |_F$ denote the absolute value on F, normalized so that $|\varpi|_F = q^{-1}$.

We begin by describing our results on the Waldspurger model. Let $G_2 = GL(2, F)$, and let T_2 be a maximal torus in G_2. Then T_2 is the connected component of the identity in a group of orthogonal similitudes of degree two corresponding to some quadratic form. If (π_2, V_{π_2}) is an irreducible admissible representation of G_2, and if $\sigma : T_2(F) \to \mathbb{C}$ is a character, then there exists at most one linear functional $W : V_{\pi_2} \to \mathbb{C}$ (up to scalar multiplication) such that

$$W(\pi_2(t)\,v) = \sigma(t)\,W(v) \tag{1.1}$$

for all $t \in T_2$ and $v \in V_{\pi_2}$. This is proved when $\sigma = 1$ in Waldspurger [Wa1], Proposition $9'$, and the proof in the general case is identical (as pointed out in [Wa2], Lemme 8). In order for such a functional to exist, since T_2 contains the center Z_2 of G_2, it is necessary that the restriction of σ to Z_2 match the central character of π_2. We will call a functional satisfying (1.1) a *Waldspurger functional.* The *Waldspurger model* for π_2 will be the space of all functions of the form $g \mapsto W(\pi_2(g)v)$ with $v \in V_{\pi_2}$.

First let us consider the case where $T_2 = T_2^{\mathrm{a}}$ is nonsplit. We will limit ourselves to the case where T_2^{a} has the form

$$\left\{ \begin{pmatrix} x & y \\ \epsilon y & x \end{pmatrix} | x^2 - y^2 \epsilon \neq 0 \right\}, \tag{1.2}$$

where $\epsilon \in \mathcal{O}^\times$ is a nonsquare. In this case, let $T_2^{\mathrm{a}}(\mathcal{O}) = T_2^{\mathrm{a}} \cap GL(2, \mathcal{O})$. We will further assume that σ is trivial on $T_2^{\mathrm{a}}(\mathcal{O})$. Since T_2^{a} is generated by $T_2^{\mathrm{a}}(\mathcal{O})$ and by the center of G_2, on which σ is to agree with the central character of π_2, it follows that σ is uniquely determined by these conditions. (We note that if T_2^{a} and σ are obtained by localizing global data then these conditions will be satisfied locally almost everywhere at places where the global quadratic form defining T_2 is nonsplit.)

Suppose that π_2 is in the unramified principal series. Then the contragredient representation of π_2 is also spherical, and the $GL(2, \mathcal{O})$-fixed vector in the contragredient representation is clearly $T_2^{\mathrm{a}}(\mathcal{O})$-invariant. Since it also has the correct transformation property with respect to the center of G_2, it is thus a Waldspurger functional. Thus if ϕ is the spherical vector in V_{π_2}, the function $g \mapsto W(\pi_2(g)\phi)$ is the spherical function for π_2, which is given by the Macdonald formula (see [Ca1]).

One of our results will be a formula analogous to the Macdonald formula for the *split* Waldspurger functional. Thus let $T_2 = T_2^{\mathrm{s}}$ be a split torus of G_2. Specifically, we may take T_2^{s} to be the group of diagonal matrices in G_2; also, let B_2 be the Borel subgroup consisting of upper triangular matrices in G_2.

Let us construct a Waldspurger functional for the unramified principal series. Let ξ_1, ξ_2 be unramified quasicharacters of $F^\times$, and let ξ be the character of B_2 given by

$$\xi\left(\begin{pmatrix} a & b \\ & d \end{pmatrix} \right) = \xi_1(a)\xi_2(d). \tag{1.3}$$

Suppose $\pi_2 = \mathrm{Ind}(\xi)$ (we also write $\pi_2 = \mathrm{Ind}(\xi_1, \xi_2)$) is the representation obtained by normalized induction from the character ξ. Thus V_{π_2} consists of the complex-valued locally constant functions f on G_2 such that

$$f(bg) = \xi(b)\, \delta_{B_2}^{1/2}(b)\, f(g) \tag{1.4}$$

for all $b \in B_2$, $g \in G_2$, where δ_{B_2} is the modular character of B_2, and π_2 is the right regular representation.

Let σ be an unramified quasicharacter of $F^\times$. Extend σ to T_2^{s} (we use the same letter) by the formula $\sigma\left(\left(\begin{smallmatrix} a & b \\ & b \end{smallmatrix}\right)\right) = \sigma(a)\,\xi_1\xi_2(b)$. Then a Waldspurger functional on V_{π_2} is defined as follows. Suppose $f \in V_{\pi_2}$. Let

$$
\begin{aligned}
W(f) &= \int\limits_{Z_2\backslash T_2^{\mathrm{s}}} f\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} t\right)\,\sigma^{-1}(t)\,d^\times t \\
&= \int\limits_{F^\times} f\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\,\sigma^{-1}(a)\,d^\times a.
\end{aligned}
\tag{1.5}
$$

The Haar measure on $F^\times$ is normalized so that the measure of $\mathcal{O}^\times$ is 1. Suppose that $\xi_i(\varpi) = \gamma_i$ for $i = 1, 2$, and that $\sigma(\varpi) = \tau$. As we shall show in Section 2 below, the integral (1.5) is absolutely convergent in the region

$$
\left|\gamma_1\tau^{-1}\right| < q^{1/2}, \qquad \left|\gamma_2^{-1}\tau\right| < q^{1/2}. \tag{1.6}
$$

For these representations, (1.1) holds.

Let ϕ_ξ be the $K_2 = GL(2, \mathcal{O})$-fixed vector in V_{π_2} such that $\phi_\xi(I_2) = 1$. Define a function $Wa_\xi\colon G_2 \to \mathbb{C}$ by the formula

$$
Wa_\xi(g) = W(\pi_2(g)\phi_\xi).
$$

This function is analogous to the Whittaker function obtained from the standard Whittaker functional.

Our main result on the Waldspurger functional gives the analytic continuation of the function $Wa = Wa_\xi$ to the full space of unramified quasicharacters ξ_1, ξ_2, and σ, and an explicit formula for its value. To describe this, note that

$$
Wa_\xi(t_2 g \kappa_2) = \sigma(t_2)\,Wa(g) \tag{1.7}
$$

for all $t_2 \in T_2^{\mathrm{s}}$, $\kappa_2 \in K_2$. Hence it suffices to determine Wa_ξ on a set of coset representatives for the double cosets $T_2^{\mathrm{s}}\backslash G_2/K_2$. Using the Iwasawa decomposition, it follows that a set of coset representatives is given by the matrices

$$
\eta_k = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \varpi^k & 0 \\ 0 & 1 \end{pmatrix},
$$

with $k \geq 0$. In Section 2 we shall prove

Theorem 1.1. *Suppose that (1.6) holds. Let*

$$
\mathcal{W}\mathrm{a}_\xi(g) = \frac{(1-\gamma_1\tau^{-1}q^{-1/2})(1-\gamma_2^{-1}\tau q^{-1/2})}{1-\gamma_1\gamma_2^{-1}q^{-1}} Wa_\xi(g).
$$

Then $\mathfrak{Wa}_\xi$ is given by the formula

$$\text{(1.8)}\quad \mathfrak{Wa}_\xi(\eta_k) = (1-q^{-1})^{-1} q^{-k/2} \times$$
$$\left[\gamma_1^k \frac{(1-\gamma_2\tau^{-1}q^{-1/2})(1-\gamma_1^{-1}\tau q^{-1/2})}{1-\gamma_1^{-1}\gamma_2} + \gamma_2^k \frac{(1-\gamma_1\tau^{-1}q^{-1/2})(1-\gamma_2^{-1}\tau q^{-1/2})}{1-\gamma_1\gamma_2^{-1}}\right].$$

In particular, the function $\mathfrak{Wa}_\xi$, originally defined as an integral when the inequalities (1.6) hold, has holomorphic continuation to all $\gamma_1, \gamma_2, \tau \in \mathbb{C}^\times$, and is invariant under the interchange of γ_1 and γ_2.

Note that if $\gamma_1 = \gamma_2$, this must be interpreted correctly: both the numerator and the denominator in (1.8) vanish, but their ratio is analytic, so the formula still makes sense.

We also address the analytic continuation of the Waldspurger functionals. Let Λ be the domain of $(\gamma_1, \gamma_2, \tau) \in (\mathbb{C}^\times)^3$ such that $\tau\gamma_1^{-1} \neq q^{1/2}$ and $\tau\gamma_2^{-1} \neq q^{-1/2}$. We note that for fixed ξ_1 and ξ_2, the space $\mathrm{Ind}(\xi)$ may be identified with the space $C^\infty\big((B_2 \cap K_2)\backslash K_2\big)$ of locally constant functions on K_2 which are left invariant by $B_2 \cap K_2$; indeed, such a function may be uniquely extended to a function on G_2 satisfying (1.4), and every element of $\mathrm{Ind}(\xi)$ arises uniquely from an element of $C^\infty\big((B_2 \cap K_2)\backslash K_2\big)$ in this way. We shall also prove in Section 2:

Theorem 1.2. *Fix an element $f \in C^\infty\big((B_2 \cap K_2)\backslash K_2\big)$. For $(\gamma_1, \gamma_2, \tau)$ satisfying (1.6), f may be extended uniquely to an element of $\mathrm{Ind}(\xi)$, and the function $W(f)$ defined by (1.5) may thus be regarded as an analytic function of three variables $(\gamma_1, \gamma_2, \tau)$. This function has analytic continuation to all of Λ, so the domain of definition of $W : \mathrm{Ind}(\xi) \to \mathbb{C}$ may be extended to all ξ for which the parameters are in Λ. If $(\gamma_1, \gamma_2, \tau) \in \Lambda$, then W defines a (possibly zero) Waldspurger functional on $\mathrm{Ind}(\xi)$.*

The analytic continuation assertion here can actually be deduced from Theorem 1.1, since if ξ is in general position then the representation π_2 is irreducible, and hence every element of $\mathrm{Ind}(\xi)$ is a linear combination of right translates of the spherical vector. Thus there is some overlap between these two theorems. In fact, however, we shall give a direct proof of Theorem 1.2, independent from Theorem 1.1.

Next let us describe our results concerning the Bessel models. For $r \geq 2$ let $SO(r, F)$ denote the split group of determinant one orthogonal matrices

$$SO(r,F) = \{g \in SL(r,F) \mid (gx, gy) = (x,y) \quad \text{for all} \quad x, y \in F^r\},$$

where $(\ ,\ )$ is the quadratic form

$$(x,y) = \sum_{i=1}^{r} x_i\, y_{r+1-i}.$$

Let $G = SO(2n+1, F)$, and let U be the subgroup of G consisting of upper triangular unipotent matrices whose center 3×3 block is the identity. An element of U is of the form $u = (u_{ij})$ with $u_{ij} = 0$ for $i, j = n, n+1, n+2$ and $i \neq j$. Let ψ be a character of F of conductor $\mathcal{O}$. Given $S = (a, b, c) \in \mathcal{O}$ such that $b^2 + 2ac \neq 0$, define a character θ_S of U by the formula

$$\theta_S(u) = \psi(u_{12} + u_{23} + \cdots + u_{n-2,n-1} + au_{n-1,n} + bu_{n-1,n+1} + cu_{n-1,n+2}). \tag{1.9}$$

Let T be the subgroup of G consisting of the matrices of the form

$$\begin{pmatrix} I_{n-1} & & \\ & g & \\ & & I_{n-1} \end{pmatrix}$$

which, acting by conjugation, stabilize θ_S. This constrains g to lie in a suitable torus in $SO(3)$. Thus T is a torus in G, which may be either split or nonsplit over F, depending on S. Note that T normalizes U, and hence $R := TU$ is again a subgroup of G. Let λ be a character of T, and extend θ_S to a character of R by $\theta_S(tu) = \lambda(t)\,\theta_S(u)$.

Let $\pi\colon G \to \mathrm{End}(V_\pi)$ be an admissible representation of G. Then a *Bessel functional* on π is a linear functional $B\colon V_\pi \to \mathbb{C}$ such that

$$B(\pi(tu)v) = \theta_S(tu)\,B(v), \tag{1.10}$$

for all $t \in T$, $u \in U$, and $v \in V_\pi$. As shown by Novodvorsky [No], if π is irreducible then the dimension of the space of such functionals is at most 1. Note also that, in view of the isomorphism of $SO(3, F)$ and $PGL(2, F)$, a Bessel functional on $SO(3, F)$ may be identified with a Waldspurger functional.

One may similarly define a Bessel functional on the larger group of orthogonal similitudes; however, since this group is the direct product of G with its center, there is no gain in generality by doing so.

Once again we shall consider this notion for the unramified principal series. Let $\chi_1, \cdots, \chi_n$ be unramified quasicharacters of $F^\times$. We shall consider the principal series representation of G

$$\pi = \mathrm{Ind}(\chi_1, \cdots, \chi_n).$$

In our notation, this will be the representation on space of locally constant functions Ψ on G which satisfy

$$\Psi(bg) = \delta_B^{1/2}(b) \left(\prod_{i=1}^{n} \chi_i(y_i) \right) \cdot \Psi(g) \tag{1.11}$$

for all

$$b = \begin{pmatrix} y_1 & * & * & * & * & * & * \\ & \ddots & * & * & * & * & * \\ & & y_n & * & * & * & * \\ & & & 1 & * & * & * \\ & & & & y_n^{-1} & * & * \\ & & & & & \ddots & * \\ & & & & & & y_1^{-1} \end{pmatrix}$$

in the standard Borel subgroup B of G. Here

$$\delta_B(b) = \left| \prod_{i=1}^{n} y_i^{2n-2i+1} \right|_F$$

is the modular character of B_F. The group action is by right translation. Let us write $\alpha_i = \chi_i(\varpi)$. If the α_i are in general position ($\alpha_i \neq \alpha_j^{\pm 1}$ for $i \neq j$ and $\alpha_i^2 \neq 1$), then the isomorphism class of this representation is invariant under permutations of the α_i, as well as transformations of the form $\alpha_i \mapsto \alpha_i^{\pm 1}$. Also, let us write K for the standard maximal compact subgroup $SO(2n+1, \mathcal{O})$ of G.

We may construct a Bessel functional explicitly as follows. There are two cases: T nonsplit, and T split. In the split case, we shall take λ unramified, i.e. identically one on $T \cap K$.

Suppose first that T is a nonsplit torus. We assume that $a \in \mathcal{O}^\times$. Write $T(\mathcal{O})$ for the subgroup $T \cap K$. For a permutation s in the symmetric group S_{2n+1}, we shall also use s to denote the corresponding signed permutation matrix in $SL(2n+1, F)$ (the matrix with $\mathrm{sgn}(s)$ in the $(s(i), i)$ position and 0 elsewhere). Let $w_1 = (1, 2n+1)(2, 2n) \cdots (n-1, n+3)$. Suppose $\Psi \in V_\pi$. Then we define

$$B(\Psi) = \int_{T(\mathcal{O})} \int_{U} \Psi(w_1 u t)\, \theta_S(u)^{-1}\, du\, dt, \tag{1.12}$$

and this is a Bessel functional on V_π. The integral is absolutely convergent if the quasicharacters χ_i are in a suitable region. Indeed, since $T(\mathcal{O})$ is compact, comparing with the standard intertwining operator T_{w_1} defined in Section 3 below (see Lemma 3.1), one finds that if

$$|\alpha_1| < \cdots < |\alpha_{n-1}| < \min(|\alpha_n|, |\alpha_n^{-1}|), \tag{1.13}$$

then the integral (1.12) is absolutely convergent.

For the second case, suppose that T is instead split. We may suppose that $S = (0, 1, 0)$. Let us introduce the following notation. For $x \in F$, let us write $n(x)$ for the unipotent matrix

$$n(x) = \begin{pmatrix} I_{n-1} & & & & \\ & 1 & x & -x^2/2 & \\ & & 1 & -x & \\ & & & 1 & \\ & & & & I_{n-1} \end{pmatrix},$$

and for $a \in F^\times$, let us write $t(a)$ for the diagonal matrix in T given by

$$t(a) = \begin{pmatrix} I_{n-1} & & & & \\ & a & & & \\ & & 1 & & \\ & & & a^{-1} & \\ & & & & I_{n-1} \end{pmatrix}.$$

The matrices $t(a)$ give all of T. Also we write $\lambda(a) = \lambda(t(a))$, $\beta = \lambda(\varpi)$. Let $w_0 = w_1(n, n+2)$ be (a representative for) the long element of the Weyl group Ω. Then for π as above, and $\Psi \in V_\pi$, we define

$$B(\Psi) = \int_{F^\times} \int_{U} \Psi(w_0\, n(1)\, u\, t(a))\, \theta_S(u)^{-1}\, \lambda^{-1}(a)\, du\, d^\times a. \tag{1.14}$$

Then this is a Bessel functional on V_π. Once again, the Haar measure on $F^\times$ is normalized so that the measure of $\mathcal{O}^\times$ is 1. As we shall show in Section 3 below, the integral (1.14) is absolutely convergent if

$$|\alpha_1| < \cdots < |\alpha_{n-1}| < \min(|\alpha_n|, |\alpha_n^{-1}|), \qquad |\alpha_n| < q^{1/2} \min(|\beta|, |\beta^{-1}|). \tag{1.15}$$

Our first pair of results concerns the functional equations satisfied by B. We shall show that, in both cases, the Bessel functional may be extended to all characters $\chi = (\chi_1, \cdots, \chi_n)$ by a variation of the familiar process whereby the standard intertwining operators are analytically continued, and has a functional equation under certain transformations of these characters χ. More precisely, the Weyl group Ω of G acts on the characters $\chi = (\chi_1, \cdots, \chi_n)$, or what is the same thing in the unramified case, on the parameters $\alpha_1, \cdots, \alpha_n$. In terms of these parameters, Ω is the group of transformations of $(\alpha_1, \cdots, \alpha_n) \in (\mathbb{C}^\times)^n$ generated by

$$(\alpha_1, \ldots, \alpha_{n-1}, \alpha_n) \mapsto (\alpha_1, \ldots, \alpha_{n-1}, \alpha_n^{-1}),$$

and by the action of the symmetric group S_n on $(\alpha_1, \cdots, \alpha_n)$. The cardinality of Ω is $2^n n!$.

Let $\Phi_\chi \in V_\pi$ be the standard nonramified vector. This is the unique function in V_π taking value one on K. We define a function $H = H_\chi : G \to \mathbb{C}$ by

$$H(g) = B(\pi(g)\,\Phi_\chi).$$

This function is once again analogous to the Whittaker function obtained from the standard Whittaker functional.

Theorem 1.3. *Suppose that T is nonsplit. Then the function H_χ, originally defined as an integral when*

$$|\alpha_1| < \cdots < |\alpha_{n-1}| < \min(|\alpha_n|, |\alpha_n^{-1}|),$$

has a meromorphic continuation to all nonzero complex $\alpha_1, \ldots, \alpha_n$. Moreover the function

$$\mathcal{H}_\chi(g) = \prod_{1 \le i < j \le n} (1 - \alpha_i \alpha_j q^{-1})^{-1} (1 - \alpha_i \alpha_j^{-1} q^{-1})^{-1}\, H_\chi(g)$$

is invariant under the action of Ω on the α_i, and is holomorphic for all $(\alpha_1, \ldots, \alpha_n) \in (\mathbb{C}^\times)^n$.

Theorem 1.4. *Suppose that T is split. Then the function H_χ, originally defined as an integral when*

$$|\alpha_1| < \cdots < |\alpha_{n-1}| < \min(|\alpha_n|, |\alpha_n^{-1}|), \qquad |\alpha_n| < q^{1/2} \min(|\beta|, |\beta^{-1}|),$$

has a meromorphic continuation to all nonzero complex $\alpha_1, \ldots, \alpha_n, \beta$. Moreover the function

$$\mathcal{H}_\chi(g) = \frac{\prod_{i=1}^n (1 - \alpha_i \beta q^{-1/2})(1 - \alpha_i \beta^{-1} q^{-1/2})}{\prod_{1 \le i < j \le n} (1 - \alpha_i \alpha_j q^{-1})(1 - \alpha_i \alpha_j^{-1} q^{-1}) \prod_{i=1}^n (1 - \alpha_i^2 q^{-1})}\, H_\chi(g)$$

is invariant under the action of Ω on the α_i, and is holomorphic for all $(\alpha_1, \ldots, \alpha_n) \in (\mathbb{C}^\times)^n$ and β satisfying $q^{-1/2} < \min(|\beta|, |\beta|^{-1})$.

The proofs of Theorems 1.3 and 1.4 are given in Section 3 below.

Our next pair of results gives an explicit formula for $\mathcal{H}_\chi(g)$. As in the case of the Waldspurger model (see (1.7)), since

$$\mathcal{H}_\chi(rg\kappa) = \theta_S(r) \mathcal{H}_\chi(g) \tag{1.16}$$

for all $r \in R$, $\kappa \in K$, it suffices to determine $\mathcal{H}$ on a set of coset representatives for $R\backslash G/K$. Let $k = (k_1, \cdots, k_n)$ be a vector of integers with $k_1 \ge 0$. If T is nonsplit then one finds, using the Iwasawa and Cartan decompositions (compare Sugano [Su], Lemma

2-4), that a set of coset representatives for $R\backslash G/K$ is given by the diagonal matrices of the form

$$d_k = \mathrm{diag}(\varpi^{k'_n}, \varpi^{k'_{n-1}}, \cdots, \varpi^{k'_1}, 1, \varpi^{-k'_1}, \cdots, \varpi^{-k'_{n-1}}, \varpi^{-k'_n}),$$

with

$$k'_i = k_1 + \cdots + k_i.$$

If T is split, then one finds instead that a set of coset representatives is given by the matrices g_k of the form

$$g_k = \begin{cases} d_k & \text{if } k_1 = 0 \\ n(1)\, d_k & \text{if } k_1 > 0. \end{cases}$$

For convenience, we write $h(k_1, \cdots, k_n)$ for the quantity $\mathcal{H}(d_k)$ (resp. $\mathcal{H}(g_k)$).

¿From equation (1.16) it follows that $H(g) = 0$ unless θ_S is identically one on $R \cap gKg^{-1}$. A short calculation shows that this condition implies that $h(k_1, \cdots, k_n) = 0$ unless each $k_i \geq 0$.

Let $\mathcal{A}$ be the alternator $\sum_{w\in\Omega}(-1)^{\mathrm{length}(w)}\, w$ in the group algebra $\mathbb{C}[\Omega]$. Let $\Delta = (-1)^n \mathcal{A}(\alpha_1^n \alpha_2^{n-1} \cdots \alpha_n)$. According to Weyl's identity for $Sp(2n, \mathbb{C})$

$$\Delta = \prod_{i=1}^{n} \alpha_i^{-1+i-n}(1-\alpha_i^2) \prod_{1\leq i<j\leq n} (1-\alpha_i\alpha_j)(1-\alpha_i\alpha_j^{-1}). \tag{1.17}$$

Also let

$$e_k = -\frac{1}{2}\sum_{i=1}^{n} \left(n^2 - (i-1)^2\right) k_i.$$

Then the evaluation of $h(k_1, \cdots, k_n)$ is given by

Theorem 1.5. *Suppose T is nonsplit and $k_i \geq 0$ for $i = 1$ to n. Then*

$$h(k_1, \cdots, k_n) = q^{e_k}(1+q^{-1})^{-1}\, \Delta^{-1}\, \mathcal{A}\left(\prod_{i=1}^{n} \alpha_{n+1-i}^{-k'_i - i}(1-\alpha_i^2 q^{-1}) \right).$$

In particular, $\mathcal{H}_\chi(I_{2n+1}) = 1$.

Theorem 1.6. *Suppose T is split and $k_i \geq 0$ for $i = 1$ to n.*

(1) *If $k_1 = 0$ then*

$$h(0, k_2, \cdots, k_n) = q^{e_k}\, \Delta^{-1} \times \mathcal{A}\left(\alpha_n^{-1} \prod_{i=1}^{n-1} \alpha_i^{-k'_{n+1-i}-n-1+i}(1-\alpha_i\beta q^{-1/2})(1-\alpha_i\beta^{-1}q^{-1/2}) \right).$$

(2) *If* $k_1 > 0$ *then*

$$h(k_1, k_2, \cdots, k_n) = q^{e_k}(1-q^{-1})^{-1}\,\Delta^{-1} \times \mathcal{A}\left(\prod_{i=1}^{n} \alpha_{n+1-i}^{-k_i'-i}(1-\alpha_i\beta q^{-1/2})(1-\alpha_i\beta^{-1}q^{-1/2})\right).$$

In particular, the function $\mathcal{H}_\chi$ *is holomorphic for all* $(\alpha_1, \ldots, \alpha_n, \beta) \in (\mathbb{C}^\times)^{n+1}$, *and* $\mathcal{H}_\chi(I_{2n+1}) = 1$.

These theorems are proved in Section 4 below.

Let us finally formulate the meromorphic continuation of the Bessel functional to all values of χ. We will formulate this result only in the split case; the nonsplit case is nearly identical. Suppose that $\Psi \in C^\infty\big((B\cap K)\backslash K\big)$. Given χ, there is a unique extension Ψ_χ of Ψ to an element of $\mathrm{Ind}(\chi)$ satisfying (1.11).

Theorem 1.7. *There exists a dense open subset* Γ *of* $(\mathbb{C}^\times)^{n+1}$ *such that there exists a Bessel functional* B *on* V_π *for all* $(\alpha_1, \cdots, \alpha_n, \beta) \in \Gamma$*; if (1.15) is satisfied, this functional agrees with that defined by (1.14); and if* $\Psi \in C^\infty\big((B\cap K)\backslash K\big)$, *then* $B(\Psi_\chi)$ *is a meromorphic function of* $\alpha_1, \cdots, \alpha_n, \beta$, *whose polar set is contained in the complement of* Γ.

As with Theorem 1.2, there is some overlap between this result and our previous Theorems. We will prove Theorem 1.7 by means of a theorem of Bernstein [Be] in Section 5. Actually Bernstein's theorem implies that the complement of Γ may be taken to be a countable union of hyperplanes.

In view of the isomorphism of $SO(5,F)$ to the projective group of symplectic similitudes $PGSp(4,F)$, the theorems above imply the functional equation and the explicit formula for the Bessel model on $G_4 := GSp(4)$. The Bessel model on $GSp(4,F)$ was first studied by Novodvorsky and Piatetski-Shapiro [NP]. A formula for the generating series of this model was given by Sugano in [Su], Proposition 2-5. To state the explicit formula, consider the group

$$G_4 = \left\{ g \in GL(4,F) \mid {}^t g \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} g = \nu(g) \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}, \nu(g) \in F^\times \right\}.$$

Let $S \in M_2(\mathcal{O})$ be a nonsingular symmetric matrix $S = \begin{pmatrix} c & b/2 \\ b/2 & a \end{pmatrix}$. Let U_4 be the unipotent radical of the Siegel parabolic

$$U_4 = \left\{ u = \begin{pmatrix} I_2 & X \\ 0 & I_2 \end{pmatrix} \mid X = {}^t X \right\},$$

and let θ_S be the character of U_4 given by $\theta_S(u) = \psi(\mathrm{tr}(SX))$. Let T be the torus in G_4 consisting of those matrices of the form

$$\begin{pmatrix} h & 0 \\ 0 & \det(h)\,{}^t h^{-1} \end{pmatrix}$$

which, acting by conjugation, stabilize θ_S; thus $h \in G_2$ is required to satisfy ${}^t hSh = \det(h)S$. Then T normalizes U, and hence $R := TU$ is a subgroup of G_4. Let λ be a character of T, and once again extend θ_S to a character of R by $\theta_S(tu) = \lambda(t)\,\theta_S(u)$.

If π is an admissible representation of G_4, a Bessel functional on π is once again a linear functional $B\colon V_\pi \to \mathbb{C}$ such that

$$B(\pi(tu)v) = \theta_S(tu)\,B(v),$$

for all $t \in T$, $u \in U$, and $v \in V_\pi$. If π transforms by a central character, then since T contains the center Z_4 of G_4, this notion requires that $\lambda\,|_{Z_4}$ match this character.

Let $\pi = \mathrm{Ind}(\chi)$ be the principal series representation obtained by normalized induction from a character χ of the standard Borel subgroup of G_4. For χ in a suitable domain a Bessel functional may be obtained by integration as above. If $\Phi_\chi \in V_\pi$ is the standard nonramified vector, then define $H = H_\chi\colon G_4 \to \mathbb{C}$ by the formula $H(g) = B(\pi(g)\Phi_\chi)$.

To describe the explicit formula for this function, define parameters α_1 to α_4 by

$$\chi\begin{pmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \alpha_1 \qquad \chi\begin{pmatrix} \varpi & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{pmatrix} = \alpha_2$$

$$\chi\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{pmatrix} = \alpha_3 \qquad \chi\begin{pmatrix} 1 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{pmatrix} = \alpha_4.$$

The Weyl group Ω_4 acts on these parameters through all permutations of the α_i which preserve the relation $\alpha_1\alpha_3 = \alpha_2\alpha_4$. Let $\mathcal{A}_4$ be the alternator $\sum_{w\in\Omega_4}(-1)^{\mathrm{length}(w)}\,w$ in the group algebra $\mathbb{C}[\Omega_4]$. Observe moreover that the function H is completely determined by its values on the elements

$$a_{k,l} = \begin{pmatrix} \varpi^{k+2l} & & & \\ & \varpi^{k+l} & & \\ & & 1 & \\ & & & \varpi^l \end{pmatrix}$$

with $k, l \geq 0$ if T is nonsplit, and by its values on the elements

$$b_{k,l} = \begin{cases} a_{k,0} & \text{if } l = 0 \\ m(1)a_{k,l} & \text{otherwise} \end{cases}$$

with $k, l \geq 0$ if T is split, where we set

$$m(1) = \begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{pmatrix}.$$

Then we have

Corollary 1.8. *Suppose that T is nonsplit and $a \in \mathcal{O}^\times$.*

(1) *The function H_χ may be defined by continuation for all unramified characters χ. Moreover the function*

$$\mathcal{H}_\chi(g) = (1 - \alpha_1\alpha_2^{-1}q^{-1})^{-1}(1 - \alpha_2\alpha_3^{-1}q^{-1})^{-1}H_\chi(g)$$

is holomorphic and invariant under the action of Ω.

(2) *The function $\mathcal{H}_\chi$ is given by the explicit formula*

$$\mathcal{H}_\chi(a_{k,l}) = (1 + q^{-1})^{-1}q^{-3k/2-2l}\frac{\mathcal{A}_4\left(\alpha_3^{k+l+2}\alpha_2^l\alpha_4^{-1}(1 - \alpha_1\alpha_3^{-1}q^{-1})(1 - \alpha_4\alpha_2^{-1}q^{-1})\right)}{\mathcal{A}_4\left(\alpha_3^2\alpha_4^{-1}\right)}$$

valid for $k, l \geq 0$.

In the split case, let us suppose without loss that $a = c = 0$, $b = 1$. Let β_1, β_2 be the parameters

$$\beta_1 = \lambda\left(\begin{pmatrix} \varpi & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{pmatrix}\right), \qquad \beta_2 = \lambda\left(\begin{pmatrix} 1 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{pmatrix}\right).$$

We are concerned with characters χ such that $\alpha_1\alpha_3 = \beta_1\beta_2$. Then we have

Corollary 1.9. *Suppose that T is split.*

(1) *The function H_χ may be defined by continuation for all unramified characters χ, λ such that $\chi\,|_{Z_4} = \lambda\,|_{Z_4}$. Moreover the function*

$$\mathcal{H}_\chi(g) = \frac{\prod_{i,j=1}^2(1 - \alpha_i\beta_j^{-1}q^{-1/2})}{(1 - \alpha_1\alpha_2^{-1}q^{-1})(1 - \alpha_1\alpha_3^{-1}q^{-1})(1 - \alpha_2\alpha_3^{-1}q^{-1})(1 - \alpha_2\alpha_4^{-1}q^{-1})}H_\chi(g)$$

is holomorphic and invariant under the action of Ω.

(2) *The function $\mathcal{H}_\chi$ is given by the explicit formula*

$$\mathcal{H}_\chi(b_{k,0}) = q^{-3k/2} \frac{\mathcal{A}_4\left(\alpha_3^{k+2}\alpha_4^{-1}\prod_{j=1}^{2}(1-\alpha_1\beta_j^{-1}q^{-1/2})\right)}{\mathcal{A}_4\left(\alpha_3^2\alpha_4^{-1}\right)}$$

valid for $k \geq 0$ and by

$$\mathcal{H}_\chi(b_{k,l}) = (1-q^{-1})^{-1}\, q^{-3k/2-2l} \frac{\mathcal{A}_4\left(\alpha_3^{k+l+2}\alpha_2^{l}\alpha_4^{-1}\prod_{i=1,4;j=1,2}(1-\alpha_i\beta_j^{-1}q^{-1/2})\right)}{\mathcal{A}_4\left(\alpha_3^2\alpha_4^{-1}\right)}$$

valid for $k \geq 0$, $l > 0$.

2. The Waldspurger Functional. We begin the study of the split Waldspurger functional with the following Lemma.

Lemma 2.1. *Suppose that the inequalities (1.6) hold. Then the integral (1.5) is absolutely convergent.*

Proof. Let $f \in \mathrm{Ind}(\xi)$ be given. Since K_2 is compact, there is a number C such that $|f(\kappa)| \leq C$ for all $\kappa \in K_2$. One has the matrix identities

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \begin{cases} \begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ a & 1 \end{pmatrix} & \text{if } |a|_F \leq 1 \\ \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & a^{-1} \end{pmatrix} & \text{if } |a|_F > 1, \end{cases} \tag{2.1}$$

where the last matrix in each case is in K_2. Since $f \in \mathrm{Ind}(\xi)$, one obtains the inequalities

$$\left| f\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\right| \leq \begin{cases} C|a|_F^{1/2}|\xi_1(a)| & \text{if } |a|_F \leq 1 \\ C|a|_F^{-1/2}|\xi_2(a)| & \text{if } |a|_F > 1. \end{cases}$$

Thus one sees that the integral (1.5) is majorized by the sum of the integrals

$$\int_{|a|_F \leq 1} |\xi_1(a)|\,|\sigma^{-1}(a)|\,|a|_F^{1/2}\, d^\times a$$

and

$$\int_{|a|_F > 1} |\xi_2(a)|\,|\sigma^{-1}(a)|\,|a|_F^{-1/2}\, d^\times a.$$

But when the inequalities (1.6) hold, each of these integrals is an absolutely convergent geometric series. The Lemma follows. ∎

We turn to the proof of Theorem 1.1. The proof makes essential use of ideas of Casselman and Shalika [CS], and of Banks [Ba].

Let us define certain elements of the representation $\pi = \pi_2 = \mathrm{Ind}(\xi)$ as follows. Let ϕ_ξ be the normalized K_2-fixed vector, and for $k \geq 0$, let

$$F_k(g) = \int_{\mathcal{O}} \phi_\xi\left(g \begin{pmatrix} 1 & z \\ & 1 \end{pmatrix} \begin{pmatrix} \varpi^k & \\ & 1 \end{pmatrix}\right) dz, \tag{2.2}$$

$$\zeta_k(g) = \phi_\xi\left(g \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \begin{pmatrix} \varpi^k & \\ & 1 \end{pmatrix}\right).$$

It is easy to see that in the integral (2.2), the integrand has constant value when z has constant valuation; and if z is a unit, this constant value is $\zeta_k(g)$. Thus

$$(1 - q^{-1})\zeta_k(g) = F_k(g) - \int_{\varpi\mathcal{O}} \phi_\xi\left(g \begin{pmatrix} 1 & z \\ & 1 \end{pmatrix} \begin{pmatrix} \varpi^k & \\ & 1 \end{pmatrix}\right) dz.$$

A simple change of variables shows that the second integral equals

$$q^{-1}\, \pi\begin{pmatrix} \varpi & \\ & 1 \end{pmatrix} F_{k-1}(g),$$

and so applying the Waldspurger functional W, we obtain

$$(1 - q^{-1})W(\zeta_k) = W(F_k) - q^{-1}\tau W(F_{k-1}). \tag{2.3}$$

As in Casselman [Ca1] and Casselman and Shalika [CS], the vectors F_k are fixed by the Iwahori subgroup $\mathcal{B}_2$ of G_2. We remind the reader of the Casselman basis of the Iwahori fixed vectors of G_2. Assuming that ξ is regular, we define linear functionals T_w for w a Weyl group representative of G_2 by

$$T_w f = \int_{N \cap wNw^{-1} \backslash N} f(w^{-1}n)\, dn,$$

where N is the group of upper triangular unipotent matrices in G_2. These functionals are linearly independent on the Iwahori fixed vectors, and the Casselman basis f_w is defined by $T_w f_{w'} = \delta(w, w')$ (Kronecker delta) for w and w' in the Weyl group. If w_0 is (a representative of) the long element of the Weyl group, then

$$f_{w_0}(g) = \begin{cases} \phi_\xi(g) & \text{if } g \in B_2 w_0 \mathcal{B}_2, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

The element f_1 is given by a more complicated formula, and we do not need to know it. Since F_k is an Iwahori fixed vector, we can write

$$F_k = c(1,\xi)\, f_1 + c(w_0,\xi)\, f_{w_0},$$

and by definition of the f_w, $c(w,\xi) = T_w F_k$. It is easy to see (and proved in Casselman and Shalika [CS]) that

$$T_1 F_k = q^{-k/2}\,\gamma_1^k, \qquad T_{w_0} F_k = (1-q^{-1}\gamma_1\gamma_2^{-1})(1-\gamma_1\gamma_2^{-1})^{-1}\,q^{-k/2}\,\gamma_2^k. \tag{2.5}$$

Thus

$$W(F_k) = W(f_1)\,q^{-k/2}\,\gamma_1^k + (1-q^{-1}\gamma_1\gamma_2^{-1})(1-\gamma_1\gamma_2^{-1})^{-1}\,q^{-k/2}\,\gamma_2^k\,W(f_{w_0}).$$

Now using (2.3), we find that

$$\begin{aligned}(1-q^{-1})W(\zeta_n) = {} & (1-q^{-1/2}\tau\gamma_1^{-1})W(f_1)\,q^{-k/2}\,\gamma_1^k + \\ & (1-q^{-1/2}\tau\gamma_2^{-1})(1-q^{-1}\gamma_1\gamma_2^{-1})(1-\gamma_1\gamma_2^{-1})^{-1}\,q^{-k/2}\,\gamma_2^k\,W(f_{w_0}).\end{aligned}$$

We may compute $W(f_{w_0})$ explicitly. By definition this equals

$$\int_{F^\times} f_{w_0}\left(\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\begin{pmatrix} 1 & 1\\ & 1\end{pmatrix}\begin{pmatrix} a & \\ & 1\end{pmatrix}\right)\sigma(a)^{-1}\,d^\times a,$$

and it follows from (2.4) that the integrand here equals

$$\phi_\xi\left(\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\begin{pmatrix} 1 & 1\\ & 1\end{pmatrix}\begin{pmatrix} a & \\ & 1\end{pmatrix}\right)$$

if $|a| \ge 1$, zero otherwise. Thus

$$W(f_{w_0}) = \int_{|a|\ge 1} \phi_\xi\left(\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\begin{pmatrix} 1 & 1\\ & 1\end{pmatrix}\begin{pmatrix} a & \\ & 1\end{pmatrix}\right)\sigma(a)^{-1}\,d^\times a.$$

If $|a| = q^k$, $k \ge 0$, then the integrand is readily evaluated using (2.1), and equals $\tau^k\,\gamma_2^{-k}\,q^{-k/2}$, so, assuming (1.6), the last integral is absolutely convergent and equals $(1-q^{-1/2}\tau\gamma_2^{-1})^{-1}$. Thus

$$\begin{aligned}(1-q^{-1})W(\zeta_n) = {} & (1-q^{-1/2}\tau\gamma_1^{-1})W(f_1)\,q^{-k/2}\,\gamma_1^k + \\ & (1-q^{-1}\gamma_1\gamma_2^{-1})(1-\gamma_1\gamma_2^{-1})^{-1}\,q^{-k/2}\,\gamma_2^k.\end{aligned}$$

We have not yet proved the Theorem since we have not evaluated $W(f_1)$. Note however that the Theorem will follow if we prove that $\mathcal{W}a_\xi$ is invariant under the interchange of γ_1 and γ_2; indeed, we may assume without loss of generality that ξ is regular, so $\gamma_1 \neq \gamma_2$. Then, because the two functions $q^{-k/2}\gamma_1^k$ and $q^{-k/2}\gamma_2^k$ of k are linearly independent, the unknown value of $W(f_1)$ will be determined.

We will show that $\mathcal{W}a_\xi(1) = 1$. This is sufficient: indeed, recalling Waldspurger's theorem on the uniqueness of the model, there is a unique spherical vector in this unique model which is normalized to equal 1 at $g = 1$, if $\mathcal{W}a_\xi(1) = 1$ then clearly $\mathcal{W}a_\xi$ must be this vector; then, since the isomorphism class of the representation $\pi_2 = \mathrm{Ind}(\xi)$ is unchanged when we interchange γ_1 and γ_2, $\mathcal{W}a_\xi$ is thus invariant under this interchange. By definition

$$Wa_\xi(1) = \int_{F^\times} \phi_\xi\left(\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right)\sigma(a)^{-1}\,d^\times a,$$

and in this integral, the integrand is constant when a has constant valuation. (This is not true for $Wa_\xi(\eta_k)$, which is the reason for the somewhat elaborate proof of this Theorem!) Once again applying (2.1), we find that the integrand equals $\tau^{-k}\gamma_1^k q^{-k/2}$ if $|a| = q^{-k}$, $k \geq 0$, or $\tau^{-k}\gamma_2^k q^{k/2}$ if $k < 0$. Assuming (1.6) holds, it is then simple to sum the two geometric series and check that $\mathcal{W}a_\xi(1) = 1$.

This completes the proof of Theorem 1.1 ∎

We turn now to the proof of Theorem 1.2. Fix an element $f \in C^\infty\big((B_2 \cap K_2)\backslash K_2\big)$, and extend f to $\mathrm{Ind}(\xi)$. Then, since f is locally constant, it follows from (2.1) that the integrand on the right in (1.5) equals

$$\begin{cases} |a|^{1/2}\xi_1(a)\sigma^{-1}(a)\,f\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } |a| \text{ is sufficiently small;} \\ |a|^{-1/2}\xi_2(a)\sigma^{-1}(a)\,f\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } |a| \text{ is sufficiently large.} \end{cases}$$

Hence the integral (1.5) is equal to an integral over a compact set plus two integrals giving geometric series, whose values have analytic continuation to the region Λ. This gives the analytic continuation of W, and it only remains to be seen that it represents a Waldspurger functional. Thus we must show that with $t \in T_2^s(F)$, $W(\pi_2(t)\,f - \sigma(t)\,f) = 0$. It is clear that this is true when (1.6) is satisfied, and that the left side is analytic, so this is true for all $(\gamma_1, \gamma_2, \tau) \in \Lambda$. This completes the proof of Theorem 1.2. ∎

3. Proof of the Analytic Continuation and Functional Equation for the Bessel Model. In this Section we shall prove (most of) Theorems 1.3 and 1.4. The proof of the continuation in the parameters α_i and of the functional equation is based on homomorphisms from $GL(2,F)$ into G, similarly to Jacquet's proof of the analytic continuation and functional equation of the Whittaker functions on Chevalley groups [Ja]. Jacquet's method suffices to give most of the functional equations, but an extra step is needed (different in the nonsplit and split cases). One then applies Hartog's theorem. At this point, we will have proved Theorems 1.3 and 1.4 except for one point, namely the meromorphic continuation in the split case outside the region $q^{-1/2} < \min(|\beta|, |\beta|^{-1})$. This meromorphic continuation follows from the explicit formula in Theorem 1.6, or from Theorem 1.7; so for this minor point, the proof will be completed in subsequent sections.

Our proof will follow to the extent possible the notation and organization of [BFG], where a similar method was used to study another unique functional.

First, we recall two well-known lemmas. To give the first, let $\omega \in \Omega$ be a Weyl group element represented by the permutation matrix w (we shall frequently abuse the notation and write $w \in \Omega$). Let N denote the full subgroup of upper triangular unipotent matrices in G. Given a character χ as above, let ${}^w\chi$ be the character satisfying ${}^w\chi(a) = \chi(w^{-1}aw)$ for all diagonal $a \in G$. Define the intertwining operator $T_w\colon \mathrm{Ind}(\chi) \to \mathrm{Ind}({}^w\chi)$ by the integral

$$(3.1) \qquad (T_w\Psi)(g) = \int\limits_{N_w\backslash N} \Psi(w^{-1}ng)\, dn,$$

where $N_w = N \cap wNw^{-1}$. Given an unramified character χ as above, let a_α be the diagonal matrix in G

$$a_\alpha = (\alpha_1, \cdots, \alpha_n, 1, \alpha_n^{-1}, \cdots, \alpha_1^{-1}).$$

Order the roots of $SO(2n+1, \mathbb{C})$ so that N corresponds to the positive ones. Then one has ([Ca2], Section 6.4)

Lemma 3.1. *The intertwining integral (3.1) is absolutely convergent if $|r(a_\alpha)| < 1$ for all positive roots r of $SO(2n+1, \mathbb{C})$ such that $w(r) < 0$. Moreover, T_w varies holomorphically with χ and has meromorphic continuation to the space of all unramified characters.*

The second lemma concerns the $G_2 = GL(2, F)$ Whittaker function. As in Section 1, given two unramified quasicharacters ξ_1, ξ_2 of $F^\times$, $\xi_i(\varpi) = \gamma_i$, define a character ξ of the standard Borel subgroup of G_2 by equation (1.3). Let $\pi_2 = \mathrm{Ind}(\xi)$ be the normalized induced representation of G_2, and ϕ_ξ be the K_2-fixed vector such that $\phi_\xi(I_2) = 1$. If $|\gamma_1| > |\gamma_2|$, let Wh_ξ be the Whittaker function

$$(3.2) \qquad Wh_\xi(g) = \int\limits_F \phi_\xi\left(\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) \psi(x)\, dx.$$

Then one has

Lemma 3.2.

(1) *The Whittaker function Wh_ξ, originally defined by the integral (3.2) when $|\gamma_1| > |\gamma_2|$, has a meromorphic continuation to all nonzero complex γ_1, γ_2. Moreover, the function*

$$\mathfrak{Wh}_\xi(g) = (1 - \gamma_1\gamma_2^{-1}q^{-1})^{-1} Wh_\xi(g)$$

is holomorphic in $(\mathbb{C}^\times)^2$ and is invariant under the interchange of γ_1 and γ_2.

(2) *Let $\gamma_1' = (1+\epsilon)\max(|\gamma_1|, |\gamma_2|)$, $\gamma_2' = (1+\epsilon)^{-1}\min(|\gamma_1|, |\gamma_2|)$, where $\epsilon \geq 0$ is chosen so that $\gamma_1' \neq \gamma_2'$. Define corresponding unramified quasicharacters ξ_i' for $i = 1, 2$ by $\xi_i'(\varpi) = \gamma_i'$. Then*

$$|Wh_\xi(g)| \ll |\phi_{\xi'}(g)|$$

uniformly in g, where $\phi_{\xi'}$ is the normalized K_2-fixed vector in $\mathrm{Ind}(\xi_1', \xi_2')$.

Just as the continuation of the Waldspurger function Wa_ξ is deduced from its evaluation, Lemma 3.2 may be deduced from the explicit evaluation of the Whittaker function Wh_ξ. Assuming $|\gamma_1| > |\gamma_2|$, a computation shows that the integral (3.2) is zero if $g = \left(\begin{smallmatrix} \varpi^k & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with $k < 0$, and is given by

$$Wh_\xi\left(\begin{pmatrix} \varpi^k & 0 \\ 0 & 1 \end{pmatrix}\right) = q^{-k/2}(1 - \gamma_1\gamma_2^{-1}q^{-1})\,\frac{\gamma_1^{k+1} - \gamma_2^{k+1}}{\gamma_1 - \gamma_2}$$

if $k \geq 0$. The Lemma then follows.

We give next a third Lemma, concerning the convergence of the integrals which arise in the consideration of the split case. Let χ be an unramified character as above. Let $w \in \Omega$ be a given permutation, and factor w^{-1} as $w^{-1} = {w'}^{-1}(n, n+2)$. Set $U_w = U \cap w^{-1}Uw$. Also, define the integer g and the complex number α by the equations

$$\delta_B\left({w'}^{-1}t(\varpi)w'\right) = q^{-g}, \qquad \chi\left({w'}^{-1}t(\varpi)w'\right) = \alpha.$$

Lemma 3.3. *For $\Psi \in \mathrm{Ind}(\chi)$, the integral*

$$\int_{F^\times}\int_{U_w\backslash U} \Psi(w^{-1}n(1)\,u\,t(a))\,\lambda^{-1}(a)\,du\,d^\times a \tag{3.3}$$

is absolutely convergent provided

$$|\alpha| < q^{g/2}\min(|\beta|, |\beta^{-1}|)$$

and provided that the intertwining integral $T_{w'}(\Psi)$ converges absolutely.

Proof. We may interchange u and $t(a)$ in the above integral. We have $w^{-1}n(1)\,t(a)\,u = {w'}^{-1}(n,n+2)\,n(1)\,t(a)\,u$. A computation similar to (2.1) shows that

$$(n,n+2)\,n(1)\,t(a) = \begin{cases} t(a)\,n(a^{-1})\,\kappa_1(a) & \text{if } |a|_F \le 1 \\ t(a^{-1})\,\kappa_2(a) & \text{if } |a|_F > 1, \end{cases}$$

with $\kappa_1(a)$, $\kappa_2(a)$ in K of the form

$$\begin{pmatrix} I_{n-1} & & \\ & * & \\ & & I_{n-1} \end{pmatrix}.$$

Since matrices of this form normalize U, we may move the $\kappa_i(a)$ to the right in the integral (3.3). Using the condition $\Psi \in \mathrm{Ind}(\chi)$ and comparing with the definition (3.1) of the intertwining operator $T_{w'}$, one sees that the integral (3.3) is absolutely bounded by the sum of the two integrals

$$\int\limits_{|a|_F \le 1} |\alpha|^{\mathrm{ord}(a)}\,|\lambda^{-1}(a)|\,|a|_F^{g/2}\,|(T_{w'}\Psi)(\kappa_1(a))|\,d^\times a$$

and

$$\int\limits_{|a|_F > 1} |\alpha|^{-\mathrm{ord}(a)}\,|\lambda^{-1}(a)|\,|a|_F^{-g/2}\,|(T_{w'}\Psi)(\kappa_2(a))|\,d^\times a.$$

Since $(T_{w'}\Psi)(g)$ is a locally constant function, it is absolutely bounded on K. Hence both integrals are bounded by geometric series, which converge under the hypotheses of the Lemma. ∎

This proof may also be rephrased by using the isomorphism of $PGL(2,F)$ with $SO(3,F)$ to write the integral (3.3) as the Waldspurger integral of an intertwining integral, and then applying Lemmas 2.1 and Lemma 3.1.

Observe that applying Lemma 3.3 with $w = w_0$, so that $w' = w_1$, $g = 1$ and $\alpha = \alpha_n$, one finds that the integral (1.14) is absolutely convergent in the region (1.15), as claimed.

To prove Theorems 1.3 and 1.4 we now proceed in two stages. First, using the standard functional equation for the $GL(2)$ p-adic Whittaker function (Lemma 3.2), we shall establish functional equations under the transpositions $(j,j+1) \in \Omega$, $1 \le j \le n-1$. The proofs of these results also give the analytic continuation to certain unions of Weyl chambers properly larger than the original region of convergence. Then, using Theorem 1.1 in the split case and the invariance of the G_2-spherical function in $\mathrm{Ind}(\xi_1,\xi_2)$ under the interchange of ξ_1 and ξ_2 in the nonsplit case, we obtain a functional equation for the Weyl group element taking α_n to α_n^{-1} and fixing the rest. Since these two elements generate Ω, these steps imply that $\mathcal{H}_\chi$ is invariant under Ω.

To obtain the functional equations under the transpositions $(j, j+1)$, $1 \le j \le n-1$, let ι_j be the embedding of $GL(2, F)$ into G given by

$$g \mapsto \begin{pmatrix} I_{j-1} & & & & \\ & \det(g)^{-1}g & & & \\ & & I_{2n-2j-1} & & \\ & & & g^\sharp & \\ & & & & I_{j-1} \end{pmatrix},$$

where $g^\sharp = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} g \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$. Let w denote w_0 if T is split, and w_1 if T is nonsplit. Let $v_1 = (j, j+1)(2n-j+1, 2n-j+2)$, and factor $w = v_1 v_2$. Let B_2 and U_2 be, as above, the subgroups of $GL(2, F)$ consisting respectively of upper triangular and of upper triangular unipotent matrices, and let B_j and U_j be the subgroups of G given by

$$B_j = \iota_j(B_2), \qquad U_j = v_2^{-1}\, \iota_j(U_2)\, v_2.$$

For $u_j \in U_j$, write $v_2\, u_j = \iota_j(\hat{u}_j)\, v_2$. Let U_j' and B_j' be the complementary subgroups in U and B to U_j and B_j, respectively, so that $U = U_j U_j'$ and $B = B_j B_j'$ (uniquely). Note that v_1 and $\iota_j(U_2)$ normalize B_j' and fix $\Phi_\chi|_{B_j'}$. Let $u \in U$, $t \in T$, $a \in G$. Factor $u = u_j u_j'$, $u_j \in U_j$, $u_j' \in U_j'$. Applying the Iwasawa decomposition, for $g \in G$, we may write $v_2 u_j' t g = b_j' b_j \kappa$, with $b_j \in B_j$, $b_j' \in B_j'$, $\kappa \in K$ (we suppress the dependence of b_j' and b_j on u_j', t, and g from the notation).

Suppose first that T is nonsplit. Then with the above notation we have

$$\begin{aligned} w_1 u t g &= v_1 v_2 u_j u_j' t g \\ &= v_1 \iota_j(\hat{u}_j)\, v_2 u_j' t g \\ &= v_1 \iota_j(\hat{u}_j)\, b_j' b_j \kappa. \end{aligned}$$

Since Φ_χ is right K-invariant, we may thus express $H_\chi(g)$ as the iterated integral

$$H_\chi(g) = \int\limits_{T(\mathcal{O})} \int\limits_{U_j'} \left[\int\limits_{U_j} \Phi_\chi(v_1 \iota_j(\hat{u}_j) b_j)\, \theta_S(u_j)^{-1}\, du_j \right] \Phi_\chi(b_j')\, \theta_S(u_j')^{-1}\, du_j'\, dt. \tag{3.4}$$

However for $h \in G$, it follows from (1.11) that the function $\phi\colon GL(2, F) \to \mathbb{C}$ given by $\phi(a) = \Phi_\chi(\iota_j(a)h)$ is in the space $\mathrm{Ind}(\chi_{j+1}^{-1}\mu^{j-n}, \chi_j^{-1}\mu^{j-n})$, where μ is the quasicharacter of $F^\times$ given by $\mu(x) = |x|_F$. Accordingly the inner integral in (3.4) is a constant multiple of the $GL(2)$ Whittaker function associated to this representation. Applying Lemma 3.2, we obtain the analytic continuation of the function

$$\frac{1}{1-\alpha_j\alpha_{j+1}^{-1}q^{-1}}H_\chi(g)$$

to the region C_j of $(\mathbb{C}^\times)^n$ defined by the inequalities

$$\begin{aligned}(3.5)\quad |\alpha_1| < \cdots < |\alpha_{j-1}| < \min(|\alpha_j|, |\alpha_{j+1}|) \le \max(|\alpha_j|, |\alpha_{j+1}|)\\ < |\alpha_{j+2}| < \cdots < |\alpha_{n-1}| < \min(|\alpha_n|, |\alpha_n^{-1}|)\end{aligned}$$

if $1 \le j < n-1$, and defined by the inequalities

$$(3.6)\qquad |\alpha_1| < |\alpha_2| < \cdots < |\alpha_{n-2}| < \min(|\alpha_{n-1}|, |\alpha_n|) \le \max(|\alpha_{n-1}|, |\alpha_n|) < 1$$

if $j = n-1$, and the invariance of this function under $(j, j+1)$ there. The region C_j is obtained by replacing the inner integral by the estimate given in Lemma 3.2, part (2), and comparing with the intertwining operator $T_{v_2^{-1}}$, whose convergence is given in Lemma 3.1. The difference in inequalities is due to this comparison; note that for $1 \le j < n-1$, v_2 is a product of transpositions

$$v_2 = (1, 2n+1)\cdots(j-1, 2n-j+3)(j, 2n-j+1)(j+1, 2n-j+2)(j+2, 2n-j)\cdots(n-1, n+3),$$

while for $j = n-1$, v_2 includes a four-cycle

$$v_2 = (1, 2n+1)\cdots(n-2, n+4)(n-1, n+2, n+3, n).$$

Also, let us remark that if $1 \le j < n-1$, then the region C_j properly contains the original region of convergence C_0, while if $j = n-1$, Hartog's theorem gives at once the analytic continuation of H_χ to $C_0 \cup C_{n-1}$.

Suppose now that T is split. Let u, u_j, u_j', $\hat{u}_j$ be as above; note that $U_j = \iota_j(U_2)$. We have

$$\begin{aligned}w_0 n(1) u t(a) &= v_1 v_2 u_j (u_j^{-1} n(1) u_j) u_j' t(a)\\ &= v_1 \iota_j(\hat{u}_j) v_2 (u_j^{-1} n(1) u_j) u_j' t(a).\end{aligned}$$

Moreover, for $u_j = u_j(x) := \iota_j\left(\begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\right)$, a calculation shows that $u_j^{-1}n(1)u_j = n(1)u_j''$ with $u_j'' \in U_j'$ and $\theta_S(u_j)^{-1}\theta_S(u_j'') = \psi(-x)$. Using the Iwasawa decomposition, write $v_2 n(1) u_j' t(a) g = b_j' b_j \kappa$, with $b_j \in B_j$, $b_j' \in B_j'$, $\kappa \in K$ (once again we suppress from the notation the dependence of b'_j and b_j on u'_j, $t(a)$, and g). Then the integral (1.14) representing $H_\chi(g)$ becomes

$$H_\chi(g) = \int\limits_{F^\times}\int\limits_{U'}\left[\int\limits_F \Phi_\chi(v_1 u_j(-x) b_j)\,\psi(-x)\,dx\right]\Phi_\chi(b'_j)\,\theta_S(u'_j)^{-1}\,\lambda^{-1}(a)\,du\,d^\times a.$$

The inner integral is once again a $GL(2)$ Whittaker function. Applying Lemma 3.2, we obtain the analytic continuation of the function

$$\frac{1}{1-\alpha_j\alpha_{j+1}^{-1}q^{-1}}H_\chi(g),$$

to the region C'_j of $(\mathbb{C}^\times)^n$ defined by requiring the inequalities (3.5) and in addition the inequality

$$|\alpha_n| < q^{1/2}\min(|\beta|, |\beta^{-1}|)$$

if $1 \le j < n-1$, and defined by requiring the inequalities (3.6) and in addition the inequality

$$\min(|\alpha_{n-1}|, |\alpha_n|) < q^{3/2}\min(|\beta|, |\beta^{-1}|)$$

if $j = n-1$. We also obtain the invariance of this function under $(j, j+1)$ there. The region C'_j is obtained by replacing the inner integral by the estimate given in Lemma 3.2, part (2), and applying Lemma 3.3. For later use, we denote the original region of convergence given by (1.15) as C'_0.

It remains to obtain the functional equation under the interchange $\alpha_n \leftrightarrow \alpha_n^{-1}$. To obtain this, let ι_n denote the homomorphism of $GL(2,F)$ into G given by

$$\iota_n\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right) = \frac{1}{ad-bc}\begin{pmatrix} (ad-bc)I_{n-1} & & & & \\ & a^2 & ab & -b^2/2 & \\ & 2ac & ad+bc & -bd & \\ & -2c^2 & -2cd & d^2 & \\ & & & & (ad-bc)I_{n-1}\end{pmatrix}. \tag{3.7}$$

Consider first the case that T is nonsplit. Since w_1 fixes T, the integral (1.12) representing $H_\chi(g)$ is expressed as an iterated integral

$$H_\chi(g) = \int\limits_U\left[\int\limits_{T(\mathcal{O})}\Phi_\chi(tw_1ug)\,dt\right]\theta_S(u)^{-1}\,du. \tag{3.8}$$

Now for $h \in G$, the function $\phi: GL(2,F) \to \mathbb{C}$ given by $\phi(a) = \Phi_\chi(\iota_n(a)h)$ is in the space $\mathrm{Ind}(\chi_n, \chi_n^{-1})$. Recalling that ι_n gives the isomorphism between $PGL_2(F)$ and $SO(3,F)$, one sees that the inner integral in (3.8) is a multiple of the nonsplit Waldspurger functional on $PGL_2(F)$, evaluated on a suitable right-translate of the spherical vector in this induced space. As explained in Section 1, it is thus a multiple of the spherical function in $\mathrm{Ind}(\chi_n, \chi_n^{-1})$. By the invariance of this spherical function under the interchange of χ_n and χ_n^{-1}, we conclude that $H_\chi(g)$ is invariant under $\alpha_n \leftrightarrow \alpha_n^{-1}$ in the domain (1.13).

Consider instead the case that T is split. Factor $w_0 = w_1 w_2$ with $w_2 = (n, n+2)$. In this case the integral (1.14) is an iterated integral of the form

$$H_\chi(g) = \int\limits_U \left[\int\limits_{F^\times} \Phi_\chi(w_2\, n(1)\, t(a)\, w_1\, u\, g)\, \lambda^{-1}(a)\, d^\times a \right] \theta_S(u)^{-1}\, du.$$

Once again using the homomorphism ι_n, one sees that the inner integral is a multiple of the split Waldspurger functional computed in Section 2, applied to a function in $\mathrm{Ind}(\chi_n, \chi_n^{-1})$. Note that this integral is absolutely convergent provided $|\alpha_n| < q^{1/2}\min(|\beta|, |\beta^{-1}|)$. Applying Theorem 1.1 and arguing as above, we obtain the invariance of the function

$$\frac{(1-\alpha_n\beta q^{-1/2})(1-\alpha_n\beta^{-1}q^{-1/2})}{(1-\alpha_n^2 q^{-1})}\, H_\chi(g),$$

under $\alpha_n \leftrightarrow \alpha_n^{-1}$ in the domain (1.15).

To complete the proof of Theorem 1.3, let C be the set of $\alpha = (\alpha_1, \cdots, \alpha_n) \in (\mathbb{C}^\times)^n$ such that at most one of the equalities $|\alpha_i| = |\alpha_j|$, $1 \le i < j \le n$, and $|\alpha_i| = |\alpha_j|^{-1}$, $1 \le i \le j \le n$, is satisfied. It may be deduced from Hartog's theorem that any analytic function on C can be extended to an analytic function on $(\mathbb{C}^\times)^n$, and so it is sufficient to extend the function $\mathcal{H}_\chi(a)$ to the domain C in such a way that the corresponding functional equations are satisfied on C. Now if $\alpha \in C$, then there is an $\omega \in \Omega$ such that $\omega\alpha$ is in C_k for some k, $0 \le k \le n-1$. We then define $\mathcal{H}_\chi(a)$ to equal $\mathcal{H}_{{}^\omega\chi}(a)$. This is well defined by the above discussion. It is apparent that this extends $\mathcal{H}_\chi(a)$ to an analytic function of C satisfying the corresponding functional equations, as required. This completes the proof of Theorem 1.3. ∎

The proof of Theorem 1.4 is similar. Suppose first that λ is chosen so that $q^{-1/2} < \min(|\beta|, |\beta|^{-1})$. Then using the regions C_j' in place of C_j, the argument in the paragraph above gives the analytic continuation of $\mathcal{H}_\chi$ to $\alpha \in (\mathbb{C}^\times)^n$, and the functional equation there. This completes the proof of Theorem 1.4, except, as we have noted, for the point of the meromorphic continuation to all α_i and β, and this point is a consequence of the explicit formula in Theorem 1.6 or of Theorem 1.7. ∎

4. Explicit Formulas for the Bessel Model. The evaluation of the Bessel model given in Theorems 1.5 and 1.6 is obtained by applying the method of Casselman and Shalika [Ca1], [CS]. This method was also used in Section 2. A similar evaluation is carried out [BFG]; however, the particulars are different in the case at hand, and the cases T nonsplit and T split are once again different from each other. Throughout the proofs we shall assume that χ is regular, i.e. ${}^{\omega}\chi \neq \chi$ for all nonidentity $\omega \in \Omega$. The general case then follows from the analytic continuation given in Theorems 1.3 and 1.4.

Let $\mathcal{B}$ be the Iwahori subgroup of K, consisting of integral matrices which are upper triangular invertible mod$\varpi\mathcal{O}$. It follows from the Iwasawa decomposition of G and the Bruhat decomposition over $\mathcal{O}/\varpi\mathcal{O}$ that the space of right-Iwahori-fixed vectors $\mathrm{Ind}(\chi)^{\mathcal{B}}$ is $|\Omega|$-dimensional; moreover, a basis is given by the functions ϕ_w defined by

$$\phi_w(bw'^{-1}b_1) = \begin{cases} \Phi_\chi(b) & \text{if } w = w'; \\ 0 & \text{otherwise,} \end{cases}$$

where $b \in B$, $w \in \Omega$, $b_1 \in \mathcal{B}$ (note that this differs from Casselman's notation in [Ca1], but is consistent with [BFG]).

If χ is regular, then it is shown in [Ca1], Section 3, that the linear functionals on $\mathrm{Ind}(\chi)^{\mathcal{B}}$ given by $f \mapsto (T_w f)(I_{2n+1})$ are linearly independent. Here T_w is the intertwining operator defined in (3.1). Let f_w, $w \in \Omega$, be the dual basis, characterized by

$$(T_w f_{w'})(I_{2n+1}) = \begin{cases} 1 & \text{if } w = w'; \\ 0 & \text{otherwise.} \end{cases}$$

Suppose first that T is nonsplit. Let $d = d_k$ be as given in Section 1, and suppose that all $k_i \geq 0$. Let F_d be the function

$$F_d(g) = \int_{T(\mathcal{O})} \int_{U \cap K} \Phi_\chi(gutd)\, du\, dt.$$

Clearly $F_d \in \mathrm{Ind}(\chi)$. Let us show that F_d is right Iwahori invariant; then we may write

$$F_d(g) = \sum_{w \in \Omega} R(d, w; \chi)\, f_w. \tag{4.1}$$

To prove the Iwahori invariance, we first note that by Jacquet's first lemma (see the proof of Proposition 2.5 of Casselman [Ca1]), F_d is (right) invariant under the group Σ of elements of $SO(2n+1, \mathcal{O})$ whose lower entries are divisible by ϖ, and whose middle 3×3 block is the identity; and also, if ι_n is as in (3.7), it is clear that F_d is invariant by $\iota_n\big(T_2^{\mathrm{a}}(\mathcal{O})\big)$. Thus

$$F_d(bg\sigma) = (\delta_B^{1/2}\chi)(b) F_d(g) \tag{4.2}$$

when $b \in B(F)$ and $\sigma \in \iota_n\big(T_2^{\mathrm{a}}(\mathcal{O})\big)\Sigma$. We will deduce invariance by the group $\iota_n(K_2)\Sigma$ (which contains the Iwahori subgroup) from the fact that the canonical map

$$B\backslash G/\iota_n(T_2^{\mathrm{a}}(\mathcal{O}))\Sigma \to B\backslash G/\iota_n(K_2)\Sigma \tag{4.3}$$

is a bijection. To see this, we note that since $\iota_n(K_2)\Sigma$ contains the Iwahori subgroup, every double coset in $B\backslash G/\iota_n(K_2)\Sigma$ contains a Weyl group element w; so what we must show is that $Bw\iota_n(T_2^{\mathrm{a}}(\mathcal{O}))\Sigma = Bw\iota_n(K_2)\Sigma$. We consider an element $bw\iota_n(k)\sigma$ of the right side, where $b \in B$, $k \in K_2$ and $\sigma \in \Sigma$. We can write $k = \beta^+t^+ = \beta^-t^-$ where $t^\pm$ are in $T_2^{\mathrm{a}}(\mathcal{O})$, β^+ is upper triangular and β^- is lower triangular. Then one of $w\iota_n(\beta^\pm)w^{-1}$ is upper triangular, so $bw\iota_n(k)\sigma = b'w\iota_n(t^\pm)\sigma$ where $b' = bw\iota_n(\beta^\pm)w^{-1}$, which shows that this element lies in $Bw\iota_n(T_2^{\mathrm{a}}(\mathcal{O}))\Sigma$. Thus (4.3) is a bijection. We may now prove the Iwahori invariance of F_d. If $g \in G$ and $\sigma \in \iota_n(K_2)\Sigma$, we write $g\sigma = bg\sigma'$ with $b \in B$ and $\sigma' \in \iota_n\big(T_2^{\mathrm{a}}(\mathcal{O})\big)\Sigma$. We note that b is conjugate to $\sigma\sigma'^{-1} \in K$, and so the eigenvalues of b are units, and $\chi(b) = 1$. It thus follows from (4.2) that $F_d(g\sigma) = F_d(bg\sigma') = F_d(g)$.

Let us compute the coefficient $R(d, w; \chi)$. It equals

$$\begin{aligned}(T_wF_d)(I_{2n+1}) &= \int\limits_{N_w\backslash N}\int\limits_{T(\mathcal{O})}\int\limits_{U\cap K} \Phi_\chi(w^{-1}nutd)\,du\,dt\,dn \\ &= \int\limits_{T(\mathcal{O})} T_w(\Phi_\chi)(td)\,dt.\end{aligned}$$

However, let Φ^+ denote the set of positive roots of $SO(2n+1, \mathbb{C})$, and if $r \in \Phi^+$, let ι_r be the corresponding embedding of $SL(2, F)$ into G, and

$$a_r = \iota_r\begin{pmatrix} \varpi & \\ & \varpi^{-1}\end{pmatrix}.$$

Then it is shown in [Ca1], Theorem 3.1, that

$$T_w(\Phi_\chi) = c_w(\chi)\,\Phi_{{}^w\chi},$$

where $\Phi_{{}^w\chi}$ is the standard nonramified vector in $\mathrm{Ind}({}^w\chi)$, and the coefficient $c_w(\chi)$ is given by

$$c_w(\chi) = \prod_{\substack{r\in\Phi^+\\ w(r)<0}} \left(\frac{1-q^{-1}\,\chi(a_r)}{1-\chi(a_r)}\right). \tag{4.4}$$

We find that

$$R(d, w; \chi) = c_w(\chi)\,\sigma_{{}^w\chi}(d),$$

where

$$\sigma_{{}^w\chi}(d) = \int\limits_{T(\mathcal{O})} \Phi_{{}^w\chi}(td)\,dt.$$

Let ι_n be the homomorphism from $GL(2,F)$ into G given by equation (3.7). Factor $d = d'\iota_n(d'')$ (all these matrices depending on the k_i) with

$$d' = \mathrm{diag}(\varpi^{k_1'}, \cdots, \varpi^{k_2'}, 1, 1, 1, \varpi^{-k_2'}, \cdots, \varpi^{-k_n'})$$

and

$$d'' = \begin{pmatrix} \varpi^{k_1} & \\ & 1 \end{pmatrix}.$$

Then one sees that

$$\sigma_{{}^w\chi}(d) = {}^w\chi\delta_B^{1/2}(d') \int\limits_{T(\mathcal{O})} \Phi_{{}^w\chi}(t\,\iota_n(d''))\,dt.$$

Now if

$${}^w(\chi_1, \cdots, \chi_n) = (\chi_1', \cdots, \chi_n'),$$

then the function $g \mapsto \Phi_{{}^w\chi}(\iota_n(g))$ lies in the induced space $\mathrm{Ind}(\chi_n', {\chi_n'}^{-1})$. Recalling that ι_n identifies $PGL(2,F)$ with $SO(3,F)$ and applying the analysis of the nonsplit Waldspurger functional in Section 1, one concludes that this last integral is equal to the spherical function on $GL(2,F)$ in this induced space, evaluated at d''. This value is given by the Macdonald formula (see [Ca1]). Substituting this formula, we obtain

$$\sigma_{{}^w\chi}(d) = (1+q^{-1})^{-1}\, q^{e_k} \prod_{i=2}^{n} \chi_{n+1-i}'(\varpi^{k_i'}) \tag{4.5}$$
$$\times \frac{(\chi_n'(\varpi) - {\chi_n'}^{-1}(\varpi)q^{-1})\,\chi_n'(\varpi)^{k_1} - ({\chi_n'}^{-1}(\varpi) - \chi_n'(\varpi)q^{-1})\,\chi_n'(\varpi)^{-k_1}}{\chi_n'(\varpi) - {\chi_n'}^{-1}(\varpi)}.$$

This completes the evaluation of $R(d, w; \chi)$.

If $f \in \mathrm{Ind}(\chi)$ with χ dominant, let

$$L(f) = \int\limits_U f(w_1 u)\,\theta_S(u)^{-1}\,du.$$

This integral converges absolutely by comparison with $T_{w_1}(f)$. Then, by (1.12) and (4.1), we have

$$H_\chi(d) = L(F_d) = \sum_{w\in\Omega} R(d, w; \chi)\,L(f_w). \tag{4.6}$$

It is clear from this formula and our evaluation of $R(d,w;\chi)$ that $h(k_1,\cdots,k_n)$ is a linear combination of functions of the α_i which lie in exactly one Ω-orbit. To complete the proof of Theorem 1.5, we shall compute the full contribution to $h(k_1,\cdots,k_n)$ of the terms on the right of (4.6) with $w=w_0$, $w=w_1$. By the computation of $R(d,w;\chi)$ given above, no other $w\in\Omega$ contributes a rational function of the form

$$\alpha_1^{-k'_n}\cdots\alpha_n^{-k'_1}$$

times a function independent of the k_i. Hence, by Theorem 1.3, we will have computed one piece of the Ω-orbit, and the final formula is then the Ω-symmetric sum of these terms, taking into account the normalizing factor in the functional equation of Theorem 1.3.

To compute the contribution of the $w=w_0,w_1$ terms to (4.6), we need the following lemma relating the two bases for the Iwahori fixed vectors, $\{\phi_w\}$ and $\{f_w\}$.

Lemma 4.1.

(1) $\phi_{w_0}=f_{w_0}$.
(2) $\phi_{w_1}=f_{w_1}+\frac{(1-q^{-1})\alpha_n^2}{1-\alpha_n^2}f_{w_0}$.
(3) $L(f_{w_0})=0$.
(4) $L(f_{w_1})=1$.

Proof. For $w'\in\Omega$ we may write

$$\phi_{w'}(g)=\sum_{w\in\Omega}c_\chi(w,w')\,f_w(g). \tag{4.7}$$

where the coefficients c_χ are given by

$$\begin{aligned}c_\chi(w,w')&=T_w(\phi_{w'})(I_{2n+1})\\&=\int\limits_{N_w\backslash N}\phi_{w'}(w^{-1}n)\,dn.\end{aligned}$$

This integral is zero unless $w^{-1}n\in Bw'^{-1}\mathcal{B}$ for some $n\in N_w\backslash N$. If $w'=w_0$, it is easy to see that this may happen only for $w=w_0$ and $n\in N\cap K$; part (1) follows. Similarly, if $w'=w_1$, then $c_\chi(w,w')=0$ unless $w=w_0$ or $w=w_1$, and $c_\chi(w_1,w_1)=1$. Finally, to determine $c_\chi(w_1,w_0)$, it suffices to take g in (4.7) to be of the form $\iota_n(g')$, with $g'\in GL(2,F)$. In that case the ϕ_{w_i} and f_{w_i} necessarily match the analogously defined $GL(2)$ functions in $\mathrm{Ind}(\chi_n,\chi_n^{-1})$, and the determination of this coefficient follows from the analogous statement on $GL(2,F)$, which is equivalent to (2.5).

To prove the remaining parts of the Lemma, it suffices to show that

$$L(\phi_{w_0})=0,\qquad L(\phi_{w_1})=1. \tag{4.8}$$

However, for $w \in \Omega$,

$$L(\phi_w) = \int_U \phi_w(w_1 u)\, \theta_S(u)^{-1}\, du.$$

A matrix calculation shows that $w_1 u \in Bw^{-1}\mathcal{B}$ is impossible if $w = w_0$, and holds when $w = w_1$ if and only if $u \in U \cap K$. Thus (4.8) holds. ∎

Combining (4.6), Lemma 4.1, and the evaluation of $R(d, w; \chi)$, one finds that

$$H_\chi(d) = c_{w_1}(\chi)\, \sigma^{w_1}\chi(d) + \sum_{w \in \Omega; w \neq w_0, w_1} R(d, w; \chi)\, L(f_w). \tag{4.9}$$

The final value for $\mathcal{H}_\chi(d)$ may now be obtained by taking the first term on the right hand side of (4.9), multiplying by the normalizing factor in the functional equation of Theorem 1.3, and symmetrizing with respect to the action of Ω on the parameters α_i. To do this, observe that (4.4) gives

$$c_{w_1}(\chi) = \prod_{1 \leq i < j \leq n} \frac{(1 - \alpha_i \alpha_j q^{-1})(1 - \alpha_i \alpha_j^{-1} q^{-1})}{(1 - \alpha_i \alpha_j)(1 - \alpha_i \alpha_j^{-1})} \prod_{1 \leq i < n} \frac{1 - \alpha_i^2 q^{-1}}{1 - \alpha_i^2}. \tag{4.10}$$

Substituting this formula and (4.5) into (4.9) and making use of Weyl's identity (1.17), the explicit formula follows.

This completes the proof of Theorem 1.5. ∎

We turn to the proof of Theorem 1.6. Suppose that T is split, so that the Bessel functional B is given by equation (1.14). Recall that N denotes the unipotent radical of the standard Borel subgroup of G. Let $N(\mathcal{O}) = N \cap K$. Given $k = (k_1, \cdots, k_n)$ with all $k_i \geq 0$, define

$$P_k(g) = \int_{N(\mathcal{O})} \Phi_\chi(g n d_k)\, dn.$$

Then using the bijection (4.3), an argument similar to the one given there demonstrates that $P_k \in \mathrm{Ind}(\chi)^{\mathcal{B}}$.

Though we will ultimately use the Casselman-Shalika method, we first establish the following Lemma, which implies that the determination of $h(k_1, \cdots, k_n)$ follows from the determination of the quantities $B(P_k)$. For convenience, let us set

$$H(k) := H_\chi(g_k) \qquad B(k) := B_\chi(P_k).$$

Then we have

Lemma 4.2. *Let $k = (k_1, \cdots, k_n)$ with all $k_i \geq 0$.*

(1) *Suppose $k_1 = 0$. Then $H(k) = B(k)$.*
(2) *Suppose $k_1 > 0$. Then*

$$H(k) = (1 - q^{-1})^{-1} \left(B(k) - q^{-1}\beta B(k_1 - 1, k_2 + 1, k_3, \cdots, k_n)\right).$$

Proof. Comparing the definitions,

$$B(k) = \int\limits_{\mathcal{O}} B\left(\pi\left(n(x)d_k\right)\Phi_\chi\right)\, dx.$$

A matrix calculation shows that if $x \in \mathcal{O}$, then

$$n(x)d_k = \begin{cases} d_k\, n(\varpi^{-k_1}x) & \text{if } x \in \varpi^{k_1}\mathcal{O} \\ t(x)\, n(1)\, d_{k_1-m, k_2+m, k_3, \cdots, k_m}\, t(\varpi^m x^{-1}) & \text{if } x \in \varpi^m \mathcal{O}^\times, 0 \leq m < k_1. \end{cases}$$

If $x \in \varpi^{k_1}\mathcal{O}$ then $n(\varpi^{-k_1}x) \in K$; moreover Φ_χ is K-fixed. Factoring

$$d_k = t(\varpi^{k_1})\, d_{0, k_1+k_2, k_3, \cdots, k_m}$$

and using property (1.10), one thus obtains

$$\int\limits_{\varpi^{k_1}\mathcal{O}} B\left(\pi\left(n(x)d_k\right)\Phi_\chi\right)\, dx = q^{-k_1}\beta^{k_1} H(0, k_1 + k_2, k_3, \cdots, k_n).$$

Similarly if $x \in \varpi^m \mathcal{O}^\times$, then $t(\varpi^m x^{-1}) \in K$. Arguing similarly, one then finds that

$$(4.11) \quad B(k) = q^{-k_1}\beta^{k_1} H(0, k_1 + k_2, k_3, \cdots, k_n) + (1 - q^{-1}) \sum_{m=0}^{k_1 - 1} q^{-m}\beta^m H(k_1 - m, k_2 + m, k_3, \cdots, k_n).$$

If $k_1 = 0$, Lemma 4.2, part (1), follows at once from (4.11). For $k_1 > 0$, equation (4.11) also implies that

$$(4.12) \quad q^{-1}\beta B(k_1 - 1, k_2 + 1, k_3, \cdots, k_n) = q^{-k_1}\beta^{k_1} H(0, k_1 + k_2, k_3, \cdots, k_n) + (1 - q^{-1}) \sum_{m=1}^{k_1 - 1} q^{-m}\beta^m H(k_1 - m, k_2 + m, k_3, \cdots, k_n).$$

Subtracting (4.12) from (4.11), part (2) of the Lemma follows. ■

Write now P_k in terms of the Casselman basis $\{f_w\}$:

$$P_k = \sum_{w\in\Omega} S(k,w;\chi)\, f_w.$$

The coefficients $S(k,w;\chi)$ are given by

$$\begin{aligned} S(k,w;\chi) &= \int_{N_w\backslash N} P_k(w^{-1}n)\, dn \\ &= \int_{N_w\backslash N} \Phi_\chi(w^{-1}nd_k)\, dn \\ &= c_w(\chi)\, ({}^w\chi\delta_B^{1/2})(d_k), \end{aligned} \tag{4.13}$$

where $c_w(\chi)$ is defined by equation (4.4) above.

In this case, unlike the nonsplit case, no two terms for different w contribute the same rational function of the α_i. Hence it suffices to determine $B(f_{w_0})$; the value of $B(P_k)$ is then obtained by symmetrization, using the functional equation of Theorem 1.4. The value of $B(f_{w_0})$ is given by

Lemma 4.3. *Suppose* $|\alpha_n\beta| < q^{1/2}$. *Then*

$$B(f_{w_0}) = \left(1 - \alpha_n\beta q^{-1/2}\right)^{-1}.$$

Proof. Suppose $|\alpha_n\beta| < q^{1/2}$. Since T normalizes U and fixes θ_S,

$$B(f_{w_0}) = \int_{F^\times}\int_U f_{w_0}(w_0\, n(1)\, t(a)\, u)\, \theta_S(u)^{-1}\, \lambda^{-1}(a)\, du\, d^\times a.$$

Suppose that $w_0 n(1)t(a)u \in Bw_0\mathcal{B}$. Then $(t(a^{-1})n(1)t(a))\, u \in w_0Bw_0\mathcal{B}$. This relation implies that $u \in U\cap K$. Thus

$$B(f_{w_0}) = \int_{F^\times} f_{w_0}(w_0\, n(1)\, t(a))\;\; \lambda^{-1}(a)\, d^\times a.$$

Suppose first that $a\in F^\times$ satisfies $|a|_F < 1$. Then one sees by a matrix calculation that in fact $w_0n(1)t(a)$ is never is the cell $Bw_0\mathcal{B}$. If instead $|a|_F \geq 1$, then

$$w_0\, n(1)\, t(a) = w_0\, t(a)\, (t(a)^{-1}n(1)t(a)),$$

and the last factor $t(a)^{-1}n(1)t(a)$ is in $\mathcal{B}$. Since f_{w_0} is right $\mathcal{B}$-invariant, this gives

$$\begin{aligned} f_{w_0}(w_0 n(1) t(a)) &= f_{w_0}(w_0 t(a) w_0^{-1} \cdot w_0) \\ &= \chi \delta_B^{1/2}\left(t(a^{-1})\right). \end{aligned}$$

Hence

$$\begin{aligned} B(f_{w_0}) &= \sum_{m=0}^{\infty} \alpha_n^m \, \beta^m \, q^{-m/2} \\ &= \left(1 - \alpha_n \beta q^{-1/2}\right)^{-1}, \end{aligned}$$

as claimed. ∎

To conclude the proof of Theorem 1.6, observe that

$$B(k) = B_\chi(P_k) = \int_{\mathcal{O}} H_\chi(n(x) d_k)\, dx.$$

By Theorem 1.4, this satisfies a functional equation under the action of Ω on the α_i. Similarly to (4.10), (4.4) gives

$$c_{w_0}(\chi) = \prod_{1 \le i < j \le n} \frac{(1 - \alpha_i \alpha_j q^{-1})(1 - \alpha_i \alpha_j^{-1} q^{-1})}{(1 - \alpha_i \alpha_j)(1 - \alpha_i \alpha_j^{-1})} \prod_{1 \le i \le n} \frac{1 - \alpha_i^2 q^{-1}}{1 - \alpha_i^2}.$$

Substituting this expression into (4.13) to obtain $S(k, w_0; \chi)$, and symmetrizing, one obtains the pleasant formula

$$\begin{aligned} &\frac{\prod_{i=1}^n (1 - \alpha_i \beta q^{-1/2})(1 - \alpha_i \beta^{-1} q^{-1/2})}{\prod_{1 \le i < j \le n} (1 - \alpha_i \alpha_j q^{-1})(1 - \alpha_i \alpha_j^{-1} q^{-1}) \prod_{i=1}^n (1 - \alpha_i^2 q^{-1})} B(k) \\ &\qquad = q^{e_k} \Delta^{-1} \times \\ &\mathcal{A}\left((1 - \alpha_n \beta^{-1} q^{-1/2}) \alpha_1^{-k'_n - n} \prod_{i=1}^{n-1} \alpha_{n+1-i}^{-k'_i - i} (1 - \alpha_i \beta q^{-1/2})(1 - \alpha_i \beta^{-1} q^{-1/2}) \right). \end{aligned}$$

Then applying Lemma 4.2, one obtains without difficulty the expression for $h(k)$ given in Theorem 1.6. (Note that if $k_1 = 0$ then in fact one obtains the formula

$$\begin{aligned} &h(0, k_2, \cdots, k_n) = q^{e_k} \, \Delta^{-1} \\ &\times \mathcal{A}\left(\alpha_n^{-1} (1 - \alpha_n \beta^{-1} q^{-1/2}) \prod_{i=1}^{n-1} \alpha_i^{-k'_{n+1-i} - n - 1 + i} (1 - \alpha_i \beta q^{-1/2})(1 - \alpha_i \beta^{-1} q^{-1/2}) \right). \end{aligned}$$

However, expanding the second factor, the term

$$\mathcal{A}\left(\alpha_n^{-1}(-\alpha_n\beta^{-1}q^{-1/2})\prod_{i=1}^{n-1}\alpha_i^{-k'_{n+1-i}-n-1+i}(1-\alpha_i\beta q^{-1/2})(1-\alpha_i\beta^{-1}q^{-1/2})\right)$$

is independent of α_n, and hence its alternator is zero. Thus one is in fact left with the expression of Theorem 1.6, part (1).)

This completes the proof of Theorem 1.6. ■

5. Continuation of the Bessel Functionals: The Application of Bernstein's Theorem. Bernstein [Be] gave a powerful new method for the meromorphic continuation of functionals satisfying a suitable uniqueness property; in the case at hand, this uniqueness is the uniqueness of the Bessel model, which, as we have already noted, was proved by Novodvorsky [No]. Bernstein's result will appear as an appendix to a book in progress of Cogdell and Piatetski-Shapiro; in the meantime, a statement (but no proof) may be found in Gelbart and Piatetski-Shapiro [GP]. In this Section we will show that Bernstein's theorem implies Theorem 1.7. We assume familiarity with either [Be] or its paraphrase in [GP].

Let $X = C^\infty\big((B\cap K)\backslash K\big)$. Given the α_i, we may identify X with V_π by extending an element $\Psi \in X$ uniquely to an element of $\Psi_\chi \in V_\pi$ satisfying (1.11). Let $D = (\mathbb{C}^\times)^{n+1}$. If $(\alpha_1,\cdots,\alpha_n,\beta)\in D$, we consider functionals B which satisfy the following two conditions:

(1) B is a Bessel functional,
(2) $B(\Phi_\chi) = 1$.

We claim that these conditions may be expressed by a polynomial system of equations (in Bernstein's sense) in the parameters $(\alpha_1,\cdots,\alpha_n,\beta)$. To see this, note that if $\Psi\in X$ and (with the notation as in (1.10)) if $t\in T$, $u\in U$, then there exist $\Psi_j\in X$ and polynomial functions f_j ($j=1,\cdots,N$) of the α_i and β such that $\pi(tu)\Psi_\chi = \sum f_j\Psi_{j,\chi}$. Thus (1.10) may be expressed by the polynomial equation

$$\sum_{j=1}^{N} f_j B(\Psi_j) - \theta_S(tu)B(\Psi) = 0.$$

It is also evident that condition (2) above is a polynomial equation. By Novodvorsky's theorem, the solution, if it exists, is unique, and we have proved that a solution exists on a non-empty open subset of D. Consequently Bernstein's theorem is applicable, yielding the meromorphic continuation of the functional B in the sense made precise by Theorem 1.7. ■

6. Bessel Periods of Eisenstein Series. In this Section we present a global application of these formulas. (In fact, the application makes use of only a particular case of them, namely the formula (6.6) for the value of the local Bessel functional at the identity.) Accordingly, we now let F be a global field and $\mathbb{A}$ be its ring of adeles. Let $\pi = \otimes \pi_v$ be a cuspidal automorphic representation of $GL(n, \mathbb{A})$. Let P be the standard maximal parabolic subgroup of $G = SO(2n+1)$ with Levi factor $GL(n)$. Denoting by δ_P the modular character of P, let $f_s \in \mathrm{Ind}_{P_{\mathbb{A}}}^{G(\mathbb{A})}(\pi \otimes \delta_P^{s-1/2})$, and let

$$E(g, s, f_s) = L_S(\pi, 2n(s-1/2)+1, \vee^2) \sum_{\gamma \in P_F \backslash G_F} f_s(\gamma g) \tag{6.1}$$

be the Eisenstein series attached to f_s. Here S is a finite set of places including the archimedean ones and those where π is ramified, and $L_S(\pi, s, \vee^2)$ is the partial symmetric square L-function, which is the normalizing factor of the Eisenstein series. Let a, b and c be elements of F such that $b^2 + 2ac$ is not a square. Let Q be the standard parabolic subgroup of G with Levi factor $GL(1) \times \cdots \times GL(1) \times SO(3)$, let U be the unipotent radical of Q, and let ψ be a nontrivial character of $\mathbb{A}/F$. Let θ be the character of $U(\mathbb{A})$ defined by (1.9). Then $Q(\mathbb{A})$ acts on $U(\mathbb{A})$ and hence on its character group by conjugation; let $R(\mathbb{A})$ be the subgroup of the stabilizer consisting of elements whose projection to the Levi factor of $Q(\mathbb{A})$ lies in the embedded $SO(3, \mathbb{A})$. We may naturally extend θ to a character of $R(\mathbb{A})$. Then $R(\mathbb{A})$ is the group of adelic points for an algebraic group R which is the semidirect product of U with a one-dimensional torus T, which for simplicity we are assuming nonsplit—this is our hypothesis that $b^2 + 2ac$ is a nonsquare. In this case, there exists a unique quadratic field extension K of F over which T splits. Let $\eta = \otimes \eta_v$ be the quadratic Hecke character of F attached to K. In this Section we will prove

Theorem 6.1. *The integral*

$$\int_{R(F) \backslash R(\mathbb{A})} E(r, s, f_s)\, \theta(r)\, dr \tag{6.2}$$

is an Euler product, whose local factor at a good place v equals

$$L\big(n(s-1/2)+1/2, \pi_v\big)\, L\big(n(s-1/2)+1/2, \pi_v \otimes \eta_v\big). \tag{6.3}$$

If $n = 2$ this result is essentially due to Böcherer [Bö] and Mizumoto [Mi] (these authors consider holomorphic Siegel modular forms on $PGSp_4$ over $\mathbb{Q}$, but the unramified local computation is the same for general base field and infinity type). The precise conditions to make v good are described below.

We turn to the proof of Theorem 6.1. Unfolding the integral, we see that (6.2) equals

$$L_S(\pi, 2n(s-1/2)+1, \vee^2) \sum_{\gamma \in P_F \backslash G_F / R_F} \int_{R_{\mathbb{A}}^{\gamma} \backslash R_{\mathbb{A}}} \int_{R_F^{\gamma} \backslash R_{\mathbb{A}}^{\gamma}} f_s(\gamma u r)\, \theta(ur)\, du\, dr,$$

where R^γ denotes the algebraic group $R \cap \gamma^{-1} P \gamma$. Using the fact that π is cuspidal, one may show that only the open orbit in $P\backslash G/R$ contributes, and we may take the representative

$$\gamma = \begin{pmatrix} & & I_n \\ & (-1)^n & \\ I_n & & \end{pmatrix}.$$

Then $\gamma R^\gamma \gamma^{-1} = U$ is a maximal unipotent subgroup in $GL(n)$, the Levi factor of P, and the character $\gamma u \gamma^{-1} \mapsto \theta(u)$ is nondegenerate; indeed, since $b^2 + 2ac$ is a nonsquare, $a \neq 0$. It follows that

$$W_s(g) = \int_{U_F\backslash U_{\mathbb{A}}} f_s(ug)\,\theta(\gamma^{-1} u \gamma)\,du$$

lies in $\mathrm{Ind}_{P_{\mathbb{A}}}^{G_{\mathbb{A}}}(\mathcal{W}_\pi \otimes \delta_P^{s-1/2})$, where $\mathcal{W}_\pi$ is the Whittaker model of π (relative to the appropriate character of its maximal unipotent group). Writing

$$\mathrm{Ind}_{P_{\mathbb{A}}}^{G_{\mathbb{A}}}(\mathcal{W}_\pi \otimes \delta_P^{s-1/2}) = \bigotimes_v \mathrm{Ind}_{P(F_v)}^{G(F_v)}(\mathcal{W}_{\pi,v} \otimes \delta_P^{s-1/2})$$

as a restricted tensor product over all places v of F, where $\mathcal{W}_{\pi,v}$ is the local Whittaker model of π_v, there is no loss of generality in assuming that $W_s(g)$ is a pure tensor; thus we write $W_s(g) = \prod_v W_{s,v}(g_v)$. The integral (6.2) thus equals

$$L_S(\pi, 2n(s-1/2)+1, \vee^2) \int_{R_{\mathbb{A}}^\gamma\backslash R_{\mathbb{A}}} W_s(\gamma r)\,\theta(r)\,dr$$

and hence is factorizable, with local factor

$$L(\pi_v, 2n(s-1/2)+1, \vee^2) \int_{R^\gamma(F_v)\backslash R(F_v)} W_{s,v}(\gamma r)\,\theta(r)\,dr. \tag{6.4}$$

We compute this local factor at a good place. More precisely, suppose that the finite place v does not ramify in K. Then we compute (6.4) for any nonramified principal series π_v with Satake parameters $\alpha_1, \cdots, \alpha_n$, under the assumption that $W_{s,v}$ is the unramified spherical vector in $\mathrm{Ind}_{P(F_v)}^{G(F_v)}(\mathcal{W}_{\pi,v} \otimes \delta_P^{s-1/2})$, normalized so that $W_{s,v}(1) = 1$. To carry out this computation we first make the further assumption that $|\alpha_i| < |\alpha_{i+1}|$; the general case then follows by analytic continuation. (Note that the assumption $|\alpha_i| < |\alpha_{i+1}|$ is unrealistic for a representation π_v which is a local component of an automorphic representation π, as this condition violates the Ramanujan conjecture.) According to the results of Casselman and Shalika [CS], we may then write

$$W_{s,v}(g) = \prod_{1\le i<j\le n} (1 - \alpha_i \alpha_j^{-1} q_v^{-1})^{-1} \int_{U(F_v)} \Phi_{\chi,v}\left(\begin{pmatrix} J_n & & \\ & 1 & \\ & & J_n \end{pmatrix} ug\right) \theta(\gamma^{-1} u \gamma)\,du,$$

where q_v is the cardinality of the residue field at v, J_n is the $n \times n$ matrix with ones on the sinister diagonal and zeros elsewhere, and $\Phi_{\chi,v}$ is the normalized spherical vector in the representation denoted (in the notation of Section 1) $\mathrm{Ind}(\chi_1, \cdots, \chi_n)$, where $\chi_i(\varpi_i) = \alpha_i\, q^{-n(s-1/2)}$. Substituting this into (6.4), we obtain precisely the integral (1.12) if the place v is inert in K, that is, if T is a nonsplit torus in F_v. If on the other hand the place v splits in K, we obtain the integral (1.14). To see this, one must remember to conjugate the torus so as to make it diagonal; and conjugating w_1 in this way produces $w_0 n(1)$ as in (1.14). One sees that, in either case, the local factor (6.4) is equal to

$$\prod_{1 \le i \le j \le n} \left(1 - \alpha_i \alpha_j\, q_v^{-1-2n(s-1/2)}\right)^{-1} \prod_{1 \le i < j \le n} \left(1 - \alpha_i \alpha_j^{-1} q_v^{-1}\right)^{-1} H_\chi(1). \tag{6.5}$$

By Theorems 1.5 and 1.6, $\mathcal{H}_\chi(1) = 1$, so by Theorems 1.3 and 1.4 (with $\alpha_i\, q_v^{-n(s-1/2)}$ replacing α_i, and $\beta = 1$),

(6.6) $H_\chi(1) =$

$$\prod_{1 \le i < j \le n} \left(1 - \alpha_i \alpha_j\, q_v^{-1-2n(s-1/2)}\right) \left(1 - \alpha_i \alpha_j^{-1} q_v^{-1}\right) \prod_{i=1}^{n} \frac{1 + \eta_v(\varpi_v) \alpha_i\, q_v^{-n(s-1/2)-1/2}}{1 - \alpha_i\, q_v^{-n(s-1/2)-1/2}},$$

where the quadratic character $\eta_v(\varpi_v)$ equals 1 if v splits in K, and -1 if v is inert. Substituting this formula into (6.5) and simplifying, we obtain the local factor (6.3) for v. ∎

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