ON CALCULATIONS OF ZEROS OF VARIOUS L-FUNCTIONS

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Introduction. As we have shown several years ago [Y2], zeros of $L(s,\Delta)$ and $L^{(2)}(s,\Delta)$ can be calculated quite efficiently by a certain experimental method. Here Δ denotes the cusp form of weight 12 with respect to $SL(2,\mathbf{Z})$ and $L(s,\Delta)$ (resp. $L^{(2)}(s,\Delta)$) denotes the standard (resp. symmetric square) L-function attached to Δ. The purpose of this paper is to show that this method can be applied to a wide class of L-functions so that we can obtain precise numerical values of their zeros.[1]

We organize this paper as follows. In §1, we shall describe basic features of our method of calculation, which is repeated applications of partial summation. In §2, we shall study the r-th symmetric power L-function $L^{(r)}(s,\Delta)$ attached to Δ. Since the cases $r = 1, 2$ are discussed in [Y2], we shall exclusively treat the cases $r = 3, 4$. In §3, we shall study the L-functions attached to modular forms of half integral weight. These L-functions do not have Euler products. Naturally the Riemann hypothesis fails for them; we shall find many zeros off the critical line, though major part of zeros lie on the critical line. We shall also calculate the location of these zeros off the critical line. Though there is some hope to find relations among zeros of L-functions of two modular forms which are in the Shimura correspondence, no explicit results came out so far.

In §4, we shall study L-functions attached to Hecke characters of non-A_0 type of real quadratic fields. D.A. Hejhal showed great interest to make experiments in this case, since coefficients are non-computable combinatorially; hence there is a slight possibility that the Riemann hypothesis may break down for these L-functions. We have made experiments on 44 cases summarized in Table 4.3; so far no counterexamples are found.

In §5, we shall study the Artin L-function attached to a 4-dimensional non-monomial representation of Gal $(\overline{\mathbf{Q}}/\mathbf{Q})$. In §6, we shall discuss the controle of error estimates in our calculation. In §7, we shall consider the explicit formula for the L-function attached to a modular form of weight 8 with respect to $\Gamma_0(2)$. We shall compare both sides of the explicit formula numerically. In §8, we shall present sample programs to compute values of L-functions, which may be convenient for the reader. In §9, we shall formulate a conjecture which emerged during the process of our experiments.

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[1] After the publication of [Y2], H. Ishii [Is] published a table of zeros of standard L-functions attached to modular forms for 15 cases. It also comes to the author's notice that a program of the calculation of zeros of $L(s,\Delta)$ is included in "Mathematica" package, following the method of [Y2].

Most of sections have attached tables to show results explicitly. Concerning actual computations, we have used "UBASIC " created by Y. Kida. (It was not available when we wrote [Y2].) The calculation was done by personal computers which are not necessarily so fast. However our experiments extended over long time (about three years) and UBASIC is quite fast (compared with some other softwares) for numerical calculations, the author thinks that our tables are fairly extensive.

A motivation in these calculations has been to find non-trivial "functorial" properties which may exist among zeros of L-functions, as was hinted in [Y1]. Though our experiments are not successful in this regard, conjectures stemmed from them are formulated in §9.

We can pursue these calculations still further. The topics which may be included in this paper are:

1) The Hasse-Weil zeta functions of algebraic curves, for example $y^2 = x^5 - x + 1$.
2) The Dirichlet series $\displaystyle\sum_{n=1}^{\infty} \frac{n\alpha - [n\alpha] - 1/2}{n^s}$ studied by Hecke [H], where α is a real irrational number.
3) Applications to Riemann-Siegel type formulas.
4) Calculations of critical values of L-functions.

Our results on these topics are still fragmentary, so the full discussion should be postponed to future occasions.

Notation. For a complex number z, we denote by $\Re(z)$ (resp. $\Im(z)$) the real (resp. imaginary) part of z. The letter q stands for $\exp(2\pi\sqrt{-1}z)$ when it is clear from the context. For modular forms, we follow the notation in Shimura [Sh1].

§1. An overview on our method of calculations

Let

$$L(s) = \sum_{n=1}^{\infty} a_n n^{-s} \tag{1.1}$$

be a Dirichlet series which is absolutely convergent when $\Re(s) > \sigma$ for some $\sigma > 1$. In this paper, we shall consider only such $L(s)$ which can be analytically continued to the whole complex plane as an entire function and satisfies a functional equation of the form

$$R(k - \bar{s}) = \kappa\overline{R(s)}. \tag{1.2}$$

Here κ is a constant of absolute value 1, $k > 0$,

$$R(s) = N^s \prod_{i=1}^{m} \Gamma(b_i s + c_i) L(s)$$

with $N > 0$, $b_i > 0$, $c_i \in \mathbf{C}$. We note that (1.2) is equivalent to

$$R(k - s) = \kappa R(s) \tag{1.3}$$

if $a_n \in \mathbf{R}$, $c_i \in \mathbf{R}$ for all n and i. Put $\kappa^{-1} = \kappa_1^2$ with some $\kappa_1 \in \mathbf{C}$. By (1.2), we have

$$\kappa_1 R(s) \in \mathbf{R} \qquad \text{if} \quad \Re(s) = k/2. \tag{1.4}$$

Take any $\delta > 0$. For $T > 0$, let $N(\delta, T)$ denote the number of zeros of $R(s)$ counted with multiplicity in the domain

$$-\delta \leq \Re(s) \leq k + \delta, \qquad 0 \leq \Im(s) \leq T.$$

Let D be the rectangle whose vertices are $-\delta$, $k+\delta$, $k+\delta+iT$, $-\delta+iT$ and let C denote the contour ∂D taken in positive direction. By the argument principle, we have

$$N(\delta, T) = \frac{1}{2\pi i} \int_C \frac{R'(s)}{R(s)}\, ds, \tag{1.5}$$

assuming that neither zeros nor poles of $R(s)$ lie on C. Let C_1 denote the portion of C from $k/2$ to $k/2+iT$. By the functional equation (1.2), we obtain

$$N(\delta, T) = \pi^{-1} \Delta \arg R(s) = \pi^{-1} \Delta(\arg N^s \prod_{i=1}^{m} \Gamma(b_i s + c_i)) + \pi^{-1} \Delta(\arg L(s)), \tag{1.6}$$

where $\Delta\arg$ denotes the variation of the argument on C_1, i.e., from $s = k/2$ to $k/2 + iT$ along $k/2$ to $k+\delta$, $k+\delta$ to $k+\delta+iT$, $k+\delta+iT$ to $k/2+iT$. Set[2]

$$\vartheta(T) = \Delta \arg\, (N^s \prod_{i=1}^{m} \Gamma(b_i s + c_i)).$$

Assume

$$\Re(c_i) > -b_i k/2 \qquad \text{for} \quad 1 \leq i \leq m.$$

Then since $b_i > 0$, $N^s \prod_{i=1}^m \Gamma(b_i s + c_i)$ has neither zeros nor poles in the domain $\Re(s) \geq k/2$. Hence $\vartheta(T)$ is equal to the variation of the argument of $N^s \prod_{i=1}^m \Gamma(b_i s + c_i))$ on the line segment $[k/2, k/2+iT]$. We note that $\vartheta(T)$ can be computed in high precision very easily using Stirling's formula (cf. [WW], p. 252) combined with the relation $\Gamma(s+1) = s\Gamma(s)$. We obtain

$$N(\delta, T) = \pi^{-1} \vartheta(T) + \pi^{-1} \Delta(\arg L(s)). \tag{1.7}$$

Now let us consider the case when $R(s)$ has zeros in $(-\delta, k+\delta)$. Let r denote the number of zeros of $R(s)$ i.e., of $L(s)$, in this interval counted with multiplicity. Then (1.7) holds with the modification

$$N(\delta, T) - \frac{r}{2} = \pi^{-1} \vartheta(T) + \pi^{-1} \Delta(\arg L(s)). \tag{1.8}$$

[2] When it is clear from the context, we shall use $\vartheta(T)$ for the "phase factor" of this type in the following sections without further explanation.

Here $\Delta(\arg L(s))$ is counted by dividing C_1 into a finite number of paths removing real zeros of $L(s)$ and summing the variations of the argument of $L(s)$ on each of them. The validity of (1.8) can be seen by modifying C by small semi-circles which detour the real zeros of $L(s)$.

Throughout the paper, to compute $L(s)$, we shall employ our method given in [Y2], which is repeated applications of Abel's partial summation. Set

$$s_n^{(0)} = a_n, \qquad u_n^{(0)} = n^{-s}$$

and define $s_n^{(l)}$, $u_n^{(l)}$ recursively by

$$s_n^{(l)} = \sum_{m=1}^{n} s_m^{(l-1)}, \qquad u_n^{(l)} = u_n^{(l-1)} - u_{n+1}^{(l-1)}, \qquad l \geq 1. \tag{1.9}$$

Put $S_N^{(l)} = \sum\limits_{n=1}^{N} s_n^{(l)} u_n^{(l)}$. Then we have

$$S_N^{(l)} = S_N^{(l-1)} - s_N^{(l)} u_{N+1}^{(l-1)}. \tag{1.10}$$

As we have seen in [Y2], in several cases, $S_N^{(l)}$ seems to approximate $L(s)$ amazingly well when we choose N and l sufficiently large. In the succeeding sections, we shall present various types of L-functions which can be treated in more or less similar fashion. The efficacy of our method seems to depend strongly on the arithmetical nature of the coefficients a_n of a Dirichlet series $L(s)$.

We shall conclude this section by technical remarks concerning actual computations of $S_N^{(l)}$. As the first step, we should construct a table of a_n. For Dirichlet series considered in this paper, this step can be achieved rather easily. Since we can compute $S_N^{(l)}$ from $S_N^{(0)}$ by (1.10), the computation of $S_N^{(0)} = \sum\limits_{n=1}^{N} a_n n^{-s}$ is the substantial and the most time consuming part of our calculation. However usually $s_n^{(l)}$ becomes very large and $u_n^{(l)}$ very small when l increases. Therefore it is indispensable to perform the actual computation in high precision. For $u_n^{(l)}$, the following formula (1.11) should preferably be used than to compute it directly from the definition.

$$\begin{aligned} u_N^{(l)} = N^{-s} \sum_{k=l}^{\infty} \Big(\sum_{m=1}^{l} (-1)^m \binom{l}{m} m^k \Big) \\ (-1)^k \frac{s(s+1)\cdots(s+k-1)}{k!} N^{-k}, \end{aligned} \qquad N > l \geq 1. \tag{1.11}$$

If we replace $\sum_{k=l}^{\infty}$ by $\sum_{k=l}^{L}$, the error is less than $2^l \frac{|s(s+1)\cdots(s+L)|}{(L+1)!} (\frac{l}{N})^{L+1} |N^{-s}|$ if $\Re(s) \geq -L-1$.

§2. L-functions attached to symmetric tensor representations of $GL(2)$

Let $f(z) = \sum_{n=1}^{\infty} c_n e^{2\pi i n z} \in S_k(SL(2, \mathbf{Z}))$ be a normalized common eigenfunction of Hecke operators. The L-function $L(s, f) = \sum_{n=1}^{\infty} c_n n^{-s}$ attached to f converges absolutely when $\Re(s) > \frac{k+1}{2}$ and has the Euler product

$$L(s,f) = \prod_p (1 - c_p p^{-s} + p^{k-1-2s})^{-1}.$$

Put

$$1 - c_p X + p^{k-1} X^2 = (1 - \alpha_p X)(1 - \beta_p X)$$

with α_p, $\beta_p \in \mathbf{C}$, where X is an indeterminate. For a positive integer r, we define an Euler product

$$L^{(r)}(s,f) = \prod_p [(1 - \alpha_p^r p^{-s})(1 - \alpha_p^{r-1} \beta_p p^{-s}) \cdots (1 - \beta_p^r p^{-s})]^{-1} \tag{2.1}$$

which converges absolutely when $\Re(s) > \frac{r(k-1)}{2} + 1$. It is conjectured that $L^{(r)}(s,f)$ can be analytically continued to the whole complex plane as an entire function and satisfies a functional equation. The conjectural functional equation of $L^{(r)}(s,f)$ takes the following form (cf. Serre [Se]). If r is odd, put $r = 2m-1$,

$$R^{(r)}(s,f) = (2\pi)^{-ms} \prod_{i=0}^{m-1} \Gamma(s - i(k-1)) L^{(r)}(s,f), \tag{2.2}$$

$$\epsilon_r = (\sqrt{-1})^{m + (k-1)m^2}. \tag{2.3}$$

If r is even, put $r = 2m$,

$$\begin{aligned} R^{(r)}(s,f) = \pi^{-s/2} (2\pi)^{-ms} \Big(\prod_{i=0}^{m-1} \Gamma(s - i(k-1)) \Big) \\ \Gamma\Big(\frac{s - m(k-1) + \delta}{2}\Big) L^{(r)}(s,f), \end{aligned} \tag{2.4}$$

$$\epsilon_r = (\sqrt{-1})^{m + (k-1)m(m+1) + \delta}, \tag{2.5}$$

where $\delta = 0$ (resp. 1) if m is even (resp. odd). Then the functional equation

$$R^{(r)}(s,f) = \epsilon_r R^{(r)}(r(k-1) + 1 - s, f) \tag{2.6}$$

is predicted. A quick way to see (2.6) is as follows. Let M_f be the motive of rank 2 over $\mathbf{Q}$ attached to f. We see that the Hodge realization of M_f corresponds to the two dimensional representation

$$\rho = \mathrm{Ind}(\psi_k; W_{\mathbf{C}} \longrightarrow W_{\mathbf{R},\mathbf{C}})$$

of $W_{\mathbf{R},\mathbf{C}}$. Here $W_{\mathbf{C}} = \mathbf{C}^{\times}$, $W_{\mathbf{R},\mathbf{C}}$ is the Weil group of $\mathbf{C}$ over $\mathbf{R}$ and ψ_k is the quasi-character $\psi_k(x) = x^{-(k-1)}$ of $W_{\mathbf{C}}$. Let $\sigma_r : GL(2) \longrightarrow GL(r+1)$ be the symmetric tensor representation of degree r and put $\rho_r = \sigma_r \circ \rho$. Then we find

$$(2.7) \quad \rho_r \cong \oplus_{i=0}^{m-1} \mathrm{Ind}(x \longrightarrow x^{-(r-i)(k-1)} \bar{x}^{-i(k-1)};\ W_{\mathbf{C}} \longrightarrow W_{\mathbf{R},\mathbf{C}}), \quad r = 2m-1,$$

$$(2.8) \quad \begin{aligned} \rho_r \cong &\oplus_{i=0}^{m-1} \mathrm{Ind}(x \longrightarrow x^{-(r-i)(k-1)} \bar{x}^{-i(k-1)};\ W_{\mathbf{C}} \longrightarrow W_{\mathbf{R},\mathbf{C}}) \\ &\oplus \{(x \longrightarrow |x|^{-m(k-1)} (\mathrm{sgn}\ x)^{m(k-1)}) \circ t\}, \quad r = 2m, \end{aligned}$$

where t denotes the transfer map from $W_{\mathbf{R},\mathbf{C}}$ to $\mathbf{R}^{\times}$. The gamma factor and the constant ϵ_r of the functional equation can be calculated as the usual gamma factor and the constant attached to the representation ρ_r of $W_{\mathbf{R},\mathbf{C}}$; hence we obtain (2.2) $\sim$ (2.6).

We refer the reader to Shahidi [Sha1], [Sha2] for what are known on these symmetric power L-functions, in more general cases.

Let $\Delta(z) = q \prod_{n=1}^{\infty} (1-q^n)^{24} \in S_{12}(SL(2,\mathbf{Z}))$, $q = e^{2\pi\sqrt{-1}z}$. The calculation of zeros of $L^{(r)}(s,\Delta)$ for $r = 1,\ 2$ is given in [Y2]. We consider the case $r \geq 3$. To compute $L^{(r)}(s,\Delta)$, we modify our summation method slightly in the following way. Fix r, choose $v = v_r > 0$ and set

$$L^{(r)}(s,\Delta) = \sum_{n=1}^{\infty} a_n n^{-s} = \sum_{n=1}^{\infty} (a_n n^{-v}) n^{-(s-v)}.$$

Put

$$s_n^{(0)} = a_n n^{-v}, \qquad u_n^{(0)} = n^{-(s-v)}$$

and define $s_n^{(l)}$, $u_n^{(l)}$ recursively by (1.9). We set $S_N^{(l)} = \sum_{n=1}^{N} s_n^{(l)} u_n^{(l)}$. It turns out that a suitable choice of v depending on r yields good results. We can interpret this as the neutralization of the effect of extremely large value of $s_n^{(l)}$ and extremely small value of $u_n^{(l)}$.

As the first example, let $r = 3$. We take $v = 8$. For $s = 17 + it$, $t = 20$, the values of

$$R_j = \Re(\exp(i\vartheta(t)) S_N^{(j)}), \qquad I_j = \Im(\exp(i\vartheta(t)) S_N^{(j)})$$

are given in Table 2.1. In Table 2.2, we give the values of t_n the n-th zero of $L^{(3)}(s,\Delta)$, $s = 17 + it$ on the critical line for $0 \leq t \leq 40$.

Next we apply our summation method to $L^{(4)}(s,\Delta)$ taking $v = 12$. For $s = \frac{45}{2} + it$, $t = 10$, the values of

$$R_j = \Re(\exp(i\vartheta(t)) S_N^{(j)}), \qquad I_j = \Im(\exp(i\vartheta(t)) S_N^{(j)})$$

are given in Table 2.3. In Table 2.4, we give the values of u_n the n-th zero of $L^{(4)}(s,\Delta)$, $s = \frac{45}{2} + iu$ on the critical line for $0 \leq u \leq 20$.

We can see, by the same technique as will be given in §3 and §4, that the Riemann hypothesis holds for $L^{(3)}(s,\Delta)$ (resp. $L^{(4)}(s,\Delta)$) in the range $0 \leq \Im(s) \leq 40$ (resp. $0 \leq \Im(s) \leq 20$) and that the zeros $17 + it_n$ (resp. $\frac{45}{2} + iu_n$) are simple.

§3. Modular forms of half integral weight

Put

$$\theta(z)=\sum_{n\in\mathbf{Z}}\mathbf{e}(n^2z)=1+2\sum_{n=1}^{\infty}q^{n^2},\qquad \eta(z)=\mathbf{e}(z/24)\prod_{n=1}^{\infty}(1-\mathbf{e}(nz)).$$

By Shimura, [Sh2], (4.1), we have

$$\dim S_8(\Gamma_0(2))=1,\qquad \dim S_{9/2}(\Gamma_0(4))=1$$

and $(\eta(z)\eta(2z))^8$ (resp. $\theta(z)^{-3}\eta(2z)^{12}$) spans $S_8(\Gamma_0(2))$ (resp. $S_{9/2}(\Gamma_0(4))$). Put

$$f(z)=(\eta(z)\eta(2z))^8=\sum_{n=1}^{\infty}a_nq^n,\qquad g(z)=\theta(z)^{-3}\eta(2z)^{12}=\sum_{n=1}^{\infty}c_nq^n,$$
$$L(s,f)=\sum_{n=1}^{\infty}a_nn^{-s},\qquad L(s,g)=\sum_{n=1}^{\infty}c_nn^{-s},$$
$$R(s,f)=2^{s/2}(2\pi)^{-s}\Gamma(s)L(s,f),\qquad R(s,g)=2^{s}(2\pi)^{-s}\Gamma(s)L(s,g).$$

Then f and g are in the Shimura correspondence; $L(s,f)$ and $L(s,g)$ can be analytically continued to the whole complex plane as entire functions and satisfy the functional equations

$$R(s,f)=R(8-s,f),\qquad R(s,g)=R(9/2-s,g). \tag{3.1}$$

This example is described in detail in [Sh2], §4. For $t>0$, let $\vartheta_f(t)$ (resp. $\vartheta_g(t)$) denote the variation of the argument of $2^{s/2}(2\pi)^{-s}\Gamma(s)$ (resp. $2^{s}(2\pi)^{-s}\Gamma(s)$) from 4 to $4+it$ (resp. $9/4$ to $9/4+it$).

For $L(s,f)$, $s=4+it$, $t=100$, the values of

$$R_j=\Re(\exp(i\vartheta_f(t))S_N^{(j)}),\qquad I_j=\Im(\exp(i\vartheta_f(t))S_N^{(j)})$$

are given in Table 3.1.

For $L(s,g)$, $s=\frac{9}{4}+it$, $t=100$, the values of

$$R_j=\Re(\exp(i\vartheta_g(t))S_N^{(j)}),\qquad I_j=\Im(\exp(i\vartheta_g(t))S_N^{(j)})$$

are given in Table 3.2.

By our method, we can compute zeros of $L(s,f)$ and of $L(s,g)$ on the critical line with sufficient accuracy observing sign changes of $e^{i\vartheta_f(t)}L(4+it,f)$ and $e^{i\vartheta_g(t)}L(\frac{9}{4}+it,g)$. In Table 3.3, we list the n-th zero t_n of $L(s,f)$, $s=4+it$ in the range $0\le t\le 100$. In Table 3.4, we list the n-th zero u_n of $L(s,g)$, $s=\frac{9}{4}+iu$ in the range $0\le u\le 100$.

Now let us examine the Riemann hypothesis for $L(s,f)$. We see $f(iy)>0$ for $y>0$ by the product expansion of the η-function. By the integral representation

$$(2\pi)^{-s}\Gamma(s)L(s,f)=\int_0^{\infty}f(iy)y^{s-1}\,dy,$$

we see that $L(s,f)>0$ for $s>0$. For $T>0$, let $N(T)$ denote the number of zeros of $L(s,f)$ counted with multiplicity in the domain

$$|\Re(s)-4|<1/2, \qquad 0\leq \Im(s)\leq T.$$

By (1.7) taking $\delta=1/2$, we have

$$N(T)=\pi^{-1}\vartheta_f(T)+\pi^{-1}\Delta\arg(L(s,f)). \tag{3.2}$$

Since $L(s,f)\neq 0$ if $\Re(s)-4\geq 1/2$, $\Delta\arg(L(s,f))$ equals the variation of the argument of $L(s,f)$ along the line segments $L_1=[4,4+\frac{1}{2}+\mu]$, $L_2=[4+\frac{1}{2}+\mu,4+\frac{1}{2}+\mu+iT]$, $L_3=[4+\frac{1}{2}+\mu+iT,4+iT]$ for any $\mu>0$. Take $T=100$, $\mu=1$. Then we have $\pi^{-1}\vartheta_f(100)=69.0171\cdots$. Hence if we can show $|\Delta\arg(L(s,f))|<\pi/2$, we can conclude $N(T)=69$. For this purpose, it suffices to show that $\Re(L(s,f))>0$ when $s\in L_i$, $i=1,\ 2,\ 3$. For L_1, we have shown this fact above. For L_2, this fact can be proved as in [Y2], §4. For L_3, we divide it into 150 small intervals and appeal to our heuristic calculation. We have observed

$$\Re(L(s,f))>0.83 \qquad \text{on} \quad L_3.$$

Thus we conclude that the Riemann hypothesis holds for $L(s,f)$ when $0\leq\Im(s)\leq 100$. All the zeros are simple.

Now let us consider zeros of $L(s,g)$. We have $g(iy)>0$ for $y>0$ since $\theta(iy)>0$, $\eta(iy)>0$ for $y>0$. By the integral representation

$$(2\pi)^{-s}\Gamma(s)L(s,g)=\int_0^\infty g(iy)y^{s-1}\,dy,$$

we see that $L(s,g)>0$ for $s>0$. For $T>0$, let $N(T)$ denote the number of zeros of $L(s,g)$ counted with multiplicity in the domain[3]

$$|\Re(s)-9/4|\leq 2, \qquad 0\leq\Im(s)\leq T.$$

By (1.7) taking $\delta=2$, we have

$$N(T)=\pi^{-1}\vartheta_g(T)+\pi^{-1}\Delta\arg(L(s,g)).$$

Here $\Delta\arg(L(s,g))$ denotes the variation of the argument of $L(s,g)$ along the line segments $L_1=[9/4,17/4]$, $L_2=[17/4,17/4+iT]$, $L_3=[17/4+iT,9/4+iT]$. Take $T=100$. Then we have $\pi^{-1}\vartheta_g(100)=79.1885\cdots$. Dividing L_2 and L_3 into small intervals, we have observed $\pi^{-1}\Delta(\arg L(s,g))=0.8114\cdots$. Thus we obtain $N(T)=80$. On the otherhand, we have obtained only 54 zeros on the critical line. Therefore, assuming that these zeros are simple, there must exist 13 zeros in the right-hand side of the critical line: $9/4<\Re(s)\leq 17/4$, $0\leq\Im(s)\leq 100$. These zeros, together with those in $100\leq\Im(s)\leq 150$, are given in Table 3.5.

[3] The zero free region of $L(s,g)$ is non-trivial. Here we content ourselves by regarding $|\Re(s)-9/4|\leq 2$ is "sufficiently wide".

Our method of calculation of these exceptional zeros is as follows. Let us consider a box B given by $d_1 \le \Re(s) \le d_2$, $h_1 \le \Im(s) \le h_2$. By the argument principle, we can determine whether $L(s,g)$ has a zero incide B or not. First we find a box B in which $L(s,g)$ has zeros by trial and error. Then dividing B into sub-boxes and applying the principle above successively, we can obtain a good approximation for a zero inside of B.[4]

The Riemann hypothesis does not hold for $L(s,g)$. This should be of no surprise since $L(s,g)$ does not have an Euler product.

We shall study one more example of modular forms of half integral weight. For $k \ge 1$, put

$$G_k(z) = \frac{1}{2}\zeta(1-k) + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \qquad \sigma_{k-1}(n) = \sum_{d|n, d>0} d^{k-1}.$$

Let

$$\delta(z) = \frac{60}{2\pi i}(2G_4(4z)\theta'(z) - G_4'(4z)\theta(z)) = \sum_{n=1, n\equiv 0,1 \mod 4}^{\infty} c(n)q^n.$$

Then we have $\delta \in S_{13/2}(\Gamma_0(4))$ (cf. Kohnen-Zagier [KZ], p. 177) and δ corresponds to $\Delta(z) = \eta(z)^{24} \in S_{12}(SL(2,\mathbf{Z}))$ under the Shimura correspondence. The values of $c(n)$ can easily be computed by

$$c(n) = \omega(\sqrt{n}) \cdot n + 120 \sum_{m=1}^{[\frac{n-1}{4}]} \omega(\sqrt{n-4m})\sigma_3(m)(2n-9m) - 15n\sigma_3(n/4),$$

where

$$\omega(x) = \begin{cases} 1 & \text{if} \quad x \in \mathbf{Z}, \\ 0 & \text{if} \quad x \notin \mathbf{Z}. \end{cases}$$

Let

$$\delta_0(z) = \sum_{n \equiv 0 \mod 4} c(n) q^{n/4}.$$

Then we have $\delta_0 \in S_{13/2}(\Gamma_0(4))$ and

$$\delta(\frac{i}{4y}) = -\sqrt{2}y^{6+1/2}\delta_0(iy), \qquad y > 0.$$

(cf. [KZ], p. 190.) Put

$$h_\pm(z) = (\delta \mp 2^{-6}\delta_0)(z), \qquad R(s, h_\pm) = 2^s(2\pi)^{-s}\Gamma(s)L(s,h_\pm).$$

Then we have

$$h_\pm(\frac{i}{4y}) = \pm(2y)^{6+1/2}h_\pm(iy), \qquad y > 0.$$

[4] It is more efficient to use a variant of the Newton method once we get a rough approximation. After finding a precise location, final check should be done by the method described above.

Hence we obtain the functional equations for the entire functions $R(s, h_{\pm})$:

$$R(\frac{13}{2} - s, h_{\pm}) = \pm R(s, h_{\pm}).$$

We have computed zeros of $L(s, h_{\pm})$ on the critical line $\Re(s) = 13/4$ in the range $0 \leq \Im(s) \leq 100$. The results are given in Table 3.6. There $u_n^{\pm}$ denotes the n-th zero of $L(s, h_{\pm})$ for $s = \frac{13}{4} + iu$.

In Table 3.7, we also list zeros of $L(s, h_{\pm})$ not on the critical line which are searched in the region $13/4 < \Re(s) \leq 25/4$, $0 \leq \Im(s) \leq 100$; $\rho_i^{\pm}$ denotes a zero of $L(s, h_{\pm})$. It is a very interesting phenomenon that $L(s, h_{-})$ has much more zeros outside of the critical line compared with $L(s, h_{+})$.

§4. L-functions attached to Hecke characters of infinite order of real quadratic fields

In this and the next section, we shall study two types of L-functions which are closely related to algebraic number fields. We can still apply our method of calculation described in §1 efficiently. However the situation changes drastically. The repeated application of partial summation does not yield good results beyond rather limited number of times. Thus our calculation cannot be as accurate as in the case of modular forms treated in §2 and §3.

Let k be a real quadratic field. For simplicity, we assume that the class number of k is 1. Let D be the discriminant and ϵ be the fundamental unit of k. Let $k_A^{\times}$ denotes the idele group of k. For a finite place v of k, let k_v denote the completion of k at v and $\mathfrak{O}_v$ denote the ring of integers of k_v. Since k is of class number 1, we have

$$k_A^{\times} = k^{\times}(\prod_v \mathfrak{O}_v^{\times} \times \mathbf{R}^{\times} \times \mathbf{R}^{\times}) \tag{4.1}$$

where v extends over all finite places of k. Let $\chi = \prod_v \chi_v$ be an unramified unitary Hecke character of $k_A^{\times}$. Let σ_1 (resp. σ_2) be the identical (resp. non-identical) isomorphism of k into $\mathbf{R}$ and let ∞_1 (resp. ∞_2) be the corresponding archimedean place of k. As unitary characters of $\mathbf{R}^{\times}$, χ_{∞_1} and χ_{∞_2} take the following form:

$$\chi_{\infty_j}(x) = \mathrm{sgn}(x)^{m_j} |x|^{iv_j} \quad \text{for} \quad x \in K_{\infty_j}^{\times} \cong \mathbf{R}^{\times}, \quad j = 1, 2, \tag{4.2}$$

where $m_j = 0$ or 1, $v_j \in \mathbf{R}$. By (4.1), we see that χ is completely determined by χ_{∞_1} and χ_{∞_2}. Since χ is trivial on $k^{\times}$, we must have $\chi(x) = 1$ for all $k^{\times} \cap (\prod_v \mathfrak{O}_v^{\times} \times \mathbf{R}^{\times} \times \mathbf{R}^{\times})$, which is the group of units of k. Therefore we have

$$(-1)^{m_1 + m_2} = 1, \qquad \mathrm{sgn}(\epsilon^{\sigma_1})^{m_1} \mathrm{sgn}(\epsilon^{\sigma_2})^{m_2} |\epsilon^{\sigma_1}|^{iv_1} |\epsilon^{\sigma_2}|^{iv_2} = 1. \tag{4.3}$$

It is easy to see that (4.3) is a necessary and sufficient condition for χ, which is determined by χ_{∞_1} and χ_{∞_2}, to be a Hecke character of $k_A^{\times}$. By (4.3), we have $m_1 = m_2$. Put $m = m_1$. Then (4.3) is equivalent to

$$|\epsilon|^{i(v_1 - v_2)} = \mathrm{sgn}(N(\epsilon))^m. \tag{4.4}$$

Let χ_* be the associated ideal character of k. If (α), $\alpha \in k^\times$ is a prime ideal, we have, by definition

$$\begin{aligned}\chi_*((\alpha)) &= \chi((\cdots,1,\cdots,\alpha,\cdots,1,\cdots)) \\ = \chi((\alpha^{-1},\cdots,1,\cdots,\alpha^{-1},\cdots)) &= \mathrm{sgn}(N(\alpha))^m \cdot (|\alpha^{\sigma_1}|^{iv_1}|\alpha^{\sigma_2}|^{iv_2})^{-1}.\end{aligned}$$

Here $(\cdots,1,\cdots,\alpha,\cdots,1,\cdots) \in k_A^\times$ denotes the idele whose (α)-component is α and all the other components are 1. Hence we have

$$\chi_*((\alpha)) = \mathrm{sgn}(N(\alpha))^m |\alpha^{\sigma_1}|^{-iv_1}|\alpha^{\sigma_2}|^{-iv_2} \qquad \text{for every} \quad \alpha \in k^\times, \tag{4.5}$$

$L(s,\chi) = L(s,\chi_*) = \sum_{(\alpha)} \chi_*((\alpha))N((\alpha))^{-s}$. Put

$$\begin{aligned}R(s,\chi) &= |D|^{s/2}\pi^{-(s+m)}\pi^{-i(v_1+v_2)/2}\Gamma((s+m+iv_1)/2) \\ &\qquad\qquad \Gamma((s+m+iv_2)/2)L(s,\chi), \\ R(s,\chi^{-1}) &= |D|^{s/2}\pi^{-(s+m)}\pi^{i(v_1+v_2)/2}\Gamma((s+m-iv_1)/2) \\ &\qquad\qquad \Gamma((s+m-iv_2)/2)L(s,\chi^{-1}).\end{aligned}$$

Then the functional equation is (cf. Weil [W], Langlands [LL])

$$R(s,\chi) = (-1)^m\chi_*((d))R(1-s,\chi^{-1}), \tag{4.6}$$

where (d) denotes the different of k over $\mathbf{Q}$. Since $\overline{R(s,\chi)} = R(\bar{s},\chi^{-1})$, we can put (4.6) in the form of (1.2):

$$R(1-\bar{s},\chi) = (-1)^m\chi_*((d))\overline{R(s,\chi)}. \tag{4.7}$$

We get

$$\overline{\chi_*((d))^{-1/2}R(s,\chi)} = (-1)^m\chi_*((d))^{-1/2}R(s,\chi) \qquad \text{if} \quad \Re(s) = 1/2. \tag{4.8}$$

Hence $\chi_*((d))^{-1/2}R(s,\chi)$ takes real or pure imaginary values on the critical line according as $m = 0$ or 1. We also note that we may assume $v_2 = 0$ without losing any generality since the choice of v_2 can be taken into account as the shift of the variable s. Then, if $m = 0$, we have $v_1 = -2n\pi/\log\epsilon$ with $n \in \mathbf{Z}$ by (4.4). We denote this Hecke character by χ_n. By (4.5), we have

$$(\chi_n)_*((\alpha)) = |\alpha|^{2n\pi i/\log\epsilon}, \qquad \alpha \in k^\times. \tag{4.9}$$

If $m = 1$, by (4.4), we have $v_1 = -2n\pi/\log\epsilon$ or $v_1 = -(2n+1)\pi/\log\epsilon$ with $n \in \mathbf{Z}$ according as $N(\epsilon) = 1$ or $N(\epsilon) = -1$. We denote this Hecke character by χ'_n. By (4.5), we have

$$(\chi'_n)_*((\alpha)) = \begin{cases} \mathrm{sgn}(N(\alpha))|\alpha|^{2n\pi i/\log\epsilon} & \text{if} \quad N(\epsilon) = 1, \\ \mathrm{sgn}(N(\alpha))|\alpha|^{(2n+1)\pi i/\log\epsilon} & \text{if} \quad N(\epsilon) = -1. \end{cases} \tag{4.10}$$

As our first example, we take $k = \mathbf{Q}(\sqrt{2})$. We have $\epsilon = \sqrt{2}+1$, $(d) = (\sqrt{2})^3$, $|D| = 8$. We are going to study $L(s,\chi_1)$ applying our summation method. For $s = \frac{1}{2}+it$, $t = 15$ and 50, the values of

$$R_j = \Re(\chi_*((d))^{-1/2}\exp(i\vartheta_0(t))S_N^{(j)}), \qquad I_j = \Im(\chi_*((d))^{-1/2}\exp(i\vartheta_0(t))S_N^{(j)})$$

are given in Tables 4.1a and in 4.1b respectively. Here $\chi_*((d))^{-1/2} = \sqrt{2}^{-3\pi i/\log\epsilon}$, $\vartheta_0(t) = \arg(8^{s/2}\pi^{-s}\pi^{2\pi i/\log\epsilon}\Gamma((s-2\pi i/\log\epsilon)/2)\Gamma(s/2))$. From this table, it is evident that R_j's for higher j do not give good results. We can judge, from the values of $|I_j|$, R_2 gives the best result, then R_3, R_1 in this order. We must be more cautious than in §2 and §3 about the accuracy of the value $e^{i\vartheta_0(t)}L(s,\chi_1)$. For example, let $t = 15$. We empirically judge that $e^{i\vartheta_0(t)}L(s,\chi_1) = 2.17375$ with error $\approx 10^{-5}$ from R_2 and I_2. We have constructed Table 4.2 in which zeros on the critical line $\Re(s) = 1/2$ are listed in the range $|\Im(s)| \le 30$. Here, for $n \ge 1$, t_n (resp. t_{-n}) denotes the n-th zero of $L(s,\chi_1)$, $s = 1/2+it$ on the critical line for $0 \le t \le 30$ (resp. $0 \le -t \le 30$).

Let us examine the Riemann hypothesis for $L(s,\chi_1)$ in the range $0 \le \Im(s) \le 100$. We have observed 84 sign changes of $e^{i\vartheta_0(t)}L(\frac{1}{2}+it,\chi_1)$ for $0 \le t \le 100$. For $T > 0$, let $N(T)$ denote the number of zeros of $L(s,\chi_1)$ counted with multiplicity in the domain

$$0 < \Re(s) < 1, \qquad 0 \le \Im(s) \le T.$$

Taking $\delta = 1/2$ in (1.7) , we get

$$N(T) = \pi^{-1}\vartheta(T) + \pi^{-1}\Delta\arg(L(s,\chi_1)). \tag{4.11}$$

Since $L(s,\chi_1) \ne 0$ if $\Re(s) \ge 1$, $\Delta\arg(L(s,\chi_1))$ equals the variation of the argument of $L(s,\chi_1)$ along the line segments $L_1 = [\frac{1}{2}, 1+\mu]$, $L_2 = [1+\mu, 1+\mu+iT]$, $L_3 = [1+\mu+iT, \frac{1}{2}+iT]$ for any $\mu > 0$. Take $T = 100$ and $\mu = 1$. We have $\pi^{-1}\vartheta(100) = 84.8864\cdots$. Hence if we can show $-\pi < \Delta\arg(L(s,f)) < 0$, then we can conclude that $N(100) = 84$. For L_1, we divide it into 15 intervals. We observed that $L(s,\chi_1)$ moves from $0.3482+0.0712\sqrt{-1}$ to $0.8011+0.0969\sqrt{-1}$ keeping $\Re(L(s,\chi_1)) > 0$. For L_2, we can show without difficulty that $\Re(L(s,\chi_1)) > 0$ on L_2. For L_3, we divide it into 150 small intervals. We observed that $L(s,\chi_1)$ moves from $0.8159+0.1227\sqrt{-1}$ to $-0.0110-0.0072\sqrt{-1}$ when s moves from $2+100\sqrt{-1}$ to $\frac{1}{2}+100\sqrt{-1}$; $L(s,\chi_1)$ never crossed the half line $\Im(L(s,\chi_1)) = 0$, $\Re(L(s,\chi_1)) \le 0$. Hence we get $N(100) = 84$. The Riemann hypothesis holds and all zeros of $L(s,\chi_1)$ are simple zeros in this range.

By (4.11), we should have

$$N(100) = 84.8864\cdots - \pi^{-1}\arctan(\frac{712}{3482}) - (1-\pi^{-1}\arctan(\frac{72}{110})) = 84.0067\cdots.$$

The error is about 6.7×10^{-3} and this is much bigger than the usual error inherent in our calculations. The reason is that $L(s,\chi_1)$ takes rather small value at $\frac{1}{2}+100\sqrt{-1}$; such an error can be made much smaller in the following way. We take $T = 101$. We have observed 86 sign changes of $e^{i\vartheta_0(t)}L(\frac{1}{2}+it,\chi_1)$ for $0 \le t \le 101$.

We have $\pi^{-1}\vartheta(101) = 86.0881\cdots$. We divide $[2+101\sqrt{-1}, \frac{1}{2}+101\sqrt{-1}]$ into 150 small intervals. We observed that $L(s,\chi_1)$ moves from $0.9322+0.2563\sqrt{-1}$ to $1.7137-0.1288\sqrt{-1}$. Hence by (4.11), we have

$$N(101) = 86.0881\cdots - \pi^{-1}\arctan(\frac{712}{3482}) - \pi^{-1}\arctan(\frac{1288}{17137}) = 86$$

with error less than 10^{-4}.

In Table 4.3, we have listed 44 examples of $L(s,\chi_n)$ for which we made experiments in the range $0 \le \Im(s) \le T$; $N(T)$ denotes the number of zeros of $L(s,\chi_n)$ in the domain $0 < \Re(s) < 1$, $0 \le \Im(s) \le T$. We found that all zeros in the ranges of Table 4.3 lie on the critical line and are simple.

§5. Artin L-functions

Let k be the minimal splitting field of the irreducible polynomial $f(X) = X^5 - X + 1$ over $\mathbf{Q}$. Then $k \supset \mathbf{Q}(\sqrt{19\cdot 151}) = k_0$, k is unramified over k_0, $\mathrm{Gal}(k/\mathbf{Q}) \cong S_5$, $\mathrm{Gal}(k/k_0) \cong A_5$. The discriminant Δ of a root of $f(X)$ is $19\cdot 151$. This example is due to E. Artin (cf. Lang [LG], p. 121). Let ρ be an irreducible 4-dimensional representation of S_5 whose character χ_ρ is given as follows.

conjugacy class	(1)	(12)	(123)	(1234)	(12)(34)	(12)(345)	(12345)
χ_ρ	4	2	1	0	0	-1	-1

Since S_5 does not have a subgroup of index 4, ρ is not monomial. We have $L(s,\rho) = \prod_p L_p(s,\rho)$ for $\Re(s) > 1$ with the Euler p-factor $L_p(s,\rho)$. We can compute $L_p(s,\rho)$ as follows. First we assume that a prime number p is unramified in k, i.e., $p \neq 19$, 151. Then we see easily that $L_p(s,\rho)^{-1}$ equals

$$\begin{array}{rll}
(1-p^{-s})^4 & \text{if} & \sigma_p=\{(1)\}, \\
(1-p^{-s})^3(1+p^{-s}) & \text{if} & \sigma_p=\{(12)\}, \\
(1-p^{-s})^2(1+p^{-s}+p^{-2s}) & \text{if} & \sigma_p=\{(123)\}, \\
(1-p^{-s})(1+p^{-s})(1+p^{-2s}) & \text{if} & \sigma_p=\{(1234)\}, \\
1+p^{-s}+p^{-2s}+p^{-3s}+p^{-4s} & \text{if} & \sigma_p=\{(12345)\}, \\
(1-p^{-s})^2(1+p^{-s})^2 & \text{if} & \sigma_p=\{(12)(34)\}, \\
(1-p^{-s})(1+p^{-s})(1+p^{-s}+p^{-2s}) & \text{if} & \sigma_p=\{(12)(345)\}.
\end{array}$$

Here σ_p denotes the Frobenius conjugacy class of p and $\{\tau\}$ denotes the conjugacy class of $\tau \in S_5$.

Let $p = 19$ or 151. Let $I_\mathfrak{p}$ denote the inertia group of a prime factor $\mathfrak{p}$ of p in k. By definition, we have

$$L_p(s,\rho)^{-1} = \det(1 - (\rho(\sigma_\mathfrak{p})|V^{I_\mathfrak{p}}) \cdot p^{-s}). \tag{5.1}$$

Here V denotes the representation space of ρ, $V^{I_\mathfrak{p}}$ the subspace of $I_\mathfrak{p}$-fixed vectors and $\sigma_\mathfrak{p}$ a Frobenius of $\mathfrak{p}$ which is determined modulo $I_\mathfrak{p}$. Since k is unramified over k_0, it is obvious that $|I_{\mathfrak{p}}| = 2$, $I_{\mathfrak{p}} \not\subseteq \mathrm{Gal}(k/k_0) \cong A_5$. Hence we may assume that $I_{\mathfrak{p}}$ is generated by (12) choosing a suitable $\mathfrak{p}$ lying over p. Let $D_{\mathfrak{p}}$ denote the decomposition group of $\mathfrak{p}$. Then $D_{\mathfrak{p}} \triangleright I_{\mathfrak{p}}$ and $D_{\mathfrak{p}}/I_{\mathfrak{p}}$ is generated by $\sigma_{\mathfrak{p}} \mod I_{\mathfrak{p}}$. We have

$$N_{S_5}(I_{\mathfrak{p}}) = I_{\mathfrak{p}} \times S_3,$$

where $N_{S_5}(I_{\mathfrak{p}})$ denotes the normalizer of $I_{\mathfrak{p}}$ in S_5 and S_3 denotes the permutation group on three letters $\{3,4,5\}$.

Let $p = 19$. Then

$$f(X) \equiv (X-6)^2(X^3 + 12X^2 + 13X + 9) \mod 19$$

is the factorization of $f(X) \mod p$ into irreducible factors in $(\mathbf{Z}/p\mathbf{Z})[X]$. Therefore the residue field extension $\mathfrak{O}_k/\mathfrak{p}$ of $\mathbf{Z}/p\mathbf{Z}$ contains the cubic extension of $\mathbf{Z}/p\mathbf{Z}$, where $\mathfrak{O}_k$ denotes the ring of integers of k. Hence we immediately obtain $D_{\mathfrak{p}}/I_{\mathfrak{p}} \cong \mathbf{Z}/3\mathbf{Z}$ and that $\sigma_{\mathfrak{p}}$ may be taken as $(345) \in S_3$. Now we find easily that

$$L_p(s,\rho)^{-1} = 1 - p^{-3s} \qquad \text{if} \quad p = 19.$$

Let $p = 151$. Then

$$f(X) \equiv (X-39)^2(X-9)(X^2 + 87X + 61) \mod 151$$

is the factorization of $f(X) \mod p$ into irreducible factors in $(\mathbf{Z}/p\mathbf{Z})[X]$. By a similar consideration as above, we find that $D_{\mathfrak{p}}/I_{\mathfrak{p}} \cong \mathbf{Z}/2\mathbf{Z}$ and that $\sigma_{\mathfrak{p}}$ may be taken as $(34) \in S_3$. We obtain

$$L_p(s,\rho)^{-1} = (1 + p^{-s})(1 - p^{-s})^2 \qquad \text{if} \quad p = 151.$$

Let $f(\rho)$ denote the Artin conductor of ρ. We easily obtain

$$f(\rho) = 19 \cdot 151.$$

For example, let $p = 19$ and $\mathfrak{p}$ be as above. We have shown $\mathrm{Gal}(k_{\mathfrak{p}}/\mathbf{Q}_{19}) \cong D_{\mathfrak{p}} \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$. We find that the restriction of ρ to $D_{\mathfrak{p}}$ splits into a direct sum of four one dimensional representations of $D_{\mathfrak{p}}$ such that three of them are unramified and one is ramified. Hence the exponent of 19 in $f(\rho)$ is 1.

We take an isomorphism σ of k into $\mathbf{C}$ and let $c \in \mathrm{Gal}(k/\mathbf{Q}) \cong S_5$ be the restriction of the complex conjugation to k. Then $c \in \mathrm{Gal}(k/k_0) \cong A_5$. Hence c is conjugate to $(12)(34)$ in S_5. Let $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ be identified with the decomposition group $\langle c \rangle$ of the archimedean place of k which corresponds to σ. The restriction of ρ to $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ splits into a direct sum of two trivial representations and two non-trivial representations. Therefore the Gamma factor to go with $L(s,\rho)$ is given by (cf. Langlands [LL])

$$(\pi^{-s/2}\Gamma(s/2))^2(\pi^{-(s+1)/2}\Gamma((s+1)/2))^2.$$

Put

$$R(s,\rho) = (19 \cdot 151)^{s/2}\pi^{-2s}\Gamma(s/2)^2\Gamma((s+1)/2)^2 L(s,\rho).$$

Since ρ is equivalent to its contragredient, we have the functional equation

$$R(s,\rho) = \kappa R(1-s,\rho), \tag{5.2}$$

where $\kappa = \pm 1$ is the Artin root number attached to ρ. Let ψ be the additive character of $\mathbf{Q}_A/\mathbf{Q}$ such that

$$\begin{aligned} \psi_\infty(x) &= \exp(2\pi\sqrt{-1}x), \qquad x \in \mathbf{Q}_\infty \cong \mathbf{R}, \\ \psi_p(x) &= \exp(-2\pi\sqrt{-1}\,\mathrm{Fr}(x)), \qquad x \in \mathbf{Q}_p, \end{aligned}$$

where Fr denotes the fractional part of x. By a theorem of Langlands, we have

$$\kappa = \prod_v \epsilon(\frac{1}{2}, \rho_v, \psi_v)$$

with the ϵ-factor defined in [LL]. By the above considerations, we easily get

$$\begin{aligned} \epsilon(\frac{1}{2}, \rho_p, \psi_p) &= \begin{cases} 1 & \text{if} \quad p \neq 19, 151, \\ i & \text{if} \quad p = 19 \text{ or } 151, \end{cases} \\ \epsilon(\frac{1}{2}, \rho_\infty, \psi_\infty) &= i^2. \end{aligned}$$

Hence we obtain

$$\kappa = 1. \tag{5.3}$$

The values of

$$R_j = \Re(\exp(i\vartheta(t))S_N^{(j)}), \qquad I_j = \Im(\exp(i\vartheta(t))S_N^{(j)})$$

for $L(s,\rho)$, $s = \frac{1}{2} + it$, $t = 5$ are given in Table 5.1. Here

$$\vartheta(t) = \arg((19 \cdot 151)^{s/2}\pi^{-2s}\Gamma(s/2)^2\Gamma((s+1)/2)^2), \qquad s = \frac{1}{2} + it.$$

In Table 5.2, we give the values of u_n, the n-th zero of $L(s,\rho)$, $s = \frac{1}{2} + iu$ on the critical line for $0 \leq u \leq 10$.

§6. Estimation of errors in our calculations

The most serious defect of our method of calculation is that we do not have rigorous controle of error estimates. In previous sections, we regarded the magnitude of $\Im(e^{i\vartheta(t)}S_N^{(l)})$ (resp. $\Re(e^{i\vartheta(t)}S_N^{(l)})$) as a rough measure of errors from the true value, when $e^{i\vartheta(t)}L(\sigma+it)$ should be real (resp. pure imaginary). In this section, we shall present several data which support this practice.

Suppose that the functional equation (1.2) for $L(s)$ holds. Then we have $e^{i\vartheta(t)}L(k/2+it) \in \mathbf{R}$, $t \in \mathbf{R}$ where $\vartheta(t) = \arg(\kappa_1 N^s \prod_{i=1}^m \Gamma(b_i s + c_i))$, $s = k/2 + it$ in the notation of (1.4). Take $0 < t_1 < t_2$ so that $e^{i\vartheta(t)}$ rotates on the unit circle exactly once when t moves from t_1 to t_2. We expect that $\max_{t_1 \le t \le t_2} |\Im(e^{i\vartheta(t)} S_N^{(l)})|$ can be used as the measure of errors. More explicitly, it seems plausible that

$$|\Re(e^{i\vartheta(t)} S_N^{(l)}) - L(k/2+it)| \le 10 \max_{t_1 \le t \le t_2} |\Im(e^{i\vartheta(t)} S_N^{(l)})|. \tag{6.1}$$

In examples below, we use $\Re(S_M^{(p)})$ as a substitute for $L(k/2+it)$ taking large M and p (except for in Example 6), and examine the ratio of two terms in (6.1) for $S_N^{(l)}$ taking relatively small N and l. The results are given in Table 6.1.

Example 1. We take the primitive form $f \in S_8(\Gamma_0(2))$ and consider the L-function $L(s,f)$, $s = 4+it$. When t moves from 97.9 to 100, $e^{i\vartheta(t)}$ rotates on the unit circle approximately once. We calculated the ratio

$$r_{5N}^{(1)} = \frac{\max_{97.9 \le t \le 100}(|\Re(e^{i\vartheta(t)} S_{2000}^{(5N)} - e^{i\vartheta(t)} S_{10000}^{(35)})|)}{\max_{97.9 \le t \le 100}(|\Im(e^{i\vartheta(t)} S_{2000}^{(5N)})|)}$$

for $1 \le N \le 6$, dividing $[97.9, 100]$ into 21 intervals of length 0.1.

Example 2. We take $g \in S_{9/2}(\Gamma_0(4))$ as in §3. When t moves from 98.2 to 100, $e^{i\vartheta(t)}$ rotates on the unit circle approximately once. We calculated the ratio

$$r_{5N}^{(2)} = \frac{\max_{98.2 \le t \le 100}(|\Re(e^{i\vartheta(t)} S_{2000}^{(5N)} - e^{i\vartheta(t)} S_{10000}^{(35)})|)}{\max_{98.2 \le t \le 100}(|\Im(e^{i\vartheta(t)} S_{2000}^{(5N)})|)}$$

for $1 \le N \le 6$, dividing $[98.2, 100]$ into 18 intervals of length 0.1.

Example 3. We take $\Delta \in S_{12}(SL(2,\mathbf{Z}))$ and consider the L-function $L^{(3)}(s,\Delta)$, $s = 17+it$, attached to the third symmetric power representation of $GL(2)$. When t moves from 17.5 to 20, $e^{i\vartheta(t)}$ rotates on the unit circle approximately once. We calculated the ratio

$$r_{5N}^{(3)} = \frac{\max_{17.5 \le t \le 20}(|\Im(e^{i\vartheta(t)} S_{2000}^{(5N)} - e^{i\vartheta(t)} S_{10000}^{(35)})|)}{\max_{17.5 \le t \le 20}(|\Re(e^{i\vartheta(t)} S_{2000}^{(5N)})|)}$$

for $1 \le N \le 6$, dividing $[17.5, 20]$ into 25 intervals of length 0.1. In this example, we have normalized $\vartheta(t)$ as in §2 so that $e^{i\vartheta(t)} L^{(3)}(17+it, \Delta)$ is pure imaginary.

Example 4. We consider the L-function $L^{(4)}(s,\Delta)$, $s = 45/2+it$, attached to the fourth symmetric power representation of $GL(2)$. When t moves from 7.2 to 10, $e^{i\vartheta(t)}$ rotates on the unit circle approximately once. We calculated the ratio

$$r_{5N}^{(4)} = \frac{\max_{7.2 \le t \le 10}(|\Re(e^{i\vartheta(t)} S_{2000}^{(5N)} - e^{i\vartheta(t)} S_{10000}^{(35)})|)}{\max_{7.2 \le t \le 10}(|\Im(e^{i\vartheta(t)} S_{2000}^{(5N)})|)}$$

for $1 \le N \le 6$, dividing $[7.2, 10]$ into 28 intervals of length 0.1.

Example 5. We consider the Artin L-function treated in §5. When t moves from 4 to 5.8, $e^{i\vartheta(t)}$ rotates on the unit circle approximately once. We calculated the ratio

$$r_{2N}^{(5)} = \frac{\max_{4\le t\le 5.8}(|\Re(e^{i\vartheta(t)}S_{10000}^{(2N)} - e^{i\vartheta(t)}S_{100000}^{(10)})|)}{\max_{4\le t\le 5.8}(|\Im(e^{i\vartheta(t)}S_{10000}^{(2N)})|)}$$

for $0 \le N \le 5$, dividing $[4, 5.8]$ into 18 intervals of length 0.1.

Example 6. We consider the Hecke L-function $L(s, \chi_1)$ for $k = \mathbf{Q}(\sqrt{2})$ treated in §4. When t moves from 47.9 to 50, $e^{i\vartheta(t)}$ rotates on the unit circle approximately once. We calculated the ratio

$$r_N^{(6)} = \frac{\max_{47.9\le t\le 50}(|\Re(e^{i\vartheta(t)}S_{10000}^{(N)} - e^{i\vartheta(t)}S_{100000}^{(2)})|)}{\max_{47.9\le t\le 50}(|\Im(e^{i\vartheta(t)}S_{10000}^{(N)})|)}$$

for $0 \le N \le 5$, dividing $[47.9, 50]$ into 21 intervals of length 0.1.

§7. A comparison with the explicit formula

We take the new form $f \in S_8(\Gamma_0(2))$ treated in §3. Let π be the irreducible unitary automorphic representation of $GL(2, \mathbf{Q}_A)$ which corresponds to f. We have

$$L_f(s, \pi) = L(s + \frac{7}{2}, f),$$

where $L_f(s, \pi)$ denotes the finite part of the Jacquet-Langlands L-function attached to π. Let

$$L_f(s, \pi) = \prod_p \left[(1 - \alpha_p p^{-s})(1 - \beta_p p^{-s})\right]^{-1}$$

be the Euler product of $L_f(s, \pi)$. For $p = 2$, the Euler 2-factor degenerates so that $\alpha_2 = -1/\sqrt{2}$, $\beta_2 = 0$. For $p \ne 2$, we have $|\alpha_p| = |\beta_p| = 1$ by the Ramanujan-Petersson conjecture proved by P. Deligne. We have the explicit formula

$$\begin{aligned}\sum_p \sum_{1\le n, p^n\le x}{}' (\alpha_p^n + \beta_p^n)\log p = &-\lim_{T\to+\infty} \sum_{|\Im(\rho)|<T} \frac{x^\rho}{\rho} - \frac{L_f'(0, \pi)}{L_f(0, \pi)} \\ &+ \log(\frac{\sqrt{x}+1}{\sqrt{x}-1}) - 2x^{-1/2} - \frac{2}{3}x^{-3/2} - \frac{2}{5}x^{-5/2}\end{aligned} \tag{7.1}$$

for $x > 1$. Here $\sum'$ means that the term $(\alpha_p^n + \beta_p^n)\log p$ should be multiplied by $1/2$ when $p^n = x$; ρ extends over zeros of $L_f(s, \pi)$ such that $0 < \Re(\rho) < 1$. This formula can be shown in the usual way as in Ingham [In], p. 77–80. The last term

$$g(x) := \log(\frac{\sqrt{x}+1}{\sqrt{x}-1}) - 2x^{-1/2} - \frac{2}{3}x^{-3/2} - \frac{2}{5}x^{-5/2},$$

which is equal to

$$2\sum_{k=3}^{\infty} \frac{x^{-(2k+1)/2}}{2k+1},$$

represents the contribution of the trivial zeros of $L_f(s,\pi)$; they are at $s=-\frac{7}{2}$, $-\frac{9}{2}$, $-\frac{11}{2}$, $\cdots$.

Now it seems very interesting to compare both sides of (7.1) numerically using the zeros of $L(s,f)$ given in Table 3.3. We approximate

$$\lim_{T\to+\infty}\sum_{|\Im(\rho)|<T}\frac{x^\rho}{\rho}$$

by

$$h_{69}(x):=\sum_{n=1}^{69}\sqrt{x}(t_n^2+\frac{1}{4})^{-1}[\cos(t_n\log x)+2t_n\sin(t_n\log x)] \tag{7.2}$$

with t_n given in Table 3.3. We have

$$L_f(0,\pi)=L(\frac{7}{2},f)=0.5942254156\cdots,\quad L'_f(0,\pi)=L'(\frac{7}{2},f)=0.1875716234\cdots,$$
$$\frac{L'_f(0,\pi)}{L_f(0,\pi)}=0.3156573558\cdots.$$

To obtain these values, we simply applied our repeated abel summation technique as before, though more rigorous evaluation could be made in this case.

In Figure 7.1, we drawed the graphs of the "step function"

$$\sum_p\sum_{1\le n,p^n\le x}{}'(\alpha_p^n+\beta_p^n)\log p$$

and $-h_{69}(x)+g(x)-0.3156$ for $1.1\le x\le 20$. (We have used "Mathematica" to make Figure 7.1.) The coincidence seems fine.

§8. Sample programs

In this section, we shall present a few sample programs to compute $L(s,f)$, $f\in S_8(\Gamma_0(2))$ (cf. §3). All programs, which are ready to be executed, are written in UBASIC created by Y. Kida. Let

$$f(z)=(\eta(z)\eta(2z))^8=\sum_{n=1}^\infty a_nq^n.$$

Using Program A, we can compute a_n for $1\le n\le M$ for any M, $1\le M\le 10^4$. From line 50 to 150, the coefficients $A(n)$ in $\eta(z)=q^{1/24}\sum_{n=1}^\infty A(n)q^{n-1}$ are computed for $1\le n\le M$ using Euler's formula

$$\eta(z)=q^{1/24}\prod_{n=1}^\infty(1-q^n)=q^{1/24}\sum_{n=-\infty}^\infty(-1)^nq^{n(3n+1)/2}.$$

From line 160 to 220, the coefficients $B(n)$ in $\eta(z)\eta(2z)=q^{1/8}\sum_{n=1}^\infty B(n)q^{n-1}$ are computed for $1\le n\le M$ by $B(I)=\sum_{J+2L=I+2}A(J)A(L)$. From line 270 to 390, the expansion of $\eta(z)\eta(2z)$ is raised to the eighth power; the final result will be stored in the data file "wt8 ".

Program B computes the value of $e^{i\vartheta_f(t)}L(4+it,f)$ for $t=100$. To save the space, this program gives the values of $e^{i\vartheta_f(t)}S_N^{(l)}$ for $N=2000$, $0\le l\le 40$. By point 15 command (line 20), UBASIC gives the precision to the 70-th digit. From line 300 to 580, the value of $e^{i\vartheta_f(t)}$ will be computed and stored in the variable THE. The calculation proceeds as follows. We have

$$\vartheta_f(t)=\arg(2^{s/2}(2\pi)^{-s}\Gamma(s))_{s=4+it}=(\frac{1}{2}\log 2-\log 2\pi)t+\arg\Gamma(4+it).$$

We also have

$$\arg\Gamma(z)=\arg\Gamma(z+1)-\arctan(\Im(z)/\Re(z)),\quad \Re(z)>0,\quad \arg\Gamma(z)=\Im(\log\Gamma(z)).$$

By these formulas, it suffices to compute $\log\Gamma(z+100)$ for $z=4+it$. We have the asymptotic expansion (cf. [WW], p. 252)

$$\log\Gamma(z)\sim(z-\frac{1}{2})\log z-z+\frac{1}{2}\log 2\pi+\sum_{r=1}^{\infty}\frac{(-1)^{r-1}B_r}{2r(2r-1)z^{2r-1}},\tag{8.1}$$

where B_r denotes the r-th Bernoulli number. In (8.1), we use the terms up to $r=10$. We can get an approximation of $e^{i\vartheta_f(t)}$ which is accurate at least to the 40-th digit (cf. [WW], p. 252).

From line 660 to 740, the values of

$$S_N^{(0)}=\sum_{n=1}^{N}a_n n^{-s},\quad s_n^{(l)}=\sum_{n=1}^{N}s_n^{(l-1)},\quad 1\le l\le 40,\quad (s_n^{(0)}=a_n)$$

are computed. From line 770 to 1020, the values of $S_N^{(l)}$ are computed for $1\le l\le 40$ using (1.10) and (1.11). We use the approximation

$$u_N^{(l)}=N^{-s}\sum_{k=l}^{100}(\sum_{m=1}^{l}(-1)^m\binom{l}{m}m^k)(-1)^k\frac{s(s+1)\cdots(s+k-1)}{k!}N^{-k}.$$

The error from the truncation by 100 is negligible (cf. §1). The variable $U(K)$ stands for $(-1)^k\frac{s(s+1)\cdots(s+k-1)}{k!}$; the variable Co stands for $\sum_{m=1}^{l}(-1)^m\binom{l}{m}m^k$; the variable $T(I)$ stands for

$$\sum_{k=l}^{100}(\sum_{m=1}^{l}(-1)^m\binom{l}{m}m^k)(-1)^k\frac{s(s+1)\cdots(s+k-1)}{k!}N^{-k}.$$

In the line 1000, $X=S_N^{(I)}$ is multiplied by $e^{i\vartheta_f(t)}$.

Program A

```
10 word 8
20 point 2
30 dim A(10000),B(10000)
40 input M
50 A1=sqrt(24*M+1)
60 M1=int((A1+1)/6)
70 A(1)=1
80 for I=1 to M1
90 I1=I−2∗int(I/2)
100 I2=1−2∗I1
110 J=int((3∗I∗I+I)/2)+1
120 A(J)=A(J)+I2
130 J=int((3∗I∗I−I)/2)+1
140 A(J)=A(J)+I2
150 next I
160 for I=1 to M
170 I1=int((I+1)/2)
180 for L=1 to I1
190 J=I−2∗L+2
200 B(I)=B(I)+A(J)∗A(L)
210 next L
220 next I
230 for I=1 to M
240 A(I)=B(I)
250 B(I)=0
260 next I
270 K=1
280 for I=1 to M
290 print K,I
300 for J=1 to I
310 B(I)=B(I)+A(J)∗A(I+1−J)
320 next J
330 next I
340 for I=1 to M
350 A(I)=B(I)
360 B(I)=0
370 next I
380 K=K+1
390 if K<4 then goto 280
400 open ”wt8” for output as ♯1
410 for I=1 to M
420 print I,A(I)
430 print ♯1,A(I)
440 next I
450 close ♯1
```

```
460 end
```

Program B

```
10 word 70
20 point 15
30 dim Bn(20),Bd(20),C(2000),T(100),U(200),Sm(50)
40 Ab=40
50 Bn(1)=1
60 Bd(1)=6
70 Bn(2)=1
80 Bd(2)=30
90 Bn(3)=1
100 Bd(3)=42
110 Bn(4)=1
120 Bd(4)=30
130 Bn(5)=5
140 Bd(5)=66
150 Bn(6)=691
160 Bd(6)=2730
170 Bn(7)=7
180 Bd(7)=6
190 Bn(8)=3617
200 Bd(8)=510
210 Bn(9)=43867
220 Bd(9)=798
230 Bn(10)=174611
240 Bd(10)=330
250 A=1/sqrt(3)
260 P=6*atan(A)
270 S=4+100*♯i
280 Ss=S
290 T=im(S)
300 Th=T*(log(2)/2−log(2*P))
310 U=0
320 for I=1 to 100
330 S1=re(S)
340 S2=im(S)
350 if S1>S2 goto 380
360 U=U−(P/2)+atan(S1/S2)
370 goto 390
380 U=U−atan(S2/S1)
390 S=S+1
400 next I
410 Th=Th+U
420 T=im(S)
430 R=re(S)
```

```
440 X1=atan(T/R)
450 X2=(log(R*R+T*T))/2
460 X3=X2+X1*♯i
470 X4=(S-(1/2))*X3
480 Th=Th+im(X4)-T
490 K=10
500 S1=S
510 for I=1 to K
520 X1=Bn(I)/(2*I*(2*I-1)*Bd(I)*S1)
530 Th=Th+im(X1)
540 S1=-S1*S*S
550 next I
560 X1=int(Th/(2*P))
570 X2=Th-2*P*X1
580 The=exp(X2*♯i)
590 S=Ss
600 X=0
610 open "wt8" for input as ♯1
620 for M=1 to 2000
630 input ♯1,C(M)
640 next M
650 close ♯1
660 for M=1 to 2000
670 Sm(1)=Sm(1)+C(M)
680 for I=2 to Ab
690 Sm(I)=Sm(I)+Sm(I-1)
700 next I
710 X1=log(M)
720 X2=exp(-X1*S)
730 X=X+C(M)*X2
740 next M
750 Z=The*X
760 print Z
770 N=2001
780 A=log(N)
790 N1=exp(-A*S)
800 U(1)=-S
810 for K=2 to 100
820 U(K)=-U(K-1)*(S+K-1)/K
830 next K
840 for I=1 to Ab
850 T(I)=0
860 if I>1 then goto 890
870 X=X-Sm(1)*N1
880 goto 1000
890 Ii=I-1
900 for K=Ii to 100
```

```
 910 Co=-Ii
 920 A1=-Ii
 930 for L=2 to Ii
 940 A1=-A1*(Ii-L+1)/L
 950 Co=Co+(A1*(L∧K))
 960 next L
 970 T(I)=T(I)+Co*U(K)/(N∧K)
 980 next K
 990 X=X-Sm(I)*N1*T(I)
1000 Z=The*X
1010 print I,Ss,Z
1020 next I
1030 end
```

§9. Conjectures

In this section, we shall discuss a few conjectures which emerged during the process of experiments: No non-trivial coincidences of zeros of two L-functions attached to two non-equivalent irreducible λ-adic representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ are found. We shall use the framework of automorphic representations of $GL(n, \mathbf{Q}_A)$ to formulate this fact in more general case.

Let π be an irreducible unitary cuspidal automorphic representation of $GL(n, \mathbf{Q}_A)$. The contragredient representation $\tilde{\pi}$ to π is equivalent to the complex conjugate representation $\bar{\pi}$ of π and we have the functional equations:

$$L(s, \pi) = \epsilon(s, \pi) L(1 - s, \bar{\pi}), \qquad L(s, \bar{\pi}) = \overline{L(\bar{s}, \pi)}.$$

Let ω_π be the central character of π. For $s \in \mathbf{C}$, set $\nu^s(x) = |x|_A^s$, $x \in \mathbf{Q}_A^\times$ where $|x|_A$ denotes the idele norm of x. We can find a $t \in \mathbf{R}$ so that $\omega_\pi \nu^{it}$ is a character of $\mathbf{Q}_A^\times$ of finite order. Since $\omega_\pi \nu^{it}$ is the central character of $\pi \otimes (\nu^{it/n} \circ \det)$ and $L(s + \frac{it}{n}, \pi) = L(s, \pi \otimes (\nu^{it/n} \circ \det))$, we may assume, without losing substantial generality, that ω_π is of finite order.

Conjecture 9.1. *Let π_1 and π_2 be irreducible unitary cuspidal automorphic representations of $GL(n, \mathbf{Q}_A)$ with the central characters ω_{π_1} and ω_{π_2}. We assume that π_1 is not equivalent to π_2 and that ω_{π_1} and ω_{π_2} are of finite order. Then $L(s, \pi_1)$ and $L(s, \pi_2)$ have no common zeros in the critical strip $0 < \Re(s) < 1$ except for $s = 1/2$.*

Remark. If we replace $\mathbf{Q}$ by an algebraic number field, the assertion is obviously false.

As a variant of 9.1, we can formulate a conjecture on L-functions of motives. Let E be an algebraic number field of finite degree. Let M_1 and M_2 be motives over $\mathbf{Q}$ with coefficients in E of pure weights w_1 and w_2, of ranks n_1 and n_2 respectively. We assume $w_1 = w_2$ and put $w = w_1$. Fix an embedding σ of E into $\mathbf{C}$ and let $L(s, M_1)$ (resp. $L(s, M_2)$) be the L-function of M_1 (resp. M_2) with respect to σ. For a finite place λ of E, let $\rho_i : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow GL(n_i, E_\lambda)$, $i = 1, 2$ be the λ-adic representation obtained from the λ-adic realization of M_i. We assume meromorphic continuation of $L(s, M_i)$, $i = 1, 2$ to the whole complex plane.

Conjecture 9.2. *Assume that ρ_i, $i = 1, 2$ are absolutely irreducible and that ρ_1 is not equivalent to ρ_2 for a finite place λ of E. Then $L(s, M_1)$ and $L(s, M_2)$ have no common zeros in the critical strip $\frac{w}{2} < \Re(s) < \frac{w}{2} + 1$ except for $s = (w+1)/2$.*

Remark. We understand that a pole of order k is a zero of order $-k$. Conjecture 9.2 implies the (usual) Artin conjecture except for $s = 1/2$.

Remark. The condition on ρ_i implies that M_1 and M_2 are simple motives with coefficients in $\overline{\mathbf{Q}}$. D. Blasius observed that the analogous conjecture for motives over a finite field is true under the Tate conjecture (cf. Milne [M], p. 415, Proposition 2.6).

Table 2.1

N	I_0	R_0	I_5	R_5	I_{10}	R_{10}
1000	3.95	-9.3×10^{-2}	3.9415443	1.3×10^{-3}	3.9237277	3.0×10^{-3}
2000	4.02	3.3×10^{-2}	3.9248413	4.6×10^{-3}	3.9238190	-1.1×10^{-5}
4000	3.90	1.0×10^{-1}	3.9233635	-2.1×10^{-4}	3.9242209	-1.8×10^{-5}
6000	3.88	-3.9×10^{-3}	3.9239218	-3.6×10^{-4}	3.9242178	-3.7×10^{-6}
8000	3.85	-1.4×10^{-2}	3.9245956	6.6×10^{-5}	3.9241989	5.7×10^{-6}
10000	3.92	2.5×10^{-2}	3.9240769	-8.5×10^{-7}	3.9242054	-3.2×10^{-7}

N	I_{15}	R_{15}	I_{20}	R_{20}
1000	3.92361007	-1.3×10^{-4}	3.924178604	-4.1×10^{-4}
2000	3.92423322	-3.8×10^{-5}	3.924207205	4.9×10^{-6}
4000	3.92420427	1.8×10^{-6}	3.924203509	-1.8×10^{-7}
6000	3.92420338	4.6×10^{-7}	3.924203748	-2.9×10^{-8}
8000	3.92420375	-1.9×10^{-7}	3.924203739	7.2×10^{-9}
10000	3.92420370	2.4×10^{-8}	3.924203738	-1.0×10^{-9}

N	I_{25}	R_{25}	I_{30}	R_{30}
1000	3.92438069843	8.8×10^{-6}	3.92417001558	2.0×10^{-4}
2000	3.92419997465	1.2×10^{-6}	3.92420393248	-9.9×10^{-7}
4000	3.92420378031	2.6×10^{-8}	3.92420373301	3.0×10^{-9}
6000	3.92420373864	1.5×10^{-9}	3.92420373759	-1.7×10^{-10}
8000	3.92420373800	-2.7×10^{-10}	3.92420373818	-8.9×10^{-13}
10000	3.92420373814	5.8×10^{-11}	3.92420373812	-1.0×10^{-11}

N	I_{35}	R_{35}
1000	3.923939180351	4.8×10^{-5}
2000	3.924204540724	8.0×10^{-7}
4000	3.924203740852	-3.7×10^{-9}
6000	3.924203738244	7.9×10^{-11}
8000	3.924203738121	-4.4×10^{-12}
10000	3.924203738135	2.1×10^{-12}

Table 2.2

n	t_n	n	t_n	n	t_n
1	0	2	4.1558656464	3	5.5491219562
4	8.1117756122	5	10.8952834492	6	12.0523651120
7	13.4542992617	8	14.9275108496	9	16.3036898019
10	17.7350625418	11	18.837088412	12	20.551890978
13	21.752187480	14	22.93715924	15	23.33859940
16	23.97767239	17	25.79365179	18	27.1212236
19	27.8904904	20	28.6462091	21	30.100668
22	30.884244	23	31.730116	24	32.248613
25	33.84677	26	34.08053	27	35.12990
28	36.04356	29	36.9637	30	38.2333
31	39.1512	32	39.7944		

Table 2.3

N	R_0	I_0	R_5	I_5	R_{10}	I_{10}
1000	-3.04	-7.7×10^{-2}	-2.9558	1.4×10^{-2}	-2.95033	9.3×10^{-4}
2000	-2.85	3.1×10^{-3}	-2.9629	-3.5×10^{-3}	-2.95651	5.4×10^{-4}
4000	-2.95	-9.7×10^{-3}	-2.9588	7.3×10^{-4}	-2.95642	-8.7×10^{-5}
6000	-2.95	6.3×10^{-2}	-2.9561	-1.4×10^{-3}	-2.95659	9.4×10^{-5}
8000	-2.98	6.9×10^{-2}	-2.9558	-9.9×10^{-4}	-2.95660	4.8×10^{-5}
10000	-2.97	1.8×10^{-2}	-2.9567	-2.3×10^{-4}	-2.95659	1.1×10^{-5}

N	R_{15}	I_{15}	R_{20}	I_{20}
1000	-2.956381	-2.8×10^{-3}	-2.9591449	-4.1×10^{-4}
2000	-2.956387	-3.6×10^{-4}	-2.9568292	-9.6×10^{-5}
4000	-2.956613	-2.3×10^{-5}	-2.9565854	4.1×10^{-6}
6000	-2.956592	-3.8×10^{-6}	-2.9565951	2.5×10^{-6}
8000	-2.956593	-1.7×10^{-6}	-2.9565940	6.7×10^{-7}
10000	-2.956593	-2.3×10^{-6}	-2.9565932	1.2×10^{-7}

N	R_{25}	I_{25}	R_{30}	I_{30}
1000	-2.9584634	3.5×10^{-3}	-2.95201141	3.7×10^{-3}
2000	-2.9566395	1.9×10^{-4}	-2.95638520	1.7×10^{-5}
4000	-2.9565937	-7.5×10^{-6}	-2.95659865	2.3×10^{-6}
6000	-2.9565925	1.3×10^{-7}	-2.95659358	-2.0×10^{-7}
8000	-2.9565932	-1.8×10^{-8}	-2.95659347	4.3×10^{-8}
10000	-2.9565934	-1.0×10^{-7}	-2.95659342	1.3×10^{-8}

N	R_{35}	I_{35}
1000	-2.951829903	1.3×10^{-4}
2000	-2.956587888	-1.9×10^{-4}
4000	-2.956589949	3.4×10^{-6}
6000	-2.956593420	1.6×10^{-7}
8000	-2.956593344	2.0×10^{-8}
10000	-2.956593405	-5.0×10^{-9}

Table 2.4

n	u_n	n	u_n	n	u_n	n	u_n
1	2.3864500	2	4.3752457	3	6.0435487	4	7.571907
5	8.841633	6	10.605890	7	11.437474	8	12.76622
9	13.76869	10	15.2075	11	15.6182	12	16.9663
13	18.0078	14	18.874				

Table 3.1

N	R_0	I_0	R_5	I_5	R_{10}	I_{10}
1000	-1.686	8.7×10^{-2}	-1.77582	1.6×10^{-2}	-1.782116715	1.7×10^{-3}
2000	-1.835	-2.9×10^{-2}	-1.78432	-1.4×10^{-3}	-1.783663963	-1.9×10^{-5}
4000	-1.758	-6.5×10^{-3}	-1.78364	-1.6×10^{-4}	-1.783642384	-1.0×10^{-6}
6000	-1.820	-5.2×10^{-3}	-1.78366	2.0×10^{-5}	-1.783642948	8.6×10^{-8}
8000	-1.760	-1.1×10^{-2}	-1.78364	-2.0×10^{-5}	-1.783642816	-2.7×10^{-8}
10000	-1.786	8.2×10^{-3}	-1.78363	1.1×10^{-5}	-1.783642826	6.5×10^{-9}

N	R_{15}	I_{15}	R_{20}	I_{20}
1000	-1.783149679719	9.4×10^{-5}	-1.7835293644621857	-2.4×10^{-5}
2000	-1.783644577360	-3.2×10^{-8}	-1.7836428890128260	8.1×10^{-9}
4000	-1.783642829247	-2.2×10^{-9}	-1.7836428272010046	-1.3×10^{-11}
6000	-1.783642827253	8.0×10^{-11}	-1.7836428271539760	4.5×10^{-13}
8000	-1.783642827162	-6.1×10^{-12}	-1.7836428271544534	-7.0×10^{-15}
10000	-1.783642827150	-1.4×10^{-12}	-1.7836428271544125	-8.3×10^{-16}

N	R_{25}	I_{25}
1000	$-1.783627194118125867\,2$	-1.3×10^{-5}
2000	-1.7836428277732280167	1.1×10^{-9}
4000	-1.7836428271546296417	1.4×10^{-15}
6000	-1.7836428271544148349	9.0×10^{-16}
8000	-1.7836428271544160390	2.0×10^{-18}
10000	-1.7836428271544160184	-6.2×10^{-19}

N	R_{30}	I_{30}
1000	$-1.7836430434865235424107$	-3.5×10^{-6}
2000	$-1.7836428270926205946250$	5.0×10^{-11}
4000	$-1.7836428271544153839298$	1.1×10^{-15}
6000	$-1.7836428271544160178686$	-1.6×10^{-18}
8000	$-1.7836428271544160181358$	2.7×10^{-20}
10000	$-1.7836428271544160181708$	-3.8×10^{-22}

N	R_{35}	I_{35}
1000	$-1.783643596072585309230549\,3$	-2.3×10^{-7}
2000	$-1.7836428271502571300252154$	-3.7×10^{-13}
4000	$-1.7836428271544160011102849$	4.2×10^{-18}
6000	$-1.7836428271544160181784305$	-8.5×10^{-21}
8000	$-1.7836428271544160181689502$	1.0×10^{-23}
10000	$-1.7836428271544160181690199$	2.9×10^{-25}

Table 3.2

N	R_0	I_0	R_5	I_5	R_{10}	I_{10}
1000	3.43	-3.9×10^{-1}	3.3563	-3.5×10^{-1}	3.2408605	-3.5×10^{-1}
2000	3.23	8.9×10^{-2}	3.1032	6.7×10^{-3}	3.0945490	2.3×10^{-4}
4000	3.10	2.6×10^{-2}	3.0906	-3.1×10^{-4}	3.0913811	2.3×10^{-6}
6000	3.12	4.1×10^{-2}	3.0912	3.7×10^{-4}	3.0914333	2.3×10^{-6}
8000	3.07	-4.2×10^{-3}	3.0914	-1.7×10^{-4}	3.0914340	-3.9×10^{-7}
10000	3.12	-1.8×10^{-2}	3.0914	-1.3×10^{-4}	3.0914342	-1.6×10^{-7}

N	R_{15}	I_{15}	R_{20}	I_{20}
1000	3.0854245832	-3.1×10^{-1}	2.923318726655	-2.4×10^{-1}
2000	3.0919697213	-3.7×10^{-5}	3.091486059342	-6.5×10^{-5}
4000	3.0914327625	-3.9×10^{-8}	3.091434224098	4.0×10^{-8}
6000	3.0914342708	6.1×10^{-8}	3.091434237097	5.7×10^{-10}
8000	3.0914342301	-3.7×10^{-9}	3.091434236714	-1.0×10^{-11}
10000	3.0914342356	-3.6×10^{-10}	3.091434236735	-1.3×10^{-12}

N	R_{25}	I_{25}	R_{30}	I_{30}
1000	2.80762049069122	-1.1×10^{-1}	2.78668118045678635	7.4×10^{-2}
2000	3.09143198592104	-2.2×10^{-5}	3.09143105396801472	-3.5×10^{-6}
4000	3.09143423720919	2.4×10^{-9}	3.09143423678556380	3.9×10^{-11}
6000	3.09143423674457	-5.1×10^{-12}	3.09143423673870285	-1.7×10^{-13}
8000	3.09143423673861	2.3×10^{-13}	3.09143423673865144	1.9×10^{-15}
10000	3.09143423673865	1.0×10^{-14}	3.09143423673865098	6.2×10^{-17}

N	R_{35}	I_{35}
1000	2.91398635132232829470	2.7×10^{-1}
2000	3.09143341359018088974	1.4×10^{-7}
4000	3.09143423673979921383	-2.4×10^{-12}
6000	3.09143423673865010310	-1.1×10^{-15}
8000	3.09143423673865095777	-1.9×10^{-18}
10000	3.09143423673865095240	-1.3×10^{-20}

Table 3.3

n	t_n	n	t_n	n	t_n
1	8.2720409199	2	11.3959869930	3	14.8616932015
4	17.1783243050	5	19.2124566315	6	20.8274294554
7	23.4659374198	8	25.2726883522	9	27.0035774491
10	28.1569222690	11	30.2145623343	12	31.6193141164
13	33.7856279775	14	34.9435854723	15	36.5559515067
16	37.6356026748	17	39.1608229256	18	40.6589300308
19	42.8804581030	20	43.2736012304	21	44.9765395474
22	46.4176568046	23	47.2517710599	24	48.7821808287
25	50.3519022325	26	51.5688981695	27	53.1356287828
28	54.0717837181	29	55.0990003336	30	56.4089955139
31	57.5391214415	32	59.1986375433	33	60.1739007171
34	61.6441827270	35	62.8146545420	36	63.4247884022
37	65.1023702197	38	66.0180646898	39	66.8050237006
40	68.6802278238	41	69.8132058342	42	70.7502185552
43	71.9861530156	44	72.7927328082	45	74.1137296216
46	74.8761895173	47	76.2796967025	48	77.4608764665
49	78.7319975717	50	79.5372511477	51	80.8499015926
52	81.9286308045	53	82.6995529553	54	83.4681179192
55	85.2402769759	56	85.6802121224	57	87.2830188249
58	88.5094955323	59	89.2377130355	60	90.0534073382
61	91.4472572430	62	92.0496894589	63	93.3566370961
64	94.2221147468	65	95.3044565474	66	96.6527715250
67	97.7264314003	68	98.4244540180	69	99.4730638315

Table 3.4

n	u_n	n	u_n	n	u_n
1	12.9399446108	2	15.1248640287	3	17.2775088490
4	21.9119654118	5	23.7124474310	6	27.6868648494
7	29.1470584255	8	31.1315265360	9	31.9862854000
10	33.6323734231	11	35.7361264638	12	38.1008875317
13	39.9548075690	14	41.3629251312	15	43.0030848131
16	43.7924301232	17	49.3874980802	18	50.3892911690
19	51.9883497256	20	53.3610715851	21	55.5058736308
22	57.1306190068	23	58.5145765119	24	59.2810504632
25	60.8114113807	26	61.7177742037	27	62.3299217969
28	65.1200148215	29	66.3871768599	30	67.7658152255
31	68.5636482576	32	70.0994795387	33	71.9139076205
34	73.4598285562	35	74.4219698604	36	75.9259426071
37	76.7219513797	38	80.2370604179	39	80.9795116625
40	82.2141987387	41	84.1686266809	42	85.4558525934
43	86.4596989555	44	87.5017176801	45	88.7989284271
46	90.8930274655	47	91.6094970880	48	93.0648834307
49	93.8455295242	50	94.4090447654	51	95.8637476000
52	96.5025102751	53	97.8219970593	54	98.9086789539

Table 3.5

n	ρ_n	n	ρ_n
1	$3.2308208282 + 8.9496290911\,i$	2	$3.0144204971 + 19.1670355895\,i$
3	$3.1880664988 + 26.3033849287\,i$	4	$3.1549639910 + 36.6242398231\,i$
5	$2.7150409653 + 45.1719799932\,i$	6	$2.4938210677 + 47.5816502442\,i$
7	$3.3624175212 + 54.4320525502\,i$	8	$2.9077749773 + 64.2513434784\,i$
9	$3.1556119321 + 71.9344926377\,i$	10	$2.4066868777 + 78.1144688947\,i$
11	$2.9102154501 + 82.3890698796\,i$	12	$2.8870784016 + 89.7875787849\,i$
13	$3.3672596002 + 99.9194124003\,i$	14	$2.7073645379 + 107.1592978688\,i$
15	$2.7721770492 + 110.2613188689\,i$	16	$3.1645227049 + 117.3896482956\,i$
17	$2.5547542302 + 126.5001914198\,i$	18	$2.8669588498 + 128.2070045099\,i$
19	$3.1444059195 + 135.3354942155\,i$	20	$3.3550056102 + 145.3938901873\,i$

Table 3.6

n	u_n^+	n	u_n^+	n	u_n^+
1	5.6185671952	2	9.3587692608	3	12.0264936925
4	13.7962357292	5	16.0894994874	6	17.7847280762
7	19.4575970308	8	21.8594962784	9	22.5758373316
10	24.1962387103	11	26.0225517432	12	27.3617234087
13	29.1281915898	14	29.9334498107	15	31.7848563053
16	33.0414393853	17	33.9453293541	18	35.7534640159
19	36.6630191145	20	38.2010379041	21	39.7144586843
22	40.6779144658	23	41.3507131813	24	43.0237415354
25	44.5137568744	26	45.2712575072	27	46.4744214908
28	47.8221666146	29	49.0105055856	30	49.9881789218
31	51.5155718913	32	51.6631984921	33	53.5375246718
34	54.4042145877	35	55.5750957955	36	58.8270009777
37	59.7418610167	38	61.2006909669	39	61.7440856002
40	63.2789136582	41	64.1010371646	42	64.8912913186
43	66.3286609040	44	67.5430514279	45	70.1770012608
46	71.4492522675	47	72.3027663172	48	73.1469069994
49	74.5424808567	50	75.2004527728	51	76.3094730233
52	77.0682809138	53	78.2931224271	54	79.2581319133
55	80.0330330946	56	81.1315223512	57	81.9704731391
58	83.0719604785	59	84.1964169499	60	85.2011952740
61	85.7778751490	62	86.9407183317	63	87.4479355411
64	88.7059787721	65	89.9368381341	66	90.6181312264
67	91.4583733416	68	92.7393858316	69	93.0881396596
70	94.5091774704	71	95.4769845208	72	96.1116129723
73	96.9317027531	74	97.8372552255	75	99.1010053953
76	99.8881597950				
n	u_n^-	n	u_n^-	n	u_n^-
1	0	2	24.4022873037	3	26.6418851276
4	29.3670678246	5	33.5747954436	6	35.2863556538
7	38.5418813017	8	40.0447318001	9	44.8539372315
10	46.8484465576	11	50.0699839799	12	52.1256566323
13	54.2746979473	14	55.4033486176	15	56.5807632829
16	58.8640673277	17	60.9985910184	18	63.8456398387
19	65.1978869599	20	71.4691922483	21	72.8978373808
22	75.0463737748	23	76.6579317410	24	78.6419254744
25	80.0462305996	26	83.8815457436	27	85.2027989008
28	86.6795992346	29	88.7015447955	30	90.5599843766
31	92.9081333908	32	94.5315054431	33	96.4777165784
34	97.7100184923	35	99.3454122963		

Table 3.7

n	ρ_n^+	n	ρ_n^+
1	$3.3591319232 + 57.3250633340\,i$	2	$3.5001503209 + 68.5679322965\,i$
n	ρ_n^-	n	ρ_n^-
1	$5.7692647648 + 8.9956889852\,i$	2	$4.8476735625 + 14.0858508094\,i$
3	$5.3846794177 + 18.2757545274\,i$	4	$4.2067408135 + 20.6821222248\,i$
5	$5.3861276711 + 26.6587658619\,i$	6	$4.5402410961 + 31.5997171480\,i$
7	$5.7862146670 + 36.6524415925\,i$	8	$4.3281324452 + 41.9810464020\,i$
9	$5.2171816339 + 44.9332286844\,i$	10	$4.7664043254 + 48.8240853600\,i$
11	$5.6527005509 + 54.2712676115\,i$	12	$4.6702375442 + 60.1227186672\,i$
13	$5.2007749392 + 63.8919812818\,i$	14	$3.7792244502 + 66.9023711945\,i$
15	$3.5790185110 + 69.3185823254\,i$	16	$5.7686816361 + 72.2279984677\,i$
17	$4.6599017691 + 76.8392451729\,i$	18	$5.3421407237 + 81.9810448650\,i$
19	$3.3026675429 + 82.2734272886\,i$	20	$3.9986981918 + 88.8368300771\,i$
21	$4.9626158230 + 90.6464501774\,i$	22	$4.7401362618 + 94.8547536615\,i$
23	$5.8412001878 + 99.7795574645\,i$		

Table 4.1a

N	R_0	I_0	R_1	I_1	R_2	I_2
1000	2.22	6.3×10^{-3}	2.1777	-1.1×10^{-2}	2.173977	2.6×10^{-4}
5000	2.14	-2.2×10^{-2}	2.1700	3.7×10^{-3}	2.173544	-1.2×10^{-4}
10000	2.18	-1.3×10^{-2}	2.1753	1.1×10^{-3}	2.173833	-1.0×10^{-4}
30000	2.19	-2.2×10^{-2}	2.1729	-6.9×10^{-4}	2.173759	-1.0×10^{-5}
100000	2.17	1.2×10^{-2}	2.1735	1.0×10^{-4}	2.173747	-4.7×10^{-6}

N	R_3	I_3	R_4	I_4	R_5	I_5
1000	2.167862	-6.2×10^{-3}	2.20484	-5.4×10^{-2}	2.49186	7.1×10^{-2}
5000	2.173880	-6.8×10^{-4}	2.17935	-1.6×10^{-4}	2.18302	2.7×10^{-2}
10000	2.173937	1.8×10^{-4}	2.17267	1.6×10^{-3}	2.16380	-2.8×10^{-3}
30000	2.173700	-2.3×10^{-6}	2.17367	-3.7×10^{-4}	2.17551	-9.3×10^{-4}
100000	2.173745	-5.9×10^{-6}	2.17378	-5.1×10^{-5}	2.17406	9.7×10^{-5}

N	R_6	I_6	R_7	I_7	R_8	I_8
1000	2.352	1.2	−1.442	2.2	−7.975	−6.1
5000	2.082	7.3×10^{-2}	1.835	-1.7×10^{-1}	2.195	−1.0
10000	2.170	-4.1×10^{-2}	2.293	-6.2×10^{-2}	2.480	2.2×10^{-1}
30000	2.179	5.3×10^{-3}	2.165	2.4×10^{-2}	2.101	1.0×10^{-2}
100000	2.173	1.3×10^{-3}	2.169	1.8×10^{-3}	2.164	-7.2×10^{-3}

Table 4.1b

N	R_0	I_0	R_1	I_1	R_2	I_2
1000	3.263	8.3×10^{-2}	3.3048	6.1×10^{-2}	3.28176	2.6×10^{-2}
5000	3.305	-1.1×10^{-2}	3.2708	-1.8×10^{-2}	3.26879	-8.6×10^{-4}
10000	3.245	6.4×10^{-3}	3.2612	-4.8×10^{-3}	3.26487	6.8×10^{-4}
30000	3.263	2.9×10^{-2}	3.2697	9.0×10^{-4}	3.26616	1.7×10^{-4}
100000	3.276	-8.2×10^{-3}	3.2665	5.8×10^{-4}	3.26610	4.9×10^{-5}

N	R_3	I_3	R_4	I_4	R_5	I_5
1000	3.38022	2.5×10^{-2}	3.524	2.2	−33.06	8.3
5000	3.26878	6.2×10^{-3}	3.080	7.2×10^{-2}	1.75	−2.9
10000	3.26486	-2.7×10^{-3}	3.329	-3.1×10^{-2}	3.88	1.0
30000	3.26672	-1.6×10^{-4}	3.271	1.2×10^{-2}	3.06	1.0×10^{-1}
100000	3.26609	6.8×10^{-5}	3.264	-1.3×10^{-3}	3.28	-3.1×10^{-2}

Table 4.2

n	t_n	n	t_n	n	t_n	n	t_n
1	10.2562	2	13.6866	3	15.9599	4	17.038
5	19.026	6	20.017	7	22.472	8	23.745
9	25.351	10	26.229	11	27.561	12	28.847
13	29.986						
−1	−3.12740	−2	−6.5577	−3	−8.8310	−4	−9.9092
−5	−11.8976	−6	−12.888	−7	−15.343	−8	−16.616
−9	−18.222	−10	−19.101	−11	−20.433	−12	−21.718
−13	−22.857	−14	−24.859	−15	−25.892	−16	−26.850
−17	−28.418	−18	−28.927				

Table 4.3

k	n	T	$N(T)$	k	n	T	$N(T)$
$\mathbf{Q}(\sqrt{2})$	1	100	84	$\mathbf{Q}(\sqrt{2})$	-1	100	93
$\mathbf{Q}(\sqrt{2})$	2	100	82	$\mathbf{Q}(\sqrt{2})$	-2	100	96
$\mathbf{Q}(\sqrt{2})$	3	100	81	$\mathbf{Q}(\sqrt{2})$	-3	100	98
$\mathbf{Q}(\sqrt{2})$	4	100	80	$\mathbf{Q}(\sqrt{2})$	-4	100	100
$\mathbf{Q}(\sqrt{2})$	5	100	78	$\mathbf{Q}(\sqrt{2})$	-5	100	101
$\mathbf{Q}(\sqrt{2})$	10	100	79	$\mathbf{Q}(\sqrt{2})$	-10	100	108
$\mathbf{Q}(\sqrt{5})$	1	100	75	$\mathbf{Q}(\sqrt{5})$	-1	100	88
$\mathbf{Q}(\sqrt{5})$	2	100	72	$\mathbf{Q}(\sqrt{5})$	-2	100	92
$\mathbf{Q}(\sqrt{5})$	3	100	71	$\mathbf{Q}(\sqrt{5})$	-3	100	95
$\mathbf{Q}(\sqrt{5})$	4	100	71	$\mathbf{Q}(\sqrt{5})$	-4	100	97
$\mathbf{Q}(\sqrt{5})$	5	100	71	$\mathbf{Q}(\sqrt{5})$	-5	100	99
$\mathbf{Q}(\sqrt{5})$	10	100	93	$\mathbf{Q}(\sqrt{5})$	-10	100	106
$\mathbf{Q}(\sqrt{19})$	1	50	51	$\mathbf{Q}(\sqrt{19})$	-1	50	52
$\mathbf{Q}(\sqrt{19})$	2	50	50	$\mathbf{Q}(\sqrt{19})$	-2	50	53
$\mathbf{Q}(\sqrt{29})$	1	100	107	$\mathbf{Q}(\sqrt{29})$	-1	100	112
$\mathbf{Q}(\sqrt{29})$	2	100	105	$\mathbf{Q}(\sqrt{29})$	-2	100	114
$\mathbf{Q}(\sqrt{29})$	3	100	104	$\mathbf{Q}(\sqrt{29})$	-3	100	116
$\mathbf{Q}(\sqrt{29})$	4	100	102	$\mathbf{Q}(\sqrt{29})$	-4	100	117
$\mathbf{Q}(\sqrt{29})$	5	100	102	$\mathbf{Q}(\sqrt{29})$	-5	100	118
$\mathbf{Q}(\sqrt{29})$	10	100	99	$\mathbf{Q}(\sqrt{29})$	-10	100	123
$\mathbf{Q}(\sqrt{31})$	1	40	41	$\mathbf{Q}(\sqrt{31})$	-1	40	42
$\mathbf{Q}(\sqrt{67})$	1	30	32	$\mathbf{Q}(\sqrt{67})$	-1	30	33

Table 5.1

N	R_0	I_0	R_2	I_2	R_4	I_4
1000	3.09	5.3×10^{-1}	3.334	4.5×10^{-1}	3.483	4.9×10^{-1}
5000	3.31	-5.5×10^{-2}	3.317	-2.4×10^{-2}	3.335	-3.2×10^{-3}
10000	3.42	1.5×10^{-1}	3.417	-7.8×10^{-3}	3.389	-1.3×10^{-2}
30000	3.45	-1.4×10^{-1}	3.385	-2.3×10^{-2}	3.380	-6.2×10^{-3}
100000	3.25	-4.8×10^{-2}	3.375	1.6×10^{-3}	3.383	1.2×10^{-3}

N	R_6	I_6	R_8	I_8	R_{10}	I_{10}
1000	3.9634	6.4×10^{-1}	4.03999	6.2×10^{-1}	4.792707	5.2×10^{-1}
5000	3.3419	3.6×10^{-2}	3.38817	4.1×10^{-2}	3.400866	6.3×10^{-2}
10000	3.3825	-1.0×10^{-2}	3.37107	-4.5×10^{-3}	3.377550	4.8×10^{-3}
30000	3.3803	1.9×10^{-4}	3.38367	1.4×10^{-3}	3.383935	3.6×10^{-4}
100000	3.3839	1.2×10^{-4}	3.38369	-8.5×10^{-5}	3.383657	-8.2×10^{-6}

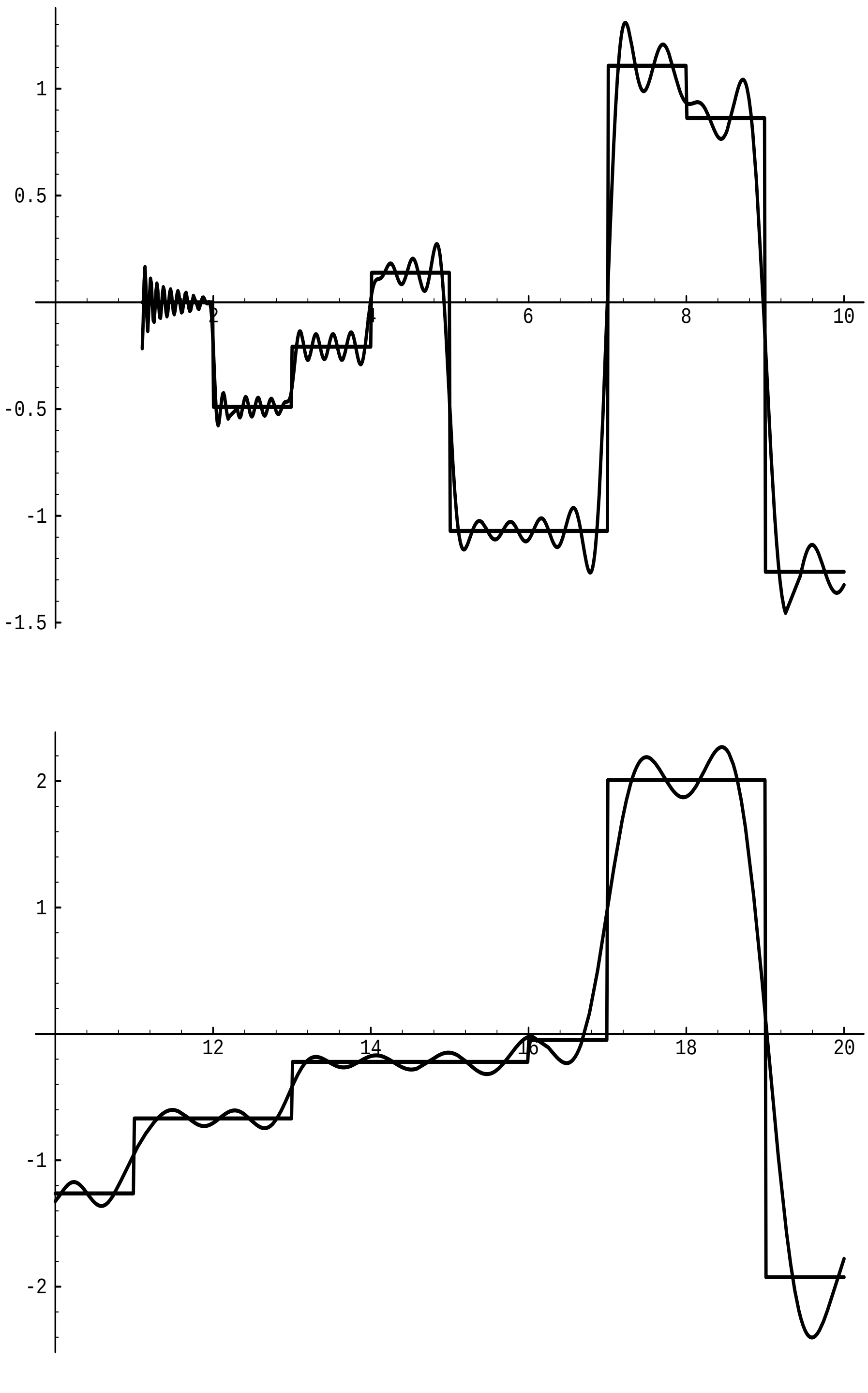
Table 5.2

n	u_n	n	u_n	n	u_n	n	u_n	n	u_n
1	2.79373	2	4.0887	3	5.362	4	5.887	5	7.03
6	7.46	7	7.90	8	8.8	9	9.6		

Table 6.1

N	$r_N^{(1)}$	$r_N^{(2)}$	$r_N^{(3)}$	$r_N^{(4)}$	N	$r_N^{(5)}$	N	$r_N^{(6)}$
5	0.96	0.98	0.93	0.92	0	1.03	0	0.98
10	0.95	1.05	1.12	0.84	2	0.82	1	0.99
15	1.07	1.06	0.85	1.03	4	0.81	2	1.01
20	1.07	1.09	1.23	1.07	6	1.19	3	0.99
25	1.06	0.90	1.25	0.87	8	1.39	4	1.01
30	0.88	0.88	0.80	1.22	10	0.67	5	0.94

FIGURE 7.1



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