R-GROUPS AND ELLIPTIC REPRESENTATIONS FOR SIMILITUDE GROUPS

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Abstract. The tempered spectrum of the similitude groups of non-degenerate symplectic, hermitian, or split orthogonal forms defined over p-adic groups of characteristic zero is studied. The components of representations induced from discrete series of proper parabolic subgroups are classified in terms of R-groups. Multiplicity one is proved. The tempered elliptic spectrum is identified, and the relation between elliptic characters appearing in a given induced representation is determined. Those irreducible tempered representations which are not elliptic and not fully induced from elliptic tempered representations are described.

Introduction. We continue our program of studying the explicit description of the tempered spectrum of reductive groups defined over a p-adic field of characteristic zero. The problem can be broken down into two steps. The first step is to classify the discrete series representations of G and of all its Levi subgroups. The second step is to determine the components of those representations which are parabolically induced from a discrete series representation of a proper parabolic subgroup P of G. It is this second step that we shall concern ourselves with. This problem also has two distinct parts. The first is to determine the criteria for an induced representation to be reducible when P is a maximal proper parabolic subgroup. This involves the computation of Plancherel measures. The second part is to use knowledge of the rank one Plancherel measures to construct the Knapp-Stein R-group. This gives one a combinatorial algorithm for determining the structure of the induced representations. Those irreducible tempered representations whose characters fail to vanish on the regular elliptic set are called elliptic tempered representations, and are of particular interest.

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In [**6**,**7**,**10**] we determined the possible R-groups for Sp_{2n}, SO_n, U_n, and SL_n. Here we built upon the work of Keys [**18**,**19**]. Arthur gives a criteria for determining the elliptic tempered representations in terms of R-groups in [**2**]. In [**15**] Herb described the elliptic tempered representations for Sp_{2n} and SO_n. For SO_{2n} she exhibited irreducible tempered representations which are neither elliptic nor irreducibly induced from an elliptic tempered representation of a proper parabolic subgroup. Such representations do not exist for $F=\mathbb{R}$, [**21**], and this is the first known example of them in the p-adic case. Using some results of Herb and Arthur, we described the elliptic representations for SL_n and the quasi-split unitary groups U_n [**6**,**10**]. The problem of determining the zeros of Plancherel measures has been explored by Shahidi in [**24**,**25**,**26**,**27**,**28**,**29**]. In [**29**] Shahidi determined the reducibility criteria for the Siegel parabolic subgroups of Sp_{2n} and SO_n. In [**24**] he determined the criteria for parabolics of SO_{2n} whose Levi component is of the form $GL_k\times SO_{2m}$ where k is even. In [**9**] we determined the criteria for the Siegel parabolics of U_n. Our joint work with Shahidi [**12**] has determined this criteria for symplectic and quasi-split orthogonal groups in the case where $\mathbf{M}=GL_k\times\mathbf{G}(m)$, with k even. Our continuing joint work with Shahidi will will examine other maximal parabolic subgroups of classical groups.

Here we compute the possible R-groups for the similitude group of a non-degenerate symplectic, symmetric, or quasi-split hermitian form, defined over F. We show that all the R-groups are elementary two groups (cf. Theorem 2.6). Moreover, the R-groups are all contained in the subgroup of the Weyl group consisting of sign changes. For GU_{2n} this extends the work of Keys [**19**]. We then undertake the computation of the elliptic tempered representations. Suppose $\mathbf{M}$ is a Levi subgroup of $\mathbf{G}$. Then, for some collection of positive integers $m_1,\ldots,m_r$, and some $m\geq 0$, we have $M=\mathbf{M}(F)\simeq GL_{m_1}\times\cdots\times GL_{m_r}\times G(m)$, where $G(m)=GSp_{2m}$, GU_{2m}, GO_{2m+1}, GU_{2m+1}, or GO_{2m} as appropriate (cf. Section 2). If σ is a discrete series representation of M, then the induced representation $i_{G,M}(\sigma)$ has elliptic constituents if and only if the longest possible sign change $w_0=C_1\ldots C_r$ is in $R(\sigma)$ (cf. Proposition 3.3). We make this more explicit on a case by case basis (cf. Theorems 3.4 and 3.6). If $\mathbf{G}=GO_{2n+1}$ or GU_{2n+1}, then every irreducible tempered representation of $G = \mathbf{G}(F)$ is either elliptic, or is irreducibly induced from an elliptic tempered representation of a proper Levi subgroup (cf. Theorem 3.4). For $\mathbf{G} = GU_{2n}$, GSp_{2n}, or GO_{2n}, this statement fails to be true. We classify those irreducible tempered representations which are non-elliptic, and are not irreducibly induced from elliptic tempered representations (cf. Theorems 3.4 and 3.6).

We also show that when an induced representation has elliptic components, then the elliptic characters of those components satisfy a particularly nice relation. More specifically, suppose κ is an irreducible representation of the R-group R, and π_κ is the irreducible subrepresentation of $i_{G,M}(\sigma)$ attached to κ [**2**,**19**]. Let Θ^e_κ be the restriction of the character of π_κ to the regular elliptic set. Then $\Theta^e_\kappa = \kappa(C_1 \ldots C_r)\Theta^e_1$, where 1 is the trivial character (cf. Theorem 3.9). This result is similar to the ones obtained by Herb in [**15**], and is in fact motivated by that work.

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Section 1 Preliminaries.

Suppose F is a nonarchimedean local field of characteristic zero and residual characteristic q. Let $\mathbf{G}$ be a connected reductive quasi–split algebraic group defined over F. We denote by G the F-rational points of $\mathbf{G}$. We let G^e be the collection of regular elliptic elements of G, i.e., the set of regular elements whose centralizers in G are compact modulo the center of G.

We denote by $\mathcal{E}_c(G)$ the collection of equivalence classes of irreducible admissible representations of G. We make no distinction between an irreducible admissible representation π of G, and its class $[\pi]$ in $\mathcal{E}_c(G)$. If $\pi \in \mathcal{E}_c(G)$, then the distribution character Θ_π of π is given by a locally integrable function [**13**]. We let Θ^e_π be the restriction of Θ_π to G^e. We say that π is elliptic if $\Theta^e_\pi \neq 0$. We let $\mathcal{E}_2(G)$, $\mathcal{E}_t(G)$, and $\mathcal{E}_e(G)$ be the collection of discrete series, tempered, and tempered elliptic classes in $\mathcal{E}_c(G)$, respectively. Then $\mathcal{E}_2(G) \subset \mathcal{E}_e(G) \subset \mathcal{E}_t(G)$.

We fix a maximal torus $\mathbf{T}$ of $\mathbf{G}$, and let $\mathbf{T}_d$ be the maximal split subtorus of $\mathbf{T}$. Denote by $\Phi = \Phi(\mathbf{G}, \mathbf{T}_d)$ the set of roots of $\mathbf{T}_d$ in $\mathbf{G}$. Let Δ be a choice of simple roots in Φ, and let $\Phi^+ = \Phi^+(\mathbf{G}, \mathbf{T}_d)$ be the positive roots with respect to this choice of Δ. The choice of Δ also determines a Borel subgroup $\mathbf{B} = \mathbf{TU}$ of $\mathbf{G}$. If $\theta \subseteq \Delta$, then we let $\mathbf{A}_\theta$ be the split subtorus of $\mathbf{T}_d$ determined by θ. Then $\mathbf{A}_\theta$ is the split component of a unique parabolic subgroup $\mathbf{P}_\theta = \mathbf{M}_\theta \mathbf{N}_\theta$ of $\mathbf{G}$ containing $\mathbf{B}$.

Suppose $\mathbf{A} = \mathbf{A}_\theta$ for some θ. Then $\mathbf{M} = \mathbf{M}_\theta = Z_{\mathbf{G}}(\mathbf{A})$ is a Levi subgroup of $\mathbf{G}$ with split component $\mathbf{A}$. We denote by $W(\mathbf{G}, \mathbf{A})$ the Weyl group $N_{\mathbf{G}}(\mathbf{A})/\mathbf{M}$. If $w \in W(\mathbf{G}, \mathbf{A})$, we let $\tilde{w} \in N_{\mathbf{G}}(\mathbf{A})$ be a representative for w.

Suppose $(\sigma, V) \in \mathcal{E}_2(M)$. Let

$$V(\sigma) = \{f \in C^\infty(G, V) | f(mng) = \sigma(m)\delta_P^{1/2}(m)f(g),\ \forall m \in M,\ n \in N,\ g \in G\}.$$

Here δ_P denotes the modular function of P. The unitarily induced representation, $\mathrm{Ind}_P^G(\sigma)$, of G on $V(\sigma)$ is given by right translation. Since the class of $\mathrm{Ind}_P^G(\sigma)$ depends only on $\mathbf{M}$ and not on the choice of $\mathbf{N}$, we may denote its class by $i_{G,M}(\sigma)$. We now recall the theory of R-groups which allows us to determine the structure of $\mathrm{Ind}_P^G(\sigma)$. For more details see [**7**,**11**,**18**,**25**,**30**,**31**]. For $\sigma \in \mathcal{E}_2(M)$, we let $W(\sigma) = \{w \in W(\mathbf{G}, \mathbf{A}) | \tilde{w}\sigma = \sigma\}$. Here $\tilde{w}\sigma(m) = \sigma(\tilde{w}^{-1}m\tilde{w})$. Since the class of $\tilde{w}\sigma$ is independent of the choice of $\tilde{w}$, we may denote it by $w\sigma$. Let $\Phi(\mathbf{P}, \mathbf{A})$ be the reduced roots of $\mathbf{A}$ in $\mathbf{P}$. For $\beta \in \Phi(\mathbf{P}, \mathbf{A})$, we let $\mathbf{A}_\beta$ be the subtorus of $\mathbf{A}$ defined by β. Let $\mathbf{M}_\beta = Z_{\mathbf{G}}(\mathbf{A}_\beta)$. Then $\mathbf{M}$ is a maximal Levi subgroup of $\mathbf{M}_\beta$. Let $\mathbf{N}_\beta = \mathbf{M}_\beta \cap \mathbf{N}$, and ${}^*\mathbf{P}_\beta = \mathbf{M}\mathbf{N}_\beta$. Then ${}^*\mathbf{P}_\beta$ is a maximal parabolic subgroup of $\mathbf{M}_\beta$. Let $\mu_\beta(\sigma)$ be the Plancherel measure of σ with respect to β [**14**,**31**]. The value of μ_β is given by ratios of certain values of Langlands L-functions [**28**]. For our purposes, it is enough to know that $\mu_\beta(\sigma) = 0$ if and only if $W(\mathbf{M}_\beta, \mathbf{A}) \cap W(\sigma) \neq \{1\}$ and $i_{M_\beta,M}(\sigma)$ is irreducible. Let $\Delta' = \{\beta \in \Phi(\mathbf{P}, \mathbf{A}) | \mu_\beta(\sigma) = 0\}$. Then $\pm\Delta'$ is a subroot system of Φ [**5**, VI,

§2, Proposition 9]. Thus, the group W' generated by the reflections determined by the elements of Δ' is a subgroup of $W(\sigma)$. Let $R = R(\sigma) = \{w \in W(\sigma) | \ w\beta > 0, \ \forall \beta \in \Delta'\}$.

Theorem 1.1 (Knapp–Stein, Silberger [20,30,32]). *For any $\sigma \in \mathcal{E}_2(M)$ we have $W(\sigma) = R \ltimes W'$. Moreover, the commuting algebra $C(\sigma)$ of $\mathrm{Ind}_P^G(\sigma)$ has dimension $|R|$.* $\square$

We call R the Knapp–Stein R-group attached to σ. Let $w \in W(\sigma)$, and choose $T_w : V \to V$ which defines an isomorphism between σ and $\tilde{w}\sigma$. Let $w_1, w_2 \in R$, and define $\eta(w_1, w_2)$ by $T_{w_1 w_2} = \eta(w_1, w_2) T_{w_1} T_{w_2}$. Then $\eta\colon R \times R \to \mathbb{C}^\times$ is a 2–cocycle.

Theorem 1.2 (Keys [19]).

(a) *The commuting algebra $C(\sigma)$ of $\mathrm{Ind}_P^G(\sigma)$ is isomorphic to $\mathbb{C}[R]_\eta$, the group algebra of R twisted by the cocycle η.*

(b) *Suppose η is a coboundary. Let $\hat{R}$ be the set of equivalence classes of irreducible representations of R. Then there is a natural one-to-one correspondence, $\kappa \longmapsto \pi_\kappa$, between $\hat{R}$ and the equivalence classes of subrepresentations of $i_{G,M}(\sigma)$ such that $\dim \kappa = \dim \mathrm{Hom}_G(\pi_\kappa, i_{G,M}(\sigma))$. In particular, if R is abelian and η is a coboundary, then $C(\sigma) \simeq \mathbb{C}[R]$, and $i_{G,M}(\sigma)$ decomposes as $|R|$ inequivalent irreducible subrepresentations.* $\square$

Let $\mathfrak{a} = \mathrm{Hom}(X(\mathbf{M})_F, \mathbb{Z})$ be the real Lie algebra of $\mathbf{A}$. (Here $X(\mathbf{M})_F$ is the collection of F-rational characters of $\mathbf{M}$.) For $w \in W(\mathbf{G}, \mathbf{A})$, we let $\mathfrak{a}_w = \{H \in \mathfrak{a} | w \cdot H = H\}$. Let $\mathbf{Z}$ be the split component of $\mathbf{G}$, and let $\mathfrak{z}$ be its real Lie algebra. Suppose $\sigma \in \mathcal{E}_2(M)$, and let R be the R-group of σ. We let $\mathfrak{a}_R = \bigcap_{w \in R} \mathfrak{a}_w$. In [**2**] Arthur gives an explicit criteria determining when $i_{G,M}(\sigma)$ has an elliptic subrepresentation. We give a weak version of his results which is sufficient for our needs.

Theorem 1.3 (Arthur [2]). *Suppose R is abelian and η is a coboundary.*

Then the following are equivalent:

(a) $i_{G,M}(\sigma)$ *has an elliptic constituent;*

(b) *all the constituents of* $i_{G,M}(\sigma)$ *are elliptic;*

(c) *there is a* $w \in R$ *with* $\mathfrak{a}_w = \mathfrak{z}$. $\square$

We are also interested in which tempered representations are not elliptic, and whether they appear as irreducibly induced from tempered representations. If a tempered representation is irreducibly induced, we wish to know when the inducing representation is elliptic. The next two results allow us to determine these things. We again state a weaker version of Arthur's result (Theorem 1.4) than the one given in [**2**].

Suppose $\mathbf{M}' \supset \mathbf{M}$ is a Levi subgroup of G satisfying Arthur's compatibility condition with respect to Δ' [**2**, §2]. Then $R' = R \cap W(\mathbf{M}', \mathbf{A})$ is the R-group attached to $i_{M',M}(\sigma)$. If $\kappa' \in \hat{R}'$, then we let $\tau_{\kappa'}$ to be the irreducible component of $i_{M',M}(\sigma)$ corresponding to κ'. Let

$$\hat{R}(\kappa') = \{\kappa \in \hat{R} \mid \kappa(w) = \kappa'(w), \forall w \in R'\}.$$

We let π_κ be the irreducible constituent of $i_{G,M}(\sigma)$ corresponding to κ.

Theorem 1.4 (Arthur [2, §2]). *Suppose that* R *is abelian and* $C(\sigma) \simeq \mathbb{C}[R]$. *Then, for any* $\kappa' \in \hat{R}'$, *we have*

$$i_{G,M'}(\tau_{\kappa'}) = \bigoplus_{\kappa \in \hat{R}(\kappa')} \pi_\kappa. \quad \square$$

Proposition 1.5 (Herb [15]). *Suppose* R *is abelian and* $C(\sigma) \simeq \mathbb{C}[R]$. *Let* π *be an irreducible constituent of* $i_{G,M}(\sigma)$. *Then* $\pi = i_{G,M'}(\tau)$ *for a proper Levi subgroup* $\mathbf{M}'$ *and some* $\tau \in \mathcal{E}_t(M')$, *if and only if* $\mathfrak{a}_R \neq \mathfrak{z}$. *Moreover,* $\mathbf{M}'$ *and* τ *can be chosen so that* τ *is elliptic if and only if there is a* $w_0 \in R$ *with* $\mathfrak{a}_R = \mathfrak{a}_{w_0}$. $\square$

Section 2. The Similitude Groups and their R-groups.

We now describe the possible R-groups that can arise when $\mathbf{G}$ is one of the following groups: GSp_{2n}, GO_n, or GU_n. Thus, $\mathbf{G}$ is the similitude group of a non–degenerate symplectic, symmetric, or hermitian form. In the last case we assume that the hermitian form defines the quasi–split unitary group of rank $[\frac{n}{2}]$, and we let E/F be the quadratic extension of F over which $\mathbf{G}$ splits. Denote by $x \mapsto \overline{x}$ the Galois automorphism of E over F. We let $N(x) = x\overline{x}$ be the norm map.

Let J be the form with respect to which $\mathbf{G}$ is defined. Then, if $\mathbf{G} = GSp_{2n}$ or GO_n,

$$\mathbf{G} = \{g \in GL_n |\ {}^t g J g = \lambda(g) J \text{ for some } \lambda(g) \in \mathbf{G}_m\}.$$

For the unitary similitude groups

$$\mathbf{G} = \{g \in \mathrm{Res}_{E/F}\, GL_n |\ {}^t\overline{g} J g = \lambda(g) J, \text{ for some } \lambda(g) \in \mathrm{Res}_{E/F}\, \mathbf{G}_m\}.$$

In each case we call λ the multiplier character of $\mathbf{G}$. If ω is a character of $F^\times$, then we also denote by ω the character of $\mathbf{G}(F)$ given by $\omega \circ \lambda$. We fix the following forms for the various $\mathbf{G}$. We let

$$w_\ell = \begin{pmatrix} & & & & & 1 \\ & & & & \cdot & \\ & & & \cdot & & \\ & & \cdot & & & \\ & 1 & & & & \\ 1 & & & & & \end{pmatrix} \in M_\ell(F).$$

Then we take $J = \begin{pmatrix} 0 & w_n \\ -w_n & 0 \end{pmatrix}$ if $\mathbf{G} = GSp_{2n}$. For $\mathbf{G} = GO_n$, we take $J = w_n$. For the unitary groups, we fix an element $y \in E$ with $\overline{y} = -y$. Then we let

$$J = y \begin{pmatrix} 0 & w_n \\ -w_n & 0 \end{pmatrix} \text{ if } \mathbf{G} = GU_{2n} \text{ and}$$

$$J = \begin{pmatrix} 0 & 0 & yw_n \\ 0 & 1 & 0 \\ y(-w_n) & 0 & 0 \end{pmatrix} \text{ if } \mathbf{G} = GU_{2n+1}.$$

We will often denote GSp_{2n}, GO_{2n}, GO_{2n+1}, GU_{2n}, or GO_{2n+1} by $\mathbf{G}(n)$. We formally let $\mathbf{G}(0) = GL_1$ if $\mathbf{G} \neq GU_{2n+1}$ and $\mathbf{G}(0) = \mathrm{Res}_{E/F}\, GL_1$ if $\mathbf{G} = GU_{2n+1}$.

Let $\mathbf{T}$ be the maximal torus of diagonal elements in $\mathbf{G}$. We given an explicit description of $T = \mathbf{T}(F)$ in each case. We adopt the convention of denoting a diagonal matrix by $\mathrm{diag}\{(x_1, \ldots, x_k)\}$. If $\mathbf{G} = GSp_{2n}$ or GO_{2n}, then

$$T = \{\mathrm{diag}\{x_1, \ldots, x_n, \lambda x_n^{-1}, \ldots, \lambda x_1^{-1}\} | x_i, \lambda \in F^\times\}. \tag{2.1}$$

If $\mathbf{G} = GO_{2n+1}$, then

$$T = \{\mathrm{diag}\{x_1, \ldots, x_n, \lambda, \lambda^2 x_n^{-1}, \ldots, \lambda^2 x_1^{-1}\} | x_i, \lambda \in F^\times\} \tag{2.2.}$$

If $\mathbf{G} = GU_{2n}$, then

$$T = \{\mathrm{diag}(x_1, \ldots, x_n, \lambda \overline{x}_n^{-1}, \ldots, \lambda \overline{x}_1^{-1}) | \lambda \in F^\times, x_i \in E^\times\}, \tag{2.3}$$

While if $\mathbf{G} = GU_{2n+1}$, then

$$T = \{\mathrm{diag}\{x_1, \ldots, x_n, \lambda, N(\lambda)\overline{x}_n^{-1}, \ldots, N(\lambda)\overline{x}_1^{-1}\} | x_i, \lambda \in E^\times\}. \tag{2.4}$$

In each case we let $\mathbf{T}_d$ be the maximal F split subtorus of $\mathbf{T}$. For $\mathbf{G} = GSp_{2n}$ or GO_n, we have $\mathbf{T} = \mathbf{T}_d$. For $\mathbf{G} = GU_{2n}$, T_d is given by (2.1), while T_d for GU_{2n+1} is given by (2.2). We sometimes denote an element of T by $t(x_1, \ldots, x_n, \lambda)$.

Note that in each case $W(\mathbf{G}, \mathbf{T}_d)$ is given by $S_n \ltimes \mathbb{Z}_2^n$. We use standard cycle notation for the elements of S_n. Thus,

$$(ij)\colon t(x_1, \ldots, x_n, \lambda) \longmapsto t(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n, \lambda).$$

We let c_i be the i-th sign change,

$$c_i\colon t(x_1, \ldots, x_n, \lambda) \longmapsto t(x_1, \ldots, x_{i-1}, \lambda x_i^{-1}, \ldots, x_n, \lambda).$$

We call any product of the c_i 's a sign change. We remark that for $\mathbf{G} = SO_{2n}$, only those sign changes which are products of an even number of c_i 's are in

$W(\mathbf{G},\mathbf{T}_d)$. However, if $\mathbf{G}=GO_{2n}$, then the sign changes c_i can be represented in $\mathbf{G}$. Finally we remark that $W(\mathbf{G},\mathbf{T}_d)=\langle (ij)\rangle \ltimes \langle c_i\rangle$.

Let Δ be the choice of simple roots of $\mathbf{T}_d$ in $\mathbf{G}$ for which the associated Borel subgroup consists of upper triangle matrices. Suppose $\theta\subset\Delta$. Let $\mathbf{P}=\mathbf{P}_\theta$ be the associated parabolic subgroup, and write $\mathbf{P}=\mathbf{MN}$ for the Levi decomposition of $\mathbf{P}$.

For some positive integers $m_1,\dots,m_r$, and some $m\geq 0$ we have,

$$M=\mathbf{M}(F)\simeq GL_{m_1}\times\ldots\times GL_{m_r}\times G(m), \tag{2.5}$$

where

$$GL_{m_i}=\begin{cases} GL_{m_i}(F), & \text{if } \mathbf{G}\neq GU_n;\\ GL_{m_i}(E), & \text{if } \mathbf{G}=GU_n.\end{cases}$$

In particular, we may assume that M consists of block diagonal matrices. Let $\varepsilon\colon GL_{m_i}\to GL_{m_i}$ be given by $\varepsilon(g)={}^t\overline{g}^{-1}$ (where for $GL_{m_i}(F)$, the Galois automorphism is of course trivial), and we let $\varepsilon'(g)=w_{m_i}\varepsilon(g)w_{m_i}^{-1}$. Then we may assume that

$$M=\left\{\operatorname{diag}\{g_1,\dots,g_r,g,\lambda(g)\varepsilon'(g_r),\dots,\lambda(g)\varepsilon'(g_1)\}\,\middle|\, g_i\in GL_{m_i}, g\in G(m)\right\}.$$

We may denote an element of M by $(g_1,\dots,g_r,g)$. Note that

$$A=\mathbf{A}(F)=\{\operatorname{diag}\{x_1I_{m_1},\dots,x_rI_{m_r},I_k,\overline{x}_r^{-1}I_{m_r}\dots\overline{x}_1^{-1}I_{m_1}\}\},$$

where k is chosen appropriately to be $2m$ or $2m+1$. For $1\leq i\leq r$, we let $b_i=\sum_{j=1}^{i}m_j$, and set $b_0=0$. For $1\leq i\leq j\leq r-1$, we let $\alpha_{ij}=e_{b_i}-e_{b_j+1}$, and $\beta_{ij}=e_{b_i}+e_{b_j+1}$. For $1\leq i\leq r$, we set

$$\gamma_i=\begin{cases} 2e_{b_i}, & \text{if } G=GSp_{2n} \text{ or } GU_{2n};\\ e_{b_i}, & \text{if } G=GO_{2n+1} \text{ or } GU_{2n+1};\\ e_{b_i-1}+e_{b_i}, & \text{if } G=GO_{2n}.\end{cases}$$

Then $\Phi(\mathbf{P},\mathbf{A})=\{\alpha_{ij},\beta_{ij}\}_{1\leq i\leq j\leq r-1}\cup\{\gamma_i\}_{1\leq i\leq r}$.

Note that the Weyl group $W(\mathbf{G},\mathbf{A})$ is a subgroup of $S_r \ltimes \mathbb{Z}_2^r$. Namely, if $m_i = m_j$, then the permutation:

$$w_{ij} = \prod_{k=1}^{m_i} (b_{i-1}+k \; b_{j-1}+k)$$

is in $W(\mathbf{G},\mathbf{A})$, and the map $w_{ij} \mapsto (ij)$ gives an isomorphism of this permutation group with a subgroup of S_r. For each i, we have the sign change

$$C_i = \prod_{j=1}^{m_i} c_{b_{i-1}+j}$$

is in $W(\mathbf{G},\mathbf{A})$. We call C_i a block sign change. The C_i 's generate the subgroup $\mathbb{Z}_2^r$ in the semidirect product. Note that, for $(g_1,\dots,g_r,g) \in M$, we have

$$w_{ij}(g_1,\dots g_r,g) = (g_1,\dots g_j,\dots g_i \dots g_r,g), \text{ and}$$
$$C_i(g_1 \dots g_r,g) = (g_1,\dots \lambda(g)\varepsilon(g_i),\dots,g_r,g).$$

Suppose $\sigma \in \mathcal{E}_2(M)$. Then $\sigma \simeq \sigma_1 \otimes \sigma_2 \otimes \dots \otimes \sigma_r \otimes \rho$, for some $\sigma_i \in \mathcal{E}_2(GL_{m_i})$ and $\rho \in \mathcal{E}_2(G(m))$. Let ω_i be the central character of σ_i. Note that if $w_{ij} \in W(\mathbf{G},\mathbf{A})$, then

$$w_{ij}\sigma \simeq \sigma_1 \otimes \dots \otimes \sigma_j \otimes \dots \otimes \sigma_i \otimes \dots \otimes \sigma_r \otimes \rho.$$

If $1 \leq i \leq r$, then

$$C_i\sigma \simeq \sigma_1 \otimes \dots \otimes \sigma_i^{\varepsilon} \otimes \dots \otimes \sigma_r \otimes (\rho \otimes \omega_i).$$

Here $\sigma_i^{\varepsilon}(g_i) = \sigma_i(\varepsilon(g_i))$. Thus, $W(\sigma) \neq \{1\}$ if and only if at least one of the following holds:

$$\sigma_i \simeq \sigma_j \text{ for some } i \neq j, \tag{2.6}$$

$$\sigma_i \simeq \sigma_j^{\varepsilon} \text{ and } \rho \otimes \omega_i\omega_j \simeq \rho, \text{ or }, \tag{2.7}$$

for some subset $\{i_j\} \subseteq \{1,\dots,r\}$,

$$\sigma_{i_j} \simeq \sigma_{i_j}^{\varepsilon} \text{ for all } j \text{ and } \rho \otimes \left(\prod_j \omega_{i_j}\right) \simeq \rho. \tag{2.8}$$

Note that if (2.6) holds then $w_{ij} \in W(\sigma)$. If (2.7) holds then $w_{ij}C_iC_j \in W(\sigma)$. Finally, (2.8) implies $\prod_j C_{i_j} \in W(\sigma)$. Note that in (2.7), since $\sigma_i \simeq \sigma_j^\varepsilon$, we have

$$\omega_i = \begin{cases} \omega_j^{-1} & \text{if } \mathbf{G} \neq GU_n, \\ \bar{\omega}_j^{-1} & \text{if } \mathbf{G} = GU_n. \end{cases}$$

Since

$$\lambda(G(m)) = \begin{cases} F^\times, & \text{if } \mathbf{G} = GSp_{2n},\ GO_{2n},\ \text{or } GU_{2n}; \\ F^{x^2}, & \text{if } \mathbf{G} = GO_{2n+1}; \\ N_{E/F}E^\times, & \text{if } \mathbf{G} = GU_{2n+1}, \end{cases}$$

(see [**16**,**22**]), we see that $\omega_i\omega_j \circ \lambda = 1$ if $\sigma_i \simeq \sigma_j^\varepsilon$. Thus, (2.7) simply becomes $\sigma_i \simeq \sigma_j^\varepsilon$.

We begin the computation of the possible R-groups with the following standard result. Recall that $\Delta' = \{\beta \in \Phi(\mathbf{P}, \mathbf{A}) | \mu_\beta(\sigma) = 0\}$.

Lemma 2.1. *The reduced root* $\alpha_{ij} \in \Delta'$ *if and only if* $\sigma_i \simeq \sigma_{j+1}$. *Similarly,* $\beta_{ij} \in \Delta'$ *if and only if* $\sigma_i \simeq \sigma_{j+1}^\varepsilon$.

Proof. Note that

$$M_{\alpha_{ij}} \simeq \left(\prod_{k \neq i, j+1} GL_{m_k} \right) \times GL_{m_i + m_{j+1}} \times G(m) \simeq M_{\beta_{ij}}.$$

Thus, by the results of Ol'šanskiĭ [**23**], Bernstein–Zelevinsky [**4**,**33**], and Jacquet [**17**], $i_{M_{\alpha_{ij}},M}(\sigma)$ and $i_{M_{\beta_{ij}},M}(\sigma)$ are both irreducible for all $\sigma \in \mathcal{E}_2(M)$. Thus, $\mu_{\alpha_{ij}}(\sigma) = 0$ if and only if there is some $w \neq 1$ in $W(\mathbf{M}_{\alpha_{ij}}, \mathbf{A})$ with $w\sigma \simeq \sigma$. If $m_i \neq m_{j+1}$, then $W(\mathbf{M}_{\alpha_{ij}}, \mathbf{A}) = \{1\}$. However, if $m_i = m_{j+1}$, then $W(\mathbf{M}_{\alpha_{ij}}, \mathbf{A}) = \{w_{ij}, 1\}$, and so $\mu_{\alpha_{ij}}(\sigma) = 0$ if and only if $\sigma_i \simeq \sigma_{j+1}$. Since $\beta_{ij} = C_{j+1}\alpha_{ij}$, we see that $\mu_{\beta_{ij}}(\sigma) = 0$ if and only if $\mu_{\alpha_{ij}}(C_{ij}\sigma) = 0$, which is equivalent to $\sigma_i \simeq \sigma_{j+1}^\varepsilon$. $\square$

We now show that the R-group of any $\sigma \in \mathcal{E}_2(M)$ is an elementary 2–group. The proof of the following lemma is based on a technique of Keys [**18**].

Lemma 2.2. *Suppose* $w = sc \in R$, *with* $s \in S_r$ *and* $c \in \mathbb{Z}_2^r$. *Then* $s = 1$.

Proof. By conjugating by a sign change, we may assume that c changes the sign of at most one e_{b_i} in each orbit of s. Suppose s has a non–trivial cycle, which we assume is $(1 \dots j+1)$. If c changes no signs among $\{e_{b_1}, \dots, e_{b_{j+1}}\}$, then we see that $w\sigma \simeq \sigma$ implies $\sigma_1 \simeq \sigma_2 \simeq \dots \simeq \sigma_{j+1}$. By Lemma 2.1, we see that $\alpha_{1j} \in \Delta'$, while $w\alpha_{1j} < 0$. This contradicts the assumption that $w \in R$. Suppose that c changes the sign of $e_{b_{j+1}}$. Then $w\sigma \simeq \sigma$ implies $\sigma_1 \simeq \dots \simeq \sigma_{j+1} \simeq \sigma_1^\varepsilon$. So by Lemma 2.1, $\beta_{1j} \in \Delta'$. But $w\beta_{1j} = s\alpha_{1j} < 0$, and again we have a contradiction. Thus, $s = 1$. $\square$

Suppose that, for some subset $B \subset \{1, 2, \dots, r\}$, we have $C_B = \prod_{j \in B} C_j \in R$. Set $\omega_B = \prod_{j \in B} \omega_j$. Then, for each $j \in B$, $\sigma_j \simeq \sigma_j^\varepsilon$, and $\rho \otimes \omega_B \simeq \rho$. Let

$$N_\varepsilon F^\times = \begin{cases} N_{E/F} E^\times, & \text{if } \mathbf{G} = GU_n; \\ F^{x^2}, & \text{if } \mathbf{G} = GSp_{2n} \text{ or } GO_n. \end{cases}$$

If $\sigma_i \simeq \sigma_i^\varepsilon$ then $\omega_i|_{N_\varepsilon F^\times} = 1$. If $\mathbf{G} = GSp_{2n}$, GO_{2n}, or GU_{2n}, then $\lambda(G(m)) = F^\times$, while if $\mathbf{G} = GO_{2n+1}$, or GU_{2n+1}, then $\lambda(G(m)) = N_\varepsilon F^\times$. Thus, for these last two groups, the condition $\rho \otimes \omega_B \simeq \rho$ is trivial. Therefore, for $\mathbf{G} = GO_{2n+1}$ or GU_{2n+1}, we see that $C_B \in R$ only if $\sigma_i \simeq \sigma_i^\varepsilon$ for each $i \in B$.

Definition 2.3. Let $\sigma \in \mathcal{E}_2(GL_k)$ and $\rho \in \mathcal{E}_2(G(m))$. We say that condition $\mathcal{X}_{m,k,G}(\sigma \otimes \rho)$ holds if $i_{G(m+k), GL_k \times G(m)}(\sigma \otimes \rho)$ is reducible. Note that a necessary condition for $\mathcal{X}_{m,k,G}(\sigma \otimes \rho)$ to hold is that $\rho \otimes \omega_\sigma \simeq \rho$, where ω_σ is the central character of σ.

Definition 2.4. If $w \in W(\mathbf{G}, \mathbf{A})$, then we let $R(w) = \{\alpha \in \Phi(P, A) | w\alpha < 0\}$. Note that $w \in R$ if and only if $w \in W(\sigma)$ and $R(w) \cap \Delta' = \emptyset$.

Lemma 2.5. *Suppose* $\mathbf{G} = GO_{2n+1}$ *or* GU_{2n+1}. *Suppose* $\mathbf{P} = \mathbf{MN}$ *is a parabolic subgroup of* $\mathbf{G}$ *with* $M = \mathbf{M}(F) \simeq GL_{m_1} \times \dots \times GL_{m_r} \times G(m)$. *Let* $\sigma \in \mathcal{E}_2(M)$, *with* $\sigma \simeq \sigma_1 \otimes \dots \otimes \sigma_r \otimes \rho$. *If* $c \in R$, *and* $c = C_B = \prod_{j \in B} C_j$, *then* $C_j \in R$ *for each* $j \in B$.

Proof. Note that, for each $j \in B$, we have $R(C_j) \subset R(c)$. Thus, if $C_j\beta < 0$ then $C_B\beta < 0$, and hence $\beta \notin \Delta'$. Now it is enough to note that, since $c\sigma \simeq \sigma$, we have $\sigma_j \simeq \sigma_j^\varepsilon$ for each $j \in B$, and thus $C_j \in W(\sigma)$. Therefore, $C_j \in R$. □

In order to give an explicit description of the R-groups, we need to define some terms. Let $\rho \in \mathcal{E}_2(G(m))$, and set

$$X(\rho) = \{\chi \in \left(F^\times/N_\varepsilon F^\times\right)^\wedge | \rho \otimes \chi \simeq \rho\}.$$

Let

$$J_1(\sigma) = \{i | \mathcal{X}_{m,m_i,G}(\sigma_i \otimes \rho) \text{ holds}\}.$$

For each $i \in J_1(\sigma)$, we have $\omega_i \in X(\rho)$. For each $\chi \in (F^\times/N_\varepsilon F^\times)^\wedge \setminus X(\rho)$, let

$$J_\chi(\sigma) = \{i | \sigma_i \simeq \sigma_i^\varepsilon \text{ and } \omega_i|_{F^\times} = \chi\}.$$

For $\chi = 1$ or $\chi \notin X(\rho)$, we set

$$I_\chi(\sigma) = \{i \in J_\chi(\sigma) | \sigma_i \not\simeq \sigma_j, \forall i > j\},$$

and let $d_\chi = d_\chi(\sigma) = |I_\chi(\sigma)|$ be the number of inequivalent σ_i for $i \in J_\chi(\sigma)$.

For a nonempty subset $S \subset (F^\times/N_\varepsilon F^\times)^\wedge$, we say that S is **minimally ρ-trivial** if $\prod_{\chi \in S} \chi \in X(\rho)$, and $\prod_{\chi \in S'} \chi \notin X(\rho)$ for any $\emptyset \subsetneq S' \subsetneq S$. We let

$$\Lambda(\sigma) =$$
$$\{S \subset (F^\times/N_\varepsilon F^\times)^\wedge \setminus X(\rho) \big| S \text{ is minimally } \rho - \text{trivial and } d_\chi(\sigma) \geq 1, \forall \chi \in S\}.$$

Let $\{S_1, \ldots, S_k\} \subseteq \Lambda(\sigma)$. We let

$$X(\{S_1, \ldots, S_k\}) = \{\chi \mid \chi \in S_j \text{ for an odd number of } j\}.$$

We say that $\Lambda(\sigma)'$ is a **basis** for $\Lambda(\sigma)$ if, for every $S \in \Lambda(\sigma)$, there is some $\{S_1, \ldots, S_k\} \subset \Lambda(\sigma)'$ with $S = X(\{S_1, \ldots, S_k\})$, and $\Lambda(\sigma)'$ is minimal with respect to this property.

Theorem 2.6. *Let* $\mathbf{M}$ *be a Levi subgroup of* $\mathbf{G}$, *with*

$$M \simeq GL_{m_1} \times \ldots \times GL_{m_r} \times G(m).$$

Suppose $\sigma \in \mathcal{E}_2(M)$, *with* $\sigma \simeq \sigma_1 \otimes \ldots \otimes \sigma_r \otimes \rho$. *Let* $R = R(\sigma)$.

(a) *If* $\mathbf{G} = GO_{2n+1}$ *or* GU_{2n+1}, *then* $R \simeq \mathbb{Z}_2^d$, *with* $d = d_1$, *i.e.,* d *is the number of inequivalent* σ_i *such that* $\mathcal{X}_{m,m_i,G}(\sigma \otimes \rho)$ *holds.*

(b) *If* $\mathbf{G} = GSp_{2n}$, GO_{2n}, *or* GU_{2n}, *then* $R \simeq \mathbb{Z}_2^d$ *with*

$$d = d_1 + \sum_{\substack{\chi \notin X(\rho) \\ d_\chi \geq 1}} (d_\chi - 1) + |\Lambda(\sigma)'|,$$

for any basis $\Lambda(\sigma)'$ *of* $\Lambda(\sigma)$.

Proof. (a) By Lemma 2.5, is enough to show that $C_i \in R$ if and only if $i \in I_1(\sigma)$. Suppose $C_i \in W(\sigma)$. Note that $R(C_i) = \{\alpha_{ij}, \beta_{ij}\}_{i \leq j \leq r-1} \cup \{\gamma_i\}$. Note that if $\sigma_{j+1} \simeq \sigma_i$, for some $j \geq i$, then α_{ij} and $\beta_{ij} \in \Delta'$, and thus $C_i \notin R$. Direct computation shows that $M_{\gamma_i} \simeq (\prod_{k \neq j} GL_{m_k}) \times G(m + m_i)$. Since $C_i \sigma \simeq \sigma$, we see that $\gamma_i \in \Delta'$ if and only if $\mathcal{X}_{m,m_i,G}(\sigma_i \otimes \rho)$ does not hold. Thus, $C_i \in R$ if and only if $\mathcal{X}_{m,m_i,G}(\sigma_i \otimes \rho)$ holds and $\sigma_i \not\simeq \sigma_j$ for all $j > i$, i.e., if and only if $i \in I_1(\sigma)$.

(b) Suppose $B \subseteq \{1, 2, \ldots, r\}$ with $C_B = \prod_{j \in B} C_j \in R$. By the argument of part (a), we see that, for each $j \in B$, $\sigma_k \not\simeq \sigma_j$ for all $k > j$. Note again that $M_{\gamma_j} \simeq (\prod_{k \neq j} GL_{m_k}) \times G(m + m_j)$. Therefore, $\gamma_j \in \Delta'$ if and only if $C_j \sigma \simeq \sigma$ and $\mathcal{X}_{m,m_j,G}(\sigma \otimes \rho)$ fails. Since $C_j \sigma \simeq \sigma$ if and only if $\sigma_j \simeq \sigma_j^{\varepsilon}$, and $\omega_j \in X(\rho)$, we see that those $j \in B$ for which $\omega_j \in X(\rho)$ are all elements of $I_1(\sigma)$. Let

$$R_1 = R_1(\sigma) = \langle C_j \bigm| j \in I_1(\sigma) \rangle.$$

Then we have just shown that $R_1(\sigma) \subseteq R(\sigma)$. If $j \in B$ and $\omega_j \notin X(\rho)$, then $C_j \notin W(\sigma)$, and thus $\gamma_j \notin \Delta'$. Let $\chi \in (F^\times / N_\varepsilon F^\times)^\wedge \backslash X(\rho)$, and define

$$R_\chi(\sigma) = \langle C_i C_j | i, j \in I_\chi(\sigma) \rangle.$$

If $C_iC_j \in R_\chi(\sigma)$, then $\omega_i\omega_j|_{F^\times} = 1$, so $C_iC_j \in W(\sigma)$. Note that $R(C_iC_j) = R(C_i) \cup R(C_j)$. Moreover, by the definition of $I_\chi(\sigma)$, and Lemma 2.1, we see that, for all $k \geq i$ and $\ell \geq j$, the reduced roots α_{ik}, $\alpha_{j\ell}$, β_{ik}, and $\beta_{j\ell}$ can not be in Δ'. Since γ_i, $\gamma_j \notin \Delta'$, we see that $C_iC_j \in R$, and thus $R_\chi(\sigma) \subseteq R$.

Since each $R_\chi(\sigma) \subset R(\sigma)$, we can multiply C_B by an element C' of

$$R_1 \times \prod_{\chi \notin X(\rho)} R_\chi(\sigma)$$

so that $C_{B'} = C'C_B$ has the property that $B' \cap I_1(\sigma) = \emptyset$, and $\omega_i \neq \omega_j$ for all $i \neq j$ in B'. Since $\omega_B \in X(\rho)$, we see that $\left\{\omega_j|_{F^\times} \middle| j \in B'\right\}$ can be partitioned into minimally ρ-trivial subsets. For each $\chi \in (N_\varepsilon F^\times \backslash F^\times)^\wedge \backslash X(\rho)$, with $d_\chi > 0$, we fix an $i_\chi \in I_\chi(\sigma)$. Let $S \in \Lambda(\sigma)$, and define $C_S = \prod\limits_{\chi \in S} C_{i_\chi}$. Since $\prod\limits_{\chi \in S} \chi \in X(\rho)$, we see that $C_S \in W(\sigma)$, and by the definition of $I_\chi(\sigma)$, we see that $C_S \in R$. Suppose $C' = C_D \in R$ has the property that $\{\omega_j|_{F^\times}|j \in D\} = S$ and $|D| = |S|$. Then

$$C_D = C_S \cdot \prod_{j \in D} C_j \; C_{i_{\omega_j}},$$

and each $C_jC_{i_{\omega_j}} \in R_{\omega_j}(\sigma)$. Thus, all the factors in the product are in R. Fix a basis $\Lambda(\sigma)'$ for $\Lambda(\sigma)$. Suppose $S \in \Lambda(\sigma)$, but $S \notin \Lambda(\sigma)'$. Then $S = X(\{S_1, \ldots, S_k\})$ for some $\{S_1, \ldots, S_k\} \subseteq \Lambda(\sigma)'$. By the definition of C_S, and of $X(\{S_1, \ldots, S_k\})$, we have $C_S = C_{S_1}C_{S_2}\ldots C_{S_k}$. Thus, for some $\{S_1, \ldots, S_\ell\} \subseteq \Lambda(\sigma)'$, and some $C'' \in \prod\limits_{\chi \notin X(\rho)} R_\chi(\sigma)$, we have $C_{B'} = C_{S_1}\ldots C_{S_\ell} \cdot C''$. Therefore, if we define

$$R'(\sigma) = \langle C_S | S \in \Lambda(\sigma)'\}\rangle,$$

then we have seen that

$$R = \left(\prod_\chi R_\chi(\sigma)\right) \cdot R'(\sigma).$$

Moreover, by considering central characters, this is a direct product. Therefore,

$$R \simeq \mathbb{Z}_2^{d_1} \times \left(\prod_{\chi \notin X(\rho)} \mathbb{Z}_2^{d_\chi - 1}\right) \times \mathbb{Z}_2^{|\Lambda(\sigma)'|},$$

as claimed. $\square$

Remark. If $G = GU_{2n}$, then $N_\varepsilon F^\times = N_{E/F}E^\times$ is of index two in $F^\times$, and so the local class field theory character $\omega_{E/F}$ is the only non-trivial element of $(F^\times/N_\varepsilon F^\times)^\wedge$. Therefore, the group R' is trivial, and $R \simeq \mathbb{Z}_2^d$, with $d = d_1$, or $d_1 + d_{\omega_{E/F}} - 1$ depending on whether or not $d_{\omega_{E/F}} = 0$ or not. We note the similarity between the R-group structure for GU_{2n} and that for SO_{2n}. We will see that the theory of elliptic representations for GU_{2n} also parallels that that of SO_{2n}.

Section 3 Elliptic Representations.

We now describe the tempered elliptic representations of G. We first show that the 2–cocycle η of R described in Section 1 is trivial. We use an idea from [**26**].

Lemma 3.1. *Let $\mathbf{G} = GSp_{2n}$, GO_n, or GU_n. Suppose $\mathbf{M}$ is a Levi subgroup of G and $\sigma \in \mathcal{E}_2(M)$. Denote by R the R-group associated to $i_{G,M}(\sigma)$. Then the commuting algebra $C(\sigma)$ is isomorphic to $\mathbb{C}[R]$.*

Proof. Suppose that $\mathbf{G} \neq GO_{2n}$. Let $\mathbf{G}_1(n) = \{g \in \mathbf{G}(n) | \lambda(g) = 1\}^\circ$. That is, $\mathbf{G}_1(n)$ is the symplectic, special orthogonal, or quasi-split unitary group associated to the form J. Let $\mathbf{M}_1$ be the Levi component of $\mathbf{G}_1(n)$ given by $\mathbf{M}_1 = \mathbf{M} \cap \mathbf{G}_1(n)$. If

$$\mathbf{M} = GL_{m_1} \times \cdots \times GL_{m_r} \times \mathbf{G}(m),$$

then

$$\mathbf{M}_1 = GL_{m_1} \times \cdots \times GL_{m_r} \times \mathbf{G}_1(m),$$

Let $\mathbf{P}_1 = \mathbf{M}_1\mathbf{N}$. Suppose that τ is an irreducible subrepresentation of $\sigma|_{M_1}$. Then $\tau \simeq \sigma_1 \otimes \cdots \otimes \sigma_r \otimes \rho_1$, for some irreducible component ρ_1 of $\rho|_{G_1(m)}$. Let $A(\nu, \tau, w)$ and $A(\nu, \sigma, w)$ be the standard intertwining operators attached to either σ and τ, respectively. [**19**,**25**,**28**]. Since $N \subset G_1(n)$, $(A(\nu, \sigma, w)f)|_{G_1} = A(\nu, \tau, w)f|_{G_1}$ for any $f \in \mathrm{Ind}_P^G(\tau)$. Thus, for each reduced root β, we have $\mu_\beta(\sigma) = \mu_\beta(\tau)$. We let $\mathcal{A}'(\tau, w)$ and $\mathcal{A}'(\sigma, w)$ be the normalized self intertwining operators (see [**10**,**15**]). Then these operators satisfy

$$\mathcal{A}'(\sigma, w_1w_2) = \eta(w_1, w_2)\mathcal{A}'(\sigma, w_1)\mathcal{A}'(\sigma, w_2)$$

for each $w_1, w_2 \in R(\sigma)$, and

$$\mathcal{A}'(\tau, w_1w_2) = \eta(w_1, w_2)\mathcal{A}'(\tau, w_1)\mathcal{A}'(\tau, w_2)$$

for $w_1, w_2 \in R(\tau)$.

Note that each element of R can be represented by an element in G_1. Suppose that $C_B \in R$. Then for each $i \in B$, $\sigma_i \simeq \sigma_i^{\varepsilon}$. Thus, by the results of [**7**] and [**6**], we have $C_B \in W(\tau)$. If $C_B\beta < 0$, then $\mu_\beta(\sigma) \neq 0$, and thus $\mu_\beta(\tau) \neq 0$. Consequently, $C_B \in R(\tau)$, i.e., $R(\sigma) \subset R(\tau)$. Now let $w_1, w_2 \in R(\sigma)$, and $f \in \operatorname{Ind}_P^G(\sigma)$, and $f_1 = f|_{G_1}$. Then

$$\eta(w_1, w_2)\mathcal{A}'(\sigma, w_1)\mathcal{A}'(\sigma, w_2)f|_{G_1} = \mathcal{A}'(w_1w_2, \sigma)f|G_1.$$

Thus,

$$\eta(w_1, w_2)\mathcal{A}'(w_1, \tau)\mathcal{A}'(w_2, \tau)f_1 = \mathcal{A}'(w_1w_2, \tau)f_1.$$

On the other hand, the results of [**6**,**15**] show that

$$\mathcal{A}'(w_1, \tau)\mathcal{A}'(w_2, \tau) = \mathcal{A}'(w_1w_2, \tau),$$

and thus $\eta(w_1, w_2) = 1$.

We are left to prove the lemma for $\mathbf{G} = GO_{2n}$. The problem with the above argument is that there may be R-group elements which are not elements of the Weyl group attached to the corresponding parabolic subgroup of SO_{2n}. We can remedy this by looking at restriction to M_1 as above, and inducing to O_{2n}. Let $\mathbf{G}_1 = SO_{2n}$, and take $\mathbf{M}_1 = \mathbf{M}\cap\mathbf{G}_1$, and $\mathbf{P}_1 = \mathbf{P}\cap\mathbf{G}_1$ as above. Let $\mathbf{G}_2 = O_{2n}$. If $m > 0$, then for each i with m_i odd, we choose the representative for C_i as in [**7**]. Then C_i also represents an element of $W(\mathbf{G}_1, \mathbf{A}_1)$, where this is the obvious Weyl group. Suppose $\sigma \in \mathcal{E}_2(M)$, and τ is an irreducible subrepresentation of $\sigma|_{M_1}$. In [**8**] we examine the decomposition of the representation

$$\operatorname{Ind}_{P_1}^{G_2}(\tau) = \operatorname{Ind}_{G_1}^{G_2}\left(\operatorname{Ind}_{P_1}^{G_1}(\tau)\right).$$

In [**1**], Arthur extends the definition of the unnormalized intertwining operators $A(\nu,\tau,w)$ to the case of disconnected groups whose component group is cyclic. Consider these operators for $\mathbf{G}_2$. Since $\mathbf{N}\subset\mathbf{G}_2$, this integral operator is again the same one which gives the unnormalized operators $A(\nu,\sigma,w)$ for GO_{2n}. In the general case, Arthur did not prove the analytic continuation of these operators. However, since $A(\nu,\sigma,w)$ can be analytically continued, and $A(\nu,\sigma,w)f|_{G_2} = A(\nu,\tau,w)(f|_{G_2})$, we see that indeed the operators for G_2 can be analytically continued.

Let $C_B\in R$. Then C_B can be represented in G_2, and is a representative for an element of $W(\mathbf{G}_2,\mathbf{A}_1)$. Moreover, from the results of [**7**,**8**], $C_B\tau\simeq\tau$. We claim that $C_B\in R_{G_2}(\tau)$, where this last object is the R-group given in [**8**] which determines the structure of $\operatorname{Ind}_{P_1}^{G_2}(\tau)$. It is enough to show that if $i\in B$, then $\mu_{\gamma_i}(\tau)\neq 0$. If C_i represents an element of $W(\mathbf{G}_1,\mathbf{A}_1)$, then the argument used for GO_{2n+1}, GSp_{2n}, and GU_n above shows that $\mu_{\gamma_i}(\tau)=\mu_{\gamma_i}(\sigma)$. Thus, we are reduced to the case where m_i is odd, and $m=0$. Then, since $W(\mathbf{M}_{\gamma_i},\mathbf{A}_1)=\{1\}$, we have $\mu_{\gamma_i}(\tau)\neq 0$, as pointed out in [**7**]. Thus, $C_B\in R_{G_2}(\sigma)$.

Now, the operators $\mathcal{A}'(w,\sigma)$ restrict to elements $\mathcal{A}'(\tau,w)$ of the commuting algebra of $\operatorname{Ind}_{P_1}^{G_2}(\tau)$. Thus, the cocycle η also determines the composition of these operators. However, these operators can also be defined using operators which intertwine τ and $w\tau$, and η is also the cocycle that arises in this way. The argument given in Proposition 2.3 of [**15**] shows that, we can choose T'_w for each w in $R_{G_2}(\tau)$, satisfying $T'_w\tau=w\tau T'_w$, with $T'_{w_1w_2}=T'_{w_1}T'_{w_2}$. But this shows that

$$\mathcal{A}'(w_1w_2,\tau)=\mathcal{A}'(w_1,\tau)\mathcal{A}'(w_2,\tau),$$

and therefore $\eta\equiv 1$. This completes the lemma. $\square$

Corollary 3.2. *For any $\sigma\in\mathcal{E}_2(M)$, the induced representation $i_{G,M}(\sigma)$ decomposes as a direct sum of $|R|$ inequivalent representations.* $\square$

Proposition 3.3. *Let $\mathbf{G}=GSp_{2n}$, GO_n, or GU_n. Suppose $\mathbf{M}$ is a Levi subgroup of $\mathbf{G}$, with $M\simeq GL_{m_1}\times\ldots\times GL_{m_r}\times G(m)$. Let $\sigma\in\mathcal{E}_2(M)$ with*

$\sigma \simeq \sigma_1 \otimes \ldots \otimes \sigma_r \otimes \rho$. *Suppose* R *is the* R*-group attached to* $i_{G,M}(\sigma)$. *Then the following are equivalent:*

(a) $i_{G,M}(\sigma)$ *has an elliptic constituent;*

(b) *all the constituent of* $i_{G,M}(\sigma)$ *are elliptic;*

(c) $C_1 \ldots C_r \in R$.

Proof. By Lemma 3.1, and Theorems 1.3 and 2.6, conditions (a) and (b) are equivalent. Furthermore, both are equivalent to $\mathfrak{a}_w = \mathfrak{z} = \{0\}$ for some $w \in R$. Note that

$$\mathfrak{a} = \{\mathrm{diag}\{x_1 I_{m_1}, \ldots , x_r I_{m_r}, 0_k, -x_r I_{m_r}, \ldots , -x_1 I_{m_1}\} | x_i \in \mathbb{R}\},$$

where $k = 2m$ or $2m+1$ as appropriate. We may denote an element of $\mathfrak{a}$ by $x = (x_1, \ldots , x_r)$. We further note that $C_i\colon (x_1, \ldots, x_i, \ldots, x_r) \mapsto (x_1, \ldots, -x_i, \ldots, x_r)$. Thus, if $C = C_B$, we have $\mathfrak{a}_C = \{(x_1, \ldots , x_r) | x_i = 0, \forall i \in B\}$. Consequently, $\mathfrak{a}_C = \{0\}$ if and only if $B = \{1, \ldots , r\}$. □

We now give more explicit criteria for $i_{G,M}(\sigma)$ to have elliptic components. In order to do this, we need to consider three cases. The first two are handled in the next result, and are similar to the results of [**15**]. We use the same type of arguments found there.

Theorem 3.4.

(a) *If* $\mathbf{G} = GO_{2n+1}$ *or* GU_{2n+1}, *then the following hold:*

(i) *Conditions (a)–(c) of Proposition 3.3 hold if and only if* $R \simeq \mathbb{Z}_2^r$.

(ii) *If* $\pi \in \mathcal{E}_t(G)$, *then either* $\pi \in \mathcal{E}_e(G)$, *or there is a proper Levi subgroup* $\mathbf{M}'$ *of* $\mathbf{G}$ *and some* $\tau' \in \mathcal{E}_e(M')$ *with* $\pi = i_{G,M'}(\tau)$.

(b) *If* $\mathbf{G} = GU_{2n}$ *then the following hold:*

(i) *Conditions (a)–(c) of Proposition 3.3 hold if and only if* $d = r - 1$ *and* $d_{\omega_{E/F}} > 0$ *is even, or* $d = r$ *and* $d_{\omega_{E/F}} = 0$.

(ii) *If* $d < r - 1$, *or* $d = r - 1$ *and* $d_{\omega_{E/F}} = 1$, *then each irreducible subrepresentation* π *of* $i_{G,M}(\sigma)$ *is of the form* $\pi = i_{G,M'}(\tau)$, *for* *some proper Levi subgroup* $\mathbf{M}'$ *of* $\mathbf{G}$, *and some* $\tau \in \mathcal{E}_t(M')$. *We can choose* $\mathbf{M}'$ *and* τ *with* $\tau \in \mathcal{E}_e(M')$ *if and only if* $d_{\omega_{E/F}}$ *is even or* $d_{\omega_{E/F}} = 1$.

(iii) *Suppose* $d = r-1$, $d_{\omega_{E/F}} \geq 3$ *is odd, and* π *is an irreducible subrepresentation of* $i_{G,M}(\sigma)$. *Then* π *cannot be irreducibly induced from a tempered representation of a proper Levi subgroup.*

Proof. (a) From Lemma 2.5, we see that if $\mathbf{G} = GO_{2n+1}$ or GU_{2n+1}, then $C_1 \dots C_r \in R$ if and only if $R = \langle C_i | 1 \leq i \leq r\rangle \simeq \mathbb{Z}_2^r$. Thus, (i) holds. Further, Lemma 2.5 implies that for any $\sigma \in \mathcal{E}_2(M)$, there is a subset B_σ of $\{1, 2, \dots, r\}$ with $R = \langle C_i | i \in B_\sigma\rangle$. Let $C = C_{B_\sigma}$. Then

$$\mathfrak{a}_R = \{(x_1, \dots, x_r) | x_i = 0, \forall i \in B_\sigma\} = \mathfrak{a}_C.$$

Thus, Theorem 1.5 implies (ii).

(b) Let $d_2 = d_{\omega_{E/F}}$. By the remark following Theorem 2.6, we know that $R \simeq \mathbb{Z}_2^{d_1+d_2-1}$ if $d_2 > 0$, and $R \simeq \mathbb{Z}_2^{d_1}$ if $d_2 = 0$. So, if $d_2 = 0$, we have $C_1 \dots C_r \in R$ if and only if $C_i \in R$ for each $1 \leq i \leq r$, which is equivalent to $R \simeq \mathbb{Z}_2^r$. On the other hand, if $d_2 > 0$, then the longest element of R consists of $d_1 + 2[\frac{d_2}{2}]$ block sign changes. Hence, if d_2 is odd, it is impossible for $C_1 \dots C_r \in R$. On the other hand, if d_2 is even, then we see that $C_1 \dots C_r \in R$ if and only if $d_1 + d_2 = r$, which is equivalent to $R \simeq \mathbb{Z}_2^{r-1}$. This proves (i).

We note that if $d < r-1$, then there is some $i \in \{1, \dots, r\}$ for which $C_i C_B \notin R$ for all $B \subseteq \{1, \dots, i-1, i+1, \dots, r\}$. Thus, for each $w \in R$, the element $E_i = (0, \dots, 1, \dots, 0)$ is in $\mathfrak{a}_w$. (Here the 1 is in the i-th coordinate.) Thus, $E_i \in \mathfrak{a}_R$, which implies $\mathfrak{a}_R \neq \{0\}$. Therefore, by Theorem 1.5, every irreducible subrepresentation π of $i_{G,M}(\sigma)$ is of the form $\pi = i_{G,M'}(\tau)$ for some $\mathbf{M}' \subsetneq \mathbf{G}$ and $\tau \in \mathcal{E}_t(M')$.

Suppose now that $d_2 = d_{\omega_{E/F}}$ is even. Let $I_2(\sigma) = I_{\omega_{E/F}}(\sigma)$. Then

$$C_0 = \prod_{i \in I_2(\sigma)} C_i$$

is in R. So if $B = I_1(\sigma) \cup I_2(\sigma)$, we have $C_B \in R$, and $\mathfrak{a}_{C_B} = \mathfrak{a}_R$. Therefore, if $d < r-1$ and d_2 is even, then we can choose $(\mathbf{M}', \tau)$, with $\tau \in \mathcal{E}_e(M')$.

Suppose that $d_2 = 1$. Then $\omega_{E/F} \notin X(\rho)$. If $\omega_1 = \omega_{E/F}$, then $C_1 C_B \notin R$, for all $B \subset \{2, \dots, r\}$. Let $C = C_{I_1(\sigma)}$. Then $\mathfrak{a}_R = \mathfrak{a}_C$, and therefore we can choose $(\mathbf{M}', \tau)$ and $\tau \in \mathcal{E}_e(M')$, with $\pi = i_{G,M'}(\tau)$.

Now suppose $d_2 \geq 3$ is odd. Without loss of generality, we assume that $I_1(\sigma) = \{k+1, k+2, \dots, k+d_1\}$, and $I_2(\sigma) = \{k+d_1+1, \dots, r-1, r\}$, with $k = r-(d_1+d_2)$. Then $\mathfrak{a}_R = \{(x_1 \dots x_r) | x_{k+1} = x_{k+2} = \dots = x_r = 0\}$. Since $C_{k+1} \dots C_r \notin R$, we see that $\mathfrak{a}_R \neq \mathfrak{a}_w$ for any $w \in R$. Therefore, π can not be of the form $i_{G,M'}(\tau)$ for some $\mathbf{M}' \subsetneq \mathbf{G}$ and $\tau \in \mathcal{E}_e(M')$. $\square$

Definition 3.5. Let $\mathbf{G} = GSp_{2n}$ or GO_{2n}. Suppose $\sigma \simeq \sigma_1 \otimes \cdots \otimes \sigma_r \otimes \rho \in \mathcal{E}_2(M)$. We set

$$O(G, \sigma) = \left\{ \chi \in \left(F^\times / F^{\times^2} \right)^\wedge \setminus X(\rho) \;\middle|\; d_\chi \text{ is odd} \right\}.$$

We also set

$$O_1(G, \sigma) = \left\{ \chi \in O(G, \sigma) \;\middle|\; d_\chi = 1, \text{ and } \chi \notin S, \ \forall S \in \Lambda(\sigma) \right\}.$$

Finally, we let $I_0(\sigma) = I_1(\sigma) \cup \bigcup_{\chi \notin X(\rho)} I_\chi(\sigma)$.

Theorem 3.6. *Suppose $\mathbf{G} = GSp_{2n}$ or GO_{2n}. Let $M \simeq GL_{m_1} \times \dots \times GL_{m_r} \times G(m)$. Assume that $\sigma \simeq \sigma_1 \otimes \cdots \otimes \sigma_r \otimes \rho \in \mathcal{E}_2(M)$ and R is the R-group of σ.*

(a) *$i_{G,M}(\sigma)$ has elliptic constituents if and only if*

(i) $O(G, \sigma)$ can be partitioned into minimally ρ-trivial subsets, and

(ii) $I_0(\sigma) = \{1, \dots, r\}$.

(b) *Suppose $\pi \in \mathcal{E}_t(G)$ is not elliptic, and $\pi \subseteq i_{G,M}(\sigma)$. Then we can find a proper Levi subgroup $\mathbf{M}'$ of $\mathbf{G}$ and a $\tau \in \mathcal{E}_t(M')$ with $\pi = i_{G,M'}(\tau)$ if and only if $I_0(\sigma) \subsetneq \{1, \dots, r\}$, or $O_1(G, \sigma)$ is nonempty. We can choose $(\mathbf{M}', \tau)$ with $\tau \in \mathcal{E}_e(M')$ if and only if $O(G, \sigma) \backslash O_1(G, \sigma)$ can be partitioned into minimally ρ-trivial subsets.*

Proof. (a) If $C_1 \dots C_r \in R$, then $I_0(\sigma) = \{1, \dots, r\}$. So condition (ii) is trivial. Suppose this is the case. For each $\chi \in O(G,\sigma)$, we choose an $i_\chi \in \{1, \dots, r\}$, with $\omega_{i_\chi} = \chi$. We then let

$$C_\chi = \prod_{i \in I_\chi(\sigma) \setminus \{i_\chi\}} C_i.$$

Since d_χ is odd, $C_\chi \in R$. For each $\chi \notin X(\rho)$ with d_χ even, we let

$$C_\chi = \prod_{i \in I_\chi(\sigma)} C_i.$$

Finally let

$$C_0 = \left(\prod_{i \in I_1(\sigma)} C_i \right) \prod_{\chi \notin X(\rho)} C_\chi.$$

Then $C_0 \in R$, and

$$C_0 = \prod_{i \notin \{i_\chi | \chi \in O(G,\sigma)\}} C_i.$$

Thus, $C_1 \dots C_r \in R$ if and only if $C' = \prod\limits_{O(G,\sigma)} C_{i_\chi} \in R$. From the proof of Theorem 2.6, we know that $C' \in R$ if and only if $\prod\limits_{O(G,\sigma)} \omega_{i_\chi} = \prod\limits_{O(G,\sigma)} \chi \in X(\rho)$. This shows that (i) and (ii) are equivalent to $C_1 \dots C_r \in R$.

(b) Note that

$$\mathfrak{a}_R = \bigcap_{w \in R} \mathfrak{a}_w \subseteq \left\{ (x_1 \dots x_r) \;\middle|\; x_i = 0, \ \forall i \in I_0(\sigma) \right\}.$$

So in particular, if $I_0(\sigma) \subsetneq \{1, \dots, r\}$, then $\mathfrak{a}_R \supsetneq \{0\}$. Thus, in this case, we can find $\mathbf{M}'$ and $\tau \in \mathcal{E}_t(M')$ as desired. Now suppose that $I_0(\sigma) = \{1, \dots, r\}$. If $\chi \notin X(\rho)$ and $\omega_i = \omega_j = \chi$ for some $i \neq j$, then $C = C_i C_j \in R$ and $\mathfrak{a}_C = \{(x_1 \dots x_r) | x_i = x_j = 0\} \subseteq \mathfrak{a}_R$. Therefore, if $d_\chi \neq 1$ for all $\chi \notin X(\rho)$ then we see that $\mathfrak{a}_R = \{0\}$, and so $i_{G,M'}(\tau) = \pi$ is impossible. So now suppose $\omega_1 = \chi$ and $\omega_i \neq \chi$ for all $i > 1$. Then there is a subset $B \subset \{2, 3, \dots, r\}$ with $C_1 C_B \in R$ if and only if $\omega_B = \chi$, which is equivalent to $\{\chi\} \cup \{\omega_j | j \in B'\} \in \Lambda(\sigma)$

for some $B' \subseteq \{2,\dots,r\}$. Thus, if no such B' exists then $\{(1,0,\dots,0)\} \in \mathfrak{a}_R$, and we can choose $\mathbf{M}'$ and $\tau \in \mathcal{E}_t(M')$ with $\pi = i_{G,M'}(\tau)$. On the other hand, suppose that, for each χ with $d_\chi = 1$, there is an $S \in \Lambda(\sigma)$ with $\chi \in S$. Then, for each $1 \leq i \leq r$, there is a subset B of $\{1,\dots,i-1,i+1,\dots,r\}$ with $C_iC_B \in R$. Thus, for each i, $\{(x_1\dots x_r)|x_i = 0\} \subseteq \mathfrak{a}_R$, which implies $\mathfrak{a}_R = \{0\}$. Therefore, finding $\mathbf{M}'$ and τ is impossible.

Finally, we assume that $\mathfrak{a}_R = \{(x_1,\dots,x_k,0,\dots,0)|x_i \in \mathbb{R}\}$, with $k \geq 1$. We need to determine when $C_{k+1}\dots C_r \in R$. Let $\chi \in O(G,\sigma)$. If $\chi \in O_1(G,\sigma)$, and $\omega_{i_\chi} = \chi$, then $i_\chi \leq k$. For all other $\chi \in O(G,\sigma)$, if $i \in I_\chi(\sigma)$, then we have $i > k$. Suppose $\chi \in O(G,\sigma)\backslash O_1(G,\sigma)$. As in the proof of part (a), we fix $i_\chi \in I_\chi(\sigma)$ and let

$$C_\chi = \prod_{i \in I_\chi(\sigma)\backslash\{i_\chi\}} C_i.$$

If $\chi \notin O(G,\sigma)$ let

$$C_\chi = \prod_{i \in I_\chi(\sigma)} C_i.$$

Set

$$C_0 = \prod_{i \in I_1(\sigma)} C_i \prod_{\chi \notin X(\rho)} C_\chi.$$

Then $C_0 \in R$. Since

$$C_{k+1}\dots C_r = C_0 \cdot \prod_{\chi \in O(G,\sigma)\backslash O_1(G,\sigma)} C_{i_\chi},$$

we see that $C_{k+1}\dots C_r \in R$ if and only if

$$\prod_{O(G,\sigma)\backslash O_1(G,\sigma)} \chi \in X(\rho),$$

which says that $O(G,\sigma) \setminus O_1(G,\sigma)$ can be partitioned into minimally ρ-trivial subsets. $\square$

Example. We note that in all previous cases where the elliptic tempered representations were determined, the following phenomenon always held: Given a parabolic subgroup $\mathbf{P} = \mathbf{MN}$ of $\mathbf{G}$, there is a fixed finite group R_M, so that, for any $\sigma \in \mathcal{E}_2(M)$, the representation $i_{G,M}(\sigma)$ has elliptic components if and only if $R(\sigma) \simeq R_M$. In fact, if $\mathbf{M}$ is a Levi subgroup that admits elliptic tempered representations, then $i_{G,M}(\sigma)$ has elliptic constituents if and only if $|R(\sigma)| = |R_M|$. With the similitude groups, we see that such R_M do not exist. We look at the following example. Suppose that $G = GSp_{6n}$ or GO_{6n} for some n, and $\mathbf{M} \simeq GL_n \times GL_n \times GL_n \times GL_1$. Let $\sigma \simeq \sigma_1 \otimes \sigma_2 \otimes \sigma_3 \otimes \rho$. In this case ρ is a one dimensional character, so $\rho \otimes \omega = \rho$ if and only if $\omega = 1$. We suppose that the σ_i are pairwise inequivalent discrete series representations of $GL_n(F)$, and that $\sigma_i \simeq \widetilde{\sigma}_i \simeq \sigma_i^{\varepsilon}$ for each i. Here $\widetilde{\sigma}_i$ is the smooth contragredient of σ_i [**3**,§7]. If $\mathcal{X}_{n,0,G}(\sigma_i \otimes \rho)$ holds for each i, then $R \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, and $i_{G,M}(\sigma)$ has eight elliptic constituents. On the other hand, if $\mathcal{X}_{n,0,G}(\sigma_1 \otimes \rho)$ holds, while $\omega_2 = \omega_3 \neq 1$, then $R = < C_1, C_2C_3 >$. Since $C_1C_2C_3 \in R$, we see that $i_{G,M}(\sigma)$ has four elliptic constituents. Finally, if the ω_i are all non-trivial, and $\omega_3 = \omega_1\omega_2$, then $R = < C_1C_2C_3 >$, and $i_{G,M}(\sigma)$ has two elliptic constituents. □

Suppose now that $\mathbf{G}$ is any of the similitude groups that we have been studying. Further suppose $\mathbf{M}$ is a Levi subgroup of $\mathbf{G}$, and $\sigma \in \mathcal{E}_2(M)$ is such that all the constituents of $i_{G,M}(\sigma)$ are elliptic. Let $\hat{R}$ be the collection of irreducible representations of R. Since R is abelian, $\dim \kappa = 1$ for all $\kappa \in \hat{R}$. We let π_κ be the irreducible subrepresentation of $i_{G,M}(\sigma)$ canonically parameterized by κ [**19**], and we denote by Θ_κ^e its character on the regular elliptic set of G. Then we see that

$$\sum_{\kappa \in \hat{R}} \Theta_\kappa^e = 0.$$

We now make this linear dependence precise by generalizing an argument of Herb [**15**]. We need some preliminary results.

Lemma 3.7. *Suppose* $i_{G,M}(\sigma)$ *has elliptic constituents, and* $R \simeq \mathbb{Z}_2^d$. *For* $1 \leq i \leq r$, *let* $\mathbf{M}_i$ *be the standard Levi component of* $\mathbf{G}$, *with split component* $\mathbf{A}_i$, *such that* $M_i \simeq GL_{m_i} \times G(n-m_i)$, *and* $W(\mathbf{G}, \mathbf{A}_i) = \langle C_i \rangle$. *Then the* R*-group* R_i *attached to* $i_{M_i,M}(\sigma)$ *is of order* $\mathbb{Z}_2^{d-1}$.

Proof. Since $i_{G,M}(\sigma)$ has elliptic constituents, $C_1 \ldots C_r \in R$, and thus $\Delta' = \emptyset$. Therefore, $\mathbf{M}_i$ satisfies Arthur's compatibility condition, which implies $R_i = R \cap W(\mathbf{M}_i, \mathbf{A})$. Suppose $R = \langle s_1, \ldots, s_d \rangle$. We assume that $s_1 = C_i C_B$ for some $B \subseteq \{1, \ldots, i-1, i+1, \ldots, r\}$. We claim that it is possible to choose $\{s_1, \ldots, s_d\}$ so that s_1 is the only generator of this form. For $j > 1$, we suppose that $s_j = C_{B_j}$. Set $s'_j = s_j$ if $i \notin B_j$, and $s'_j = s_1 s_j$ otherwise. Then $\{s_1, s'_2, \ldots, s'_d\}$ is a set of generators of the desired form. Therefore, $R_i = R \cap W(\mathbf{M}_i, \mathbf{A}) = \langle s'_2, \ldots, s'_d \rangle \simeq \mathbb{Z}_2^{d-1}$. $\square$

Lemma 3.8. *Suppose that* $M \simeq GL_{m_1} \times \cdots \times GL_{m_r} \times G(m)$. *Let* $\sigma \in \mathcal{E}_2(M)$, *and suppose that* $i_{G,M}(\sigma)$ *has elliptic constituents. We further suppose that* $R = R(\sigma) \simeq \mathbb{Z}_2^d$. *Then there is a set of generators* $\Omega = \{s_1, \ldots, s_d\}$ *for* R *so that, for each* i, $\Omega \setminus \{s_i\}$ *generates* R_{j_i}, *for some* j_i.

Proof. By Lemma 3.7 we can choose a set of generators $\{s_{11}, \ldots, s_{1d}\}$ for R with $R_{j_1} = < s_{12}, \ldots, s_{1d} >$ for some j_1. Now suppose $1 \le k < d$, and we have chosen a generating set $\Omega_k = \{s_{k1}, \ldots, s_{kd}\}$ for R with the property that, for each $1 \le i \le k$ the set $\Omega_k \setminus \{s_{ki}\}$ generates R_{j_i} for some j_i. Suppose $s_{k(k+1)} = C_B$, for some $B \subset \{1, \ldots, r\}$. Since

$$s_{k(k+1)} \in \bigcap_{i=1}^{k} R_{j_i},$$

we know that $j_i \notin B$ for each $1 \le i \le k$. Fix $j_{k+1} \in B$. Suppose that $i \ne k+1$ and $s_{ki} = C_{B_i}$. For $i \ne k+1$, we let

$$s_{(k+1)i} = \begin{cases} s_{ki} & \text{if } j_{k+1} \notin B_i \\ s_{k(k+1)} s_{ki} & \text{if } j_{k+1} \in B_i, \end{cases}$$

and take $s_{(k+1)(k+1)} = s_{k(k+1)}$. Set $\Omega_{k+1} = \{s_{(k+1)1}, \ldots, s_{(k+1)d}\}$. By our choice of Ω_k, Ω_{k+1}, and by Lemma 3.7, we see that for $1 \le i \le k$, the group R_{j_i} is generated by $\Omega_{k+1} \setminus \{s_{(k+1)i}\}$. Moreover, we have chosen the $s_{(k+1)i} = C_{D_i}$ so that $j_{k+1} \notin D_i$, for $i \neq k+1$. Thus, $R_{j_{k+1}}$ is generated by $\Omega_{k+1} \setminus \{s_{(k+1)(k+1)}\}$. Consequently, by induction, we can choose $\Omega = \Omega_d$ with the desired property. □

Theorem 3.9. *Let $G = GSp_{2n}$, GO_n, or GU_n. Suppose $\mathbf{M}$ is a Levi subgroup of $\mathbf{G}$, with $M \simeq GL_{m_1} \times \ldots \times GL_{m_r} \times G(m)$. Further suppose that $\sigma \in \mathcal{E}_2(M)$ satisfies conditions (a)–(c) of Proposition 3.3. Let $R = R(\sigma)$. Suppose $\kappa \in \hat{R}$ and that π_κ is the irreducible subrepresentation of $i_{G,M}(\sigma)$ parameterized by κ. Denote its character on G^e by Θ^e_κ. We set $w_0 = C_1 \ldots C_r$, and let $\varepsilon(\kappa) = \kappa(w_0)$. Then $\Theta^e_\kappa = \varepsilon(\kappa)\Theta^e_1$, where 1 is the trivial character.*

Proof. By Lemma 3.8, we can choose a set of generators $\Omega = \{s_1, \ldots, s_d\}$ for R with the property that $\Omega \setminus \{s_i\}$ generates R_{j_i} for some j_i. We fix a choice of such an Ω. Let $\kappa \in \hat{R}$. Denote by $s(\kappa)$ the number of $s_i \in \Omega$ for which $\kappa(s_i) = -1$. If $s(\kappa) = 0$, then $\kappa = 1$, and the claim is trivially true. Assume that the statement of the theorem is true whenever $s(\kappa) \leq s$. Suppose that $s(\kappa) = s+1$. Without loss of generality, we suppose that $\kappa(s_1) = -1$. We are assuming that $< s_2, \ldots, s_d >= R_j$ for some j. Let $\kappa_j = \kappa|_{R_j}$. If $\xi \in \hat{R}_j$, then

$$\hat{R}(\xi) = \{\chi \in \hat{R} \mid \chi|_{R_j} = \xi\} = \{\xi^+, \xi^-\},$$

where $\xi^\pm$ is determined by $\xi^\pm(s_1) = \pm 1$. Note that $\kappa = \kappa_j^-$, and consequently $s(\kappa_j^+) = s(\kappa) - 1 = s$. Therefore, by our induction hypothesis $\Theta^e_{\kappa_j^+} = \varepsilon(\kappa_j^+)\Theta^e_1$. Let τ_{κ_j} be the irreducible subrepresentation of $i_{M_j,M}(\sigma)$ parameterized by κ_j. Then, by Theorem 1.4, we have $i_{G,M_j}(\tau_{\kappa_j}) = \pi_{\kappa_j^+} \oplus \pi_{\kappa_j^-}$, and thus, $\Theta^e_\kappa = -\Theta^e_{\kappa_j^+} = -\varepsilon(\kappa_j^+)\Theta^e_1$. It is enough to show that $\varepsilon(\kappa) = -\varepsilon(\kappa_j^+)$. To see this, write $w_0 = s_1 w$, with $w \in R_j$. Now we have

$$\varepsilon(\kappa) = \kappa(w_0) = \kappa(s_1)\kappa(w) = -\kappa_j^+(s_1)\kappa_j^+(w) = -\kappa_j^+(w_0) = -\varepsilon(\kappa_j^+).$$

Therefore, $\Theta^e_\kappa = \varepsilon(\kappa)\Theta^e_1$, and the theorem follows by induction. □

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