Rational maps whose Fatou components are Jordan domains

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Abstract

We prove: If $f(z)$ is a critically finite rational map which has exactly two critical points and which is not conjugate to a polynomial, then the boundary of every Fatou component of f is a Jordan curve. If $f(z)$ is a hyperbolic critically finite rational map all of whose postcritical points are periodic, then there exists a cycle of Fatou components whose boundaries are Jordan curves. We give examples of critically finite hyperbolic rational maps f with the property that on the closure of a Fatou component Ω satisfying $f(\Omega) = \Omega$, $f|_{\partial\Omega}$ is not topologically conjugate to the dynamics of any polynomial on its Julia set.

1 Introduction

A *rational map* $f(z) = p(z)/q(z)$ where p and q are relatively prime complex polynomials determines a holomorphic map of the Riemann sphere $\widehat{\mathbb{C}}$ to itself, and so defines a holomorphic dynamical system. The *Fatou set* $J(f)$ is the set of those $z \in \widehat{\mathbb{C}}$ such that there exists a neighborhood U of z on which the iterates $\{f^n|_U\}$ form a normal family of holomorphic functions. The complement $J(f)$ of the Fatou set is called the *Julia set*. We shall assume throughout that the degree d of f is

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larger than one. The set of points $C(f)$ where the derivative of f vanishes is the set of *critical points* of f; these are the points where the local degree of the map is larger than one. Counted with multiplicity, there are $2d-2$ critical points.

A characteristic feature of the dynamics of iterated rational maps is that the behavior of the finite set of critical points under iteration strongly influences the dynamics of the map on the entire sphere. For examples of this, and basic definitions, see e.g. [Mil1]. The *postcritical set* $P(f)$ of a rational map f is defined by $P(f) = \cup_{n>0} f^n(C(f))$. The map f is said to be *critically finite* if $P(f)$ is finite. The map f is said to be *hyperbolic* if $P(f) \cap J(f) = \emptyset$.

It is known that the Julia set of a critically finite map is connected, and hence every Fatou component is an open disc. The boundaries of Fatou components for critically finite maps are known also to be locally connected. The boundaries of the Fatou components, however, need not be Jordan curves. A *Jordan domain* is a component of the complement of a Jordan curve in S^2.

In this paper we prove

Theorem 1.1 *Let $f(z)$ be a critically finite rational map with exactly two critical points, not counting with multiplicity. Then exactly one of the following possibilities holds:*

- *f is conjugate to z^d and the Julia set of f is a Jordan curve, or*
- *f is conjugate to a polynomial of the form $z^d + c, c \neq 0$, and the Fatou component corresponding to the basin of infinity under a conjugacy is the unique Fatou component which is not a Jordan domain, or*
- *f is not conjugate to a polynomial, and every Fatou component is a Jordan domain.*

Since a quadratic rational map has exactly two simple critical points, the hypothesis of Theorem 1.1 are satisfied for all postcritically finite quadratic rational maps. Theorem 1.1 confirms what had been experimentally observed in computer studies: that for many critically finite quadratic rational maps which are not polynomials, every Fatou component has Jordan curve boundary; see [Mil2].

Theorem 1.2 *Let f be a hyperbolic critically finite rational map. If every postcritical point of f is periodic, then there exists at least one cycle of Fatou components of f consisting of Jordan domains, and every Fatou component which maps onto an element of this cycle is also a Jordan domain.*

Corollary 1.3 *If $P(f)$ consists of a single superattracting cycle, then every Fatou component of f is a Jordan domain.*

A proof of Theorem 1.1 in the special case of certain critically finite quadratic maps appeared in [Ree], Section 5.4, as an ingredient in the classification of quadratic rational maps. The argument given assumes the following fact, specialized to the quadratic case:

Invariance Condition: *Let f be a critically finite rational map with exactly two critical points. Let Ω be a periodic Fatou component of period p for which $f^p|_\Omega$ is conjugate to $z \mapsto z^m$. Then for every $x \in \partial\Omega$, $|(f^p|_{\partial\Omega})^{-1}(x)| \leq m$.*

This fact need not hold for maps with three or more critical points: we give a degree three example in Section 5. These examples show (with the above notation) that $f^p|_{\partial\Omega}$ need not be topologically conjugate to the dynamics of any polynomial on its Julia set, even in the hyperbolic case. In addition, a Jordan curve in $\partial\Omega$ need not have a preimage under f^p which is contained in $\partial\Omega$. This shows that, if one associates to Ω an invariant lamination L in the sense of Thurston (see [Thu]), then L may fail to satisfy the condition of *gap invariance*. Since the writing of this paper Tan Lei and the author have obtained a description of these examples in terms of a new kind of surgery which will be the subject of a future paper; see [PT].

The process of *tuning* is a way of combining the dynamics of a rational map f with the dynamics of a polynomial p. An open question is to find conditions on a critically finite map f and a critically finite polynomial g for the tuning to be combinatorially equivalent to a rational map in the sense of Thurston. If f has a periodic Fatou component with non-Jordan curve boundary, it is generally believed that there exists some g for which the tuning has a topological obstruction to being combinatorially equivalent to a rational map. The converse is known to be false; see [Ahm], Theorem 5.11.1.

In Section 2 we state known facts from the theory of iterated rational maps which we use in the proofs of the theorems. In Section 3 we develop the main technique used in the proof: an analysis of how Jordan curves in the Julia set behave under backwards iteration. In Section 4 we prove Theorems 1.1 and 1.2. In Section 5 we give examples of non-polynomial maps which fail the Invariance Condition stated above.

2 Background

2.1 An important consequence of Montel's theorem

We will use the following proposition to control how a Jordan domain behaves under backwards iteration of a rational map.

Proposition 2.1 (Montel's Theorem) *Let f be a rational map and $U \subset \widehat{\mathbb{C}}$ be a connected open set whose complement contains at least three points. If $f^{-p}(U) \supset U$ for some $p > 0$, then U is in the Fatou set.*

Proof: For all $n \geq 0$, the images $(f^p)^n|_U(U)$ omit at least three points, and hence form a normal family of holomorphic functions by Montel's theorem. The iterates of $f|_U$ then form a normal family, and so U is in the Fatou set.

■

2.2 Postcritically finite rational maps

In this subsection we collect needed facts about critically finite maps. In particular, these maps have important expanding properties. For the definition of orbifold, the canonical orbifold associated to a critically finite map, and the definition of the associated canonical orbifold Poincaré or Euclidean metric, see [Mil1], Lemma 14.5. This metric is behaves very much like the usual Poincaré or Euclidean metric on $\widehat{\mathbb{C}} - P(f)$. Let $Q(f)$ denote the set of postcritical points which eventually land on cycles containing critical points. Then $Q(f) \subset F(f)$. The canonical orbifold metric ρ is supported on $\widehat{\mathbb{C}} - Q(f)$ and lifts under f to a metric $\tilde{\rho}$ on $\widehat{\mathbb{C}} - f^{-1}Q(f)$. With respect to the metric $\tilde{\rho}$ on $\widehat{\mathbb{C}} - f^{-1}Q(f)$ and the metric ρ on $\widehat{\mathbb{C}} - Q(f)$, the inclusion $\widehat{\mathbb{C}} - f^{-1}(Q(f)) \hookrightarrow \widehat{\mathbb{C}} - Q(f)$ is a strict contraction. We then have

Proposition 2.2 *Let f be a critically finite rational map. Then f is uniformly expanding with respect to the canonical orbifold metric ρ on the complement of any open neighborhood of $f^{-1}Q(f)$. In particular, f is uniformly expanding on $J(f)$ with respect to ρ.*

Proof: See [Mil1], Theorem 14.4.

Proposition 2.3 *Let f be a critically finite rational map. Then*

1. a Fatou component contains at most one critical point;

2. every Fatou component is eventually periodic;

3. the Julia set is connected;

4. the Julia set is the whole sphere iff there are no periodic critical points.

Proof: In a periodic Fatou component containing two or more critical points, at least one must have an infinite forward orbit, which cannot happen if the map is critically finite. This implies (1). To prove (2), we may appeal to Sullivan's No Wandering Domains Theorem [Sul], but in our case one can use expansion of the orbifold metric to give a direct argument. The point is that a wandering Fatou component must avoid a neighborhood of $Q(f)$, and so each iterate of f contributes a definite factor of expansion on such a component. Hence in a wandering sequence of components, the diameters of the components must tend to infinity with respect to the canonical orbifold metric, which is impossible. This proves (2). It now follows that every periodic Fatou component is a covering of an open disc branched over at most one point, hence every Fatou component is a disc and so (3) is proved. To prove (4), if there are no periodic critical points, then $Q(f)$ is empty and so f expands the orbifold metric at every point of the Riemann sphere. The Julia set of f is thus the entire sphere. Conversely, a periodic critical point is always in the Fatou set.

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Proposition 2.4 *Let f be a critically finite rational map and Ω a period p Fatou component. Then $\partial\Omega$ is locally connected.*

Proof: The proof is virtually identical to the proof of the well-known corresponding fact for subhyperbolic polynomials with connected Julia set; see [Mil1], Theorem 17.5. The only difference is that one uses the first return map f^p restricted to Ω in place of the polynomial restricted to its basin of infinity.

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2.3 Riemann mappings and local connectivity

A set $K \subset \mathbb{C}$ is said to be *full* if it is compact, connected, and if its complement is nonempty and connected. A full set is said to be *nondegenerate* if it is not a point.

We will need the following

Proposition 2.5 *Let K be a full nondegenerate subset of $\mathbb{C}$ whose boundary is locally connected. Let V be a bounded component of $\widehat{\mathbb{C}} - \partial K$. Then*

1. *V is a Jordan domain,*
2. *$\overline{V}$ and $\widehat{\mathbb{C}} - V$ are closed discs, and*
3. *a Jordan curve in K is contained in the closure of a unique bounded component U of $\widehat{\mathbb{C}} - \partial K$.*

The first conclusion is essentially the content of [DH1], Section 2.4.3, where it is stated without proof. For completeness, we give a proof. In what follows, Δ denotes the open unit disc $\{z : |z| < 1\}$ and $\Sigma = \widehat{\mathbb{C}} - \overline{\Delta}$.

Theorem 2.6 (Carathéodory) *Let K be a full nondegenerate set in $\mathbb{C}$. Let $\phi : (\Delta, 0) \to (\widehat{\mathbb{C}} - K, \infty)$ be a Riemann map uniformizing the complement of K in $\widehat{\mathbb{C}}$. Then ϕ extends to a continuous map $\overline{\phi} : \overline{\Delta} \to \widehat{\mathbb{C}}$ if and only if ∂K is locally connected, or if and only if K is locally connected.*

See [Mil1], Theorem 16.6 for the proof.

Let $\phi : (\Delta, 0) \to (U, z)$ be a Riemann map uniformizing an open disc U. For $t \in \mathbb{R}/\mathbb{Z}$ the *ray of angle t for ϕ* is the set $\{\phi(re^{2\pi it}) | r \in [0, 1)\}$, and is denoted by R_t. If R_t has a unique limit point x in ∂U, the ray R_t is said to *land* at x.

Theorem 2.7 *Let K be a full nondegenerate locally connected set in $\mathbb{C}$ and $U = \widehat{\mathbb{C}} - K$. Let $\phi : (\Delta, 0) \to (U, \infty)$ be a Riemann mapping. Suppose two distinct rays R_t and $R_{t'}$ of ϕ land at a common point x of ∂U. Then x separates ∂K so that each component of the complement of the Jordan curve $C = R_t \cup R_{t'} \cup \{x\}$ contains a nonempty component of $\partial K - \{x\}$.*

Proof: Since ∂K is locally connected, ϕ extends to a map $\overline{\phi}$ of the closed disc, by Carathéodory's Theorem. Suppose C failed to separate ∂K so that some component of its complement did not contain points of ∂K. Since ϕ is a homeomorphism and since rays cannot cross in U, by relabelling t and t' we may assume that for the set $W = \{re^{2\pi is} | |r| < 1, s \in (t,t')\}$, $\overline{\phi(W)} \cap K = \{x\}$. So $\overline{\phi}$ collapses the circular arc $(t,t') \subset S^1$ to the point x. But this contradicts the Theorem of F. and M. Riesz [Car], Volume II, Section 313.

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Proof of Proposition 2.5: Since K is full, its boundary is connected. Hence every bounded component of the complement of ∂K is an open disc.

Let V be a bounded component of $\mathbb{C} - \partial K$. Note first that $\overline{V} \subset K$, since K is full. We first show that ∂V is locally connected. Let $\phi : (\Sigma, \infty) \to (\widehat{\mathbb{C}} - K, \infty)$ be a Riemann map to the complement of K in $\widehat{\mathbb{C}}$. By Carathéodory's Theorem, ϕ extends to a continuous map on $\overline{\Sigma}$. Consider the equivalence relation on S^1 defined as follows: $x \sim y$ iff $\phi(x) = \phi(y)$. For each equivalence class in S^1, form its Euclidean convex hull in $\overline{\Delta}$. The convex hulls of any two distinct equivalence classes are disjoint. Let L be the union of the convex hulls of equivalence classes. Then ϕ extends to a map $\overline{\phi} : \overline{\Sigma} \cup L \to \overline{\widehat{\mathbb{C}} - K}$ by mapping the convex hull of any equivalence class $[x]$ to $\phi(x)$. Then $\partial K = \overline{\phi}(S^1 \cup L)$ and $\overline{\phi}$ is a homeomorphism from Σ to $\widehat{\mathbb{C}} - K$.

Given V, let x and y be distinct points on ∂V. Then $\overline{\phi}^{-1}(x)$ and $\overline{\phi}^{-1}(y)$ are not separated in $\overline{\Delta}$ by the preimage of any other point z in ∂K distinct from x and y. For otherwise there are rays $R_t, R_{t'}$ landing at z such that $C = R_t \cup \{z\} \cup R_{t'}$ is a simple closed curve in $\overline{\widehat{\mathbb{C}} - K}$ separating x and y, contradicting the fact that x and y lie in the boundary of a single component of $\widehat{\mathbb{C}} - \partial K$.

Let $X = \overline{\phi}^{-1}(\partial V)$, and let $\mathcal{CH}(X)$ be the Euclidean convex hull of X in $\overline{\Delta}$. Then $\mathcal{CH}(X) \supset X$, and since it is the convex hull of X its boundary is contained in ∂X. Its boundary is locally connected since the boundary of any bounded convex set is locally connected; see [New], ch. 6 section 4. Since X is closed, $\partial X \subset X$, and so $\overline{\phi}(\partial\mathcal{CH}(X)) \subset \partial V$. Moreover, $\overline{\phi}(\partial\mathcal{CH}(X)) = \partial V$. So ∂V is the continuous image of a compact locally connected set, and hence is locally connected, by Lemma 16.5 of [Mil1]. In particular, every point on ∂V is the unique limit point of some ray of ψ, where $\psi : \Delta \to V$ is a Riemann map uniformizing V.

We now show that ∂V is a Jordan curve. If ∂V is not a Jordan curve, there is a point $p \in \partial V$ which is the landing point of at least two rays for ψ. The union of these rays, together with the common landing point, gives a Jordan curve $C \subset \overline{V}$ which separates ∂V into at least two pieces, one of which lies in the bounded component W of $\widehat{\mathbb{C}} - C$ by Theorem 2.7. Then $\partial K \cap W \neq \emptyset$, hence there exist points of $\widehat{\mathbb{C}} - K$

in W. Since $\partial W \subset \overline{V} \subset K$, this implies that ∂W separates points in $\widehat{\mathbb{C}} - K$, and hence that $\widehat{\mathbb{C}} - K$ is not connected, contradicting K full.

Hence V is an open disc with Jordan curve boundary. The Schoenflies theorem implies that $\overline{V}$ and $\widehat{\mathbb{C}} - V$ are both homeomorphic to closed discs. If C is any Jordan curve contained in K, let W be the bounded component of $\widehat{\mathbb{C}} - C$. Then W is contained in the interior of K. If $\overline{W}$ is not contained in the closure of a unique bounded component V of $\widehat{\mathbb{C}} - \partial K$, then W must contain points of ∂K, which is impossible.

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2.4 Topological propositions

We will make extensive use of the following

Proposition 2.8 *If f is a rational map and U is a Jordan domain whose closure contains at most one critical value v of f, then every component V of $f^{-1}(U)$ is also a Jordan domain. If $v \in \partial U$, then $f|_{\overline{V}} : \overline{V} \to \overline{U}$ is a homeomorphism.*

Proof: Suppose first that $v \in U$. Let $U' = \overline{U} - \{v\}$. Let V' be the unique component of $f^{-1}(U')$ which contains a point of V. Then $f|_{V'} : V' \to U'$ is an unbranched covering of a closed punctured disc, and so V' is a closed punctured disc. Hence $\overline{V}$ is a closed disc, and so V is a Jordan domain.

If $v \in \partial U$, it suffices to show that $f|_{\overline{V}}$ is injective. Suppose otherwise. Then there exist $x_1, x_2 \in \partial V$ such that $f(x_1) = f(x_2) = y \in \partial U$. Choose $v \in V$, and let $\alpha_i, i = 1, 2$ be closed, embedded arcs in $\overline{V}$ whose interiors are disjoint open arcs in V and which join v to x_i. Then $f(\alpha_1 \cup \alpha_2)$ is a Jordan curve in $\overline{U}$ which intersects ∂U in exactly one point. Hence there is a component W of $V - (\alpha_1 \cup \alpha_2)$ such that $f(\partial W \cap \partial V) = y$. The set $\partial W \cap \partial V$ is not discrete, since $x_1 \neq x_2$. But this is impossible since f is holomorphic on $\widehat{\mathbb{C}}$.

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3 Jordan curves in $J(f)$

In this section we develop the techniques used in the proof of Theorems 1.1 and 1.2.

Convention: Throughout this section, $f(z)$ will denote either a postcritically finite rational map with at most one critical point in $J(f)$, or an iterate of such a map.

Given an *oriented* Jordan curve γ in the sphere, define the *inside* of γ, denoted by $\mathrm{Ins}(\gamma)$, to be the component of the complement of γ lying to the left of γ, and the *outside* $\mathrm{Out}(\gamma)$ of γ to be the component of the complement lying to the right of γ.

Let $\gamma \subset J(f)$ be an unoriented Jordan curve. A *lift* of γ we define to be a Jordan curve $\eta \subset f^{-1}(\gamma)$ such that $f|_\eta : \eta \to \gamma$ is a covering map. If $\gamma \subset J(f)$ is oriented, a *lift* of γ is an unoriented lift η of γ, equipped with an orientation so that $f|_\eta : \eta \to \gamma$ is orientation-preserving. Since f is an iterate of a map with at most one critical point in $J(f)$, a component of $f^{-1}(\gamma)$ is homeomorphic either to a Jordan curve, or to a one-point union of Jordan curves. Thus if $\deg(f) = d$, there are exactly d lifts of any Jordan curve in $J(f)$, counted with multiplicity equal to the absolute value of the degree of the map $f|_\eta : \eta \to \gamma$. We denote the set of lifts of a Jordan curve γ by $f^*(\gamma)$.

Let Ω be a Fatou component of f. It is convenient to replace the map f by a conjugate so that some point of Ω is the point at infinity. Then $\widehat{\mathbb{C}} - \Omega$ is a full locally connected subset of the plane which has empty interior if and only if Ω is the unique Fatou component of f, which in turn holds if and only if $f(z)$ is conjugate to a polynomial with a unique Fatou component which is the basin of infinity. We shall therefore make the additional assumption that $f(z)$ is not conjugate to such a polynomial.

An oriented Jordan curve γ is said to be *positively (negatively) oriented with respect to* Ω if $\Omega \subset \mathrm{Out}(\gamma)$ (respectively $\Omega \subset \mathrm{Ins}(\gamma)$). We also say that the *sign* $\mathrm{sign}(\gamma)$ of γ is positive (negative) if it is positively (negatively) oriented. We denote by γ both an oriented and an unoriented Jordan curve; in the following we will explicitly mention which is meant.

Notation:

- Let Γ denote the set of unoriented Jordan curves in the Julia set of f.
- Let A denote the set of closures of components of $\widehat{\mathbb{C}} - \Omega$. By Proposition 2.5, each element $a \in A$ is a closed disc. The proposition also implies that there is a well-defined function $p_A : \Gamma \to A$ which assigns to every $\gamma \in \Gamma$ the unique element $a \in A$ for which $\gamma \subset a$.
- Let $\Gamma^\pm$ denote the set of *oriented* Jordan curves in $J(f)$. Then $\Gamma^\pm = \Gamma^+ \cup \Gamma^-$, where Γ^+ (Γ^-) is the set of curves which are postively (respectively negatively)

oriented with respect to Ω. The function p_A extends naturally to $\Gamma^{\pm}$ by forgetting the sign and then applying p_A; the composition we denote again by p_A.

- Let $\Sigma\Gamma^{\pm}$ denote the set of infinite sequences $\{\gamma_n\}_{n=0}^{\infty}$ satisfying: $\gamma_n \in \Gamma^{\pm}$, $\gamma_{n+1} \in f^*(\gamma_n)$, and γ_{n+1} is equipped with an orientation so that $f|_{\gamma_{n+1}} : \gamma_{n+1} \to \gamma_n$ is orientation-preserving. If $\text{sign}(\gamma_n) \neq \text{sign}(\gamma_{n+1})$ we say that the sequence $\{\gamma_n\}_{n=0}^{\infty}$ has a *sign change* between γ_n and γ_{n+1}.

The idea for the proofs of our theorems is the following: consider the subset $S(\Omega)$ of $\Sigma\Gamma^{\pm}$ consisting of sequences $\{\gamma_n\}$ for which $\gamma_0 \in \partial\Omega$. We use Proposition 2.1, *Montel's theorem*, and the fact that the preimages of disjoint sets are disjoint to deduce relationships between sign changes in elements of $S(\Omega)$ and the value of the function p_A at the terms where sign changes occur. In the special cases where the hypothesis of the theorems are satisfied, this in turn will yield information about $\partial\Omega$.

It turns out, however, that the set $S(\Omega)$ is too large to be used in this manner, so we introduce a smaller space which captures the features in which we are interested. For any finite collection of disjoint unoriented Jordan curves in $J(f)$, there is at least one curve which is *outermost* in the following sense: it is not separated from Ω by any other curve in the collection. It is not unique, in general. We define $S_{out}(\Omega)$ to be the set of sequences $\{\gamma_n\}_{n=0}^{\infty} \in S(\Omega)$ such that the following holds: given any two consecutive terms γ_{n+1}, γ_n regarded as unoriented curves, γ_{n+1} is outermost among the collection of lifts of γ_n. We then say that γ_{n+1} is *outermost* among lifts of γ_n.

Let $\{\gamma_n\}_{n=0}^{\infty} \in S_{out}(\Omega)$. Fix some $n \geq 0$. If $\text{sign}(\gamma_n)$ is positive, set $U = \text{Ins}(\gamma_n)$ and $V = \text{Ins}(\gamma_{n+1})$; otherwise let $U = \text{Out}(\gamma_n)$ and $V = \text{Out}(\gamma_{n+1})$. In both cases, $U \subset \widehat{\mathbb{C}} - \Omega$. Let R be the the unique component of $f^{-1}(U)$ for which $\gamma_{n+1} \subset \partial R$.

Proposition 3.1 (Sign changes) *Given the hypotheses and notation in the preceding paragraph:*

1. *There is a sign change between γ_n and γ_{n+1} if and only if $V \supset R \supset \Omega$ (Figure 1). Furthermore, if there is a sign change,*

$$\partial R = \bigcup_{\substack{\delta \in f^*(\gamma_n) \\ \delta \text{ outermost}}} \delta.$$

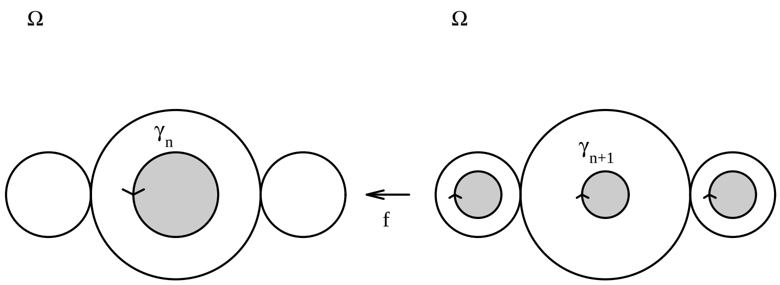

Figure 1: $U = \mathrm{Ins}(\gamma_n)$ is the shaded region on the left. R is the complement of the shaded regions on the right. Only one outermost lift $\gamma_{n+1} \subset \partial R$ is labelled.

2. *There is no sign change between γ_n and γ_{n+1} if and only if $R \subset V \subset \widehat{\mathbb{C}} - \Omega$ (Figure 2). Furthermore, if there is no sign change,*

$$f^{-1}(U) \subset \bigcup_{\substack{\delta \in f^*(\gamma_n) \\ \delta \ \ outermost}} V_\delta \ \subset \widehat{\mathbb{C}} - \Omega,$$

where $V_\delta = Ins(\delta)$ if δ is positive and $V_\delta = Out(\delta)$ otherwise.

3. *All outermost lifts of γ_n have the same sign.*

Proof: That $R \subset V$ is clear in both cases since V is a disc and $\gamma_{n+1} = \partial V \subset \partial R$.

1: Since $\partial V = \gamma_{n+1}$ is outermost among lifts of γ_n, there are no components of ∂R separating ∂V from Ω. Since $\partial R \subset \widehat{\mathbb{C}} - \Omega$ and there is a sign change, we must have $R \supset \Omega$. Since every other component of $f^{-1}(U)$ is disjoint from R, $R \supset \Omega$, and $\partial R \subset \widehat{\mathbb{C}} - \Omega$, every other component of $f^{-1}(U)$ is contained in $\widehat{\mathbb{C}} - \Omega$. Hence every other component of $f^{-1}(U)$ is separated from Ω by a lift of γ_n which is contained in ∂R. Hence any outermost lift of γ_n must be contained in ∂R. Since $R \supset \Omega$, every Jordan curve in ∂R must be outermost among lifts of γ_n. The other implication is then clear.

3: By Part 1, if there is a sign change between γ_n and γ_{n+1}, then all outermost lifts have the same sign since they comprise $\partial R \supset \Omega$. Hence all outermost lifts must have the same sign.

2: By Part 1, all outermost lifts of γ_n have the same sign. Since $U \subset \widehat{\mathbb{C}} - \Omega$ and there is no sign change between $\partial U = \gamma_n$ and $\partial V = \gamma_{n+1}$, $V \subset \widehat{\mathbb{C}} - \Omega$. Applying

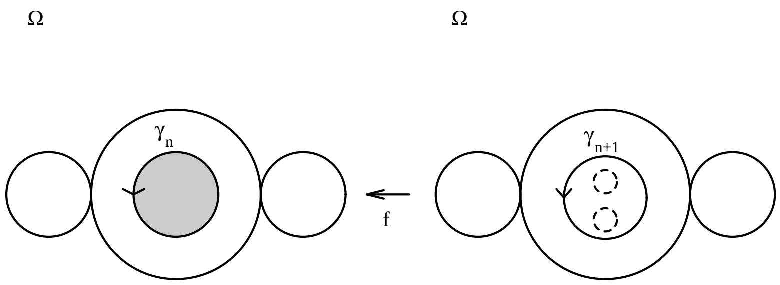

Figure 2: $U = \mathrm{Ins}(\gamma_n)$ is the shaded region on the left. R is region on the right bounded by γ_{n+1} and the two dashed Jordan curves, which are not outermost.

this to the collection of outermost lifts and using the definition of outermost proves the second assertion.

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As a corollary to the previous proposition, we have

Proposition 3.2 (Ω fixed iff no sign changes) *The following are equivalent:*

1. $f(\Omega) = \Omega$;
2. *for every sequence* $\{\gamma_n\}_{n=0}^{\infty} \in S_{out}(\Omega)$, $sign(\gamma_1) = sign(\gamma_0)$.
3. *for every sequence* $\{\gamma_n\}_{n=0}^{\infty} \in S_{out}(\Omega)$, *there are no sign changes;*

Proof: That $3 \Longrightarrow 2$ is obvious.
1 $\Longleftrightarrow$ 2: By the previous proposition, 2 holds if and only if for every $a \in A$, $f^{-1}(a) \subset \widehat{\mathbb{C}} - \Omega$. Since $\widehat{\mathbb{C}} - \Omega = \overline{\cup_{a \in A} a}$, we have that 2 holds if and only if $f^{-1}(\widehat{\mathbb{C}} - \Omega) \subset \widehat{\mathbb{C}} - \Omega$, which in turn holds if and only if $f(\Omega) \subset \Omega$. Since Ω is a Fatou component of f, this holds if and only if $f(\Omega) = \Omega$.
1 $\Longrightarrow$ 3: If there is a sign change between γ_n and γ_{n+1}, let U, R be as in Part 1 of the previous proposition. Then $f^{-1}(U) \supset \Omega$. Since $U \cap \Omega = \emptyset$, we have $f^{-1}(U) \cap f^{-1}(\Omega) = \emptyset$. Hence $f^{-1}(\Omega) \subset \widehat{\mathbb{C}} - f^{-1}(U) \subset \widehat{\mathbb{C}} - \Omega$. But then $f^{-1}(\Omega) \not\supset \Omega$, therefore $f(\Omega) \neq \Omega$.

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The next proposition relates the dynamics of curves in the boundary of a Fatou component Ω for which $f(\Omega) = \Omega$ to the dynamics inside $\widehat{\mathbb{C}} - \overline{\Omega}$.

Proposition 3.3 (Ω fixed) *Suppose Ω is forward-invariant under f. Let $E \subset Int(a)$ be a nonempty subset, and suppose $f^{-1}E \subset \cup_{i=1}^{i=r} b_i$, where the b_i's are distinct elements of A and $b_i \cap f^{-1}(E) \neq \emptyset$ for each i. Let γ_0 be the positively oriented boundary of a. If $\gamma_1 \in f^*(\gamma_0)$, then $p_A(\gamma_1) = b_i$ for some b_i.*

Remark: If $f(\Omega) = \Omega$, a Jordan curve $\gamma \subset \partial\Omega$ need not have a lift which is contained in $\partial\Omega$; see the examples in Section 5.

Proof: We may assume that γ_1 is outermost among lifts of γ_0. By Part 2 of Proposition 3.1, *Sign changes*, $f^{-1}(E) \subset f^{-1}(\mathrm{Ins}(\gamma_0)) \subset \widehat{\mathbb{C}} - \Omega$. Hence for $b \in A$, $f^{-1}(E) \cap b \neq \emptyset \iff f^{-1}(\mathrm{Ins}(\gamma_0)) \cap b \neq \emptyset$. Since $f^{-1}(\mathrm{Ins}(\gamma_0)) \subset \widehat{\mathbb{C}} - \Omega$, $f^{-1}(\mathrm{Ins}(\gamma_0)) \cap b \neq \emptyset$ if and only if there is a collection $\gamma_1^1, \gamma_1^2, ..., \gamma_1^k$ of outermost lifts of γ_0 such that $f^{-1}(\mathrm{Ins}(\gamma)) \cap b \subset \cup_{j=1}^{k} \mathrm{Ins}(\gamma_1^j)$. This proves the proposition.

∎

The next proposition refines the conclusion of the previous one in the case where the topology of the map f is simple. Note that the hypothesis is on the preimage of $\mathrm{Out}(\gamma_0)$, which contains Ω.

Proposition 3.4 (Ω fixed plus disc preimage) *Given the hypothesis in Proposition 3.3, Ω fixed, suppose further that the component V of the preimage of $Out(\gamma_0)$ containing Ω is a Jordan domain. Then $f^{-1}E \subset b$ for a unique $b \in A$, and Ω is a Jordan domain if and only if $a = b$.*

Proof: The boundary of V is the unique outermost preimage of γ_0, since V is a Jordan domain. Thus $f^{-1}E$ is contained in a unique $b \in A$, by Proposition 3.3, Ω *fixed.* If Ω is already a Jordan domain the statement is trivially satisfied; the other direction follows from the fact that if $a = b$, then $\partial V \subset a$ and so $V \supset \mathrm{Out}(\gamma_0) = f(V)$. But then V is in the Fatou set, by Proposition 2.1, *Montel's Theorem.* Since ∂V is a Jordan curve in $J(f)$, $V = \Omega$ is a Jordan domain.

∎

The next sequence of propositions treat the case when there are sign changes, i.e. when $f(\Omega) \neq \Omega$.

Proposition 3.5 **(Ω not fixed)** *Suppose $f(\Omega) \subset a_\Omega \in A$.*

1. *Let $\eta_0 = \partial a_\Omega$, equipped with positive orientation. Then there exists a negatively oriented lift η_1 of η_0 which is outermost among lifts of η_0.*
2. *If $\{\gamma_n\}_{n=0}^{\infty}$ has a sign change between γ_n and γ_{n+1}, then*
   (a) *$p_A(\gamma_n) = a_\Omega$,*
   (b) *there is a lift γ'_{n+1} of γ_n which is outermost among lifts of γ'_n such that $p_A(\gamma'_{n+1}) = a_\Omega$.*

Proof: **1:** Since $f(\Omega) \subset \text{Ins}(\eta_0) = a_\Omega$, $f^{-1}(\text{Ins}(\eta_0)) \supset \Omega$. Hence by Part 1 of Proposition 3.1, *Sign changes*, there is a negatively oriented lift η_1 of η_0 which is outermost among lifts of η_0. This proves the first assertion.

2(a): We argue by contradiction. We may assume that γ_n is postively oriented (otherwise, replace Ins with Out in what follows). If γ_n is contained in some $a \neq a_\Omega$, then $\text{Ins}(\gamma_n) \cap \text{Ins}(\eta_0) = \emptyset$, hence $f^{-1}(\text{Ins}(\gamma_n)) \cap f^{-1}(\text{Ins}(\eta_0)) = \emptyset$. Since $\Omega \subset f^{-1}(\text{Ins}(\eta_0))$, $f^{-1}(\text{Ins}(\gamma_n))$ cannot contain Ω. By Proposition 3.1, *Sign changes*, this implies that $\text{sign}(\gamma_{n+1}) = \text{sign}(\gamma_n)$.

2(b): We now prove the remaining assertion by contradiction. Again, we may assume that γ_n is positively oriented. Suppose no outermost lift of γ_n is contained in $a_\Omega = p_A(\gamma_n)$. Let R be the component of $f^{-1}(\text{Ins}(\gamma_n))$ which contains Ω. Then by Proposition 3.1, *Sign changes*, ∂R forms the collection of outermost lifts of γ_n. If no such lift is contained in $p_A(\gamma_n) = a_\Omega$, then $R \supset a_\Omega \supset \text{Ins}(\gamma_n) = f(R)$. Hence by Proposition 2.1, *Montel's theorem*, R is contained in the Fatou set of f, and this is impossible since $\partial a_\Omega \subset R \subset F(f)$ while at the same time $\partial a_\Omega \subset J(f)$.

■

As before, we now refine the conclusion of the previous proposition in the case when the topology of the map is simple.

Proposition 3.6 **(Ω not fixed plus disc preimages)** *Suppose for all oriented Jordan curves $\gamma \in J(f)$, equipped with positive orientation relative to Ω, every component of Ins(γ) is a Jordan domain. Then*

1. *If sign(γ_n) $\neq$ sign(γ_{n+1}), then $p_A(\gamma_n) = p_A(\gamma_{n+1}) = a_\Omega$, i.e. any sign changes are concentrated in a_Ω.*

2. *For any sequence* $\{\gamma_n\}_{n=0}^{\infty} \in S_{out}(\Omega)$*, if* $p_A(\gamma_n) \neq a_\Omega$*, there are no sign changes after the nth term;*

3. $(\cup_{n>0} f^n\Omega) - \Omega \subset a_\Omega$*;*

4. *If* η_0 *denotes the positively oriented boundary of* a_Ω*, then there exists a unique outermost negatively oriented lift* η_1 *of* η_0 *which is contained in* a_Ω*.*

5. $f^{-1}\Omega \subset a_\Omega$*.*

Proof:
1: By Part 1 of Proposition 3.1, *Sign changes*, there exists a component R of $f^{-1}(\mathrm{Ins}(\gamma_n))$ which contains Ω. R is a Jordan domain, by the hypothesis. Hence there is a unique outermost lift γ_{n+1}. The first statement then follows from Part 2 of Proposition 3.5, Ω *not fixed.*

2: Let $\{\gamma_n\}_{n=0}^{\infty} \in S_{out}(\Omega)$ be any sequence containing a sign change and suppose $p_A(\gamma_n) \neq a_\Omega$. Let k be the smallest postive integer such that there is a sign change between the $(n+k)$th and the $(n+k+1)$st term. We may assume that γ_{n+k} is positively oriented and γ_{n+k+1} negatively oriented (otherwise, replace Ins with Out in what follows). By hypothesis, and the assumption that there are no sign changes until the $(n+k+1)$st term, for $i = 1, 2, ..., k-1$, the unique component R_{n+i+1} of $f^{-i}(\mathrm{Ins}(\gamma_n))$ containing γ_{n+i+1} is a Jordan domain contained in $\widehat{\mathbb{C}} - \Omega$ whose boundary is γ_{n+i+1}. Since there is a sign change between γ_{n+k} and γ_{n+k+1}, by Part 1 above, $p_A(\gamma_{n+k+1}) = a_\Omega$. Let R_{n+k+1} be the unique component of $f^{-1}(R_{n+k})$ whose boundary is γ_{n+k+1}. Then since there is a sign change, by Part 1 of Proposition 3.1, *Sign changes*, $R_{n+k+1} \supset \Omega$. Hence $R_{n+k+1} \supset \mathrm{Ins}(\gamma_n) = f^{k+1}(R_{k+1})$. Proposition 2.1, *Montel's theorem*, then implies that R_{n+1} is in the Fatou set, which is impossible.

3: A consequence of Part 2 is the following: if γ_n is a positively oriented element of the sequence $\{\gamma_n\}_{n=0}^{\infty} \in S_{out}(\Omega)$, and if $p_A(\gamma_n) = a \neq a_0$, then every lift of γ_n is outermost. For since there are no sign changes, $f^{-1}(\mathrm{Ins}(\gamma_n)) \cap \Omega = \emptyset$, by Part 2 of Proposition 3.1, *Sign changes*. Every component of $f^{-1}(\mathrm{Ins}(\gamma_n))$ is a Jordan domain in $\widehat{\mathbb{C}} - \Omega$, by hypothesis. Hence no boundary component of $f^{-1}(\mathrm{Ins}(\gamma_n))$ separates another boundary component from Ω, and hence every lift of γ_n is outermost.

This observation has the following strong consequence: if $a \neq a_\Omega$, and if γ_0 is the positively oriented boundary of a, then $\cup_{n \geq 0} f^{-n}(\mathrm{Ins}(\gamma_0)) \cap \Omega = \emptyset$. We prove this by contradiction. Let $n \geq 0$ be the smallest postive integer for which $f^{-(n+1)}(a) \cap \Omega \neq \emptyset$. By induction and the result of the preceding paragraph, for every $1 \le i \le n$, every component of $f^{-i}(\mathrm{Ins}(\gamma_0))$ is a Jordan domain in $\widehat{\mathbb{C}} - \Omega$ whose boundary γ_i is outermost among lifts of $f(\gamma_i)$. If $f^{-(n+1)}(\mathrm{Ins}(\gamma_0)) \cap \Omega \neq \emptyset$, there is a finite sequence $\{\gamma_i\}_{i=0}^{n+1}$ of curves in $J(f)$ such that $\gamma_{i+1} \in f^*(\gamma_i)$, γ_{i+1} is outermost among lifts of γ_i for all $i \le n$, and $\mathrm{sign}(\gamma_{n+1}) \neq \mathrm{sign}(\gamma_n)$ (by Part 1 of Proposition 3.1), *Sign changes*. But this violates the conclusion of Part 2.

The result in the preceding paragraph implies that no component in the forward orbit of Ω can intersect an element of A which is not a_Ω.

4: By Part 1 of Proposition 3.5, Ω *not fixed*, existence is clear. Uniqueness follows since the unique component of the preimage of $\mathrm{Ins}(\eta_0)$ containing Ω is a Jordan domain.

5: By the previous step, $f^{-1}(\mathrm{Ins}(\eta_0))$ is a Jordan domain containing Ω whose boundary is contained in a_Ω. Hence $f^{-1}(\Omega) \subset f^{-1}(\mathrm{Out}(\eta_0)) \subset \widehat{\mathbb{C}} - f^{-1}(\overline{\mathrm{Ins}(\eta_0)}) \subset a_\Omega$.

∎

4 Proofs of the theorems

4.1 Proof of Theorem 2

If f is hyperbolic, there are no critical or postcritical points in $J(f)$, so we may apply the analysis in Section 3. The proof of Theorem 1.2 is then essentially a straightforward application of Proposition 3.4, Ω *fixed.*

Choose arbitrarily an element $x \in P(f)$. There is a partial ordering on $P(f)$ defined as follows: for two elements p and q of $P(f)$, $p < q$ if the boundary of the Fatou component containing q separates p from x.

Let y be any minimal element with respect to this ordering. Then y is a superattracting periodic point of period $p \ge 1$. Let Ω be the Fatou component containing y. Let $E = P(f) - \{y\}$. Since y is minimal and f is hyperbolic, $E \subset \mathrm{Ins}(\gamma_0)$, for some unique $\gamma_0 \in \partial\Omega$. Moreover, $(f^{-p}E) \cap E \neq \emptyset$. (If $p > 1$ this is obvious, since f^p must fix every point in the orbit of y; if $p = 1$, this follows since not all points in the postcritical set can land on y under one iterate of f.) Hence $f^{-p}(E) \cap \mathrm{Ins}(\gamma_0) \neq \emptyset$. Since $P(f) = P(f^p)$, the Jordan domain $\mathrm{Out}(\gamma_0)$ contains a unique critical value of f^p in its closure, since y is minimal and f is hyperbolic. Hence every component of the preimage of $\mathrm{Out}(\gamma_0)$ under f^p is a Jordan domain. Proposition 3.4, Ω *fixed plus disc preimage*, applied to f^p now shows that Ω is a Jordan domain. Since f

is hyperbolic and $P(f^p) = P(f)$, there are no elements of $P(f)$ in $\partial\Omega$. So every component Ω' of $f^{-n}(\Omega), n \geq 0$, is also a Jordan domain, since $\overline{\Omega'}$ is a branched cover of $\overline{\Omega}$ branched over at most one point which lies in the interior of Ω.

■

Proof of Corollary 3 By the above Theorem, the unique periodic cycle of Fatou components of f consists of Jordan domains. Since there are no critical points in the Julia set, there are no critical values for iterates of f in the boundaries of these Jordan domains. Hence they all pull back to Jordan domains under iterates of f.

■

4.2 Proof of Theorem 1

If $f(z)$ is conjugate to z^d, then it is well-known that $J(f) = S^1$.

Now suppose that upon conjugating by an automorphism of $\widehat{\mathbb{C}}$, f is equal to a polynomial. Let Ω be the basin of infinity. Then $J(f) = \partial\Omega$. If Ω is a Jordan domain, $J(f)$ is a Jordan curve. It then follows that there are exactly two Fatou components Ω and Ω', and these components satisfy $f^{-1}(\Omega) = \Omega$, $f^{-1}(\Omega') = \Omega'$. Since f is critically finite, this implies that f is of the form $z \mapsto z^d, d \geq 2$. If Ω is not a Jordan domain, then any other Fatou component of f is a component of $\widehat{\mathbb{C}} - \overline{\Omega}$. By Proposition 2.5, these components are all Jordan domains.

So we may assume that f is not conjugate to a polynomial or to a map of the form $z \mapsto z^d$. If there are no periodic critical points, then $J(f) = \widehat{\mathbb{C}}$, and there is nothing to prove. Otherwise, there is a periodic critical point c_1. Since there are exactly two critical points, each has multiplicity $d-1$, where $d = \deg(f)$. If the period of c_1 is equal to one, then f is conjugate to a polynomial. Hence we may assume $p \geq 2$.

Let v_1 be the image of c_1. Let Ω_0 be the Fatou component containing v_1. It suffices to show that Ω_0 is a Jordan domain whose closure contains exactly one critical value v_1. For from this it follows that every Fatou component contains at most one critical value in its closure. Since every Fatou component is eventually periodic, every Fatou component Ω' of f is a covering of a Jordan domain, branched over at most one point in its closure, and hence Ω' is a Jordan domain.

Let $\Omega_i = f^{p-i}\Omega_0, i = 1, ..., p$; note that $\Omega_p = \Omega_0$. Since the Ω_i are Fatou components, they are contained in unique components of $\widehat{\mathbb{C}} - \overline{\Omega_0}, i = 1, ..., p-1$.

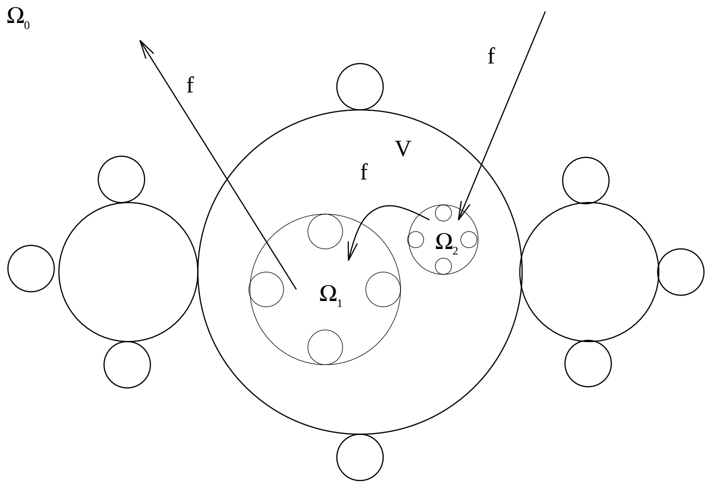

Figure 3: The $\Omega_i, i = 1, ..., p-1$ are contained in $V = a_{\Omega_0}$. Here $p = 3$.

By Proposition 3.2, Ω *fixed iff no sign changes*, since Ω is not fixed, there are sign changes in the set of sequences $S_{out}(\Omega_0)$. Let a_{Ω_0} be the component of $\widehat{\mathbb{C}} - \overline{\Omega_0}$ containing $f(\Omega_0)$, and let γ_0 be the positively oriented boundary of a_{Ω_0}.

Since there are exactly two critical points, there are exactly two critical values v_1, v_2. Since $v_1 \in \Omega_0$, a Jordan domain in $\widehat{\mathbb{C}} - \Omega$ contains at most one critical value v_2 in its closure, hence the preimage under f of every Jordan domain in $\widehat{\mathbb{C}} - \Omega_0$ is again a Jordan domain. By Proposition 3.6, Ω *not fixed plus disc preimages*, we have that $\cup_{i=1}^{p-1} \Omega_i \subset a_{\Omega_0}$. We then have a basic picture of part of the dynamics; see Figure 3.

Next, we prove

If $v_2 \in Int(a_{\Omega_0})$, then Ω_0 is a Jordan domain.

Let $D_0 = \mathrm{Out}(\gamma_0)$. Let $E = \{c_1\}$. Then $E \subset a_{\Omega_0}$ and $f^p(E) = E$. By Proposition 3.4, Ω *fixed plus disc preimage*, it suffices to prove that the unique component D_p of the preimage of D_0 under f^p which contains Ω_0 is a Jordan domain. We prove this by pulling back D_0 along the orbit of Ω_0 and using induction.

Let D_i be the component of the preimage of D_0 under f^p containing $\Omega_i, i = 0, ..., p$. We first claim that $D_1 \subset a_\Omega$. Since $v_2 \in a_{\Omega_0}$, D_0 contains exactly critical value in its closure, so D_1 is a Jordan domain. By Part 1 of Proposition 3.6, Ω

not fixed plus disc preimages, we must have $\gamma_1 = \partial D_1 \subset a_{\Omega_0}$ and its sign must be negative. Since the sign of γ_1 is negative, $\mathrm{Out}(\gamma_1) = D_1 \subset a_{\Omega_0}$.

We now use induction. Assume D_i is a Jordan domain contained in $\widehat{\mathbb{C}} - \Omega_0, i = 1, ..., n < p$. Then D_{n+1} is also a Jordan domain since D_n contains at most one critical value in its closure. A sign change between γ_n and γ_{n+1} implies that $\gamma_{n+1} \subset a_{\Omega_0}$, by Proposition 3.6, Ω *not fixed plus disc preimages*, and hence that $D_{n+1} \supset D_0 = f^{n+1}(D_{n+1})$. But this implies by Proposition 2.1, *Montel's theorem*, that Ω_0 is a Jordan domain fixed under the $(n+1)$st iterate of f, which is impossible if $n + 1 < p$. The absence of a sign change then implies that $D_{n+1} \subset \widehat{\mathbb{C}} - \Omega_0$, and so the induction proceeds. Hence D_{p-1} is a Jordan domain in $\widehat{\mathbb{C}} - \Omega_0$, and so D_p is a Jordan domain.

∎

So to prove Theorem 1.1, it suffices to prove (using the notation in the preceding discussion)

Proposition 4.1 *The critical value v_2 is contained in the open disc a_{Ω_0}.*

As an immediate consequence, we have the following corollary:

Corollary 4.2 *Let f be a critically finite rational map which has exactly two critical points, and which is not conjugate to a polynomial. Then no Fatou component of f contains two critical values in its closure.*

Proof of Proposition: We argue by contradiction. Let $D_0 = \mathrm{Out}(\gamma_0)$. Let D_i be as above. We will show that $\partial D_0 \subset \partial D_p$, and that $f^p|_{\partial D_0} : \partial D_0 \to \partial D_0$ is a homeomorphism. Since postcritically finite maps are expanding on their Julia sets with respect to the canonical orbifold metric, by Proposition 2.2, any compact connected set in $J(f)$ mapped homeomorphically onto itself is a point. This gives a contradiction.

In order to carry out the argument, we need to show that $\partial D_i \subset \partial \Omega_i \subset a_{\Omega_0}, i = 0, 1, ..., p$. This will be implied by the following lemma. (We will only need the case where X_0 is a Fatou component homeomorphic to an open disc and Y_0 is homeomorphic either to the sphere minus a finite union of disjoint closed discs, or to the sphere minus a finite union of closed discs whose boundaries meet in exactly one common point to all of them.)

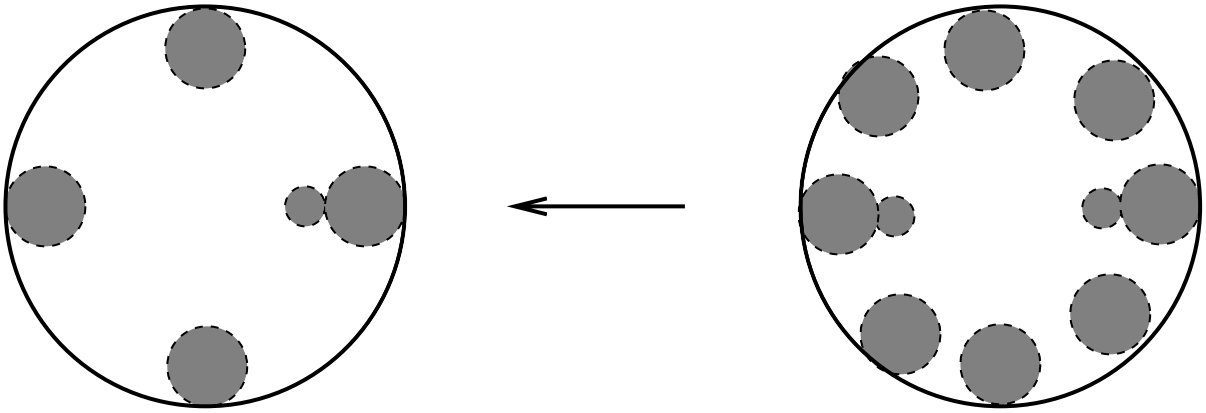

Figure 4: Y_0 and Y_1 are the large discs. X_i is the complement of the shaded discs in $Y_i, i = 1, 2$.

Lemma 4.3 *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map. Let X_0 and Y_0 be proper open subsets of $\widehat{\mathbb{C}}$with $X_0 \subset Y_0$. Suppose $\partial Y_0 \subset \partial X_0$.*

1. *If $Y_1 = f^{-1}Y_0$ and $X_1 = f^{-1}X_0$, then $\partial Y_1 \subset \partial X_1$. (See Figure 4.)*
2. *If Y_1 is a component of $f^{-1}Y_0$, if $f|_{\overline{Y_1}} : \overline{Y_1} \to \overline{Y_0}$ is a homeomorphism, and if $X_1 = (f|_{\overline{Y_1}})^{-1}(X_0)$, then $\partial Y_1 \subset \partial X_1$.*

Proof of Lemma:

1. Since f is a nonconstant rational map, it is an open map, and so for any proper open subset $Z \subset \widehat{\mathbb{C}}$, $f^{-1}\partial Z = \partial f^{-1}Z$. So $\partial Y_1 = \partial f^{-1}Y_0 = f^{-1}\partial Y_0 \subset f^{-1}\partial X_0 = \partial f^{-1}X_0 = \partial X_1$.

2. Since Y_1 is a component of $f^{-1}Y_0$, $f(\partial Y_1) \subset \partial Y_0$. Since $f : \overline{Y_1} \to \overline{Y_0}$ is a homeomorphism and $X_1 = (f|_{\overline{Y_1}})^{-1}(X_0), \partial X_1 = (f|\overline{Y_1})^{-1}(\partial X_0)$. Hence $\partial Y_1 \subset \partial X_1$.

∎

Remark: The second statement is no longer true if we drop the requirement that $\overline{Y_1}$ maps homeomorphically to $\overline{Y_0}$. For example, let Y_0 be the open unit disc, let X_0 be the open disc minus the interval [0,1), and let $f(z) = z^2$ map the Riemann sphere to itself. Let X_1 be the intersection of the upper half-plane $\{z|Im(z) > 0\}$ with the unit disc, and let Y_1 be the unit disc again. Then all other hypotheses of the lemma are satisfied but $\partial Y_1 \not\subset \partial X_1$.

Proof of Proposition 4.1, continued:

Suppose $v_2 \notin \text{Ins}(\gamma_0) = \text{Int}(a_{\Omega_0})$, and let $D_1 = f^{-1}(D_0)$.

1. We first show that $\partial D_p \subset \partial\Omega_0$.

   Since $v_1 \in \Omega_0 \subset D_0$, $D_1 = f^{-1}(D_0)$. The first case of Lemma 4.3 then applies, and so $\partial D_1 \subset \partial\Omega_1 \subset a_{\Omega_0}$. It follows that D_1 must be contained in a_{Ω_0}. For otherwise, $D_1 \supset D_0$, and so $D_1 \subset F(f)$, by Proposition 2.1, *Montel's theorem.* But then $p = 1$ and so Ω is fixed.

   When $v_2 \notin a_{\Omega_0}$, the region D_1 is homeomorphic to the complement of d disjoint closed discs. If $v_2 \in \gamma_0 = \partial a_{\Omega_0}$, D_1 is homeomorphic to the complement in the sphere of a union of d closed discs whose boundaries meet at exactly one point. Note that in both cases the boundary of D_1 consists of exactly d lifts of γ_0, each of which maps homeomorphically to γ_0. This can be seen as follows: $\widehat{\mathbb{C}} - \overline{D_0}$ is a Jordan domain which contains either no critical points in its closure, or one critical point in its boundary. Hence every component of its preimage is a Jordan domain, there are exactly d such components, and the boundary of each maps homeomorphically under f.

   We now argue by induction. Assume for $1 \leq i < p$ that D_i is contained in a_{Ω_0}, and that $\partial D_i \subset \partial\Omega_i$. We show that this implies $\partial D_{i+1} \subset \partial\Omega_{i+1}$ if $i < p$, and that D_{i+1} is contained in a_{Ω_0} if $i < p - 1$. Since $\Omega_i \subset a_{\Omega_0}$, and $D_i \subset a_{\Omega_0}$ with $\partial D_i \subset \partial\Omega_i$, D_i is contained in a_{Ω_0}, for otherwise D_i contains D_0, implying by Proposition 2.1, *Montel's theorem*, that Ω_0 is fixed under f^{i+1}. The set $V = \text{Int}(a_{\Omega_0})$ is a Jordan domain containing at most one critical value in its closure. Let V' be the unique component of $f^{-1}V$ whose closure contains D_{i+1}. Since V contains no critical values and ∂V contains at most one critical value, $f|_{\overline{V'}} : \overline{V'} \to \overline{V}$ is a homeomorphism. By restriction, $f|_{\overline{D_{i+1}}} : \overline{D_{i+1}} \to \overline{D_i}$ is also a homeomorphism. We may now apply the second case of Lemma 4.3 to conclude that $\partial D_{i+1} \subset \partial\Omega_{i+1}$. Moreover, if $i + 1 < p$, then $\Omega_{i+1} \subset a_{\Omega_0}$, and hence $\partial\Omega_{i+1} \subset a_{\Omega_0}$. $D_{i+1} \subset a_{\Omega_0}$ if $i < p - 1$, for otherwise $D_{i+1} \supset D_0 = f^{i+1}(D_{i+1})$. Proposition 2.1, *Montel's theorem*, would then imply that $D_{i+1} \subset F(f)$, whence $D_{i+1} = D_0 = \Omega_0$ and Ω_0 is periodic of period strictly less than p, a contradiction.

Hence $\partial D_p \subset \partial \Omega_p = \Omega_0$, and $D_p \supset \Omega_0$.

2. We next claim that the boundary of every component of the complement of D_{i+1} (for convienience, let us call these *boundary pieces* of D_i), $i = 0, ..., p-1$ maps injectively onto its image under f, and so that every boundary piece of D_p maps homeomorphically onto its image ∂D_0 under f^p. That this holds for $i = 0$ has already been proved. For $1 \leq i \leq p-1$, the map $f : \overline{D_{i+1}} \to \overline{D_i}$ is a homeomorphism, by the argument given in the previous paragraph.

3. We now claim that f^p maps γ_0 homeomorphically to itself, which we have shown is impossible.

   By Proposition 3.3, Ω *fixed*, applied to f^p, there must be some boundary piece of D_p contained in a_{Ω_0}. But since $\partial D_p \subset \partial \Omega$, this implies that some boundary piece of D_p is actually equal to $\partial a_{\Omega_0} = \gamma_0$. The map f restricted to a single boundary component of $D_i, i = 1, ..., p$ is a homeomorphism, by the previous step, and so the map f^p sends γ_0 homeomorphically to itself.

■

5 Examples

5.1 Examples where the Invariance Condition fails

J. Kahn, C. McMullen and the author discovered a degree four map where the Invariance Condition appears to fail: the map turned out to be

$$z \mapsto \frac{3(z-1)^3(z+3)}{3-8z+6z^2}$$

whose Julia set is given in Figure 5. The black regions just to the right and left of the pinched point are the immediate basins of attraction of a period two superattracting cycle. All black regions eventually map onto these basins. The white regions all eventually map onto the basin of infinity, which is forward-invariant.

This map is one member in a family of maps of varying degree: set

$$f_d(z) = N_d \circ p_d \circ M(z)$$

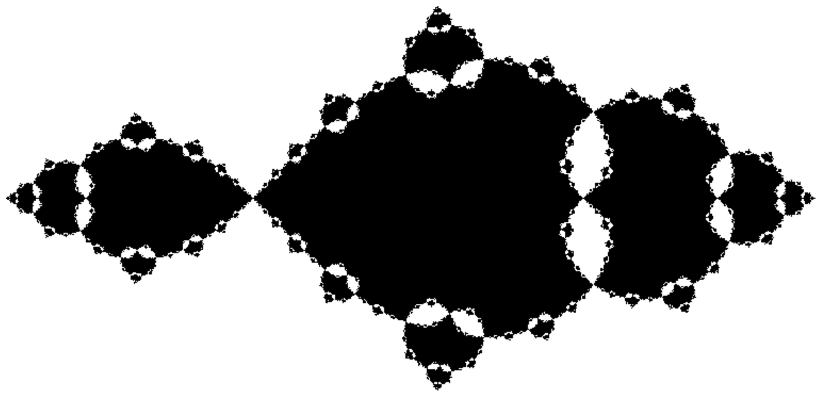

Figure 5: The degree 4 pseudo-basilica

where $M(z) = \frac{z-1}{z}$, $p_d(z) = (d-1)z^d - dz^{d-1} + 1$, and $N_d = (1-d)\frac{z-1}{z}$. For this family, infinity is a simple critical point, 1 is a critical point of local degree $d-1$, 0 is a critical point of local degree d, 1 maps to 0, 0 maps to $1-d$, and $1-d$ maps to 0. For $d = 2$ one obtains a map conjugate to $z \mapsto z^2 - 1$, whose Julia set is called the basilica. For $d \geq 3$, however, on the basin of the unbounded Fatou component Ω, the map is conjugate to $z \mapsto z^2$, but $\partial\Omega$ appears to homeomorphic to a figure-8. We refer to the Julia sets of f_d as "pseudo-basilicas".

We now give a direct argument in the case $d = 3$ which shows that the Invariance Condition fails for the boundary of the basin of infinity. For the definition of Thurston equivalence of branched coverings, and Thurston's theorem on the existence of a rational map in a given Thurston class, see e.g. [Ree] or [DH2].

5.2 The degree three pseudobasilica

Let g be the map $f_3(z) = \frac{(z+2)(z-1)^2}{\frac{3}{2}z-1}$ and Ω be the basin of infinity of g. Figure 6 is a picture of its Julia set. The point at which $\partial\Omega$ appears pinched is not a critical point of g. We give a proof that $g|_{\partial\Omega}$ fails the invariance condition which depends strongly on the degree and the fact that the map is real.

Since the point at infinity is a simple superattracting critical point and g is postcritically finite, a theorem of Böttcher ([Mil1], Theorem 6.7) implies that there exists a unique Riemann map $\phi : (\Delta, 0) \to (\Omega, \infty)$ such that $\phi(z^2) = f(\phi(z))$. In

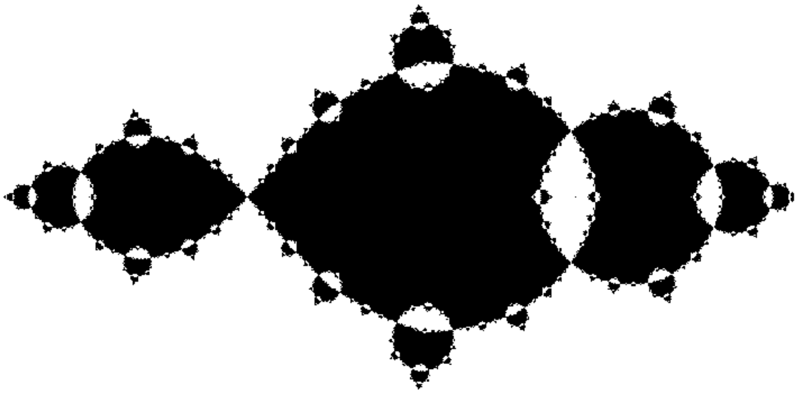

Figure 6: The pseudo-basilica in degree 3

what follows, "ray" means a ray for the map ϕ.

Step 1 We first claim that the $1/3$ and $2/3$ rays land at a common fixed point. First, we show that every point of period less than or equal to two is real. There are ten total, one of which is the point at infinity. The remainder are the finite solutions of the equation $g^2(z) - z = 0$. Two solutions are the period 2 attractors 0 and -2. Another is the fixed point 2, which is the landing point of the 0 ray in the basin of infinity. The remaining six solutions are roots of the polynomial $(4z^4 - 2z^3 - 15z^2 + 16z - 4)(2z^2 + z - 2)$, all of which are real. We next claim that the $1/3$ and $2/3$ rays in Ω land at a common fixed point p. Since g is real, it commutes with conjugation, so $R_{1/3} = \overline{R_{2/3}}$. The landing point of $R_{1/3}$ is therefore the complex conjugate of the landing point of $R_{2/3}$. But these landing points are points of period less than or equal to two, so by Step 1, they must be real and hence equal. Since the two rays are exchanged under the dynamics, the common landing point x is actually fixed under g.

Step 2 Let C be the closed curve which is the union of the point at infinity, the $1/3$ and $2/3$ rays, and p. We claim that C separates 0 from -2 in $\widehat{\mathbb{C}}$. For otherwise, one component of U of $\widehat{\mathbb{C}} - C$ is an open disc containing no elements of $P(g) = P(g^2)$. The preimages of U under g^2 are then all disjoint open discs. The curve C is fixed as an oriented curve under g^2. So for some preimage V of U under g^2, $V = U$, and so U must be contained in the Fatou set. But this then implies that C cannot separate $\partial\Omega$, contradicting Theorem 2.7.

Step 3 We next claim that there is a unique (up to combinatorial equivalence)

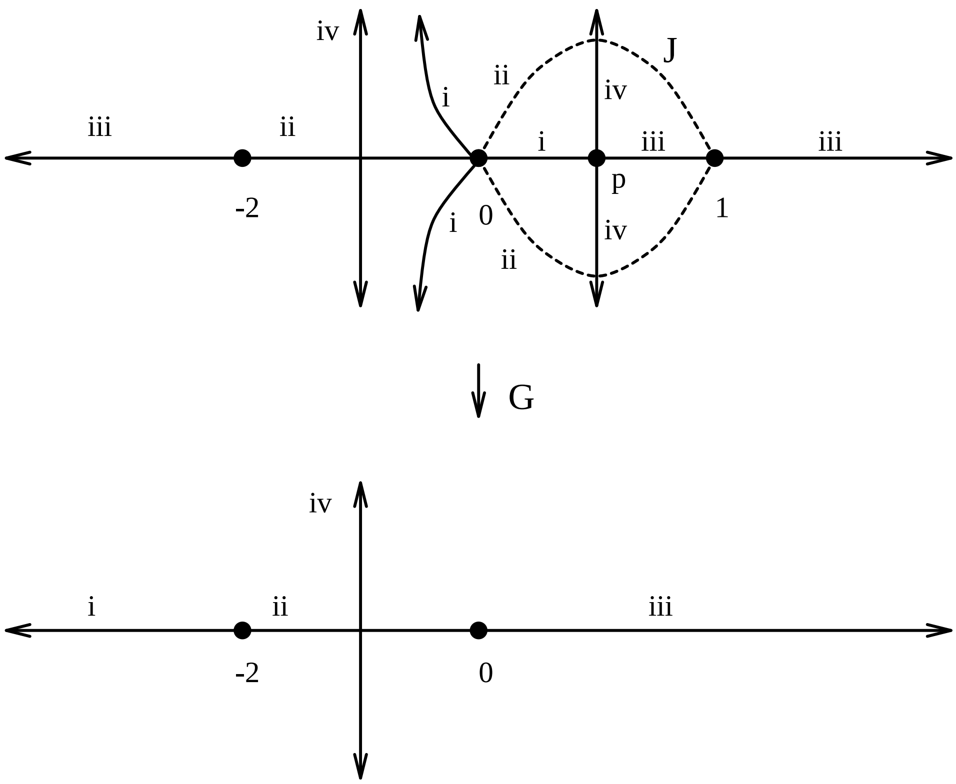

Figure 7: The branched covering G. The bounded region formed by the dashed curve J maps onto the complement of the arc numbered by (ii).

real degree three branched cover of the sphere with the same postcritical data. Since any critically finite branched covering G with $|P(G)| = 3$ and hyperbolic orbifold is Thurston equivalent to a rational map, it suffices to prove there is a unique rational map with this data. By conjugating we may assume that infinity is a fixed simple critical point, 1 maps with local degree two to zero and -2 maps with local degree one to zero. These conditions imply that g is of the form $g(z) = \frac{(z-1)^2(z+2)}{az-b}$. If we require that zero is to map with local degree three onto its image and then back to itself, there are a unique parameters a and b, namely $a = 3/2$ and $b = -1$.

Step 4 The previous step implies that the branched cover G described in Figure 7 is Thurston equivalent to the map g. The top figure is to be overlaid the bottom one to form a critically finite branched covering of the sphere to itself.

For this map, a loop γ, separating 0 from -2, and represented by the line (iv) in the bottom half of Figure 7 union the point at infinity, has two preimages. One preimage is homotopic to γ and maps by degree -1, while the other is a closed curve mapping to γ by degree two, represented as the union of the point at infinity together with the two arcs in the top figure labelled (iv) passing through the pole p.

Since G is Thurston equivalent to g, and C corresponds to the curve γ under the obvious Thurston equivalence, it follows that the curve C must also have a preimage which maps to C by degree two. Hence the endpoints of $R_{1/6}$ and $R_{5/6}$ are necessarily distinct, implying that p has three preimages in $\partial\Omega$ under g. For any polynomial $p(z)$ of degree d with basin of infinity Ω', $J(p) = \partial\Omega'$, and so $\partial\Omega'$ is totally invariant under p. If $g|_{\partial\Omega}$ were topologically conjugate to a polynomial, a generic point in $\partial\Omega$ would then have three preimages, and so $\partial\Omega$ would be totally invariant under g. Since $g(\Omega) = \Omega$, if $\partial\Omega$ is totally invariant, then so is Ω, which it is not. Hence $g|_{\partial\Omega}$ cannot be topologically conjugate to the dynamics of any polynomial on its Julia set.

Remark: A similar proof works to show that the maps f_d possess the same property. The only significant difference is that a different argument in Step 1 is required. One can prove this using the fact that γ is fixed up to homotopy relative to $P(G)$ as an unoriented curve under G, together with the fact that the maps f_d are expanding on their Julia sets.

Remark: For the map g, if Ω is the basin of infinity, the set A consists of precisely two elements. Let a_{-2}, a_0 be the closures of the components of $\widehat{\mathbb{C}} - \overline{\Omega}$ containing -2 and 0 respectively. Let $\gamma_{-2} = \partial a_{-2}$ and $\gamma_0 = \partial a_0$ with positive orientation relative to Ω. Then γ_{-2} has a unique lift $\widetilde{\gamma_{-2}}$ mapping by degree $+3$. There is an open topological arc $\alpha \subset \widetilde{\gamma_{-2}}$ mapping homeomorphically to $\gamma_{-2} - \{x\}$; this open arc is the portion of the boundary of the immediate basin of 0 lying between the landing points of the 1/6 and 5/6 rays; see Figure 6. Thus $\widetilde{\gamma_{-2}} \subset a_0$ is not contained in $\partial\Omega$. The curve γ_0 has two lifts. One is γ_{-2} which maps by degree $+1$; the other maps by degree $+2$ and is contained in a_0 but not in ∂a_0.

5.3 Other interesting examples

The family $g_r(z) = \frac{r}{d-1} N_d \circ p_d \circ M$, provide other examples of maps which appear to fail the Invariance Condition. For example, if $d = 3$, and if r is a solution of

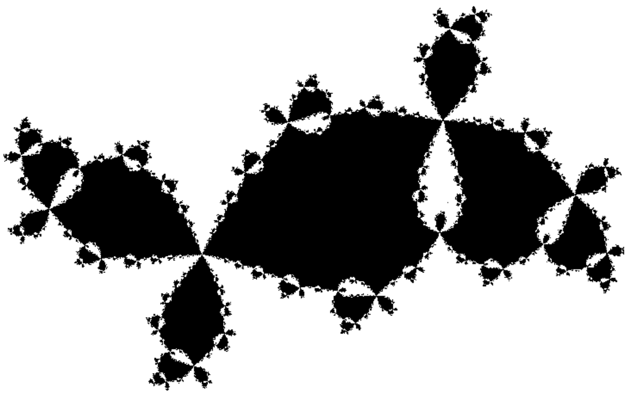

Figure 8: A degree 3 pseudo-rabbit

$g_r^3(0) = 0, g_r(0) \neq 0$, the Julia sets look quite interesting. A complex solution is $r \approx 1.34781 + 1.02885i$ and yields a "pseudorabbit", as is shown in Figure 8.

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