Eigenfunctions on the Finite Poincaré Plane
by Jinghua Kuang *

1. Introduction. Let F_q be a finite field with q elements (q odd.) Fix a non-square elemnet $\delta \in F$.

$$H_q = F_q(\sqrt{\delta}) - F_q = \{x + y\sqrt{\delta} \,|\, x, y \neq 0 \in F_q\}$$

is called the finite Poincaré plane. From [1], [2], [4], [7], we have the following facts:

1. $G = \mathrm{GL}_2(F_q)$ acts on H_q by linear transformation:
$$gz = \frac{az+b}{cz+d}, \qquad \forall g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, \forall z \in H_q.$$
Let $K = \left\{ \begin{pmatrix} a & b\delta \\ b & a \end{pmatrix} \right\}$. One may identify G/K with H_q and with $P = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \,|\, y \neq 0, x \in F_q \right\}$.
2. There is a G-invariant pseudo-distance Δ on H_q:
$$\Delta(z_1,\, z_2) = \frac{N(z_1 - z_2)}{\mathrm{Im} z_1 \cdot \mathrm{Im} z_2} \qquad \forall z_1, z_2 \in H_q,$$
where N is the norm from $F_q(\sqrt{\delta})$ to F_q and $\mathrm{Im} z = y$ if $z = x + y\sqrt{\delta}$.
3. For $a \in F_q^{\times}$ and $a \neq 4\delta$, a graph structure $X_q(\delta, a)$ is defined on H_q: (z_1, z_2) is an edge iff $\Delta(z_1, z_2) = a$. Katz [3] and Li [5] proved that $X_q(\delta, a)$ are Ramanujan graphs using different methods.

However, distributions of eigenvalues of the adjacency matrices have not been well understood. We will calculate the first and second moments of the asymptotic distribution of the eigenvalues of the adjacency matrices and provide evidence for Terras' conjecture in [7].

Let V be the space of all complex functions on H_q. Using character sums, Evans [2] constructed an orthogonal basis of V that diagonalizes all the adjacency matrices. Kuang [4], using group representation theory, also constructed such an orthogonal basis. This paper will also compare these two bases.

2. Hecke Algebra. Since K is the analogue of $\mathrm{O}(2)$ in the classical Poincaré upper half plane, it is natural to consider the Hecke algebra $H(G, K)$, which is defined as the algebra of all bi-K-invariant complex functions on G under convolution. Since K is the isotropic subgroup of $\sqrt{\delta}$, $\Delta(g, 1) = \Delta(g^{-1}, 1)$ implies that $KgK = Kg^{-1}K$. Therefore, $H(G, K)$ is commutative. Obviously, $H(G, K)$ has dimension q. Hence there exist q idempotents in $H(G, K)$.

3. Idempotents. For an irreducible representation π of G, let $\eta'_\pi(g) = \frac{1}{|K|}\sum_{k \in K} \mathrm{tr}\, \pi(kg)$. π is spherical (i.e. $\mathrm{Res}_K^G(\pi) \supset 1_K$) iff $\eta'_\pi(g) \neq 0$. From [4], we have

$$\eta'_{\pi_1} * \eta'_{\pi_2} = \begin{cases} 0, & \text{if} \pi_1 \not\simeq \pi_2; \\ \frac{|G|}{\dim(\pi)} \eta'_\pi, & \text{if} \pi_1 \simeq \pi_2 \simeq \pi. \end{cases}$$

Let ω be a multiplicative character of $F_q(\sqrt{\delta})$ of order $q^2 - 1$, χ be a multiplicative character of order $q-1$, and $s = \chi^{(q-1)/2}$. There are q irreducible spherical representations of G (see [6]):

$$\begin{aligned} &\pi = 1_G; \\ &\text{or} = \rho(s, s) = q \text{ dimensional irreducible component of } \mathrm{Ind}_B^G(s, s); \\ &\text{or} = \rho(\chi^j, \bar{\chi}^j) = \mathrm{Ind}_B^G(\chi^j, \bar{\chi}^j), \quad j = 1, \ldots, \frac{q-3}{2}; \\ &\text{or} = \rho_{\nu_j} = \text{cuspidal representation corresponding to } \nu_j = \omega^{j(q-1)}, \quad j = 1, \ldots, \frac{q-1}{2}. \end{aligned}$$

(B is the Borel subgroup of G.) Let $\eta_\pi = \frac{\dim(\pi)}{|G|}\eta'_\pi$. Let $\eta_0 = \eta_{1_G}$, $\eta_j = \eta_{\rho(\chi^j, \bar{\chi}^j)}$, $j = 1, \ldots, (q-3)/2$, $\eta_{\frac{q-1}{2}} = \eta_{\rho(s,s)}$ and $\eta_{\frac{q-1}{2}+j} = \eta_{\rho_{\nu_j}}$, $j = 1, \ldots, (q-1)/2$. Then

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Theorem 1 (Kuang [4]). *$\eta_0, \eta_1, \ldots, \eta_{q-1}$ are the q idempotents in $H(G,K)$.*

4. Representation of $H(G,K)$. $H(G,K)$ has a natural representation on $L^2(H_q) = L^2(P)$, the space of all complex functions on $H_q = P$:

$$T_\varphi(f)(p) = \frac{1}{|K|} f * \varphi(p) = \frac{1}{|K|} \sum_{h \in G} f(h)\varphi(h^{-1}p) = \sum_{p_1 \in P} f(p_1)\varphi(p_1^{-1}p),$$

for $\varphi \in H(G,K)$, $f \in L^2(P)$ and $p \in P$.

For the usual inner product $\langle , \rangle$ on $L^2(P)$, we have $\langle T_\varphi(f_1), f_2\rangle = \langle f_1, T_{\hat{\varphi}}(f_2)\rangle$, where $\hat{\varphi}(h) = \overline{\varphi(h^{-1})}$. That is, $H(G,K)$ acts on $L^2(P)$ self-adjointly. Hence, there exists a basis of simultaneous eigenfunctions of $H(G,K)$.

5. Construction of the Bases. Let ψ be a fixed non-trivial additive character of F_q. Denote $\psi_a(x) = \psi(ax)$. Define

$\chi_i(g) = \chi^i(y)$ if $gK = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} K$, $i = 1, \ldots, q-1$.

$\Psi_a(g) = \delta(y)\psi_a(x)$ if $gK = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} K$, $a \in F_q^\times$,

where $\delta(y) = 1$ if $y = 1$; $= 0$ otherwise. Then

Theorem 2 (Kuang [4]). *χ_i, $i = 1, \ldots, q-1$ and $\Psi_a * \eta_i$, $i = 1, \ldots, q-1$, $a \in F_q^\times$ are simultaneous eigenfunctions of $H(G,K)$ and make up an orthogonal basis of $L^2(P)$. Moreover, $T_{\eta_j}(\chi_i) = \chi_i$ if $\eta_j = \eta_{\rho(\chi^{\pm i}, \bar{\chi}^{\pm i})}$; $= 0$ otherwise. $T_{\eta_j}(\Psi_a * \eta_i) = \Psi_a * \eta_i$ if $i = j$; $= 0$ otherwise.*

Terras [8] constructed the following functions:

$$K_{i,\psi_a}(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}) = \psi_a(-x)\chi^i(y) \sum_{u \in F_q} \psi_a(u)\bar{\chi}^i(u^2 - \delta y^2).$$

Evans [2], based on Velasquez's work according to [9], constructed the following functions:

$$H_{t,i,\psi_a}(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}) = \frac{\psi_a(-x)}{q+1} \sum_{u \in F_q}{}^{*} \psi_a(u) \sum_{\alpha \in U_1} \omega^i(\alpha) s\left(\alpha + \frac{1}{\alpha} + \frac{\delta y}{t} + \frac{t}{\delta y} - \frac{u^2}{ty}\right),$$

where $U_1 = \{\alpha \in F_q^\times \mid N(\alpha) = 1\}$ and the asterisk indicates that when $y = \pm t$, the $u = 0$ term is multiplied by $q+1$.

Theorem 3 (Terras, Velasquez, Evans [2]). *K_{i,ψ_a}, $i = 1, \ldots, q-1$, $a \in F_q$ and H_{t,i,ψ_a}, $t \in F_q^\times$, $i = 1, \ldots, q-1$, $a \in F_q$ are simultaneous eigenfunctions of $H(G,K)$. There exist a t_0 for any fixed i and a such that K_{i,ψ_0}, $i = 1, \ldots, q-1$, K_{i,ψ_a}, $i = 1, \ldots, (q-1)/2$, $a \in F_q$ and H_{t_0,i,ψ_a} $i = 1, \ldots, (q-1)/2$, $a \in F_q^\times$ make up an orthogonal basis of $L^2(P)$.*

6. Comparison of the Bases. We now prove that the two bases constructed above are essentially same. Let us denote by π_i the spherical representation that give rise to η_i.

Theorem 4. *The following identities hold.*

$$\overline{C_i(0)} \cdot \chi_i = K_{i,1},\ i = 1, \ldots, q-1;$$

$$\Psi_a * \eta_i = \frac{\dim(\pi_i)}{|G|(q+1)} C_i(a) K_{i,\psi_{-a}},\ i = 1, \ldots, (q-1)/2;$$

$$\Psi_a * \eta_{i+(q-1)/2} = \frac{\dim(\pi_{i+(q-1)/2})}{|G|} H_{-\delta,i,\psi_{-a}},\ i = 1, \ldots, (q-1)/2,$$

where $C_i(a) = \sum_{x \in F_q} \psi_a(x)\chi^i(x^2 - \delta)$.

Proof. $K_{i,1} = \overline{C_i(0)} \cdot \chi_i$ is obvious. Since for any pair of $a, b \in F_q$, K_{i,ψ_b} and $\Psi_a * \eta_i$ are eigenfunctions of $H(G,K)$ of the same eigenvalues ($i = 1, \ldots, (q-1)/2$), $\Psi_a * \eta_i$ is a complex linear combination of K_{i,ψ_b}, ($b \in F_q$). Now,

$$\begin{aligned}
\langle K_{i,\psi_b}, \Psi_a * \eta_i \rangle &= \sum_{p \in P} K_{i,\psi_b}(p)\overline{\Psi_a * \eta_i}(p) \\
&= \sum_{p \in P} \sum_{u \in F_q} \psi_b(u - x)\chi^i(y)\overline{\chi}^i(u^2 - \delta y^2)\overline{\eta_i}(p) \sum_{v \in F_q} \psi(-(a+b)v) \\
&= 0 \quad \text{if } a + b \neq 0
\end{aligned}$$

Hence $\Psi_a * \eta_i$ is a multiple of $K_{i,\psi_{-a}}$. Using the equation (2.17) in [2], we calculate

$$\begin{aligned}
\Psi_a * \eta_i(1) &= \sum_{v \in F_q} \psi_a(u)\eta_i(\begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix}) \\
&= \frac{\dim(\pi_i)}{|G|(q+1)} C_i(a)\overline{C_i(a)}.
\end{aligned}$$

And $K_{i,\psi_{-a}}(1) = \overline{C_i(a)}$. Therefore, $\Psi_a * \eta_i = \frac{\dim(\pi_i)}{|G|(q+1)} C_i(a) K_{i,\psi_{-a}}$.

For the third identity, we use the equation (2.16) in [2]. Let $|S(z)|$ be the cardinality of the K-orbit of z in H_q. Let $\hat{i} = i + (q-1)/2$. Then,

$$\begin{aligned}
\Psi_a * \eta_{\hat{i}}(p) &= \sum_{u \in F_q} \psi_a(u)\eta_{\hat{i}}(\begin{pmatrix} y & x-u \\ 0 & 1 \end{pmatrix}) \\
&= \sum_{u \in F_q} \psi_a(x-u)\eta_{\hat{i}}(\begin{pmatrix} y & u \\ 0 & 1 \end{pmatrix}) \\
&= \psi_a(x) \sum_{u \in F_q} \psi_a(-u) \frac{\dim(\pi_{\hat{i}})}{|G|} \frac{1}{|S(u + y\sqrt{\delta})|} \sum_{\alpha \in U_1} \omega^i(\alpha) s\left(\alpha + \frac{1}{\alpha} - 2 + \frac{\Delta(u + y\sqrt{\delta}, \sqrt{\delta})}{\delta}\right) \\
&= \psi_a(x) \sum_{u \in F_q} \psi_a(-u) \frac{\dim(\pi_{\hat{i}})}{|G|} \frac{1}{|S(u + y\sqrt{\delta})|} \sum_{\alpha \in U_1} \omega^i(\alpha) s\left(\alpha + \frac{1}{\alpha} + \frac{u^2}{y\delta} - y - \frac{1}{y}\right) \\
&= \frac{\dim(\pi_{\hat{i}})}{|G|} H_{-\delta,i,\psi_{-a}}(p)
\end{aligned}$$

7. Comments and Questions. For any function f on G, define the Fourier coefficient

$$F(g; a, f) = \sum_{x \in F_q} f(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g)\psi_a(x).$$

From [4], we see $\langle \Psi_a * \eta_i, \Psi_a * \eta_i \rangle = |K| \cdot q \cdot F(1; a, \eta_i)$. So $F(1; a, \eta_i) \neq 0$. But can one evaluate $F(1; a, \eta_i)$? We have $F(1; a, \eta_i) = \Psi_{-a} * \eta_i(1)$. Hence, for $i = 1, \ldots, (q-1)/2$,

$$\begin{aligned}
F(1; a, \eta_i) &= \frac{\dim(\pi_i)}{|G|(q+1)} |C_i(-a)|^2 = \frac{\dim(\pi_i)}{|G|(q+1)} |C_i(a)|^2 \\
F(1; a, \eta_{\hat{i}}) &= \frac{\dim(\pi_{\hat{i}})}{|G|} \sum_{u \in F_q} \psi_a(u) \frac{1}{|S(u + \sqrt{\delta})|} \sum_{\alpha \in U_1} \omega^i(\alpha) s\left(\alpha + \frac{1}{\alpha} + \frac{u^2}{\delta} - 2\right)
\end{aligned}$$

However, we don't know much about $C_i(a)$ and about the last set of sums. We note that $C_i(0)$ is essentially a Jacobi sum, in fact, $C_i(0) = -\chi^i(-\delta)J(\chi^i, s)$.

8. Connection to $X_q(\delta, a)$. There are q double cosets: $D_a = KgK$, where $\Delta(g,1) = a \in F_q$. Let φ_a be the characteristic function of the set D_a. Then $A_a = \frac{1}{|K|}T_{\varphi_a}$ acting on $L^2(H_q)$ is the adjacency matrix of $X_q(\delta, a)$. Therefore each basis constructed in Section 5 diagonalizes all the adjacency matrices. Let $S_a = D_a \cap P$. Then

$$\lambda_i(a) = \frac{|G|}{\dim(\pi_i)}|S_a|\eta_i(D_a),\ i = 0, 1, \dots, q-1$$

are all the eigenvalues of A_a. Katz [3], Li [5] proved that $|\lambda_i(a)| \le 2\sqrt{q}$ for $i = 1, \dots, q-1$, which confirms that $X_a(\delta, a)$ is Ramanujan.

Fix $a \neq 4\delta \in F^\times$, Terras [7] conjectured that $\{\lambda_i(a)/\sqrt{q} \,|\, i = 1, \dots, q-1\}$ asymptotically has Sato-Tate distribution, i.e. for $E \subset [-2, 2]$,

$$\lim_{q\to\infty} \frac{1}{q-1}|\{\lambda_i(a) \,|\, \lambda_i(a)/\sqrt{q} \in E\}| = \frac{1}{2\pi}\int_E \sqrt{4-x^2}dx.$$

9. Moments. Define a $q \times q$ matrix

$$M = \left(\sqrt{\frac{|S_a|}{\dim(\pi_i)}}\eta_i(S_a)\right)_{i=0,1,\dots,q-1,\, a\in F_q}.$$

The idempotent property of η_i $(i = 0, \dots, q-1)$ implies that $MM' = \frac{1}{|K||G|}I_q$. ($M'$ is the transpose of M.) Hence, $M'M = \frac{1}{|K||G|}I_q$, that gives

$$\frac{1}{q-1}\sum_{i=0}^{q-1}\frac{\dim(\pi_i)}{q}\cdot\frac{\lambda_i(a)}{\sqrt{q}} = 0$$

$$\frac{1}{q-1}\sum_{i=0}^{q-1}\frac{\dim(\pi_i)}{q}\cdot\left(\frac{\lambda_i(a)}{\sqrt{q}}\right)^2 = \frac{|G||S_a|}{|K|q^2(q-1)}$$

That, in turn, implies

$$\lim_{q\to\infty}\frac{1}{q-1}\sum_{i=1}^{q-1}\frac{\lambda_i(a)}{\sqrt{q}} = 0$$

$$\lim_{q\to\infty}\frac{1}{q-1}\sum_{i=1}^{q-1}\left(\frac{\lambda_i(a)}{\sqrt{q}}\right)^2 = 1.$$

Therefore, the first and second moments of $\{\lambda_i(a)/\sqrt{q} \,|\, i = 1, \dots, q-1\}$ asymptotically match with those of the Sato-Tate distribution.

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Department of Mathematics
Pennsylvania State University, at Fayette
Uniontown, PA 15401
E-mail address: *kuang@math.psu.edu*