A note on tiling with integer-sided rectangles.

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Abstract

We show how to determine if a given simple rectilinear polygon can be tiled with rectangles, each having an integer side.

1 Introduction

In [6], Stan Wagon provides us with 14 proofs of the fact that if a rectangle R is tiled with rectangles, each having at least one side of integral length, then R has a side of integral length.

Rather than simply add a fifteenth proof to his list, we would like to address the much more general problem of tileability of arbitrary rectilinear polygons. In particular we will give an algorithm for deciding when a rectilinear polygon (that is, a polygon with sides parallel to the axes) can be tiled with rectangles, each having an integer side.

The proof idea comes from a neat method, due to John Conway in the case of polyominos, for dealing with tiling problems of this sort. That method is to define a group, the "tiling group" which depends on the set of tiles involved in the particular problem, and gives a necessary condition for the tileability of a given simply connected region.

This method has been used with success in many cases where the tiles have simple shapes ([5], [4], [2], [3]). Here we apply it to Wagon's problem. Surprisingly, the method resembles closely that of [5] (or [2]) in the case of dominos.

One corollary to our construction is a simple description of the set of all possible tilings of a polygon R (Theorem 4).

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2 The tiling group

Rather than define the tiling group we will simply use one of its quotient groups; this will be sufficient for our purposes. For background into defining the tiling group in a more general setting see the references.

Let S^1 denote the group $\mathbb{R}/\mathbb{Z}$. Let $G = S^1 * S^1$, the free (nonabelian) product of S^1 with itself. Don't let the size or topology on this group worry you. Despite being non-locally compact, it is a rather easy group to work with.

An element of G is a product of elements in one or the other factor, for example a typical element is

$$h(\frac{1}{3})h(\frac{1}{2})v(-.12)h(\pi)v(\sqrt{2}),$$

where the symbols $h(\cdot), v(\cdot)$ refer to elements of the first or second free factor, respectively. An element of G can be written in **reduced form** using the (confluent) identities $h(t_1)h(t_2) = h(t_1 + t_2)$, $v(t_1)v(t_2) = v(t_1 + t_2)$, and $h(0) = v(0) = e$ (the identity). The reduced form of an element is either e or an alternating sequence of $h(t)$'s and $v(t)$'s, where the $t \neq 0$ (the element can start and end in either an h or a v).

Multiplication in G is just concatenation of the corresponding expressions.

To a rectilinear polygonal path α starting from the origin we associate an element of G as follows. The path α is an ordered sequence of horizontal and vertical edges; to a horizontal edge of "length" $t \in \mathbb{R}$ (that is an edge from a point with x-coordinate a to a point with x-coordinate $t+a$) we associate the element $h(t)$. For a vertical edge from y-coordinate b to y-coordinate $b+t$ we associate the element $v(t)$. Then the element of G corresponding to a path is just the product of the elements coming from the sequence of horizontal and vertical edges of that path.

For example, the path running counterclockwise around the boundary of a $t_1 \times t_2$ rectangle with the origin in its lower left corner is:

$$h(t_1)v(t_2)h(-t_1)v(-t_2). \tag{1}$$

For a simple rectilinear polygon R having the origin on its boundary, let $\alpha(R)$ be the element of G corresponding to the path running counterclockwise around the boundary of R.

Lemma 1 *If R is tileable with rectangles, each having an integer side, then $\alpha(R) = e$ in G.*

Proof. A tiling of R gives a representation of $\alpha(R)$ as a product of "lassos", i.e. words of the form $xh(t)v(t')h(-t)v(-t')x^{-1}$ (conjugates of $[h(t), v(t')]$), where the words $[h(t), v(t')]$ are the boundaries of the tiles: this can be proved inductively on the number of tiles, taking at the inductive step a tile touching ∂R (Figure 1), see also [1].

Since each tile has an integral side, for each lasso $x[h(t), v(t')]x^{-1}$, one of t, t' is integral and so the lasso is trivial in G. Thus $\alpha(R)$ is the product of trivial elements of G and so is trivial. $\square$

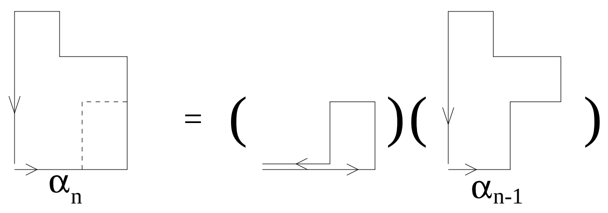

Figure 1: Inductive step in the proof that a tiling gives a product of lassos.

This lemma is a necessary condition for tileability of a region by integer-sided rectangles. It is not sufficient, as can be seen by considering for example the (untileable) polygon whose boundary word is

$$h(\frac{2}{3})v(\frac{1}{3})h(-\frac{1}{3})v(1)h(-\frac{2}{3})v(-\frac{1}{3})h(\frac{1}{3})v(-1) = e.$$

Corollary 2 *If a $t_1 \times t_2$ rectangle R is tileable with integer-sided rectangles, then either $t_1 \in \mathbb{Z}$ or $t_2 \in \mathbb{Z}$.*

Proof. The element $\alpha(R) = h(t_1)v(t_2)h(-t_1)v(-t_2)$ is trivial if and only if either $h(t_1) = e$ or $v(t_2) = e$, that is, if and only if $t_1 = 0 \bmod 1$ or $t_2 = 0 \bmod 1$. $\square$

We leave the reader to compare this proof with those of [6].

3 The algorithm

We give here an algorithm for deciding if a given simply rectilinear polygon R can be tiled with integer-sided rectangles.

Firstly, it is necessary that $\alpha(R) = e$ in G by Lemma 1.

On the group G define the distance function: for $x, y \in G$, $d(x, y) = d(xy^{-1}, e)$, where $d(x, e)$ is the length of the reduced word representing x in terms of the number of h's and v's. For example, the word in (1) has length 4 assuming neither t_1 nor t_2 is integral.

Let $x_0 \in G$ be the following base point:

$$x_0 = (h(-\frac{1}{2})v(-\frac{1}{2}))^k = h(-\frac{1}{2})v(-\frac{1}{2})h(-\frac{1}{2})\ldots v(-\frac{1}{2}),$$

where k is sufficiently large so that x_0 is far away from $\alpha(R)$ and any of its prefixes. We have $d(x_0, e) = 2k$.

We define a height function h_R on ∂R as follows. This is a function on points of ∂R taking nonnegative integer values. Let $y \in \partial R$, and α_y be the path around R counterclockwise from 0 to y. Then define $h_R(y) = d(\alpha_y, x_0)$.

Since $\alpha(R) = e$, we have $h_R(y) = d(\alpha'_y, x_0)$ where α'_y is the path from 0 to y going around R clockwise instead (α_y and α'_y are the same element of G). Similarly, if R is tiled by integer-sided rectangles, we can extend h_R to a height function on all of the edges in the tiling: define h_R on a point y on an interior edge by simply taking any path w_y along tile boundaries from the origin to that point; then $h_R(y) \stackrel{def}{=} d(w_y, x_0)$ is independent of the path w_y taken since for each tile the path around its boundary is trivial in G.

Remark. Note that on an edge z almost all the points have the same h_R-value. The exceptions are points whose h_R-value is one less; these points occur at distance exactly 1 apart on z.

To tile the region R, we use the following result.

Lemma 3 *If R is tileable, there is a tiling in which the maximum of h_R occurs on the boundary of R.*

Proof. Suppose we have a tiling and the maximum is at a point x on an edge z in the interior of R. Suppose without loss that z is vertical. Let z' be a horizontal edge meeting z at a point y. If $h_R(y) = h_R(x)$ then points on z' are higher than x, contradicting our choice of x. So $h_R(y) = h_R(x) - 1$.

By the remark which precedes this lemma, every horizontal edge meeting z is at an integer distance from y. In particular the gap between adjacent horizontal edges has integral length. So each rectangle adjacent to z has integral height.

Subdivide these rectangles using horizontal segments at each height which is an integral distance from y (Figure 2b). This divides each rectangle adjacent to z into other rectangles of height 1. Now remove the edge z completely; this has the effect of combining pairs of rectangles left and right of z into single rectangles of height 1 (Figure 2c).

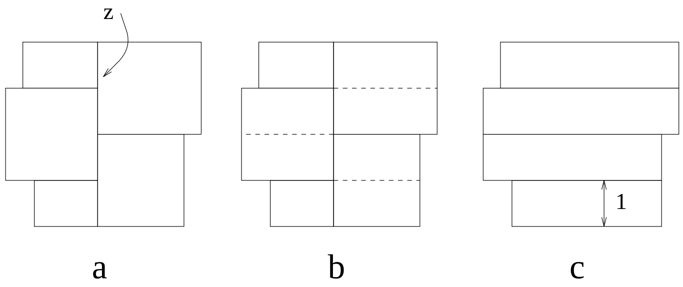

Figure 2: Removing an interior maximum.

So we have a new tiling with integer-sided rectangles. On each edge we added, the points are at a strictly lower height than x.

By repeating this process for each interior edge which contains a point of maximal height, we eventually decrease the maximum in the interior. Since the maximum takes integer values, eventually we find a tiling with the maximum on the boundary. Furthermore there is a tiling in which points on interior edges are *strictly* lower than the boundary maximum. $\square$

This lemma gives us an algorithm for tiling. First compute h_R on the boundary of R. By Lemma 1, it is well-defined. Find the maximum and minimum of h_R on the boundary of R. If the maximum minus the minimum is 1, then all edges have integral length, and so it is easy to tile.

Suppose the maximal height difference is at least 2. By the remark preceding Lemma 3, the maximum height occurs on a boundary edge s of integer length. Assume without loss that s is vertical. Put in as first tile a rectangle R_1 of side s and width r, where $r > 0$ is the smaller of:

1. the smallest value for which the height of all points on the edge s' of R_1 opposite s (extending h_R to $R \cup R'$) is less than the maximum height,

2. the smallest value at which the edge s' opposite s touches the boundary of R at a point on the interior of the edge s'.

We claim that if there is a tiling of R, there is a tiling in which this rectangle R' is a tile. By the lemma, there exists a tiling with the maximum at s whose interior edges are lower than s. The rectangles adjacent to s in this tiling must have integer height, with their horizontal edges meeting s at integer distance from the two vertices of s (since the horizontal edges must be lower than s). Furthermore these rectangles have widths at least as large as r since their vertical edges opposite s either have height lower than s or touch the boundary. So in such a tiling we can cut off each of these tiles at distance r from s so that the added rectangle R_1 is a tile in a tiling of R.

So we can now continue with the remaining untiled region $R' = R - R_1$. It may be that R' has several components; however each component is still simply connected, so we run the algorithm in each component separately. On each component, for a tiling to exist the height function must be well-defined, that is, the heights on new edges of R_1 must agree with the heights at points where they touch ∂R.

In either case 1 or 2 above in our choice of r, the total length of boundary edges where the height is maximal has decreased by an integer amount: in case 1 it has decreased by the length of s, and in case 2 it has decreased by the length of $s' \cap \partial R$ which must be a positive integer.

So after a finite number of added rectangles the maximum on the various boundaries has decreased. Since the minimum has not changed, the maximal height difference on each boundary has now decreased. Continue until this difference is equal to 1 on each component. Then the boundary must have all edges of integral length and so can be trivially tiled.

This completes the algorithm.

Example. In the polygon R of Figure 3, the counterclockwise path from e to y is the word

$$h(1)v(\frac{1}{2})h(\frac{1}{3})v(\frac{1}{2})h(\frac{1}{4})v(\frac{1}{4})h(\frac{1}{4})v(1),$$

and the clockwise path from e to y is

$$v(\frac{3}{2})h(\frac{1}{3})v(\frac{1}{2})h(\frac{1}{4})v(\frac{1}{4})h(\frac{5}{4}),$$

these are equal in G so $\alpha(R) = e$.

The heights (assuming the height of the first edge is 1) are indicated on the boundary (the height labelling an edge is the largest height of points on that edge). The lowest tiling is shown in the first figure, the highest in the second. The heights on interior edges are as shown. Our algorithm would construct the first tiling.

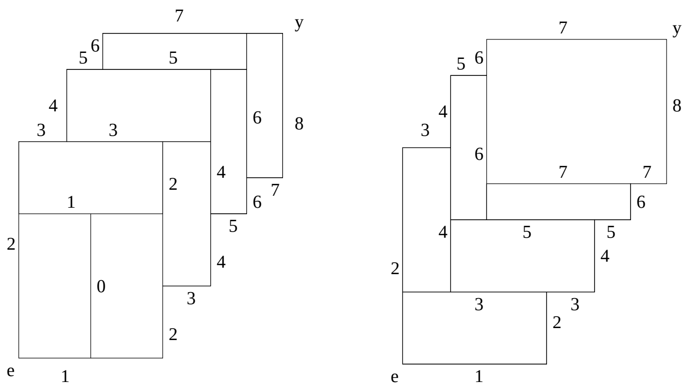

Figure 3: Lowest and highest tilings of a polygon.

A consequence of the local rearrangement of tiles as in Lemma 3 and this algorithm is:

Theorem 4 *Any two tilings of a polygon can be obtained from one another by a sequence of the two types of operations:* **(1)** *subdividing a tile into two tiles along an edge of integer length,* **(2)** *coalescing two tiles which share a common edge of integer length into one tile by removing that edge.*

Proof. Define a tiling T_0 which is that constructed by the algorithm, except don't stop when all the edges are integral but continue to apply the algorithm as before. Once all the edges are integral or half-integral, they remain so by our definition of base point x_0. So each rectangle added from this point on takes up at least area $1/2$, and thus the algorithm terminates after a finite number of steps.

We claim any tiling T can be transformed using operations (1) and (2) to this tiling T_0. By the lemma we can transform T until the interior edges have heights strictly less than the boundary maximum. Take a highest edge s on ∂R as before. We can now transform T so that R_1, the rectangle added in the algorithm, is a rectangle of T: each rectangle adjacent to s has integral height, and so can be subdivided into two legal rectangles with a vertical edge at distance r from s. Then removing the horizontal edges which cut through R_1 leaves the tile R_1.

The proof is then completed by induction on the number of tiles of T_0. $\square$

4 Polyominos

The preceding algorithm gives a bonus in the case of tiling a polyomino. Suppose we want to tile a simply connected polyomino with rectangles, each of which has a side of length n.

The algorithm of section 3 has the property that the other side-lengths of the tiles used are in the lattice generated by the edge lengths of R. In the case of polyominoes, this lattice is just $\mathbb{Z}$, and so the algorithm yields in this case a tiling with bars of length $1 \times n$ and $n \times 1$.

By stretching the vertical coordinate we can also similarly solve the case of tiling with $1 \times n$ and $m \times 1$ bars. This case was previously solved by a similar algorithm in [2].

Our algorithm runs in time linear in the area.

5 Generalization

Let A be an arbitrary subgroup of $\mathbb{R}$ and let S_A be the set of rectangles with at least one side in A. Our algorithm also works to tile with tiles in S_A. In this case the group $G = \mathbb{R}/A * \mathbb{R}/A$ loses all of its irrelevant topology (which was implicit but not explicit in our understanding of $S^1 * S^1$), but what remains is the word metric coming from the distance function $d(x, y)$ which is still well-defined.

The proofs of the lemmas extend easily to this case. For the rest of the algorithm, if A is not discrete then the choice of width r of the added tile is not well-defined. However we can take any r sufficiently small so that the tile R_1 encounters no other vertex of R, and such that s' has lower height than s. We again continue with the untiled region until the boundary height difference is 1, in which case all the edges are in A, and so tiling is trivial.

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