The Structure and Enumeration of Link Projections

Martin Bridgeman[1]

October 15, 1994

Abstract

We define a decomposition of link projections whose pieces we call atoroidal graphs. We describe a surgery operation on these graphs and show that all atoroidal graphs can be generated by performing surgery repeatedly on a family of well known link projections. This gives a method of enumerating atoroidal graphs and hence link projections by recomposing the pieces of the decomposition.

1 Introduction

The problem of enumeration of knots and links has always interested knot theorists. In this paper we introduce a method of enumerating link projections by first decomposing them into pieces called *atoroidal graphs*. We define surgery on these atoroidal graphs and show how they can be enumerated by performing surgery on a well known family of link projections. By recomposing these atoroidal graphs we can thus enumerate link projections. I have included an enumeration of atoroidal graphs to 12 crossings.

A link projection is given by a 4-valent planar graph G. To form a link we can replace each vertex of G by a crossing. To enumerate links in this way we must first enumerate link projections. It was Kirkman's success in enumerating link projections or *polyhedra* as he called them([K1],[K2]) that

[1]Research at MSRI is supported in part by NSF grant no.DMS-9022140.

formed the basis of the knot tables of both Tait([T]) and Little([L1],[L2]). In [C], Conway introduced a notation which made it possible for him to enumerate knots to 11 crossings and links to 10 crossings in a single afternoon where before it had taken years. In his paper Conway defined a *basic polyhedron* to be a polyhedron with no bigon regions and showed that every link is obtained by replacing each vertex of a basic polyhedron by a rational tangle. These basic polyhedra are closely related to the atoroidal graphs defined in this paper and can be enumerated using the enumeration of atoroidal graphs described.

The decomposition of a link projection into atoroidal graphs is achieved by cutting the projection along certain non-trivial curves. We then define surgery on an atoroidal graph giving a new atoroidal graph with one more vertex. This gives a partial ordering on atoroidal graphs where $G_1 \prec G_2$ if a surgery on G_1 results in G_2 and we show that a graph is initial if and only if it has no vertices of a given type. Using this we can list the initial objects and thus enumerate all atoroidal graphs by repeatedly performing surgery on these initial objects.

The motivation for the paper comes from orbifold theory and hyperbolic geometry but a background in these is not necessary here. For a reference see [Th]. For readers interested, these aspects are laid out in the section on orbifolds.

I would like to thank Curt McMullen, Joe Christy, Rich Schwartz and especially my advisor Bill Thurston.

2 Decomposition

Link Projections

Given a link L a general position projection of L is a 4-valent graph G embedded in S^2. As we are only considering such graphs we will use graph to mean a 4-valent graph embedded in S^2.
Let G be a graph with vertex set V.

Definition: 1 *An* n-curve *of G is a simple closed curve in $S^2 - V$ intersecting G n times.*

Let α be an n-curve($n = 0, 2, 4$) of G. Then α splits S^2 into two disks D_1, D_2. We say a component D_i is *trivial* if $D_i \cap G$ is either empty, a simple

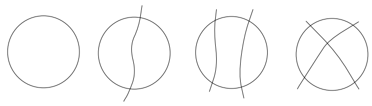

Figure 1: Trivial 0,2,4-curves

arc, two disjoint simple arcs or two arcs crossing at a single vertex(figure 1). If α has a trivial component then α is *trivial*. Otherwise α is called *non-trivial*.

Definition: 2 *A graph G is* irreducible *all n-curves($n = 0, 2$) are trivial.*

Definition: 3 *A graph G is* atoroidal *if all n-curves($n = 0, 2, 4$) are trivial.*

Decomposition

If α is a non-trivial n-curve($n = 0, 2, 4$) of G then we can decompose G along α into graphs G_1 and G_2 as follows. First cut along α, this splits the sphere into two disks D_1, D_2. To obtain the graph G_i from D_i we identify the boundary of D_i to a single point. We say that G decomposes into G_1 and G_2 along α.

We now describe the decomposition of a link projection G into atoroidal graphs. If all n-curves($n = 0, 2, 4$) in G are trivial the the decomposition is done. Otherwise decompose G into G_1 and G_2 along a non-trivial n-curve($n = 0, 2, 4$) where n is chosen to be as small as possible. Now repeat the decomposition on the resultant graphs G_1 and G_2. It is obvious that this decomposition terminates.

The decomposition along non-trivial 0-curves is especially simple, corresponding to splitting a graph into its connected components and thus when dealing with connected graphs we only need concern ourselves with non-trivial 2-curves and 4-curves.

3 Structure of Atoroidal Graphs

Almost all Atoroidal Graphs are Hyperbolic

To investigate the type of atoroidal graphs possible we consider the cell division of S^2 given by G. We call the cells, faces of G and a face F is called an n-gon if it has n vertices of G on its boundary. Any n-gon F of G gives a 2n-curve α_F by taking the boundary of a small neighborhood N_F of F.

If G has a 0-gon F then α_F is a trivial 0-curve (i.e. it bounds a disk D in $S^2 - G$) as G is irreducible. Attach a disk to the boundary of $N_F \Rightarrow G$ is just a jordan curve or the unknot projection(figure 2).

If G has a 1-gon(monogon) F then α_F is a trivial 2-curve as G is irreducible. Attach a neighborhood of an arc to $N_F \Rightarrow G$ is the graph having the form of the number 8 (figure 2).

If G has a 2-gon(bigon) F then α_F is a trivial 4-curve as G is atoroidal. Attaching a neighborhood of a vertex to $N_F \Rightarrow G$ is the projection of the trefoil and attaching a neighborhood of two parallel arcs of G to N_F can be done in 2 ways to give two possible graphs but as can be seen in figure 2 only one is also irreducible. Thus G is either the trefoil projection or the Hopf link projection.

Apart from these 4 exceptional graphs all other atoroidal graphs have faces that are at least triangular and are called *hyperbolic graphs*.

Surgery

Given any atoroidal graph G which has a face F with greater than 3 vertices then we can perform *surgery* on G to give another atoroidal graph G'. We choose edges e, e_1, e_2 of F with e_1, e_2 adjacent to e. G' is obtained by pinching together e_1, e_2, i.e. take a simple arc $\alpha \subset F$ with endpoints in the interior of e_1, e_2 resp. and homotope α to a single point. The graph G' obtained by performing surgery on G has one more vertex than G(figure 3).

Lemma: 1 *G' is atoroidal.*

Proof : Let α' be an n-curve in G' which decomposes S^2 into disks D'_1, D'_2. We can get G back by splitting open the new crossing v and as this splitting can be done in a small neighborhood of v and α' is outside such a neighborhood, we get an n-curve α in G which splits S^2 into disks D_1, D_2. Note that D'_i and

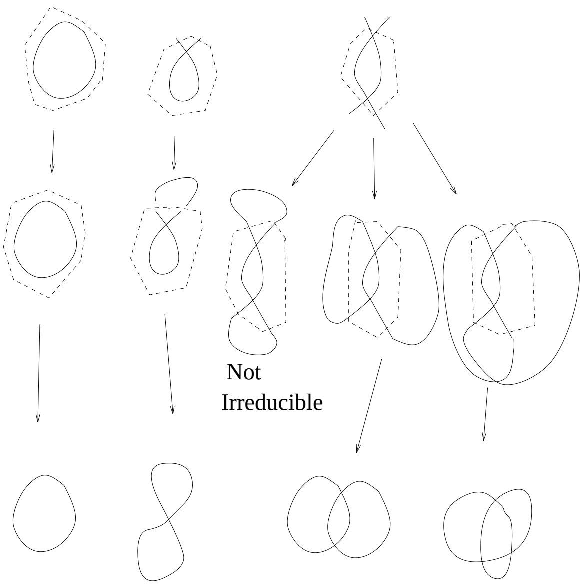

Figure 2: Exceptions

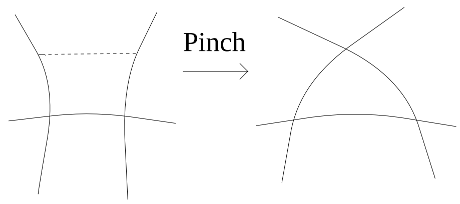

Figure 3: Surgery

D_i are either the same or the former is obtained from the latter by pinching two edges together.

As G is atoroidal then α is trivial for $n = 0, 2, 4$ and we can assume D_1 is trivial.

$n = 0, 2 \Rightarrow D_1' = D_1$ as D_1 has at most one edge intersecting it so we haven't enough edges to pinch $\Rightarrow D_1'$ trivial. $\Rightarrow \alpha'$ trivial.

$n = 4 \Rightarrow D_1$ either neighborhood of vertex or neighborhood of two non-intersecting arcs of G. If D_1 is neighborhood of vertex then the only edges that can be pinched are adjacent $\Rightarrow D_1' = D_1 \Rightarrow \alpha'$ trivial. If D_1 is neighborhood of two parallel arcs $\Rightarrow D_1' = D_1$ or D_1' is obtained by pinching the parallel arcs of D_1 together $\Rightarrow D_1'$ is neighborhood of a vertex $\Rightarrow \alpha'$ trivial $\Rightarrow G'$ is atoroidal $\square$

4 Orbifolds

This section explains how the work arises out of considering certain orbifolds associated with a link projection. In this setting the decomposition and surgery we define are the torus decomposition and dehn surgery on these orbifolds. An orbifold is a generalization of a manifold in which the space is locally modeled on R^n modulo the action of a finite group. For example if a group G acts properly discontinuously on a space M then M/G is an orbifold and is a manifold if the action is also free. For a reference on orbifolds see chapter 13 of [Th].

Associated Orbifolds

We associate two orbifolds O_G and O_G' to a graph G as follows. We consider G as a graph siting on S^2 in S^3. Let B be a ball in S^3 with boundary S^2 and V be the vertex set of G. O_G is a polyhedral orbifold with underlying space $X_{O_G} = B - V$, singular locus $\Sigma_{O_G} = S^2 - V$ and 1-dimensional singular locus $\Sigma^1_{O_G} = G - V$. The 1-dimensional singular locus is marked with D_2 indicating that any point on it is modeled by D^3/D_2 where D_2 acts by two reflections in planes meeting in right angles. O_G' has underlying space $X_{O_G'} = S^3 - V$ and singular locus $\Sigma_{O_G'} = G - V$. Here the singular locus is 1-dimensional and is marked with Z_2 to indicate any point on it is modeled on D^3/Z_2 where Z_2 acts by rotation of order two.

O'_G is the double of O_G in the sense of orbifolds.

Decomposition and Surgery

In [B] we show that the torus decomposition on the orbifolds O_G and O'_G is the decomposition we've described on G. By Andreev's theorem(see [Th]) if G is a hyperbolic graph then O_G can be realized as an ideal hyperbolic polyhedron with all dihedral angles right angles. Taking the subgroup of orientation preserving elements of $\pi_1(O_G)$ shows us that O'_G can also be realized as a hyperbolic orbifold.

We show([B]) that for any graph G the double cover of O'_G is a link compliment denoted by L_G with one component for each vertex of G and is a hyperbolic link complement iff G is a hyperbolic graph. Also if G' is obtained by surgery on G then link complement $L_{G'}$ is obtained from L_G by removing a simple closed curve, i.e. by *dehn drilling*.

5 Partial Ordering

Surgery gives atoroidal graphs a partial ordering $\prec$ by defining $G_1 \prec G_2$ iff G_2 is obtained by performing r successive surgeries on $G_1, r = 0, 1, 2, \ldots$ Note that surgery cannot be performed on any of the four exceptional atoroidal graphs and they are never the resultant graph of surgery, therefore they are isolated objects(both initial and final). Thus $\prec$ restricts to a p.o. on hyperbolic graphs. To study $\prec$ we show that the initial objects are a well known family of graphs and we can generate all atoroidal graphs by performing surgery on these initial objects.

After surgery has been performed on a graph to give a graph G with a new vertex v. The vertex v is a vertex of a triangle T which has adjacent faces F_1, F_2 meeting at v each being greater than triangular(figure 4). A vertex with this local structure we call *simple*. To find a $\bar{G}$ s.t. $\bar{G} \prec G$ we might just look for a simple vertex v and cut open at v(there is a unique way to cut open a simple vertex) but this doesn't necessarily give an atoroidal graph as the resultant may have non-trivial 4-curves(figure 5). What we will show is that if a graph has a simple vertex v belonging to a triangle T then the graph

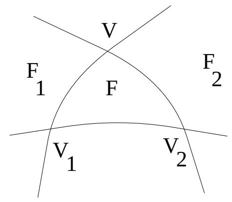

Figure 4: Simple vertex

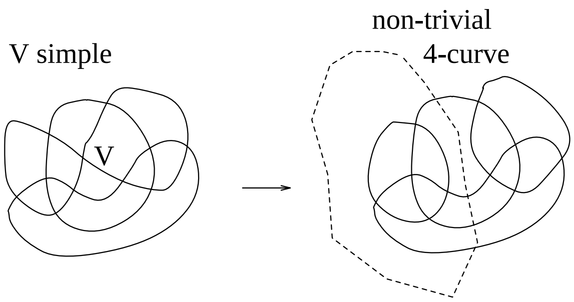

Figure 5: Cutting open at a simple vertex doesn't work

can be cut open at *some* vertex of T to give an atoroidal graph. This implies that an initial object cannot have the any simple vertices. Before proving the stated result we will use it to show what the initial objects are.

Since the exceptions are isolated, they never arise in a sequence of surgeries and all other initial objects are hyperbolic. Let G be a hyperbolic initial object(not one of the exceptions), calculating the euler number of the cell division of S^2 into faces of G we see that G has a triangular face T_1. As G is initial, T_1 has two adjacent triangular faces T_2^l, T_2^r(figure 6). Again using the fact that G is initial we have that T_2^l, T_2^r both have a neighboring triangular face other than T_1 labeled T_3^l, T_3^r respectively.

If $T_3^l = T_3^r$ then G atoroidal $\Rightarrow$ G is borromian ring projection. Also if T_3^l, T_3^r have a common vertex then G irreducible $\Rightarrow$ G is again the Borromian ring projection which we call T_3 (figure 6).

If T_3^l, T_3^r are disjoint then each has a neighboring triangular face T_4^l, T_4^r other than the previous faces T_2^l, T_2^r. These are unique as the face adjacent

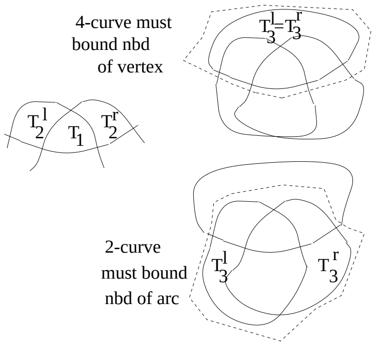

Figure 6: Initial Setup and the resulting Borromian Rings

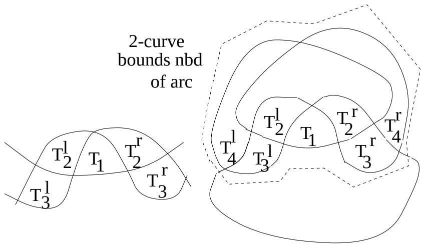

Figure 7: Next Stage and the resulting graph T_4

to both T_3^l and T_3^r is at least 4 sided. If now T_4^l, T_4^r have a common vertex then using G atoroidal $\Rightarrow$ G is of the form given in figure 7 which we call T_4.

Continuing this we get the collection of graphs $\{T_n\}_{n\geq 3}$(figure 8) which together with the exceptional atoroidal graphs are the initial objects of $\prec$ of which only $\{T_n\}_{n\geq 4}$ are non-terminal. Knowing the initial objects allows us enumerate all atoroidal graphs by performing surgery repeatedly.

Lemma: 2 *Let G be an atoroidal graph with a simple vertex v of triangle T and let G' be the graph obtained by cutting G open at v. Let e_1', e_2' be the two edges of face F' in G' pinched to get G then*
1. *G' is irreducible.*
2. *Any non-trivial 4-curve α' of G' intersects the face F' in a single arc β' which separates e_1', e_2'.*

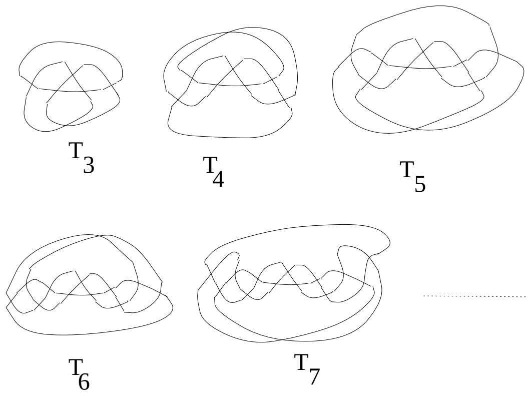

Figure 8: T_3 T_4 T_5 T_6 T_7 ...

Proof : As above let e'_1, e'_2 be the two edges of F' pinched together to give G with e' the edge adjacent to both. If $\exists$ neighborhood $N_{e'}$ of e' s.t. $N_{e'} \cap \alpha'$ is empty $\Rightarrow$ can pinch e'_1, e'_2 in $N_{e'}$ with α' giving an n-curve α in G. If $n = 0, 2, 4$ then α is trivial and splits S^2 into two disks D_1, D_2 with D_1 trivial. Similarly α' splits S^2 into D'_1, D'_2 with either $D'_1 = D_1$ or D'_1 obtained from D_1 by cutting open a crossing. In either case this gives D'_1 trivial $\Rightarrow \alpha'$ trivial.
Therefore every non-trivial n-curve ($n = 0, 2, 4$) must intersect F' in an arc β' that has one endpoint on e' and the other on another edge e'_3 of F' with $e'_3 \neq e', e'_1, e'_2$.

If $n = 0$ then α' doesn't intersect $G' \Rightarrow \alpha'$ trivial.

If $n = 2$ then α' only intersects G' at the two endpoints of β'. If we pinch e'_1, e'_2 together to get G we can do so by either leaving e'_1 fixed and pulling e'_2 through α' or vise-versa. This gives 4-curves α^r, α^l resp. in G which are identical with α' outside a neighborhood of the new vertex v and either go right or left around v as the curves approach v from inside T(figure 9). G atoroidal $\Rightarrow \alpha^r$ is trivial $\Rightarrow \alpha^r$ splits S^2 into D_1, D_2 s.t. D_1 trivial. The region containing v also contains another vertex of T so it can't be trivial. Therefore D_1 is the region containing vertex v_2 of T and is neighborhood of $v_2 \Rightarrow v$ not simple as one of faces is bigon. This contradiction implies that G' has no non-trivial 2-curves $\Rightarrow G'$ is irreducible.

If $n = 4 \Rightarrow \alpha' \cap F'$ consists of either 1 or 2 arcs. If it is 2 arcs β'_1, β'_2

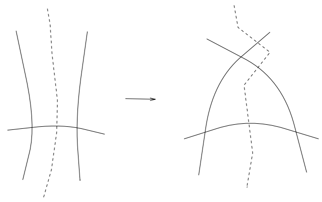

Figure 9: Pinch through α'

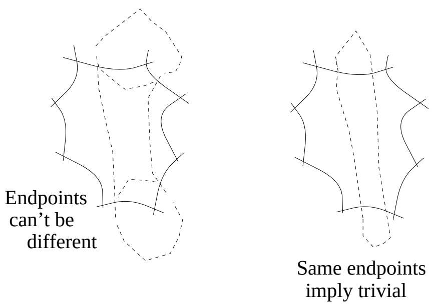

Figure 10: Non-Trivial curve can't intersect in more than one arc

then traversing around α' we have 4 connected arcs $\beta_1', \gamma_1', \beta_2', \gamma_2'$. We can join endpoints of γ_1' by another arc δ_1' in F' s.t. $\gamma_1' \cup \delta_1'$ is a 2-curve in $G' \Rightarrow$ trivial. If endpoints of γ_1' belong to different edges then both components of $S^2 - \gamma_1' \cup \delta_1'$ contain vertices which contradicts it being trivial. Therefore both γ_1', γ_2' are contained in adjacent faces to F' and α' is neighborhood of two parallel arcs of $G' \Rightarrow \alpha'$ trivial(figure 10). Therefore any non-trivial 4-curve in G' intersects F' in a single arc β' separating the two edges pinched in G' to obtain G $\square$

Definition: 4 *An n-curve α ($n \geq 4$) in G is trivial iff either*
- *α is the boundary of a neighborhood of a vertex of G*

or
- *$\exists$ arc β intersecting G at most once s.t. $\beta \cap \alpha = \partial\beta$ and $\partial\beta$ splits α into α_1, α_2 each containing at least two points of G.*

β is called a compression of α.

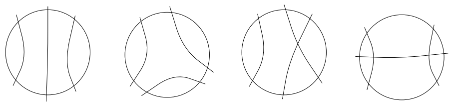

Figure 11: Only trivial 6-curves in atoroidal graph

If G is atoroidal then the only trivial 6-curves can easily shown to be those curves that split S^2 into two disks one of which has one of four types given in figure 11. Disks of this kind bounding a 6-curve are called trivial.

Lemma: 3 *If α' is a non-trivial 4-curve in G' then associated with it are two non trivial 6-curves α^l, α^r in G.*

Proof :α' intersects F' in a single arc β' separating edges e_1', e_2' and as before we can pinch e_1', e_2' together to get graph G. This can be done in two ways, either by fixing e_1' and pushing e_2' across β' or vise-versa. We get two 6-curves α^r, α^l in G both identical to α' outside a neighborhood of the new vertex v and either going right or left around the vertex inside the neighborhood of v as before(figure 9).

α^r splits S^2 into disks D_1, D_2 with D_1 containing vertices v, v_1 of T and D_2 containing vertex v_2. Therefore if D_1 is trivial then it must be the same type as the fourth disk described in figure 11. But then α' would bound a neighborhood of a vertex of G which contradicts α' non-trivial. If D_2 is trivial then it is either the same type as the third or fourth disk in figure 11. If its the third type then as before α' would bound a neighborhood of a vertex of G which contradicts α' non-trivial. If it is the fourth type then adjacent face F_2 of T would be a triangle contradicting v being simple. Therefore α^r is non-trivial and similarly α^l $\square$

Theorem 1 *If G has a simple vertex v of a triangle T then either*
- *splitting open at v gives an atoroidal graph G'*

or
- *both other vertices v_1, v_2 of T are simple and splitting at either gives an atoroidal graph.*

Proof : If G' is not atoroidal $\Rightarrow$ $\exists\alpha'$ non-trivial 4-curve in G' and α^r, α^l non-trivial 6-curves in G. Triangle T has adjacent faces F_1, F_2, F_3 with both

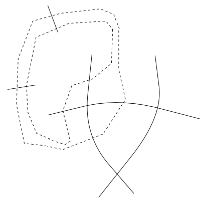

Figure 12: All vertices of T are simple

F_1, F_2 non-triangular as v is simple. If F_3 was triangular then α^r splits F_3 in two, one piece containing just one vertex say v_1 and the other containing two. Therefore α^r takes a clockwise path about v_1 from T through F_3. If instead we take an anticlockwise path α' gives us another 4-curve in G' called $\bar{\alpha}'$(figure 12). Since $\bar{\alpha}'$ doesn't intersect e' then it is trivial. Therefore it splits S^2 into disks $\bar{D}_1', \bar{D}_2'$ with $\bar{D}_1'$ trivial. α' splits S^2 into D_1', D_2' with D_1' obtained from $\bar{D}_1'$ by crossing two adjacent ends so if $\bar{D}_1'$ is neighborhood of two parallel arcs then D_1' is neighborhood of vertex and if $\bar{D}_1'$ is neighborhood of vertex then G' would contain a bigon. Therefore F_3 must be non-triangular and both v_1 and v_2 are simple.

If splitting at v_1 doesn't give an atoroidal graph then there is a non-trivial 4-curve α_1' in G_1' and non-trivial 6-curves α_1^r, α_1^l in G. Considering the 6-curves α, α_1, where $\alpha = \alpha^l$ and $\alpha_1 = \alpha_1^r$ we will show that they can be isotoped to intersect in only two points.Then by showing that they cannot intersect in the given way(figure 13) the theorem is proven.

Firstly we will show that α, α_1 can be isotoped to only intersect twice. If they intersect any more then $S^2 - \alpha \cup \alpha_1$ contains at least four regions that are disks with boundary consisting of one arc of α and α_1. As $\alpha \cup \alpha_1$ has 12 intersections with G then each of these four regions cannot have boundaries being n-curves $n \geq 4$. Therefore one of these regions D has $\gamma = \partial D$ either a trivial 0 or 2-curve and $\gamma = \beta \cup \beta_1$ where β, β_1 are arcs of α, α_1 resp.. If γ is a 0-curve then either D or D^c is a trivial disk. If D is trivial then can isotope to remove two intersections of α and α_1 by pulling β through β_1. If D^c is trivial then any of the other 3 disks with boundary consisting of one arc of α and α_1 are trivial and hence can reduce the number of intersections as in first case.

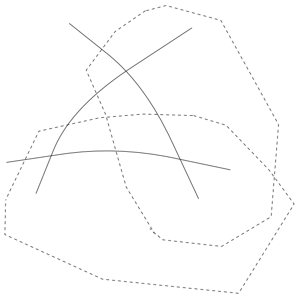

Figure 13: Two intersecting Non-trivial 6-curves

If γ is a 2-curve then each of β, β_1 intersect G as if say β didn't then it would be a compression for α_1 which contradicts α_1 being non-trivial. Therefore either D or D^c is trivial i.e. a neighborhood of an arc of G. If D is trivial then can isotope by pulling β through β_1 reducing number of intersections of α and α_1. If D^c is trivial then other 3 disks with boundary consisting of one arc of α and α_1 cannot have boundaries being n-curves $n \geq 4$ as they can have a maximum of 10 intersections with G between them. Therefore there is region $\bar{D}$ which either doesn't intersect G and thus we can isotope as before to reduce the number of intersections of α and α_1 or is neighborhood of an arc of G which can also be isotoped as before.

So we can assume α and α_1 intersect twice and divide S^2 into 4 disks. We label these disks $D_i, i = 1, \ldots, 4$ where D_1, D_2, D_3 contain v_1, v_2, v resp. and $\gamma^i = \partial D_i$. γ^i is an n_i-curve where $\sum n_i = 24$ and $n_i \geq 4$. Also $\gamma^i = \beta^i \cup \beta_1^i$ where β^i, β_1^i are arcs of α, α_1 resp.(figure 13). Note that the arcs β^i, β_1^i have duplication with each of two arcs that α or α_1 is divided repeated twice. This is for ease of labeling and can be thought of as the two sides of the same arc on α or α_1.

Case 1: If $n_2 = 4$ then D_2 is a neighborhood of v_2 and both β^2 and β_1^2 intersect G twice otherwise we get a compression of α or α_1. Therefore γ^1 is a 6-curve and inside D_1 is a 4-curve $\tilde{\gamma}^1$(figure 14). If it is boundary of neighborhood of two parallel arcs of G then this implies either F_3 is a bigon or α_1 is a trivial 6-curve. If it is boundary of neighborhood of a vertex then this would imply that F_3 was triangular. Therefore $n_2 \neq 4$.

Case 2: If $n_1 = 4$ then β_1^1 intersects G only twice as otherwise β^1 is

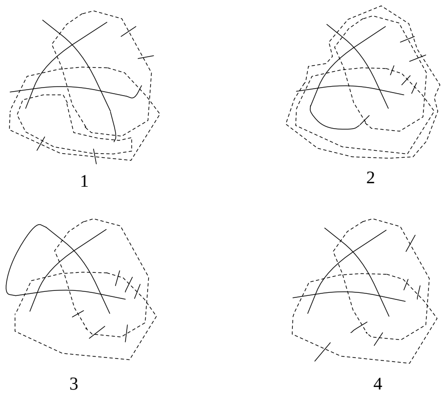

Figure 14: Cases

a compression of α_1. Therefore γ^4 is a 6-curve and D_4 contains a 4-curve $\tilde{\gamma}^4$(figure 14). If it is boundary of neighborhood of two parallel arcs of G then this implies that either F_1 is a bigon or α is trivial 6-curve. If it is the boundary of neighborhood of vertex then this implies that F_1 is triangular. Therefore $n_1 \neq 4$ and by symmetry $n_3 \neq 4$.

Case 3: If $n_4 = 4$ then D_4 must be neighborhood of parallel arcs of G which implies F_1 is a bigon(figure 14). Therefore $n_4 \neq 4$.

Case 4: Therefore $n_i = 6$ and each arc β^i, β_1^i intersects G exactly 3 times. Therefore $\tilde{\gamma}^1, \tilde{\gamma}^4$ are both 4-curves. If $\tilde{\gamma}^4$ is boundary of neighborhood of two parallel arcs of G then this implies either F_1 is a bigon or both α and α_1 have compressions, contradicting them being non-trivial(figure 14). If $\tilde{\gamma}^4$ is boundary of a neighborhood of a vertex then F_1 would be triangular. Therefore we have shown that there cannot exist 6-curves intersecting as α and α_1 do $\Rightarrow$ if α exists (i.e. G' isn't atoroidal) $\Rightarrow$ α_1 can't exist $\Rightarrow$ G'_1 is atoroidal. Similarly G'_2 is atoroidal also $\square$

We have shown that the initial objects of $\prec$ are $\{T_n\}_{n\geq 3}$ along with the 4 exceptions. T_n is the projection of the (3,n) torus link with the link having three components if 3 divides n and having one component otherwise. From this we see that T_n has symmetries taking any directed edge of one of the non-triangular faces to any other. Therefore any surgery on T_n gives the same graph which we call T_n^+.

Lemma: 4 *If $T_n \prec G$ and $T_n \neq G$ $(n > 4)$ then $T_{n-1} \prec G$.*

Proof : If $T_n \prec G$ and $T_n \neq G$ then $T_n^+ \prec G$. As in figure 7 we can pinch together edges of T_2^l to T_4^r to get T_n^+. Only one vertex of T_1 is simple so we cut open at that vertex first. This reduces the pinched T_2^l to a triangle which has only one simple vertex which we now cut open(figure 15). This resulting graph is T_{n-1}^+ so we have that $T_{n-1} \prec T_{n-1}^+ \prec T_n^+ \prec G$ □

Therefore if C_n is the set of proper descendants of T_n(i.e. $T_n \notin C_n$) then

$$C_4 \supseteq C_5 \supseteq C_6 \supseteq \cdots \supseteq C_n \cdots$$

6 Enumeration

We now have a way to enumerate atoroidal graphs up to any prescribed crossing number by performing surgery on the initial objects. Figure 16 is the enumeration of atoroidal graphs up to 12 crossings. To enumerate prime link projections we need only recombine the atoroidal graphs as follows. We choose two atoroidal graphs G_1 and G_2 with vertices v_1 and v_2 respectively. Now take the compliment of a neighborhood of each vertex and attach their boundaries, making sure to match up the strands of the graphs. In recombining we do not use the first 3 exceptions as either they have no vertices or the compliment of a neighborhood of a vertex is trivial. To enumerate the basic polyhedra of Conway the trefoil projection is also not used as the compliment of a vertex is a bigon.

References

[B] M. Bridgeman *Volume Increase under Dehn Drilling Operations.* Phd. thesis Princeton June 1994

[C] J. H. Conway *An enumeration of knots and links, and some of their algebraic properties.* Computational Problems in Abstract Algebra. (1970)

[K1] T. P. Kirkman *The enumeration, description and construction of knots of fewer than ten crossings.* Trans. Roy. Soc. Edinburgh 32(1885)

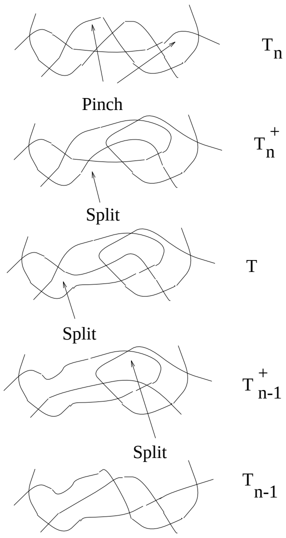

Figure 15: $T^+_{n-1} \prec T^+_n$

[K2] T. P. Kirkman *The 364 unifilar knots of ten crossings enumerated and defined.* Trans. Roy. Soc. Edinburgh 32(1885)

[L1] C. N. Little *Non-alternate ± knots, of order eight and nine.* Trans. Roy. Soc. Edinburgh 35(1889)

[L2] C. N. Little *Alternate ± knots of order 11.* Trans. Roy. Soc. Edinburgh 36(1890)

[T] P. G. Tait *On knots I, II, III (1887, 1884, 1885).* Scientfic Papers I

[Th] W. P. Thurston *The geometry and topology of three manifolds.* Princeton Lecture Notes (1979)

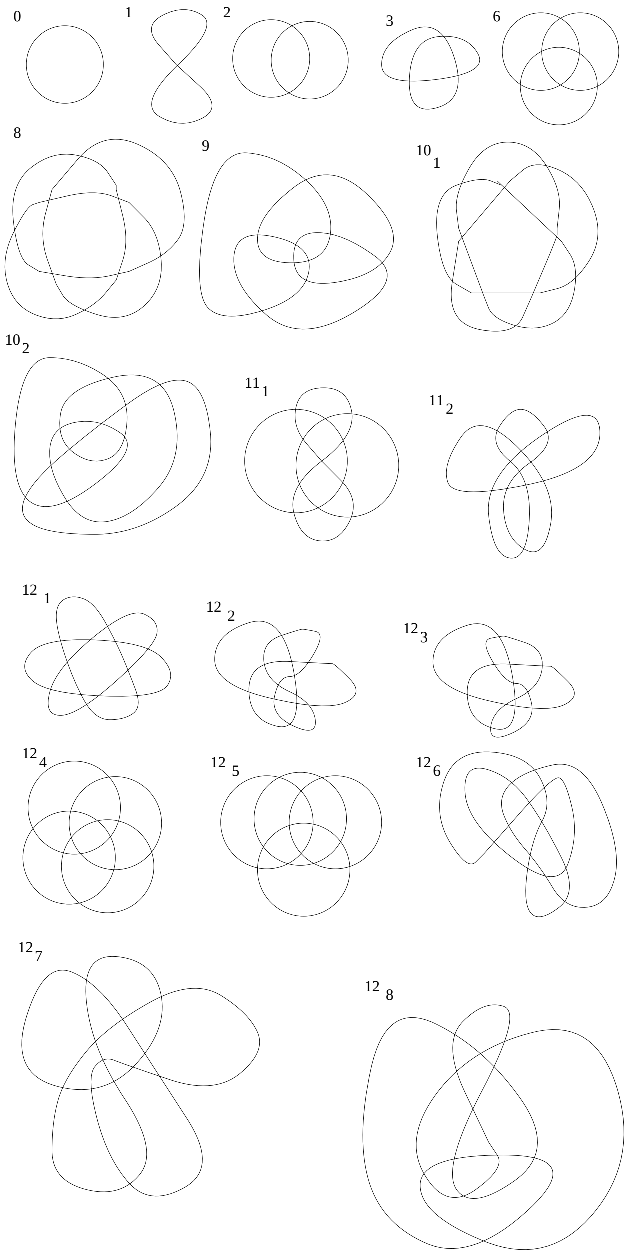

Figure 16: Atoroidal Graphs of 12 crossings or less