MEAN VALUES OF $\frac{\zeta'}{\zeta}$ AND THE GUE HYPOTHESIS

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1. Introduction

The Riemann Hypothesis states that the nontrivial zeros of the Riemann ζ-function are of the form $\rho_j = \frac{1}{2} + i\gamma_j$ with $\gamma_j \in \mathbb{R}$, but it makes no assertion about the distribution of the numbers γ_j. To study this distribution, the zero-counting function $N(T) = \#\{0 < \gamma_j < T \mid \zeta(\frac{1}{2} + \gamma_j) = 0\}$ is a natural object to consider. Von Mangoldt proved $N(T) = \frac{1}{2\pi} T \log(T/2\pi e) + O(\log T)$, and this contains much information about the individual γ_j. For example, $\gamma_j = 2\pi j/\log j + O(\log j)$, and the numbers $\tilde{\gamma} = \frac{1}{2\pi}\gamma \log\gamma$ have mean spacing 1. Montgomery [M] began a finer study of the distribution of the γ_j by considering the pair-correlation function

$$F(\alpha, T) = N(T)^{-1} \sum_{0<\gamma,\gamma'<T} T^{i\alpha(\gamma-\gamma')}\, w(\gamma - \gamma'),$$

where $w(u) = 4/(4+u^2)$. Montgomery [M] showed $F(\alpha, T) = (1 + o(1))T^{-2\alpha} \log T + \alpha + o(1)$, uniformly for $0 \le \alpha \le 1$, and he conjectured $F(\alpha, T) = 1 + o(1)$, uniformly for $1 \le \alpha \le A$ for any fixed $A > 1$. This conjecture has consequences for the distribution of primes [G1] [GG] [GM], mean values of $S(t)$ [G2], and mean values of the ζ-function [CG] [GG]. As an example, Goldston and Gonek [GG] show, assuming RH, if $a \approx 1/\log T$, then

$$\int_0^T \left|\frac{\zeta'}{\zeta}(\tfrac{1}{2} + a + it)\right|^2 dt \sim T\, \frac{1 - T^{-2a}}{4a^2} + \log^2 T \int_1^\infty (F(\alpha, T) - 1)\, T^{-2a\alpha} d\alpha. \tag{1.0}$$

Note that the first term dominates if and only if Montgomery's conjecture is true.

Perhaps the most profound aspect of Montgomery's conjecture is that it says the distribution of gaps between zeros of the ζ-function is the same as the Gaussian Unitary Ensemble distribution from random matrix theory. In other words, the gaps between zeros of $\zeta(s)$ are distributed like the gaps between eigenvalues of large random Hermitian matrices. Numerical computations of Odlyzko [Odl] have found amazingly close agreement between the actual distribution of zeros and the distribution predicted by the GUE model.

Recently, Rudnick and Sarnak [RS] have considered higher correlation functions of zeros of the ζ-function. Their result, which we describe later, strongly supports the conjecture that all correlation functions of zeros of the ζ-function agree with those of the GUE model. We refer to this as the "GUE Hypothesis," and we write GUEN for the assumption that the n-correlations of zeros of $\zeta(s)$ agree with GUE for $2 \le n \le N$. For example, GUE2 is Montgomery's conjecture.

In the spirit of (1.0), we prove,

Theorem 1. *Assume RH and GUE3. If $a \approx 1/\log T$ then*

$$\int_{-T}^{T} \left|\frac{\zeta'}{\zeta}(\tfrac{1}{2} + a + it)\right|^2 \frac{\zeta'}{\zeta}(\tfrac{1}{2} + a + it)dt \;\sim T \log T\, \frac{T^{-2a}}{2a^2}.$$

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The paper is organized as follows. In Section 2 we describe the GUE Hypothesis and the result of Rudnick and Sarnak [RS]. In Section 3 we describe the connection between GUE and other conjectures, and we discuss the situation for L-functions. In Section 4 we prove Theorem 1. In Section 5 we indicate other integrals which can be evaluated by these techniques and we discuss the limitations of these methods.

2. The GUE Hypothesis

Following Rudnick and Sarnak [RS], on RH and GUE we can evaluate expressions of the form

$$S_n(T,f) = \sum_{0<\gamma_1,\ldots,\gamma_n<T}{}' f\left(\frac{L}{2\pi}\gamma_1,\ldots,\frac{L}{2\pi}\gamma_n\right)$$

where $\sum'$ means summation over distinct indices, $L=\log T$, and f satisfies

TF1. $f(x_1,\ldots,x_n)$ is symmetric.

TF2. $f(x+(t,\ldots,t)) = f(x)$ for $t\in\mathbb{R}$.

TF3. $f(x)\to 0$ rapidly as $|x|\to\infty$ in the hyperplane $\sum x_j = 0$.

Condition TF1 is not really necessary because $S_n(T,f) = S_n(T,f_0)$, where $f_0(x) = \frac{1}{n!}\sum f(\sigma x)$, where σ runs over all permutations on n letters. Condition TF2 merely states that f is a function of the differences $x_i - x_j$. Condition TF3 assures that various integrals will converge. The factor $L/2\pi$ is introduced because the $\tilde\gamma = L\gamma/2\pi$ have mean spacing 1.

The GUE Hypothesis states that, under the conditions described above,

$$S_n(T,f) \sim \frac{1}{2\pi} TL \int_{\mathbb{R}^n} f(x) W_n(x)\delta(\overline{x})\,dx \tag{2.0}$$

where

$$W_n(x) = \det(K(x_i - x_j)), \qquad K(t) = \frac{\sin(\pi t)}{\pi t}, \qquad \overline{x} = \frac{x_1+\cdots+x_n}{n},$$

and δ is the Dirac δ-function. Rudnick and Sarnak [RS] prove (2.0) under the assumption that the Fourier transform $\hat f(\xi)$ is supported in $\sum_j|\xi_j|<2$. They actually prove the corresponding result for essentially all interesting GL_r L-functions. The only modification needed is to put $L = r\log T$ and $\hat f(\xi)$ must be supported in $\sum_j |\xi_j| < 2/r$.

3. GUE, other conjectures, and L-functions

We describe a conjecture related to GUE, and we speculate about the connection between the ζ-function and other L-functions.

The following conjecture is (7.4) in [F1].

Conjecture. *Suppose a,b,u,v are complex numbers of modulus $\ll 1/\log T$, each having real part contained in $[A/\log T, B/\log T]$. Then*

$$\int_0^T \frac{\zeta(\frac12+u+it)\,\zeta(\frac12+v-it)}{\zeta(\frac12+a+it)\,\zeta(\frac12+b-it)}\,dt = T\left(1+\left(1-T^{-(u+v)}\right)\frac{(u-a)(v-b)}{(u+v)(a+b)}\right)+o(T), \tag{3.0}$$

uniformly for fixed positive A, B.

We may differentiate both sides with respect to any of a, b, u, v by using Cauchy's theorem and integrating around circles of size $1/\log T$. This will increase the error term by a factor of $\log T$, but the main term increases by the same amount, so we get another meaningful formula. For example, differentiating with respect to u and v, and then setting $u = a$ and $v = b$ gives

$$\int_0^T \frac{\zeta'}{\zeta}(\tfrac{1}{2} + a + it)\frac{\zeta'}{\zeta}(\tfrac{1}{2} + b - it)\, dt \sim T\, \frac{1 - T^{-(a+b)}}{(a+b)^2}.$$

Put $b = a$ to get the left side of (1.0). By the comment following (1.0), this implies GUE2. The integral above is not more difficult to evaluate than the integral in (1.0), and the methods of [GG] are easily modified to give the above formula, on GUE2.

If we differentiate (3.0) with respect to u, v and a, and then set $u = a$ and $v = b$, we get

$$\int_0^T \left|\frac{\zeta'}{\zeta}(\tfrac{1}{2} + a + it)\right|^2 \frac{\zeta'}{\zeta}(\tfrac{1}{2} + b + it)dt \sim T \log T\, \frac{T^{-(a+b)}}{(a+b)^2}, \tag{3.1}$$

which implies Theorem 1. This gives an interesting consistency check between (3.0) and the GUE Hypothesis. If the methods in Section 4 could be modified to express the left side of (3.1) in terms of the triple-correlation measure $W_3(x, y, z)$, then it might be possible to show that (3.1) implies GUE3. Other formulas derived from (3.0) are given in (7.5) - (7.8) of [F1].

Let $L(s)$ be the L-function associated to a cuspidal automorphic representation of $GL_r/\mathbb{Q}$. In [RS] it is shown that one expects all of the correlation functions of the zeros of $L(s)$ to be given by the GUE model. They refer to this as the 'universality of GUE.' We now describe a general method for using known results about $\zeta(s)$ to conjecture results about $L(s)$. This can be seen as transferring GUE for $\zeta(s)$ to GUE for $L(s)$.

The Scaling Principle. *For $|s - \frac{1}{2}| \ll 1/\log t$ we have the correspondence:*

$$L(s - \tfrac{1}{2}) \qquad \longleftrightarrow \qquad \zeta(r(s - \tfrac{1}{2}))$$

$$\sum_{n \le X^r} \frac{a(n)}{n^s} \qquad \longleftrightarrow \qquad \sum_{n \le X} \frac{A(n)}{n^s}.$$

Some simple cases of this principle occur in the approximation of Dirichlet series by Dirichlet polynomials. For example, the approximate functional equation for $\zeta(s)$ consists of two Dirichlet polynomials of length $T^{\frac{1}{2}}$, while for a GL_2 L-function the polynomials have length T. Another example is Bombieri and Friedlander's [BF] result that a GL_r L-function cannot be well approximated by a single polynomial of length less than T^r.

A more interesting example can be found in mean-values near the $\frac{1}{2}$-line of the Dirichlet series multiplied by a Dirichlet polynomial. In [F2], Theorem 1 and directly following, two examples are given, and a version of the scaling principle is described at the end of that paper's introductory section.

Applying the scaling principle to (3.0) we get, after changing variables:

$$\int_0^T \frac{L(\frac12+u+it)L(\frac12+v-it)}{L(\frac12+a+it)L(\frac12+b-it)}\,dt = T\left(1+\left(1-T^{-r(u+v)}\right)\frac{(u-a)(v-b)}{(u+v)(a+b)}\right)+o(T)$$

One can use this formula to obtain the analog of (3.1), and then apply the methods of [GG] to obtain the analog of (1.0), and so obtain GUE2 for $L(s)$. In other words, (3.0) plus the scaling principle implies that the pair correlation of zeros of any L-function follow the GUE model. One could also argue that the scaling principle directly implies that the zeros of $L(s)$ behave like the (scaled) zeros of $\zeta(s)$, so GUE for $\zeta(s)$ transfers to GUE for $L(s)$.

4. Proof of Theorem 1

The proof closely follows the first part of [GG].

From the elementary identities $|z|^2z=\frac43x^3+\frac43iy^3-\frac13\overline{z}^3$ and $\overline{\zeta(s)}=\zeta(\overline{s})$ we obtain

$$\begin{aligned}\int_{-T}^T \left|\frac{\zeta'}{\zeta}(\tfrac12+a+it)\right|^2\frac{\zeta'}{\zeta}(\tfrac12+a+it)dt =&\frac43\int_{-T}^T\left(\mathrm{Re}\frac{\zeta'}{\zeta}(\tfrac12+a+it)\right)^3dt\\ &-\frac13\int_{-T}^T\frac{\zeta'}{\zeta}(\tfrac12+a+it)^3dt+\frac{4i}{3}\int_{-T}^T\left(\mathrm{Im}\frac{\zeta'}{\zeta}(\tfrac12+a+it)\right)^3dt\\ =&\frac83\int_0^T\left(\mathrm{Re}\frac{\zeta'}{\zeta}(\tfrac12+a+it)\right)^3dt-\frac23\,\mathrm{Re}\int_0^T\frac{\zeta'}{\zeta}(\tfrac12+a+it)^3dt.\end{aligned} \tag{4.0}$$

Only the first term on the last line makes a contribution, as the next Lemma shows.

Lemma 2. *On RH, if* $\frac12<\sigma<\frac34$, *then*

$$\int_0^T\frac{\zeta'}{\zeta}(\sigma+it)dt\ll\log T\log(\sigma-\tfrac12),$$

and if $n\geq 2$,

$$\int_0^T\frac{\zeta'}{\zeta}(\sigma+it)^n dt\ll\frac{\log^n T}{(\sigma-\frac12)^{n-1}}.$$

Proof: From the partial fraction formula [T],

$$\frac{\zeta'}{\zeta}(s)=A-\frac{1}{s-1}-\frac12\frac{\Gamma'}{\Gamma}\left(\frac s2+1\right)+\sum_\rho\left(\frac{1}{s-\rho}+\frac1\rho\right),$$

where $A=\frac12\log\pi-\sum_\rho\mathrm{Re}\,\rho^{-1}$, follows easily the estimate

$$\frac{\zeta'}{\zeta}(s)\ll\frac{\log t}{s-\frac12}.$$

Moving the path of integration to the 2-line, using the above estimate on the horizontal segments, and integrating term-by-term on the new path, gives the estimates in Lemma 2.

Now we evaluate the first term in (4.0). From the partial fraction formula and

$$\frac{\Gamma'}{\Gamma}(s) = \log s + O\left(\frac{1}{1+|t|}\right)$$

we get

$$\mathrm{Re}\left(\frac{\zeta'}{\zeta}(\tfrac{1}{2}+a+it)\right) = -\frac{1}{2}\log\frac{t}{2\pi} + a\Sigma_a(t) + O\left(\frac{1}{|t|+1}\right), \tag{4.1}$$

where

$$\Sigma_a(t) = \sum_\gamma \frac{1}{a^2+(t-\gamma)^2}.$$

So,

$$\int_0^T \left(\mathrm{Re}\frac{\zeta'}{\zeta}(\tfrac{1}{2}+a+it)\right)^3 dt = -\frac{1}{8}\int_0^T \log^3\frac{t}{2\pi}\,dt + \frac{3a}{4}\int_0^T \Sigma_a(t)\log^2\frac{t}{2\pi}\,dt - \frac{3a^2}{2}\int_0^T \Sigma_a(t)^2\log\frac{t}{2\pi}\,dt$$
$$+ a^3\int_0^T \Sigma_a(t)^3\,dt + O\left(\int_0^T \left(\log^2 t + a^2\Sigma_a(t)^2\right)\right)(t+1)^{-1}\,dt\Bigg). \tag{4.2}$$

It remains to evaluate the moments of $\Sigma_a(t)$.

Proposition 3. *Assume RH and GUE3. If $a \approx 1/\log T$ then*

$$\begin{aligned}
\Sigma_a(t) &\ll \frac{\log(|t|+2)}{a^2}\\
\int_0^T \Sigma_a(t)\,dt &= \frac{1}{2a}\left(T\log\frac{T}{2\pi} - T\right) + O\left(\log^2 T\log\log T\right)\\
\int_0^T \Sigma_a(t)^2\,dt &\sim \frac{1}{4a^2}T\log^2 T\left(1+\frac{1-T^{-2a}}{2a^2\log^2 T}\right)\\
\int_0^T \Sigma_a(t)^3\,dt &\sim \frac{1}{8a^3}T\log^3 T\left(1+\frac{3}{2a^2\log^2 T}\right).
\end{aligned}$$

By Proposition 3, and integration-by-parts, we can evaluate the terms in (4.2):

$$\begin{aligned}
\int_0^T \left(\mathrm{Re}\left(\frac{\zeta'}{\zeta}(\tfrac{1}{2}+a+it)\right)\right)^3 dt &\sim -\frac{1}{8}T\log^3 T + \frac{3}{8}T\log^3 T - \frac{3}{8}T\log^3 T\left(1+\frac{1-T^{-2a}}{2a^2\log^2 T}\right)\\
&\quad + \frac{1}{8}T\log^3 T\left(1+\frac{3}{2a^2\log^2 T}\right)\\
&\sim \frac{3}{8}T\log T\,\frac{T^{-2a}}{2a^2}.
\end{aligned}$$

Combining this with (4.0) and Lemma 2 gives Theorem 1. It remains to prove Proposition 3. This involves putting the expressions in Proposition 3 in a form where we can apply GUE.

Proof of Proposition 3. The first estimate follows from

$$\sum_{|\gamma-t|<1} 1 \ll \log t, \tag{4.3}$$

which follows from the zero-counting result $N(T)=\frac{1}{2\pi}T\log(T/2\pi e)+O(\log T)$.

To prove the second result, combine Lemma 2 and equation (4.1):

$$\begin{aligned}\int_0^T \Sigma_a(t)\,dt =&\frac{1}{2a}\int_0^T \log\frac{t}{2\pi}\,dt+O\left(\frac{\log T\log a}{a}\right)\\ =&\frac{1}{2a}\left(T\log\frac{T}{2\pi}-T\right)+O\left(\log^2 T\log\log T\right).\end{aligned}$$

The proof of the third result is given in [GG]; this requires GUE2. We will use essentially the same method to evaluate the last expression in Proposition 3; this requires GUE3.

Lemma 4. *If $a\gg 1/\log T$ then*

$$\int_0^T \Sigma_a(t)^n\,dt=\int_{-\infty}^{\infty}\left(\sum_{0<\gamma<T}\frac{1}{a^2+(t-\gamma)^2}\right)^n dt+O\left(\log^{3n}T\right)$$

Proof. The method used here is known as "Montgomery's trick." We truncate the sum so that it is over $0<\gamma<T$, and then extend the integral to go over all t. First note that if $1<t<T$ then

$$\sum_{\gamma\notin[0,T]}\frac{1}{a^2+(t-\gamma)^2}\ll\left(\frac{1}{T+1}+\frac{1}{T-t+1}\right)\frac{\log T}{a^2}.$$

This follows from (4.3) and $N(T)\ll T\log T$. Combining this with the first estimate in Proposition 3 shows that we may restrict all the summations to $0<\gamma<T$ with an error of $O(\log^{3n}T)$. A similar argument shows that we may extend the range of integration to go over $(-\infty,\infty)$ at the cost of an even smaller error. This proves Lemma 4.

Expanding the integrand in Lemma 4 and exchanging summation and integration gives

$$\int_0^T \Sigma_a(t)^3\,dt\sim\sum_{0<\gamma,\gamma',\gamma''<T}\int_{-\infty}^{\infty}\frac{1}{a^2+(t-\gamma)^2}\frac{1}{a^2+(t-\gamma')^2}\frac{1}{a^2+(t-\gamma'')^2}\,dt.$$

This integral can be evaluated explicitly. A straightforward calculation yields:

Lemma 5. *If $a>0$,*

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{1}{a^2+(t-\gamma)^2}\frac{1}{a^2+(t-\gamma')^2}\frac{1}{a^2+(t-\gamma'')^2}\,dt\\ &\qquad=12\pi a F_1(\gamma,\gamma',\gamma'')+\frac{\pi}{a}\left(F_2(\gamma,\gamma',\gamma'')+F_2(\gamma',\gamma'',\gamma)+F_2(\gamma'',\gamma,\gamma')\right),\end{aligned}$$

where

$$F_1(x,y,z) = \frac{1}{4a^2+(x-y)^2}\frac{1}{4a^2+(y-z)^2}\frac{1}{4a^2+(z-x)^2}$$
$$F_2(x,y,z) = \frac{1}{4a^2+(x-y)^2}\frac{1}{4a^2+(y-z)^2}.$$

This brings us to

$$\int_0^T \Sigma_a(t)^3\,dt \sim 12\pi a \sum_{0<\gamma,\gamma',\gamma''<T} F_1(\gamma,\gamma',\gamma'') + \frac{3\pi}{a}\sum_{0<\gamma,\gamma',\gamma''<T} F_2(\gamma,\gamma',\gamma''). \tag{4.4}$$

We will use the GUE Hypothesis to evaluate these sums. Everything must be expressed in terms of sums over distinct indices. For the sum on F_1 we have,

$$\begin{aligned}\sum_{0<\gamma,\gamma',\gamma''<T} F_1(\gamma,\gamma',\gamma'') &= \sum_{0<\gamma,\gamma',\gamma''<T} f_1\left(\tilde\gamma,\tilde\gamma',\tilde\gamma''\right)\\ &= \sum_{0<\gamma,\gamma',\gamma''<T}{}' f_1\left(\tilde\gamma,\tilde\gamma',\tilde\gamma''\right) + \frac{3}{4a^2}\sum_{0<\gamma,\gamma'<T}{}' f_1\left(\tilde\gamma,\tilde\gamma'\right) + \frac{1}{2^6a^6}\sum_{0<\gamma<T} 1\end{aligned}$$

where

$$\begin{aligned}f_1(x,y,z) =& \frac{L^6}{2^6\pi^6}\left(\frac{L^2a^2}{\pi^2}+(x-y)^2\right)^{-1}\left(\frac{L^2a^2}{\pi^2}+(y-z)^2\right)^{-1}\left(\frac{L^2a^2}{\pi^2}+(z-x)^2\right)^{-1}\\ f_1(x,y) =& \frac{L^4}{2^4\pi^4}\left(\frac{L^2a^2}{\pi^2}+(x-y)^2\right)^{-2}.\end{aligned}$$

Recall that $L=\log T$ and $\tilde\gamma = L\gamma/2\pi$. And for the sum on F_2 we have,

$$\begin{aligned}\sum_{0<\gamma,\gamma',\gamma''<T} F_2(\gamma,\gamma',\gamma'') &= \sum_{0<\gamma,\gamma',\gamma''<T} f_2\left(\tilde\gamma,\tilde\gamma',\tilde\gamma''\right)\\ &= \sum_{0<\gamma,\gamma',\gamma''<T}{}' f_2\left(\tilde\gamma,\tilde\gamma',\tilde\gamma''\right) + \frac{2}{4a^2}\sum_{0<\gamma,\gamma'<T}{}' f_2\left(\tilde\gamma,\tilde\gamma'\right)\\ &\quad + \sum_{0<\gamma,\gamma'<T}{}' f_1\left(\tilde\gamma,\tilde\gamma'\right) + \frac{1}{2^4a^4}\sum_{0<\gamma<T} 1\end{aligned}$$

where

$$\begin{aligned}f_2(x,y,z) =& \frac{L^4}{2^4\pi^4}\left(\frac{L^2a^2}{\pi^2}+(x-y)^2\right)^{-1}\left(\frac{L^2a^2}{\pi^2}+(y-z)^2\right)^{-1}\\ f_2(x,y) =& \frac{L^2}{2^2\pi^2}\left(\frac{L^2a^2}{\pi^2}+(x-y)^2\right)^{-1}.\end{aligned}$$

It is interesting that $\sum f_2\left(\tilde\gamma,\tilde\gamma'\right)$ is the sum encountered in [GG].

Now apply (2.0) to the expressions above. The resulting integrals can be evaluated explicitly. With the help of a computer-algebra package we find,

$$\begin{aligned}\sum_{0<\gamma,\gamma',\gamma''<T}{}' f_1(\tilde\gamma,\tilde\gamma',\tilde\gamma'') \sim \frac{3}{2\pi}TL\Bigg(&\frac{L^2}{576\,a^4}-\frac{L}{128\,a^5}+\frac{23}{1536\,a^6}-\frac{3}{256\,a^7\,L}\\ &+T^{-2a}\left(\frac{1}{96\,a^6}+\frac{1}{96\,a^7\,L}\right)\\ &+T^{-4a}\left(\frac{1}{1536\,a^6}+\frac{1}{768\,a^7\,L}\right)\Bigg)\end{aligned}$$

$$\sum_{0<\gamma,\gamma'<T}{}' f_1(\tilde\gamma,\tilde\gamma') \sim \frac{1}{\pi}TL\left(\frac{L}{64\,a^3}-\frac{1}{32\,a^4}+\frac{3}{128\,a^5\,L}-T^{-2a}\left(\frac{1}{64\,a^4}+\frac{3}{128\,a^5\,L}\right)\right)$$

$$\begin{aligned}\sum_{0<\gamma,\gamma',\gamma''<T}{}' f_2(\tilde\gamma,\tilde\gamma',\tilde\gamma'') \sim \frac{3}{2\pi}TL\Bigg(&\frac{L^2}{48\,a^2}-\frac{5\,L}{96\,a^3}+\frac{25}{384\,a^4}-\frac{7}{192\,a^5\,L}+T^{-2a}\frac{1}{24\,a^5\,L}\\ &-T^{-4a}\left(\frac{1}{384\,a^4}+\frac{1}{192\,a^5\,L}\right)\Bigg)\end{aligned}$$

$$\sum_{0<\gamma,\gamma'<T}{}' f_2(\tilde\gamma,\tilde\gamma') \sim \frac{1}{\pi}TL\left(\frac{L}{8\,a}-\frac{1}{8\,a^2}+\frac{1}{16\,a^3\,L}-T^{-2a}\frac{1}{16\,a^3\,L}\right)$$

Inserting these into our previous formulas gives

$$\int_0^T \Sigma_a(t)^3\,dt \sim \frac{1}{8a^3}TL^3\left(1+\frac{3}{2\,a^2\,L^2}\right),$$

as claimed. This completes the proof of Proposition 3, and so that of Theorem 1.

5. Limitations of the method

The GUE Hypothesis refers to summations over zeros with distinct indices. In this paper we encountered unrestricted sums, which we broke up into several sums over distinct indices. This step is not really necessary, for one can write a version of (2.0) where the summation is not restricted. In fact, all known methods of evaluating correlation functions [M] [H] [RS] proceed by first evaluating an unrestricted sum, and then subtracting out the "nondistinct" parts of the sum. See (4.1) in [RS] for the general case, and (10) in [H] for a particularly explicit version in the case of triple correlation.

It is clear that the methods of Section 4 are sufficient, assuming GUEN, to evaluate

$$\int_0^T \prod_{n=1}^{N} \mathrm{Re}\frac{\zeta'}{\zeta}\left(\tfrac12+a_n+it\right)\,dt, \tag{5.0}$$

for $a_n \approx 1/\log T$. The point is that Montgomery's trick works here, and the rest is just calculating explicit, but messy, integrals. These last computations could perhaps be simplified by dealing only with the unrestricted sums described above. There is nothing intrinsically interesting about the real part of ζ'/ζ, but by good fortune the integrand in (1.0) and in Theorem 1 can be directly related to (5.0) by simple algebraic identities. Unfortunately, this trick does not seem to work for any other integral. For example, evaluating

$$\int_0^T \left|\frac{\zeta'}{\zeta}\left(\tfrac{1}{2}+a+it\right)\right|^4 dt \tag{5.1}$$

by these methods requires evaluation of one of these integrals:

$$\int_0^T \left(\mathrm{Im}\frac{\zeta'}{\zeta}\left(\tfrac{1}{2}+a+it\right)\right)^4 dt \qquad \text{or} \qquad \int_0^T \left(\mathrm{Re}\frac{\zeta'}{\zeta}\left(\tfrac{1}{2}+a+it\right)\right)^2 \left(\mathrm{Im}\frac{\zeta'}{\zeta}\left(\tfrac{1}{2}+a+it\right)\right)^2 dt.$$

The presence of the imaginary part of ζ'/ζ makes these much more difficult than (5.0). In a few cases this difficulty can be avoided by simple tricks. For example, on GUE2,

$$\int_0^T \frac{\zeta'}{\zeta}\left(\tfrac{1}{2}+a+it\right)^2 dt \ll \log^3 T \qquad \text{and} \qquad \int_0^T \left(\mathrm{Re}\frac{\zeta'}{\zeta}\left(\tfrac{1}{2}+a+it\right)\right)^2 dt \sim T\,\frac{1-T^{-2a}}{8a^2},$$

which leads to

$$\int_0^T \left(\mathrm{Im}\frac{\zeta'}{\zeta}\left(\tfrac{1}{2}+a+it\right)\right)^2 dt \sim T\,\frac{1-T^{-2a}}{8a^2}. \tag{5.2}$$

Proving (5.2) directly seems to be difficult. One can use the partial fraction formula to give an explicit expression for $\mathrm{Im}(\zeta'/\zeta)$, but the methods of Section 4 fail to yield (5.2). In particular, Montgomery's trick does not work here. If these difficulties could be overcome then integrals like (3.1) and (5.1) could be evaluated directly. This could lead to a proof that (3.0) implies GUE3 for the ζ-function. It would also give a consistency check between (3.0) and GUE4. It is interesting that (3.0) can be used to conjecture a value for (5.1), and this can be used to deduce the value of the two integrals displayed below (5.1).

6. References

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