LOG SARKISOV PROGRAM

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Contents

The purpose of this paper is two-fold. The first is to give a tutorial introduction to the so called Sarkisov program, a 3-dimensional generalization of Castelnuovo-Nöther Theorem "untwisting" birational maps between Mori fiber spaces, which was recently established by Corti[4]. We should emphasize that though the general features were understood (cf.Matsuki[17]) after Reid[29] explained the original ideas of V.G. Sarkisov in a substantially laundered form, it is only after Corti[4] that we are beginning to understand the details of the mechanism. Here we will present a flowchart to visualize how the Sarkisov program works and also slightly simplify the proof of termination after the ingenious argument of Corti[4]: we prove there is no infinite loop in the program just observing that the Sarkisov degree decreases strictly after each untwisting and it cannot decrease infinitely many times using the boundedness of $\mathbb{Q}$-Fano d-folds $d \leq 3$ together with the ascending chain condition S_3(Local) of Alexeev[1] (cf.Shokurov[31]Kollár et al[16]). Our argument also makes it explicit that the Sarkisov program holds in arbitrary dimension n once we have Log MMP in dimension n, boundedness of $\mathbb{Q}$-Fano d-folds for $d \leq n$ and S_n(Local). The second is an attempt to give a logarithmic generalization following the philosophy of Iitaka, based upon the Log MMP (established in dimension 3 by Shokurov[31]Kawamata[9] (cf.Kollár et al[16])). The key is to understand the meaning of the genuine Sarkisov program and set up the natural and right generalization. The genuine Sarkisov program untwists any birational map between two

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Mori fiber spaces which are birationally equivalent. A naive speculation that the log Sarkisov program should untwist any birational map between two log Mori fiber spaces which are only birationally equivalent turns out to be not natural and simply does not work! In order to reach the right understanding of what the genuine Sarkisov program does we have to introduce the notion of the Sarkisov relation:Mori fiber spaces are Sarkisov related iff they are the end results of the K-MMP starting from an appropriate nonsingular projective variety. The genuine Sarkisov program untwists a birational map between two Mori fiber spaces which are Sarkisov related, by factorizing it into "links" among intermediate Mori fiber spaces, all of which including the original two are Sarkisov related. Log Mori fiber spaces are said to be Sarkisov related iff they are the end results of the $K + B$-MMP starting from one log pair consisting of a nonsingular variety and an S.N.C. divisor as a boundary. The log Sarkisov program should be the one to untwist a birational map between two log Mori fiber spaces which are Sarkisov related, by factorizing it into "links" among intermediate log Mori fiber spaces, all of which including the original two are Sarkisov related. Once this is understood, the log Sarkisov program works almost parallel to the genuine Sarkisov program in the case of kawamata log terminal singularities in arbitrary dimension, except for the verification of termination. The boundedness of a certain class of log $\mathbb{Q}$-Fano varieties becomes crucial for our argument in showing termination, just as the result of Kawamata[8] on the boundedness of $\mathbb{Q}$-Fano 3-folds was crucial in showing termination of the genuine Sarkisov program in dimension 3. While in dimension 2 we establish termination thanks to a result of Nukulin[26]Alexeev[2], in dimension 3 we have to use a conjecture by Borisov[3] at the last stage of the proof of termination. We also establish the log Sarkisov program in the case of weakly kawamata log terminal singularities in dimension 2 including its termination. But in dimension 3, the Sarkisov relation becomes quite subtle for wklt singularities and we can only discuss problems toward establishing the program including not only its temination but the general mechanism itself.

We also remark that the birational transformations among the various moduli spaces studied by M. Thaddeus and others can be put into the general frame work of the (log) Sarkisov program.

We would like to thank A. Corti, who allowed us to present many of his ideas here in logarithmic form. Most of the arguments here are taken from his paper Corti[4] and repeated for the sake of understanding of the reader. The original ideas are due entirely to V.G. Sarkisov and our indebtedness to M. Reid toward understanding them is as clear as his paper Reid[29]. The conversations with J. Kollár and J. McKernan were very helpful and critical. We would like to thank S. Mori, who gave us warm encouragement throughout.

§0. Introduction.

¿From the view point of Minimal Model Program (the so-called Mori's program), our basic strategy to understand the birational geometry of higher dimensional algebraic varieties (established in dimension 3 and conjectural for higher dimension) is divided into the following 3 steps:

1. Find "good representatives" among varieties with a given function field through MMP: We take a nonsingular projective variety X by Hironaka's resolution of singularities with a given function field. Input X into the black box called Minimal Model Program (abbreviated MMP), which produces a good representative, i.e., a minimal model or a Mori fiber space as an output, depending whether X is non-uniruled or uniruled.

$$\begin{array}{ccccc} & & X & & \\ & & \downarrow & & \\ & & \boxed{\text{MMP}} & & \\ \text{non uniruled} & \swarrow & & \searrow & \text{uniruled} \\ \text{a minimla model} & & & & \text{a Mori fiber space} \end{array}$$

2. Study the properties of "good representatives": The most important property is the Dichotomy, which says that the uniruledness should characterize the variety with Kodaira dimension $-\infty$, i.e. $\kappa = -\infty$ or $\kappa \geq 0$ depending upon whether X is uniruled or not. In dimension 3, this is a theorem. For 3-folds X with $\kappa(X) \geq 0$, the Abundance Theorem of Kawamata-Miyaoka further claims that a minimal model X_{min} has a base point free pluri-canonical system which induces the canonical morphism $\Phi_{|mK_{X_{min}}|} : X_{min} \to X_{can}$, crucial for the understanding of the global structure of X and its moduli. For 3-folds with $\kappa = -\infty$, a theorem of Miyaoka-Mori says that a Mori fiber space X_{mori} is covered by rational curves intersecting $K_{X_{mori}}$ negatively.

3. Study the relation among "good representatives": For 3-folds with $\kappa \geq 0$, the basic relation among good representatives is that minimal models in a given birational equivalence class are connected by a sequence of flops (cf. Reid[28]Kawamata[5] Kollár[15]. See also Matsuki[18] for a finer description of their relation). It is the Sarkisov program, the main theme of our paper, which describes the relation among good representatives for 3-folds with $\kappa = -\infty$, i.e., Mori fiber spaces in a given birational equivalence class.

In dimension 2, i.e., in the case of classical birational geometry of surfaces, the meaning of these 3 steps is rather straightforward.

1. Starting from a nonsingular projective surface, we keep contracting (-1)-curves (MMP in dimension 2) until we get either a surface X_{min} with the canonical

divisor $K_{X_{min}}$ being nef or a ruled surface X_{mori} over a curve (or $\mathbb{P}^2$ over a point).

2. When $\kappa(X) \geq 0$, the canonical morphism $\Phi_{|mK_{X_{min}}|} : X_{min} \to X_{can}$ from a minimal model is:

$\kappa = 2$ - a biational map to a canonically polarized surface with only rational double points

$\kappa = 1$ - an elliptic fibtration whose degeneration fibers are studied by Kodaira[14]

$\kappa = 0$ - a trivial map to a point, where we know X_{min} must be either Abelian, bielliptic, K3 or Enriques.

When $\kappa = -\infty$, the structure of a Mori fiber space is rigid and well-understood: either a $\mathbb{P}^1$-bundle over a nonsingular curve or $\mathbb{P}^2$.

3. A minimal model is unique in a fixed birational equivalence class for surfaces with $\kappa \geq 0$, while any birational map among ruled surfaces in a given birational equivalence class is decomposed into a sequence of elementary transformations by Castelnuovo-Nöther theorem.

Logarithmic Generalization

The logarithmic generalization of the basic strategy following the philosophy of Iitaka to understand the birational geometry of varieties WITH BOUNDARIES goes along the same line:

1. We take a pair (X, B_X) where X is a nonsingular projective variety and $B_X = \Sigma b_i B_i$ is a simple normal crossing divisor with $0 \leq b_i \leq 1$. Input (X, B_X) into the black box called Log MMP, which produces a log minimal model $(X_{min}, B_{X_{min}})$ with $K_{X_{min}} + B_{X_{min}}$ being nef or a log Mori fiber space $\phi : (X_{mori}, B_{X_{mori}}) \to S$ with $K_{X_{mori}} + B_{X_{mori}}$ being ϕ-negative, depending upon whether (X, B_X) is non log uniruled or log uniruled. (We say (X, B_X) is log uniruled iff it is covered by rational curves intersecting $K_X + B_X$ negatively.)

$$\begin{array}{ccc} & (X, B_X) & \\ & \downarrow & \\ & \boxed{\text{Log MMP}} & \\ \text{non log uniruled} & \swarrow \qquad \searrow & \text{log uniruled} \\ \text{a log minimal model} & & \text{a log Mori fiber space} \end{array}$$

2. Again the most important property is the Dichotomy, which says that the log uniruledness should characterize the varieties with log Kodaira dimension $-\infty$, i.e. $\kappa(K_X + B_X) = -\infty$ or $\kappa(K_X + B_X) \geq 0$ depending upon whether (X, B_X) is log uniruled or not. In dimension 3, this is a theorem. For log 3-folds (X, B_X) with $\kappa(K_X + B_X) \geq 0$, the Log Abundance Theorem of KeelMatsukiMcKernan[12]

further claims that a log minimal model $(X_{min}, B_{X_{min}})$ has a base point free pluri-log canonical system. For log 3-folds with $\kappa = -\infty$, a theorem of Miyaoka-Mori applies again to imply that a log Mori fiber space $(X_{mori}, B_{X_{mori}})$ is covered by rational curves intersecting $K_{X_{mori}} + B_{X_{mori}}$ negatively.

3. In the genuine birational geometry, we are interested in the relation among good representatives in a given birational equivalence class, where two good representatives are outcomes of one appropriate nonsingular projective variety through MMP if and only if they are birationally equivalent. This is not the case with logarithmic birational geometry. We say (two or more) good representatives are Sarkisov related iff they are outcomes through Log MMP of one appropriate log pair consisting of a nonsingular projective variety and a S.N.C. divisor as a boundary. Then log minimal models which are Sarkisov related are connected by a sequence of log flops (cf. Kollár[15]Kollár et al[16]. See §4 for a detalied discussion.) The log Sarkisov program should be the one to untwist a birational map between two log Mori fiber spaces which are Sarkisov related.

§1. Flowchart for Sarkisov Program.

In this section, we review the (genuine) Sarkisov program after Corti[4] (cf.Sarkisov[30]Reid[29]Matsuki[17]) with some simplifications and present a flowchart to visualize how it works. The aim of this section is mostly tutorial aside from simultaneously preparing the notations for the logarithmic case which goes almost parallel to the genuine case after the introduction of the Sarkisov relation (See §3.). We refer the reader to KawamataMatsudaMatsuki[11] for the general features of MMP and to Kollár et al[16] for those of Log MMP.

The Sarkisov program, in short, is an algorithm to factorize a birational map between two Mori fiber spaces, i.e., two different end results of MMP of one appropriate nonsingular projective variety, when the Kodaira dimension is $-\infty$.

Definition 1.1. *A Mori fiber space $\phi : X \to S$ is the contraction of an extremal ray with respect to K_X from a normal projective variety with only $\mathbb{Q}$-factorial terminal singularities onto a variety S, i.e., ϕ is a morphism from a normal projective variety with only $\mathbb{Q}$-factorial terminal singularitieswith connected fibers onto a normal variety S with $\dim S < \dim X$ s.t. $\rho(X/S) = 1$ and $-K_X$ is ϕ-ample.*

Theorem 1.2 (Sarkisov Program in dimension 3) (cf.Sarkisov[30]Reid[29] Corti[4]). *A birational map*

$$\begin{tikzcd} X \arrow[r, dashed, "\Phi"] \arrow[d, "\phi"'] & X' \arrow[d, "\phi'"] \\ S & S' \end{tikzcd}$$

between two Mori fiber spaces in dimension 3 is a composite of the following 4 types of links.

Links of type (I)

$$\begin{tikzcd} & Z \arrow[dl] \arrow[r, dashed] & X_1 \arrow[dd] \\ X \arrow[d] & & \\ S & & S_1 \arrow[ll] \end{tikzcd}$$

where $Z \to X$ is a K-negative extremal divisorial contraction, $Z' \dashrightarrow X_1$ a sequence of log flips with respect an appropriate log pair and $\rho(S_1/S) = 1$.

Links of type (II)

$$\begin{tikzcd} & Z \arrow[dl] \arrow[r, dashed] & Z' \arrow[dr] & \\ X \arrow[d] & & & X_1 \arrow[d] \\ S & & & S_1 \arrow[lll, "\sim"'] \end{tikzcd}$$

where $Z \to X$ and $Z' \to X_1$ are K-negative extremal divisorial contractions, $Z \dashrightarrow Z'$ a sequence of log flips with respect to an appropriate log pair.

Links of type (III) (Inverses of links of type (I))

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[dd] & Z' \arrow[dr] & \\ & & X_1 \arrow[d] \\ S \arrow[rr] & & S_1 \end{tikzcd}$$

where $Z' \to X_1$ is a K-negative extremal divisorial contraction, $X \dashrightarrow Z'$ a sequence of log flips with respect to an appropriate log pair and $\rho(S/S_1) = 1$.

Links of type (IV)

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[d] & & X_1 \arrow[d] \\ S \arrow[dr] & & S_1 \arrow[dl] \\ & T & \end{tikzcd}$$

where $X \dashrightarrow X'$ is a sequence of log flips with respect to an appropriate log pair and $\rho(S/T) = \rho(S_1/T) = 1$.

Remark 1.3.

All the intermediate Mori fiber spaces

$$\phi_k : X_k \to S_k$$

in the process of untwisting the given birational map by the Sarkisov program are the end results of K-MMP over *Spec* k starting from one appropriate nonsingular projective variety W. In order to see this, we just have to take W to be a common resolution

$$p_k : W \to X_k.$$

Then each p_k is a process of K-MMP over X_k, and thus a process of K-MMP over *Spec* k.

This fact that all the Mori fiber spaces in the process of the Sarkisov program are Sarkisov related (See §3 for the precise definition of the Sarkisov relation.) is automatic in the case of the genuine Sarkisov program and implicit in the statement. But it is the key point of understanding the log Sarkisov program.

The strategy to untwist Φ into a composite of links is to set up a good invariant, the Sarkisov degree (μ, λ, e) so that it strictly decreases after untwisting the birational map. That is to say, we would like to construct a sequence of links as below

$$\begin{array}{ccccccc} X = X_0 & \dashrightarrow X_1 & \dashrightarrow X_2 & & \cdots & & \dashrightarrow X' \\ \phi\downarrow & \phi_1\downarrow & \phi_2\downarrow & & \cdots & & \phi'\downarrow \\ S = S_0 & S_1 & S_2 & & \cdots & & S' \end{array}$$

so that the Sarkisov degree strictly decreases each time we untwist the birational map

$$(\mu, \lambda, e) = (\mu_0, \lambda_0, e_0) > (\mu_1, \lambda_1, e_1) > \cdots$$

and that it cannot decrease infinitely many times.

Definition 1.4 (Sarkisov degree). *The Sarkisov degree of a birational map between two Mori fiber spaces*

$$\begin{array}{ccc} X & \overset{\Phi}{\dashrightarrow} & X' \\ \phi\downarrow & & \downarrow\phi' \\ S & & S' \end{array}$$

with reference to the fixed Mori fiber space $\phi' : X' \to S'$ is the triplet

$$(\mu, \lambda, e)$$

of the numbers defined below, endowed with the lexicographical order.

First we take a very ample divisor A' on S' and a sufficiently divisible $\mu' \in \mathbb{N}$ such that

$$\mathcal{H}_{X'} = -\mu' K_{X'} + \phi'^* A'$$

is very ample on X'. $\mathcal{H}_X$ is the strict transform of $\mathcal{H}_{X'}$.

$\boxed{\mu : \text{the quasi-effective threshold}}$

The quasi-effective threshold μ is defined to be a positive rational number s.t.

$$\mu K_X + \mathcal{H}_X \equiv 0 \text{ over } S.$$

Note that μ' is the quasi-effective threshold for the special case Φ being the identity map of the Mori fiber space $\phi' : X' \to S'$.

In dimension 2, it is easy to see $\mu \in \frac{1}{3!}\mathbb{N}$.

$\boxed{\lambda : \text{the maximal multiplicity of an extremal ray}}$

We take a common resolution

$$\begin{array}{ccccc} & & W & & \\ & p\swarrow & & \searrow q & \\ X & & \overset{\Phi}{\dashrightarrow} & & X' \\ \phi\downarrow & & & & \downarrow\phi' \\ S & & & & S'. \end{array}$$

such that the exceptional locus of p is an S.N.C. divisor $\cup E_k$. Then taking a general member of $\mathcal{H}_{X'}$ and its strict transform $\mathcal{H}_X$ (We denote them by the same symbols $\mathcal{H}_{X'}$ and $\mathcal{H}_X$ by abuse of notation.) and writing

$$K_W = p^* K_X + \Sigma a_k E_k$$
$$q^* \mathcal{H}_{X'} = p^* \mathcal{H}_X - \Sigma b_k E_k,$$

we define

$$\lambda := \max\{\frac{b_k}{a_k}\}.$$

We remark that $\frac{1}{\lambda}$ has a more intrinsic description and is called the canonical threshold of X with respect to $\mathcal{H}_X$, i.e.,

$$\frac{1}{\lambda} = \max\{c \in \mathbb{Q}_{>0}; K_X + c\mathcal{H}_X \text{ canonical}\},$$

where $K_X + c\mathcal{H}_X$ being canonical means by definition that for some (and thus for any) common resolution

$$p : W \to X$$
$$q : W \to X'$$

such that the exceptional locus $\cup E_k$ of p is an S.N.C. divisor $\cup E_k$ (Note that the strict transform $\mathcal{H}_W$ which is nothing but the total transform $q^*\mathcal{H}_{X'}$ may assumed to be nonsingular and cross $\cup E_k$ normally.), we have

$$K_W + c\mathcal{H}_W = p^*(K_X + c\mathcal{H}_X) + \Sigma r_k E_k$$

with

$$r_k \geq 0 \text{ for } \forall k.$$

Thus λ is independent of the common resolution that we take and well-defined. We note that when X is $\mathbb{Q}$-factorial and thus p has purely one-codimensional exceptional locus, the assumption of the exceptional locus of p being an S.N.C. divisor is unnecessary.

Note also that since $\mathcal{H}_X$ has no base component of codimension one, even when $c \geq 1$ we can regard the pair $(X, c\mathcal{H}_X) = (X, \Sigma c_q B_q)$ as a canonical log pair with only klt singularities in the usual sense (cf. Kollár et al[16]) by taking general members $B_q \in \mathcal{H}_q$ and a suitable set of positive rational numbers $0 < c_q < 1$ with $\Sigma c_q = c$ and that thus $K + c\mathcal{H}$-MMP works as well as $K + \Sigma c_q B_q$-MMP.

In dimension 2, λ is nothing but the maximal multiplicity of a general member of the linear system $\mathcal{H}_X$ (We note that the linear system $\mathcal{H}_X$ consists of the strict transforms of the complete linear system $\mathcal{H}_{X'}$ and that it may not be complete itself.) at the base points $Bs(\mathcal{H}_X)$. When $Bs(\mathcal{H}_X) = \emptyset$, $\lambda = 0$ by definition.

$$\boxed{e : \text{the number of } K + \frac{1}{\lambda}\mathcal{H} - \text{crepant divisors}}$$

$$e := \begin{cases} \#\{E; a(E, \frac{1}{\lambda}\mathcal{H}_X) = 0\} \text{ if } \lambda > 0 \\ 0 \text{ if } \lambda = 0 \end{cases}$$

In dimension 2, e is nothing but the number of the base points of the linear system $\mathcal{H}_X$ with the maximal multiplicity λ when $Bs(\mathcal{H}_X) \neq \emptyset$, and $e = 0$ by definition when $Bs(\mathcal{H}_X) = \emptyset$.

Once the Sarkisov degree, which should measure the extent of untwisting, is set up, the only other ingredient we need is a criterion to judge if the untwisting is completed:

Proposition 1.5 (Nöther-Fano criterion). *A birational map Φ between two Mori fiber spaces is an isomorphism of Mori fiber spaces, i.e.,*

$$\begin{array}{ccc} X & \overset{\sim}{\to} & X' \\ \phi\downarrow & & \downarrow\phi' \\ S & \overset{\sim}{\to} & S' \end{array}$$

if $\lambda \leq \mu$ and $K_X + \frac{1}{\mu}\mathcal{H}_X$ is nef.

The proposition can be proved as an easy application of the Negativity Lemma or Hodge Index Theorem and we refer the reader to Corti[4],Theorem4.2 for a proof.

Flowchart for Sarkisov Program.

In the following, we present a flowchart to untwist a birational map

$$\begin{array}{ccc} X & \overset{\Phi}{\dashrightarrow} & X' \\ \phi\downarrow & & \downarrow\phi' \\ S & & S' \end{array}$$

between two Mori fiber spaces.

We START.

The first question to ask is:

$$\lambda > \mu?$$

According to whether the answer to this question is YES or NO, we proceed separately into the case $\lambda > \mu$ or into the case $\lambda \leq \mu$.

$$\boxed{\text{Case} : \lambda \leq \mu}$$

If $\lambda \leq \mu$, then the next question to ask is:

$$K_X + \frac{1}{\mu}\mathcal{H}_X \text{ nef?}$$

If the answer to this question is YES, then $K_X + \frac{1}{\mu}\mathcal{H}_X$ is nef and $\lambda \leq \mu$ by the case assumption. Thus the Nöther-Fano criterion applies to conclude Φ is an isomorphism of Mori fiber spaces. This leads to an

END.

If $K_X + \frac{1}{\mu}\mathcal{H}_X$ is not nef, then we construct as follows a normal projective variety T dominated by $S \to T$ s.t. $K_X + \frac{1}{\mu}\mathcal{H}_X$ is not relatively nef over T and $\rho(X/T) = 2$, so that we run $K + \frac{1}{\mu}\mathcal{H}$-MMP over T to have an untwisting link.

We pick a $K_X + \frac{1}{\mu}\mathcal{H}_X$-negative extremal ray P of $\overline{NE}(X/Spec\ k)$ s.t. the span $F := P + R$ is a 2-dimensional extremal face, where R is the K_X-negative extremal ray giving the Mori fiber space $\phi : X \to S$. F is $K_X + (\frac{1}{\mu} - \epsilon)\mathcal{H}_X$-negative for $0 < \epsilon << 1$, thus we have the contraction morphism $cont_F : X \to T$ to obtain T. Since $F \supset R$, $cont_F$ factors through S and by construction T satisfies all the desired conditions.

Now we

$$\text{Run } K + \frac{1}{\mu}\mathcal{H} - \text{MMP over } T.$$

We reach either a minimal model or a Mori fiber space (with respect to $K + \frac{1}{\mu}\mathcal{H}$ and over T).

First we show that it is

IMPOSSIBLE to reach a minimal model!

Suppose we did. Then according to whether the first nonflipping contraction is divisorial or not, we should have two different diagrams as follows:

$$\begin{tikzcd}
X \arrow[r, dashed] \arrow[d] & Z' \arrow[dr] & \\
S \arrow[dr] & & X_1 \arrow[d] \\
& T \arrow[r, "\sim"] & S_1
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[rr, dashed] \arrow[d] & & Z' = X_1 \arrow[d] \\
S \arrow[dr] & & S_1 \arrow[dl] \\
& T &
\end{tikzcd}$$

We take a general curve $\Sigma_1 \in Hilb(X_1/T)$ away from the locus of indeterminacy of the birational map $X_1 \dashrightarrow X$ (i.e., in the first case the union of the image of the exceptional divisor of the divisorial contraction and all the flipped curves and in the second case all the flipped curves). Σ_1 can be considered to lie on X and since Σ_1 is general we conclude $\phi(\Sigma_1)$ is a curve (not a point), which implies $(K_X + \frac{1}{\mu}\mathcal{H}_X) \cdot \Sigma_1 < 0$. But then

$$0 > (K_X + \frac{1}{\mu}\mathcal{H}_X) \cdot \Sigma = (K_{X_1} + \frac{1}{\mu}\mathcal{H}_{X_1}) \cdot \Sigma_1 \geq 0,$$

a contradiction!

Next suppose we

Reach a Mori fiber space $X_1 \to S_1$.

Then the next question to ask just in order to separate the types of links is:

Is the first nonflipping contraction divisorial ?

If the answer is YES, the $K + \frac{1}{\mu}\mathcal{H}$-MMP consists of a sequence of $K + \frac{1}{\mu}\mathcal{H}$-flips $X \dashrightarrow Z'$ followed by a $K + \frac{1}{\mu}\mathcal{H}$-negative divisorial contraction $Z' \to X_1$. Since $\rho(X_1/T) = 1$, $\phi_1 : X_1 \to S_1 = T$ is a $K_{X_1} + \frac{1}{\mu}\mathcal{H}_{X_1}$-negative and thus K_{X_1}-negative fiber space.

$$\begin{array}{ccccc} X & \dashrightarrow & Z' & & \\ \downarrow & & & \searrow & \\ S & & & & X_1 \\ & \searrow & & & \downarrow \\ & & T & \stackrel{\sim}{\to} & S_1 \end{array}$$

If the answer is NO, then the $K + \frac{1}{\mu}\mathcal{H}$-MMP consists of a sequence of $K + \frac{1}{\mu}\mathcal{H}$-flips $X \dashrightarrow Z'$ followed by a $K_{X_1} + \frac{1}{\mu}\mathcal{H}_{X_1}$-negative and thus K_{X_1}-negative fibering contraction $\phi_1 : Z' = X_1 \to S_1$. Since $\rho(X_1/T) = \rho(X/T) = 2$, we have $\rho(S_1/T) = 1$.

$$\begin{array}{ccccc} X & & \dashrightarrow & & Z' = X_1 \\ \downarrow & & & & \downarrow \\ S & & & & S_1 \\ & \searrow & & \swarrow & \\ & & T & & \end{array}$$

We claim in both cases X_1 has only terminal singularities. ($\mathbb{Q}$-factoriality of X_1 is automatic from construction.) Let I be the locus of indeterminacy of the birational map $X_1 \dashrightarrow X$. If E is a discrete valuation whose center on X_1 is not contained in I (and has codimension ≥ 2), then

$$a(E, X_1, \emptyset) = a(E, X, \emptyset) > 0.$$

If the center of E on X_1 is contained in I, then

$$\begin{aligned} a(E, X_1, \emptyset) &\geq a(E, X_1, \frac{1}{\mu}\mathcal{H}_{X_1}) \\ &> a(E, X, \frac{1}{\mu}\mathcal{H}_X) \geq 0. \end{aligned}$$

Thus we have the claim.

Therefore, we have a link of type (III) in the former case and a link of type (IV) in the latter.

Moreover, since $K_{X_1} + \frac{1}{\mu}\mathcal{H}_{X_1}$ is negative over S_1, we conclude in both cases

$$\mu_1 < \mu.$$

Therefore, after untwisting Φ by a link of type (III) or type (IV), we go back to the START with strictly decreased quasi-effective threshold.

$$\boxed{\text{Case} : \lambda > \mu}$$

In this case we

Take a maximal divisorial blow up $p : Z \to X$,

with respect to $K_X + \frac{1}{\lambda}\mathcal{H}_X$, i.e., p is a projective morphism from Z with only $\mathbb{Q}$-factorial terminal singularities s.t.

i) $\rho(Z/X) = 1$,

ii) the exceptional locus of p is a prime divisor E, and

iii) p is $K + \frac{1}{\lambda}\mathcal{H}$-crepant, i.e.,

$$K_Z + \frac{1}{\lambda}\mathcal{H}_Z = p^*(K_X + \frac{1}{\lambda}\mathcal{H}_X).$$

Proposition 1.5. *A maximal divisorial blow up $p : Z \to X$ with respect to $K_X + \frac{1}{\lambda}\mathcal{H}_X$ exists.*

We remark that the exceptional divisor E of p is necessarily one of the $K + \frac{1}{\lambda}\mathcal{H}$-crepant divisors $\{E_1, E_2, \cdots, E_e\}$ counted for the number e. As long as we require Z to have only terminal singularities, we can't quite specify which E_i would be the exceptional divisor. On the other hand, if we allow Z to have canonical singularities, for each E_i we can construct a maximal blow up $p_i : Z_i \to X$ (allowing Z_i to have canonical singularities) with the exceptional divisor being E_i.

Proof.

Take a resolution $Y \to X$ s.t.

a) the exceptional locus is a divisor with only S.N.C.,

b) Y dominates X' so that the strict transform $\mathcal{H}_Y$ coincides with the total transform of $\mathcal{H}_{X'}$ and that a general member $\mathcal{H}_Y$ is smooth and crosses normally with the exceptional locus.

We run the $K + \frac{1}{\lambda}\mathcal{H}$-MMP over X to get a minimal model $f : (Z', \frac{1}{\lambda}\mathcal{H}_{Z'}) \to (X, \frac{1}{\lambda}\mathcal{H}_X)$. As before, it is easy to see that Z' has only $\mathbb{Q}$-factorial terminal singularities. Since both Z' and X are $\mathbb{Q}$-factorial, the exceptional locus of f is purely one-codimensional. An easy application of the Negativity Lemma (Shokurov[30]Kollár et al[16]Corti[4]) shows that the exceptional locus is actually $\cup_{i=1}^{e} E_i$ and that f is $K + \frac{1}{\lambda}\mathcal{H}$-crepant, i.e., $K_{Z'} + \frac{1}{\lambda}\mathcal{H}_{Z'} = f^*(K_X + \frac{1}{\lambda}\mathcal{H}_X)$.

Now we run the K-MMP starting from Z' over X ending necessarily with a divisorial contraction $p : Z \to X$. It is immediate that $p : Z \to X$ is a maximal divisorial blow up with respect to $K_X + \frac{1}{\lambda}\mathcal{H}_X$. (If we want to specify the exceptional divisor E_i allowing Z to have canonical singularities, then we run $K + \frac{1}{\lambda}\mathcal{H} + \epsilon\Sigma_{j \neq i} E_j$-MMP $(0 < \epsilon << 1)$ instead.)

We also remark that in order to construct just one maximal divisorial blowup we can start from any common resolution Y which may not satisfy a) as long as X is $\mathbb{Q}$-factorial and thus the exceptional locus of $Y \to X$ is purely one codimensional.

There is another method called the “Nef Threshold Method” to construct a maximal divisorial blow up by M. Reid.

We construct a chain of

Y_i: 3-folds with only $\mathbb{Q}$-factorial terminal singularities projective over X ($Y_0 = Y$),

$\mathcal{H}_{Y_i}$: the strict transforms of $\mathcal{H}_Y$,

λ_i: (a non-decreasing sequence of) nonnegative rational numbers

s.t.

a) $\lambda_i K_{Y_i} + \mathcal{H}_{Y_i}$ is a supporting function of a face containing a K_{Y_i}-negative extremal ray R_i of $\overline{NE}(Y_i/X)$, i.e., $\lambda_i K_{Y_i} + \mathcal{H}_{Y_i}$ is relatively nef over X and

$$(\lambda_i K_{Y_i} + \mathcal{H}_{Y_i})^{\perp} \cap \overline{NE}(Y_i/X) \supset R_i,$$

b) either $Y_i \to Y_{i+1}$ is a divisorial contraction of R_i or the flip $Y_i \dashrightarrow Y_{i+1}$, and

c) the chain ends with a divisorial contraction $p : Z = Y_n \to X$ of an extremal ray R_n

$$(\lambda_i K_{Y_i} + \mathcal{H}_{Y_i})^{\perp} \cap \overline{NE}(Y_i/X) \supset R_n.$$

Then it is easy to see that $p : Z \to X$ is a maximal divisorial blow up with respect to $K_X + \frac{1}{\lambda}\mathcal{H}_X$ and $\lambda_n = \lambda$.

We construct inductively.

Suppose we have succeeded constructing the chain up to the i-th stage. Consider the nef threshold λ_i of $\mathcal{H}_{Y_i}$ with respect to K_{Y_i}

$$\lambda_i := \sup\{\nu; \nu K_{Y_i} + \mathcal{H}_{Y_i} \text{ relatively nef over } X\}.$$

Remark that since $\lambda_{i-1}K_{Y_{i-1}} + \mathcal{H}_{Y_{i-1}}$ is relatively nef over X and the contraction of R_{i-1} is $\lambda_{i-1}K_{Y_{i-1}} + \mathcal{H}_{Y_{i-1}}$-trivial, $\lambda_{i-1}K_{Y_i} + \mathcal{H}_{Y_i}$ is also relatively nef over X and thus $\lambda_i \geq \lambda_{i-1}$.

We claim that λ_i is rational and that there exists a K_{Y_i}-negative extremal ray R_i s.t.

$$(\lambda_i K_{Y_i} + \mathcal{H}_{Y_i})^{\perp} \cap \overline{NE}(Y_i/X) \supset R_i.$$

Instead of applying the Rationality Theorem (KaMaMa[11],Theorem4-1-1) whose proof only applies to ample divisors, we use a result of Kawamata[8] on the boundedness of lengths of the extremal rational curves to the relatively nef divisor $\lambda_{i-1}K_{Y_i} + \mathcal{H}_{Y_i}$. (This idea was communicated to us by J. McKernan. See KeMaMc[12].) First from the definition and the Cone Theorem, we have

$$\begin{aligned}\lambda_i &= \lambda_{i-1} + \inf\{\frac{(\lambda_{i-1}K_{Y_i} + \mathcal{H}_{Y_i}) \cdot l}{-K_{Y_i} \cdot l}\}; l : K_{Y_i} - \text{negative extremal rays}\} \\ &= \inf\{\frac{\mathcal{H}_{Y_i} \cdot l}{-K_{Y_i} \cdot l}\}; l : K_{Y_i} - \text{negative extremal rays}\}.\end{aligned}$$

The result of Kawamata tells us that for each K_{Y_i}-negative extremal ray l, there exists a rational curve L_l which generates $l = \mathbb{R}_+[L_l]$ and $0 < -K_{Y_i} \cdot L_l \leq 2 \cdot \dim X$. Thus if q_i is the $\mathbb{Q}$-factorial index of Y_i

$$q_i := \min\{z \in \mathbb{N}; zD \text{ is Cartier for all integral Weil divisors } D \text{ on } Y_i\},$$

(which coincides with the index r_i of K_{Y_i} in dimension 3), then

$$\frac{\mathcal{H}_{Y_i} \cdot l}{-K_{Y_i} \cdot l} \in \frac{1}{(r_i \cdot 2\dim X)! q_i}\mathbb{Z}_{\geq 0}.$$

Therefore, "inf" is actually attained as the minimum for some K_{Y_i}-negative extremal ray R_i and for this R_i we have

$$R_i \subset (\lambda_i K_{Y_i} + \mathcal{H}_{Y_i})^{\perp} \cap \overline{NE}(Y_i/X).$$

As for Y_{i+1}, we take either the divisorial contraction $Y_i \to Y_{i+1}$ of R_i or the flip $Y_i \dashrightarrow Y_{i+1}$ of R_i.

We go back to the discussion of the flowchart.

Now we

$$\text{Run } K + \frac{1}{\lambda}\mathcal{H} - \text{MMP over } S.$$

A priori we reach either a minimal model or a Mori fiber space (with respect to $K + \frac{1}{\lambda}\mathcal{H}$ over S).

First we show that it is

IMPOSSIBLE to reach a minimal model!

Suppose we did.

Then according to whether the first nonflipping contraction is divisorial or not we should have two different diagrams:

$$\begin{tikzcd}
 & Z \arrow[dl] \arrow[r, dashed] & Z' \arrow[dr] & \\
X \arrow[d] & & & X_1 \arrow[d] \\
S & & S_1 \arrow[ll, "\sim"'] & S_1
\end{tikzcd}$$

$$\begin{tikzcd}
 & Z \arrow[dl] \arrow[r, dashed] & X_1 \arrow[dd] \\
X \arrow[d] & & \\
S & & S_1 \arrow[ll]
\end{tikzcd}$$

We take a general curve $\Sigma \in Hilb(X/S)$ away from $p(E)$ (and thus can be considered to lie on Z) and away from all the flipping curves (and thus can be considered to lie on Z').

In the first case, we have

$$\begin{aligned}
0 &\leq (K_{X_1} + \frac{1}{\lambda}\mathcal{H}_{X_1}) \cdot q_*\Sigma \\
&= \{(K_{Z'} + \frac{1}{\lambda}\mathcal{H}_{Z'}) - aE_q\} \cdot \Sigma \ (a > 0) \\
&\leq (K_X + \frac{1}{\lambda}\mathcal{H}_X) \cdot \Sigma \\
&< (K_X + \frac{1}{\mu}\mathcal{H}_X) \cdot \Sigma = 0,
\end{aligned}$$

a contradiction!

In the second case, we have

$$\begin{aligned}
0 &\leq (K_{Z'} + \frac{1}{\lambda}\mathcal{H}_{Z'}) \cdot \Sigma \\
&= (K_X + \frac{1}{\lambda}\mathcal{H}_X) \cdot \Sigma \\
&< (K_X + \frac{1}{\mu}\mathcal{H}_X) \cdot \Sigma = 0,
\end{aligned}$$

again a contradiction!

Next suppose we

$$\text{Reach a Mori fiber space } X_1 \to S_1.$$

Then the next question to ask just in order to separate the types of links is:

$$\text{Is the first nonflipping contraction divisorial ?}$$

If the answer is YES, the $K + \frac{1}{\lambda}\mathcal{H}$-MMP consists of a sequence of $K + \frac{1}{\lambda}\mathcal{H}$-flips $X \dashrightarrow Z'$ followed by a $K+\frac{1}{\lambda}\mathcal{H}$-negative contraction $Z' \to X_1$. Since $\rho(X_1/S) = 1$, $\phi_1 : X_1 \to S_1 = S$ is a $K_{X_1} + \frac{1}{\lambda}\mathcal{H}_{X_1}$-negative and thus K_{X_1}-negative fiber space.

$$\begin{tikzcd}
 & Z \arrow[dl] \arrow[r, dashed] & Z' \arrow[dr] & \\
X \arrow[d] & & & X_1 \arrow[d] \\
S & & \overset{\sim}{\leftrightarrows} & S_1
\end{tikzcd}$$

We note that the exceptional divisors E and E_q are distinct, since otherwise X and X_1 are isomorphic in codimension one, which would imply X and X_1 are indeed isomorphic over S, but then while E_q is NOT $K_{X_1} + \frac{1}{\lambda}\mathcal{H}_{X_1}$-crepant E is $K_X + \frac{1}{\lambda}\mathcal{H}_X$-crepant, absurd!

If the answer is NO, then the $K + \frac{1}{\lambda}\mathcal{H}$-MMP consists of a sequence of $K + \frac{1}{\lambda}\mathcal{H}$-flips $X \dashrightarrow Z'$ followed by a $K_{X_1} + \frac{1}{\lambda}\mathcal{H}_{X_1}$-negative and thus K_{X_1}-negative fibering contraction $Z' = X_1 \to S_1$. Since $\rho(X_1/S) = \rho(Z/S) = 2$, we have $\rho(S_1/T) = 1$.

$$\begin{tikzcd}
 & Z \arrow[dl] \arrow[r, dashed] & X_1 \arrow[dd] \\
X \arrow[d] & & \\
S & & S_1 \arrow[ll]
\end{tikzcd}$$

In both cases, X_1 has only $\mathbb{Q}$-factorial terminal singularities and thus we have a link of type (II) or a link of type (I), respectively.

Now we study how the Sarkisov degree (μ, λ, e) changes after untwisting by a link of type (II) or type (I).

We claim that

$$\mu_1 \leq \mu$$

with equality holding only if

$$\begin{aligned}
&\text{either } \dim S_1 > \dim S \\
&\quad \text{or } \dim S_1 = \dim S \text{ and } \psi_1 \text{ is square ,}
\end{aligned}$$

i.e.,

$$\begin{tikzcd} X \arrow[r, dashed, "\psi_1"] \arrow[d, "\phi"'] & X_1 \arrow[d, "\phi_1"] \\ S & S' \arrow[l, "\pi"'] \end{tikzcd}$$

π is a birational morphism and $\psi_\eta : X_\eta \dashrightarrow (X_1)_\eta$ is an isomorphism, where η is the generic point of S.

First by definition of λ and the assumption of this case $\lambda > \mu$, it follows that

$$p^*(K_X + \frac{1}{\mu}\mathcal{H}_X) = K_Z + \frac{1}{\mu}\mathcal{H}_Z + bE$$

for some $b > 0, b \in \mathbb{Q}$.

We take a general curve $\Sigma_1 \in Hilb(X_1/S_1)$ away from the locus of indeterminacy of the birational map $X_1 \dashrightarrow Z$ (i.e., in the case of a link of type (II) the union of $q(E_q)$ and all the flipped curves and in the case of a link of type (I) the union of all the flipped curves). Then Σ_1 can be considered to lie on Z and

$$\begin{aligned} 0 &= (K_X + \frac{1}{\mu}\mathcal{H}_X) \cdot p_*\Sigma_1 \\ &= (K_Z + \frac{1}{\mu}\mathcal{H}_Z + bE) \cdot \Sigma_1 \\ &\geq (K_Z + \frac{1}{\mu}\mathcal{H}_Z) \cdot \Sigma_1 \\ &= (K_{X_1} + \frac{1}{\mu}\mathcal{H}_{X_1}) \cdot \Sigma_1, \end{aligned}$$

which implies

$$\mu_1 \leq \mu.$$

Moreover, if $\mu_1 = \mu$ and $\dim S = \dim S_1$ (which implies that $\pi : S_1 \to S$ is a birational morphism, since both field extensions $k(X)/k(S)$ and $k(X_1) = k(X)/k(S_1)$ are algebraically closed), then $E \cdot \Sigma_1 = 0$, which is equivalent to saying ϕ_1(the strict transform of E) $\neq S_1$. Therefore, ψ_1 is square.

We also claim that

$$\lambda_1 \leq \lambda$$

and

$$\text{if } \lambda_1 = \lambda \text{ then } e_1 < e.$$

First $(X_1, \frac{1}{\lambda}\mathcal{H}_{X_1})$ is canonical, since it is obtained from a canonical pair $(Z, \frac{1}{\lambda}\mathcal{H}_Z)$ through $K + \frac{1}{\lambda}\mathcal{H}$-MMP. Thus $\lambda_1 \leq \lambda$. (Note that in general canonicality may not be preserved when we contract a component of the boundary B through $K + B$-MMP. But in our case, $\mathcal{H}$'s are the strict transforms of one unique base point free system and thus canonicality is preserved.)

Moreover, if $\lambda_1 = \lambda$, then in the case of untwisting by a link of type (II)

$$K_{Z'} + \frac{1}{\lambda}\mathcal{H}_{Z'} = q^*(K_{X_1} + \frac{1}{\lambda}\mathcal{H}_{X_1}) + aE_q(a > 0)$$

implies E_q is not a $K_{X_1} + \frac{1}{\lambda}\mathcal{H}_{X_1}$-crepant divisor (and E is a divisor on X_1 and thus not exceptional) and thus

$$e_1 \le e - 1 < e.$$

In the case of untwisting by a link of type (I) E is a divisor on X_1 (and thus not exceptional) and thus we have the same conclusion.

Therefore, after untwisting by a link of type (II) or type (I), we go back to the START with strictly decreased Sarkisov degree.

The “visualization” of the flowchart can be found at the end of the paper.

§2. Termination of Flowchart.

In this section, we discuss the termination of the flowchart for the Sarkisov program, i.e., the problem of showing that there is no infinite loop in the flowchart and thus after a finite number of untwisting it gives a factorization of any given birational map between two Mori fiber spaces. Once we have (Log)-MMP in dimension n the key points of showing termination for Sarkisov program for n-folds are:

i) Discreteness of the quasi-effective thresholds μ, which follows from the boundedness of $\mathbb{Q}$-Fano d-folds $d \leq n$, and

ii) Corti[4]'s ingeneous argument to reduce the probelm to S_n(Local) when the quasi-effective threshold stabilizes.

In dimension 3, where we have all the necessary ingredients, the termination of the flowchart is a theorem by Corti[4]. The argument here is a modification of Corti[4] following a slightly simplified flowchart in the previous section. We restrict ourselves to dimension 3 in the following presentation, but we carry the argument so that it works almost verbatim in arbitrary dimension (once all the necessary but still conjectural ingredients are established).

Claim 2.1. *There is no infinite number of untwisting (successive or unsuccessive) by the links under the case $\lambda \leq \mu$.*

Proof.

Suppose there are infinitely many links (successive or unsuccessive)

$$\begin{tikzcd} X_i \arrow[r, dashed, "\psi_i"] \arrow[d, "\phi_i"'] & X_{i+1} \arrow[d, "\phi_{i+1}"] \\ S_i & S_{i+1} \end{tikzcd}$$

under the case $\lambda \leq \mu$. Note that in the case $\lambda \leq \mu$ we have $\dim S_i \geq 1$ (unless Φ_i becomes an isomorphism of Mori fiber spaces). When $\dim S_i = 2$, l being a rational curve which is a general fiber of ϕ_i, we have

$$K_{X_i} \cdot l = -2$$
$$(\mu K_{X_i} + \mathcal{H}_{X_i}) \cdot l = 0,$$

which implies

$$\mu \in \frac{1}{2}\mathbb{N}.$$

When $\dim S_i = 1$, we can take a rational curve l in a general fiber which is a Del Pezzo surface s.t.

$$K_{X_i} \cdot l = -1, -2 \text{ or } -3$$
$$(\mu K_{X_i} + \mathcal{H}_{X_i}) \cdot l = 0,$$

which implies

$$\mu \in \frac{1}{3!}\mathbb{N}.$$

Since after any link in the case $\lambda \leq \mu$ the quasi-effective threshold strictly decreases and it does not increase afyter any link in any case, we then have a strictly decreasing sequence in $\frac{1}{3!}\mathbb{N}$

$$\mu \geq \mu_1 > \mu_2 \cdots > 0,$$

a contradiction!

In general, we only have to use the boundedness of $\mathbb{Q}$-Fano d-folds for $d \leq n-1$ to derive the discreteness of μ and thus a contradiction to establish this claim.

Claim 2.2. *There is no infinite (successive) sequence of untwisting by the links under the case $\lambda > \mu$ with stationary quasi-effective threshold.*

Proof.

This is the heart of the ingeneous argument by Corti[4]. Suppose there is such an infinite sequence

$$\begin{tikzcd}
X = X_0 \arrow[r, dashed] \arrow[d, "\phi"'] & X_1 \arrow[r, dashed] \arrow[d, "\phi_1"'] & X_2 \arrow[r, dashed] \arrow[d, "\phi_2"'] & \cdots \arrow[r, dashed] \arrow[d, phantom, "\cdots"] & X_k \arrow[r, dashed, "\psi_k"] \arrow[d, "\phi_k"'] & X_{k+1} \cdots \arrow[d, "\phi_{k+1}"'] & \cdots \\
S = S_0 & S_1 \arrow[l] & S_2 \arrow[l] & \cdots \arrow[l] & S_k \arrow[l] & S_{k+1} \cdots \arrow[l] &
\end{tikzcd}$$

Since $\mu_k = \mu_{k+1}$ for each k by assumption, we have

$$\begin{aligned}
&\text{either } \dim S_{k+1} > \dim S_k \\
&\quad \text{or } \dim S_{k+1} = \dim S_k \text{ and } \psi_k \text{ is square.}
\end{aligned}$$

The first cannot happen infinitely many times, thus we may assume we have the second case for all k. Note that $\dim S_k \geq 1$, since if $\dim S_k = \dim S_{k+1} = 0$ then ψ_k being square would imply ψ_k is an isomorphism of Mori fiber spaces, which is absurd!

We also know that $\{\lambda_k\}$ is a nonincreasing sequence and since if $\lambda_k = \lambda_{k+1}$ then $e_{k+1} < e_k$, the value of λ_k cannot be stationary. Therefore, we have a sequence

$$\{\frac{1}{\lambda_k}\} \quad (\frac{1}{\lambda_k} < \frac{1}{\mu_k} = \frac{1}{\mu_0})$$

which accumulates from below to (but never equals)

$$\alpha \leq \frac{1}{\mu_0}.$$

Step 1. We claim $(X_k, \alpha\mathcal{H}_{X_k})$ and $(Z_k, \alpha\mathcal{H}_{Z_k})$ ($p_k : Z_k \to X_k$ is a maximal divisorial blowup with respect to $K + \frac{1}{\lambda}\mathcal{H}$) have only log canonical singularities for k sufficiently large (and thus we may assume this holds for $\forall k$).

Let $\alpha_k (> \frac{1}{\lambda_k})$ be the log canonical threshold of the pair X_k with respect to $\mathcal{H}_k$. If $\alpha > \alpha_k$ for infinitely many k's, then there is a strictly increasing subsequence $\{\alpha_l\}$ of log canonical thresholds accumulating to α. This contradicts S_3(Local) proved by Alexeev[1] (cf.Kollár et al[16]). The same argument applies to $(Z_k, \alpha\mathcal{H}_{Z_k})$.

Every link $X_k \dashrightarrow X_{k+1}$ is an outcome of $K + \frac{1}{\lambda_k}\mathcal{H}$-MMP over S_k (after taking a maximal divisorial blowup $p_k : Z_k \to X_k$) consisting of a finite number of $K + \frac{1}{\lambda}\mathcal{H}$-flips

$$Z_k = Z_k^0 \overset{t^0}{\dashrightarrow} Z_k^1 \overset{t^1}{\dashrightarrow} Z_k^2 \cdots \overset{t^{m-1}}{\dashrightarrow} Z_k^m,$$

possibly followed by a divisorial contraction $q_k^m : Z_k^m \to X_k^{m+1} = X_{k+1}$ (otherwise $Z_k^m = X_k^{m+1}$).

Step 2. We claim that every step

$$\begin{array}{ccccc} Z_k^i & & \dashrightarrow & & Z_k^{k+1} \\ & \searrow & & \swarrow & \\ & & X_k^{i+1} & & \end{array}$$

is a step of $K + \alpha\mathcal{H}$-MMP.

We prove this by induction on i.

First note that since $\alpha > c_k$, we have

$$K_{Z_k} + \alpha\mathcal{H}_{Z_k} = p_k^{0*}(K_{Z_k} + \alpha\mathcal{H}_{Z_k}) - aE_k (a > 0).$$

Therefore, we have

$$(K_{Z_k^0} + \alpha\mathcal{H}_{Z_k^0}) \cdot P_k^0 > 0,$$

P_k^0 being the extremal ray giving rise to the morphism p_k^0.

Suppose we have

$$(K_{Z_k^i} + \alpha\mathcal{H}_{Z_k^i}) \cdot P_k^i > 0,$$

P_k^i being the extremal ray giving rise to the morphism p_k^i.

Note that $K_{Z_k^i} + \alpha\mathcal{H}_{Z_k^i}$ is never relatively nef over S_k. We see this as follows: First $\alpha \leq \frac{1}{\mu_k} = \frac{1}{\mu_1}$. Suppose $\alpha = \frac{1}{\mu_k}$. Then

$$K_{Z_k^i} + \alpha\mathcal{H}_{Z_k^i} \equiv_{S_k} -a(\text{the strict transform of } E_k)(a > 0)$$

is never relatively nef over S_k. Suppose $\alpha < \frac{1}{\mu_k}$. Then by taking a general curve $\Sigma \in Hilb(X_k/S_k)$ away from the locus of indeterminacy of the birational map $X_k \dashrightarrow Z_k^i$ (which thus can be considered to lie on Z_k^i) we have

$$\begin{aligned}(K_{Z_k^i} + \alpha\mathcal{H}_{Z_k^i}) \cdot \Sigma &= (K_{X_k} + \alpha\mathcal{H}_{X_k}) \cdot \Sigma \\ &< (K_{X_k} + \frac{1}{\mu_k}\mathcal{H}_{X_k}) \cdot \Sigma = 0.\end{aligned}$$

This implies that

$$(K_{Z_k^i} + \alpha \mathcal{H}_{Z_k^i}) \cdot Q_k^i < 0$$

for the other extremal ray Q_k^i of 2-dimensional cone $\overline{NE}(Z_k^i/S_k)$. This proves the claim.

A consequence of this claim is that (cf.KaMaMa[11],Proposition 5-1-11)

$$a(\nu, X_1, \alpha\mathcal{H}_{X_1}) \le a(\nu, X_k, \alpha\mathcal{H}_{X_k})$$

for any discrete valuation ν of $k(X)$ and the strict inequality holds iff ψ_i is not an isomorphism at the center of ν on X_i for some $i < k$.

Step 3. We claim that $(X_k, \alpha\mathcal{H}_{X_k})$ has purely log terminal singularities for k sufficiently large (and thus we may assume this holds for $\forall k$).

Assume to the contrary that there exists infinitely many k s.t. $(X_k, \alpha\mathcal{H}_{X_k})$ is not purely log terminal, which is equivalent to saying by the consequence above that for all k there exists a valuation ν_k of $k(X)$ with

$$a(\nu_k, X_k, \alpha\mathcal{H}_{X_k}) = -1,$$

which implies again by the consequence that

$$a(\nu_k, X_1, \alpha\mathcal{H}_{X_1}) = -1$$

and that at the center $z(\nu_k, X_1)$ of ν_k on X_1, the birational map $\psi_{k-1} \circ \cdots \psi_2 \circ \psi_1 : X_1 \dashrightarrow X_k$ is an isomorphism. Thus the local (w.r.t. Zariski topology) canonical thresholds are the same

$$c(z(\nu_k, X_k), X_k, \mathcal{H}_{X_k}) = c(z(\nu_k, X_1), X_1, \mathcal{H}_{X_1}).$$

On the other hand, by definition

$$\frac{1}{\lambda_k} \le c(z(\nu_k, X_k), X_k, \mathcal{H}_{X_k})$$

and since $K_{X_k} + \alpha\mathcal{H}_{X_k}$ is not canonical at the center $z(X_k, X_k)$, we have

$$c(z(\nu_k, X_k), X_k, \mathcal{H}_k) < \alpha.$$

Therefore,

$$\frac{1}{\lambda_k} \le c(z(\nu_k, X_1), X_1, \mathcal{H}_{X_1}) < \alpha.$$

But $\{\frac{1}{\lambda_k}\}$ is a nondecreasing and nonstationary sequence converging to α and it is easy to see the set $\{c(x, X_1, \mathcal{H}_{X_1}); x \in X\}$ is finite, a contradiction!

We remark that the valuations of $k(X)$ corresponding to the E_k's are all distinct. In fact, suppose E_i and E_j coincide, and thus Z_i and Z_j are isomorphic in a neighborhood of E_i and E_j, which would imply

$$a(E_i, X_i, \alpha\mathcal{H}_{X_i}) = a(E_j, X_j, \alpha\mathcal{H}_{X_j}).$$

On the other hand, from Step 2 we have

$$a(E_i, X_i, \alpha\mathcal{H}_{X_i}) < a(E_j, X_j, \alpha\mathcal{H}_{X_j}),$$

a contradiction!

Finally we conclude the proof of the claim as follows: From Step 3 we may assume that $(X_1, \alpha\mathcal{H}_{X_1})$ has only purely log terminal singularities. But on the other hand, for infinitely many E_k with distinct corresponding discrete valuations

$$a(E_k, X_1, \alpha\mathcal{H}_{X_1}) \leq a(E_k, X_k, \alpha\mathcal{H}_{X_k}) < 0,$$

a contradiction!

In general, we only need S_n(Local) to carry out the argument for this claim.

Claim 2.3. *There is no infinite successive sequence of untwisting by the links under the case $\lambda > \mu$ with nonstationary quasi-effective threshold.*

Suppose there is such an infinite sequence

$$\begin{array}{ccccccccccccc}
X = X_0 & \dashrightarrow & X_1 & \dashrightarrow & X_2 & \dashrightarrow & \cdots & \dashrightarrow & X_k & \overset{\psi_k}{\dashrightarrow} & X_{k+1} & \cdots & \\
\phi \downarrow & & \phi_1 \downarrow & & \phi_2 \downarrow & & \cdots & & \phi_k \downarrow & & \phi_{k+1} \downarrow & & \cdots \\
S = S_0 & \leftarrow & S_1 & \leftarrow & S_2 & \leftarrow & \cdots & \leftarrow & S_k & \leftarrow & S_{k+1} & \cdots &
\end{array}$$

Case: For some k_0, $\dim S_{k_0} \geq 1$.

In this case for $\forall k \geq k_0$ we have

$$\mu_k \in \frac{1}{3!}\mathbb{N}$$

as before, and $\{\mu_k\}$ is a nonstationary and nonincreasing infinte sequence $\mu_0 \geq \mu_k > 0$, a contradiction! In general, we only need the boundedness of $\mathbb{Q}$-Fano d-folds for $d \leq n-1$ up to this point of the argument.

Finally

Case:For $\forall k$, $\dim S_k = 0$.

In this case, the X_k's are all $\mathbb{Q}$-Fano variety with $\rho(X_k) = 1$. Thus in dimension 3 Kawamata[8]'s result implies that they belong to a bounded family. Therefore, there exists $q \in \mathbb{N}$ s.t. qD is Cartier for all integral Weil divisor on X_k for $\forall k$ and there exists $r \in \mathbb{N}$ s.t. rK_{X_k} is Cartier for $\forall k$. Then another result of Kawamata[8] on the boundedness of the lengths of the extremal rational curves says that there exists a rational curve L_k on X_k s.t. $0 < -K_{X_k} \cdot L_k \leq 2 \cdot \dim X_k$, which implies

$$\mu_k \in \frac{1}{(r \cdot 2\dim X)!q}\mathbb{N}.$$

Again $\{\mu_k\}$ is a nonstationary and nonincreasing infinite sequence $\mu_0 \geq \mu_k > 0$, a contradiction!

We remark that this last step is the only place where we use the boundedness of $\mathbb{Q}$-Fano n-folds.

Cliams 2.1, 2.2 and 2.3 show that there is no infinite loop in the flowchart of the Sarkisov program.

This completes the discussion of termination of the flowchart.

§3. Log Sarkisov Program with KLT Singularities.

In this section, we try to establish the Log Sarkisov Program for untwisting a birational map between two log Mori fiber spaces with only kawamata log terminal singularities

$$\begin{tikzcd}
(X, B_X) \arrow[r, dashed, "\Phi"] \arrow[d, "\phi"'] & (X', B_{X'}) \arrow[d, "\phi'"] \\
S & S'
\end{tikzcd}$$

The guiding principle throughout is that while the genuine Sarkisov program is the one to untwist a birational map between two Mori fiber spaces obtained as two different end results of K-MMP starting from one nonsingular projective variety W

$$\begin{tikzcd}
 & W \arrow[dl, "K - \mathrm{MMP}"'] \arrow[dr, "K - \mathrm{MMP}"] & \\
X \arrow[rr, dashed, "\Phi"] \arrow[d, "\phi"'] & & X' \arrow[d, "\phi'"] \\
S & & S',
\end{tikzcd}$$

the log Sarkisov program should be the one to untwist a birational map between two log Mori fiber spaces obtained as two different end results of $K + B$-MMP starting from one log variety (W, B_W) consisting of a nonsingular projective variety W and an S.N.C. divisor B_W

$$\begin{tikzcd}
 & (W, B_W) \arrow[dl, "K + B - \mathrm{MMP}"'] \arrow[dr, "K + B - \mathrm{MMP}"] & \\
(X, B_X) \arrow[rr, dashed, "\Phi"] \arrow[d, "\phi"'] & & (X', B_{X'}) \arrow[d, "\phi'"] \\
S & & S'.
\end{tikzcd}$$

(Note that in the two diagrams above we do not require a priori the slanted arrows to be morphisms.)

While this principle does not put any restriction on the birational map Φ in the case of the genuine Sarkisov program (namely, for any birational map Φ between two Mori fiber spaces we can find a common resolution W s.t. $X \to S$ and $X' \to S'$ are two end results of K-MMP as the diagram above), this principle in the case of the log Sarkisov program allows us to consider only such birational map Φ for which a log variety as above exists. This naturally leads to the notion of the Sarkisov relation: (Two or more) Log Mori fiber spaces are Sarkisov related iff they are the end results of $K + B$-MMP starting from one log pair consisting of a nonsingular projective variety and an S.N.C. divisor as a boundary. Thus the log Sarkisov program untwists a birational map between two log Mori fiber spaces which are Sarkisov related, by factorizing it into links among intermediate Mori fiber spaces all of which including the original two we require to be also Sarkisov related. (Remark again that in the case of the genuine Sarkisov program the Sarkisov relation happens to be an equivalence relation and coincide with the usual birational equivalence.)

Once we understand what the appropriate logarithmic generalization of the Sarkisov program should be through the notion of the Sarkisov relation, the flowchart for the log Sarkisov program with klt singularities works almost parallel to that of the genuine Sarkisov program as well as termination except the very last step. In order to show that there is no infinite successive sequence of links of type (II) with nonstationary quasi-effective threshold, we have to use the conjecture by Borisov[3] in dimension 3 (cf.Nikulin[26]Alexeev[2]).

Conjecture 3.1. *Fix a nonnegative rational number* $0 \leq \epsilon < 1$. *Then the family of log* $\mathbb{Q}$*-Fano* n*-folds (a normal projective* n*-fold* X *with only* $\mathbb{Q}$*-factorial log terminal singularities (and thus automatically has only klt singularities) s.t. the anti-canonical divisor* $-K_X$ *is ample) whose discrepancies are all* $> -\epsilon$ *is bounded.*

In dimension 2, the conjecture holds (cf.Nikulin[26]Alexeev[2]) and thus we establish the log Sarkisov program for klt surfaces. The conjecture in dimension 3 for the case of Picard number 1 and $\epsilon = 0$ is the theorem of Kawamata, which completes the proof of termination for the genuine Sarkisov program for 3-folds.

In the following we discuss the log Sarkisov program with klt singularities in detail.

Definition 3.2 (cf.Kollár et al[16]). *A log pair* $(X, B_X = \Sigma b_i B_i)$ *has only kawamata log terminal singularities iff every discrete valuation of* $k(X)$ *having center on* X *has positive log discrepancy, i.e.,*

o) X *is normal,*

i) $0 \leq b_i < 1$ *for* $\forall i$, *and*

ii) there exists a log resolution $f : Y \to X$ *where all the* f*-exceptional divisors have positive log discrepancies (and thus this holds for any log resolution).*

Definition 3.3. *A log Mori fiber space* $\phi : (X, B_X) \to S$ *with only klt singularities is the contraction of an extremal ray with respect to* $K_X + B_X$ *from a log pair* (X, B_X) *consisting of a normal projective variety* X *and a divisor* B_X *with only* $\mathbb{Q}$*-factorial klt singularities onto a variety* S, *i.e.,* ϕ *is a morphism from a* $\mathbb{Q}$*-factorial klt log pair* (X, B_X) *with connected fibers onto a normal variety* S *with* $\dim S < \dim X$ *s.t.* $\rho(X/S) = 1$ *and* $-(K_X + B_X)$ *is* ϕ*-ample.*

Definition 3.4 (Sarkisov Relation). *Log Mori fiber spaces (resp. Log minimal models)*

$$(X_0, B_{X_o}), (X_1, B_{X_1}), \cdots, (X_k, B_{X_k}), \cdots, (X_l, B_{X_l})$$

are Sarkisov related iff they are all end results of $K + B$-MMP starting from one appropriate log pair (W, B_W) consisting of a nonsingular projective variety W and an S.N.C. divisor $B_W = \Sigma b_i B_i$ with $0 \leq b_i \leq 1$.

A birational map Φ between two log Mori fiber spaces (resp. log minimal models) which are Sarkisov related is by definition the one for which we have a commutative diagram

$$\begin{array}{ccc} & (W, B_W) & \\ p \swarrow & & \searrow q \\ (X, B_X) & \overset{\Phi}{\dashrightarrow} & (X', B_{X'}) \end{array}$$

where (W, B_W) is a log pair specified as above (Note that p or q may not be a morphism.)

The Sarkisov relation behaves very well for log Mori fiber spaces (or log minimal models) with only kawamata log terminal singularities.

Proposition 3.5. *Let*

$$(X_0, B_{X_o}), (X_1, B_{X_1}), \cdots, (X_k, B_{X_k}), \cdots, (X_l, B_{X_l})$$

be log Mori fiber spaces (resp. log minimal models) with only klt singularities and $0 \leq \epsilon < 1$ a rational number such that all the coefficients of the boundaries B_{X_k} are $\leq \epsilon$ and that all the discrepancies of the log pairs (X_k, B_{X_k}) are $> -\epsilon$.

Then the following are equivalent:

(i) The log Mori fiber spaces (resp. log minimla models) with klt singularities

$$(X_0, B_{X_o}), (X_1, B_{X_1}), \cdots, (X_k, B_{X_k}), \cdots, (X_l, B_{X_l})$$

are Sarkisov related, i.e., there exists a log variety (W, B_W) consisting of a nonsingular projective variety W and an S.N.C. divisor B_W as a boundary such that all the log Mori fiber spaces (resp. log minimal models) are end results of $K + B$-MMP over Spec k starting from (W, B_W).

(ii) There exists a log variety (W, B_W) consisting of a nonsingular projective variety W and an S.N.C. divisor B_W as a boundary such that all the log Mori fiber spaces (resp. log minimal models) are end results of $K + B$-MMP over Spec k starting from (W, B_W) and that

$$B_W = D_W(B_{X_0}, B_{X_1}, \cdots, B_{X_k}, \cdots, B_{X_l}) + \Sigma_{E_j \text{not appearing as a divisor on any of } X_k} \epsilon E_j,$$

where

$$D_W(B_{X_0}, B_{X_1}, \cdots, B_{X_k}, \cdots, B_{X_l}) := \Sigma d_m D_m$$

summation being taken over all divisors on W which appear as a divisor on some X_m and d_m being the coefficient of D_m in B_{X_m}.

(iii) There exists a log variety (W, B_W) consisting of a nonsingular projective variety W and an S.N.C. divisor B_W as a boundary such that each log Mori fiber space (resp. log minimal model) is dominated by a birational morphism $p_k : (W, B_W) \to (X_k, B_{X_k})$ and an end result of $K + B$-MMP over X_k starting from (W, B_W) and that

$$B_W = D_W(B_{X_0}, B_{X_1}, \cdots, B_{X_k}, \cdots, B_{X_l}) + \Sigma_{E_j \text{not appearing as a divisor on any of } X_k} \epsilon E_j.$$

Proof.

The proposition is a straightforward consequence of the following lemma, whose first claim holds not only for klt singularities but also weakly kawamata log terminal singularities (or even more generally for log canonical singularities) while the second claim only holds for klt singularities. This is why the Sarkisov relation behaves very well for klt singularities but becomes quite subtle for wklt or lc singularities. The verification of the lemma is left to the reader as an exercise.

Lemma 3.6.

(i) Let $p : (W, B_W) \to (X, B_X)$ be a projective birational morphism between $\mathbb{Q}$ factorial log varieties with klt (or more generally with wklt or lc) singularities. Then p is a process of $K + B$-MMP over X starting from (W, B_W) iff $B_X = p_*(B_W)$ and the ramification divisor R

$$K_W + B_W = p^*(K_X + B_X) + R$$

has the same support as the exceptional locus $E(p)$ of p.

(ii) Let (X, B_X) be a log pair with only $\mathbb{Q}$-factorial klt singularities and $0 \leq \epsilon < 1$ a rational number such that all the coefficients of the boundary B_X are $\leq \epsilon$ and that all the discrepancies are $> -\epsilon$. Then any projective birational morphism $p : (W, B_W) \to (X, B_X)$ from a log pair (W, B_W) consisting of a nonsingular projective variety and an S.N.C. divisor B_W

$$B_W = D_W(B_X) + \Sigma_{E_j \text{not appearing as a divisor on } X} \epsilon E_j,$$

is a process of $K + B$-MMP over X starting from (W, B_W).

Remark 3.7.

Though unfortunately the Sarkisov relation is NOT an equivalence relation in general, it is an equivalence relation for the following special classes of log pairs (X, B_X) with klt singularities: We fix $0 \leq \epsilon < 1$. The class consists of $\mathbb{Q}$-factorial projective log pairs (X, B_X) with klt singularities whose coefficients of the boundaries B_X are all equal to ϵ, and all the discrepancies of the valuations of the exceptional divisors are $> -\epsilon$.

The genuine Sarkisov program is nothing but the program for the class given by $\epsilon = 0$.

Theorem 3.8 (Log Sarkisov Program for log 3-folds with klt singularities). *Let*

$$\begin{tikzcd} (X, B_X) \arrow[r, dashed, "\Phi"] \arrow[d, "\phi"'] & (X', B_{X'}) \arrow[d, "\phi'"] \\ S & S' \end{tikzcd}$$

be a birational map between two log Mori fiber spaces in dimension 3 with only klt singularities, which are Sarkisov related.

Suppose the Borisov conjecture holds for $\mathbb{Q}$-Fano 3-folds with klt singularities and Picard number 1.

Then for any rational number $0 \leq \epsilon < 1$ such that all the coefficients in B_X or $B_{X'}$ are $\leq \epsilon$ and that all the discrepancies of (X, B_X) or $(X', B_{X'})$ are $> -\epsilon$, Φ is a composite of 4 types of links as in the genuine Sarkisov program

$$\begin{tikzcd} (X, B_X) = (X_0, B_{X_0}) \arrow[r, dashed] \arrow[d, "\phi"'] & (X_1, B_{X_1}) \arrow[r, dashed] \arrow[d, "\phi_1"'] & \cdots \arrow[r, dashed] & (X_k, B_{X_k}) \arrow[r, dashed] \arrow[d, "\phi_k"'] & \cdots \arrow[r, dashed] & (X', B_{X'}) \arrow[d, "\phi'"'] \\ S = S_0 & S_1 & \cdots & S_k & \cdots & S' \end{tikzcd}$$

such that all the log Mori fiber spaces (X_k, B_{X_k}) have only $\mathbb{Q}$-factorial klt singularities, the coefficients of B_{X_k} are $\leq \epsilon$, all the discrepancies of (X_k, B_{X_k}) are $> -\epsilon$ and that all the (X_k, B_{X_k}) are Sarkisov related. More precisely, all the log Mori fiber spaces are dominated by birational morphisms

$$p_k : (W, B_W) \to (X_k, B_{X_k})$$

from a log pair (W, B_W) consisting of a nonsingular projective 3-fold and an S.N.C. divisor B_W

$$B_W = D_W(B_X, B_{X'}) + \Sigma_{E_j \text{not appearing as a divisor either on } X \text{ or on } X'} \epsilon E_j$$

and each (X_k, B_{X_k}) is an end result of $K+B$-MMP over X_k starting from (W, B_W).

The strategy to establish the log Sarkisov program goes along the same line as the one to establish the genuine Sarkisov program, constructing (W, B_W) as above inductively as the program proceeds.

We define the log Sarkisov degree of an intermediate log Mori fiber space (X_k, B_{X_k}) which appears in the due course of untwisting a birational map

$$\begin{tikzcd} (X, B_X) \arrow[r, dashed, "\Phi"] \arrow[d, "\phi"'] & (X', B_{X'}) \arrow[d, "\phi'"] \\ S & S' \end{tikzcd}$$

between two log Mori fiber spaces which are Sarkisov related as follows.

Definition 3.9 (the log Sarkisov degree). *Let*

$$\begin{tikzcd} (X, B_X) \arrow[r, dashed, "\Phi"] \arrow[d, "\phi"'] & (X', B_{X'}) \arrow[d, "\phi'"] \\ S & S' \end{tikzcd}$$

be a birational map between two log Mori fiber spaces with only klt singularities which are Sarkisov related, and fix a rational number $0 \leq \epsilon < 1$ such that all the coefficients in B_X or $B_{X'}$ are $\leq \epsilon$ and that all the discrepancies of (X, B_X) or $(X', B_{X'})$ are $> -\epsilon$. Then the log Sarkisov degree of any intermediate log Mori fiber space $\phi_k : (X_k, B_{X_k}) \to S_k$ that appears in the due course of untwisting the birational map with reference to the fixed log Mori fiber space $\phi' : (X', B_{X'}) \to S'$ is the triplet

$$(\mu_k, \lambda_{\epsilon k}, e_{\epsilon k})$$

of the numbers defined below, endowed with the lexicographical order.

Notice that there is an auxiliary parameter ϵ, which was implicit (actually equal to 0) in the case of the genuine Sarkisov degree. Also note that the log Sarkisov degree depends not only on (X_k, B_{X_k}) and $(X', B_{X'})$ but also on the initial log Mori fiber space (X, B_X).

First we take a very ample divisor A' on S' and a sufficiently divisible $\mu' \in \mathbb{N}$ such that

$$\mathcal{H}_{X'} = -\mu'(K_{X'} + B_{X'}) + \phi'^* A'$$

is very ample on X'. $\mathcal{H}_{X_k}$ is the strict transform of $\mathcal{H}_{X'}$ on X_k.

$\boxed{\mu_k : \text{the quasi-effective threshold}}$

The quasi-effective threshold is defined exactly the same way as before, replacing K with $K+B$, namely

$$\mu_k \in \mathbb{Q}_{>0}\ \mu_k(K_{X_k}+B_{X_k})+\mathcal{H}_{X_k} \equiv 0 \text{ over } S_k$$

$\boxed{\lambda_{\epsilon k} : \text{the maximal multiplicity of an extremal ray}}$

Let (W, B_W) be a log pair consisting of a nonsingular projective variety W and an S.N.C. divisor B_W

$$B_W = D_W(B_X, B_{X'}) + \Sigma_{E_j \text{not appearing as a divisor either on } X \text{ or on } X'} \epsilon E_j$$

such that it dominates

$$p : (W, B_W) \to (X, B_X)$$
$$q : (W, B_W) \to (X', B_{X'})$$
$$p_k : (W, B_W) \to (X_k, B_{X_k})$$

by birational morphisms and that they are all processes of $K+B$-MMP over X, X' and X_k, respectively. (The existence of such (W, B_W) will be shown inductively in the course of log Sarkisov program.)

Then

$$\frac{1}{\lambda_{\epsilon k}} := \max\{c \in \mathbb{Q}_{>0}; (K_W+B_W)+c\mathcal{H}_{X_k} = p_k^*(K_{X_k}+B_{X_k})+c\mathcal{H}_{X_k}+\text{some effective divisor}\}.$$

Note that $\frac{1}{\lambda_{\epsilon k}}$ is independent of the choice of such (W, B_W) and well-defined. When $Bs(\mathcal{H}_X) = \emptyset$, $\lambda_{\epsilon k} = 0$ by definition.

We note that in general

$$\lambda_{\epsilon k} \leq \epsilon - \text{log terminal threshold of } (X_k, B_{X_k}) \text{ w.r.t. } \mathcal{H}_{X_k}.$$

The ϵ-log termianl threshold of the pair (X_k, B_{X_k}) is defined to be

$$\max\{c \in \mathbb{Q}_{>0}; (K_{V_k}+B_{V_k})+c\mathcal{H}_{V_k} = v_k^*((K_{X_k}+B_{X_k})+c\mathcal{H}_{X_k})+\text{some effective divisor}\},$$

where V_k is any nosingular projective variety which dominates both $v_k : V_k \to X_k$ and X' by birational morphisms such that the union of the exceptional locus of v_k and ${v_k}_*^{-1}(B_{X_k})$ is an S.N.C. divisor (Recall the $\mathbb{Q}$-factoriality of X_k), and

$$B_{V_k} = {v_k}_*^{-1}(B_{X_k}) + \Sigma_{E_j \text{not Apearing as a divisor on } X_k} \epsilon E_j.$$

$\boxed{e_{\epsilon k} : \text{the number of } K+B+\frac{1}{\lambda_{\epsilon k}}\mathcal{H} - \text{crepant divisors}}$

$e_{\epsilon k}$ is defined to be the number of exceptional divisors for p_k whose coefficient in the ramification divisor R is 0

$$(K_W+B_W)+\frac{1}{\lambda_{\epsilon k}}\mathcal{H}_W = p_k^*((K_{X_k}+B_{X_k})+\frac{1}{\lambda_{\epsilon k}}\mathcal{H}_{X_k})+R.$$

Again this is independent of the choice of (W, B_W) and well-defined.

Nöther-Fano Criterion for the log Sarkisov program with KLT singularities. *The birational map Φ_k between an intermediate log Mori fiber space $\phi_k : (X_k, B_{X_k}) \to S_k$ and $\phi' : (X', B_{X'}) \to S'$ is an isomorphism of log Mori fiber spaces*

$$\begin{array}{ccc} (X_k, B_{X_k}) & \overset{\Phi_k}{\tilde{\to}} & (X', B_{X'}) \\ \phi_k \downarrow & & \downarrow \phi' \\ S_k & \tilde{\to} & S' \end{array}$$

if $\lambda_{\epsilon k} \leq \mu_k$ and $(K_{X_k} + B_{X_k}) + \frac{1}{\lambda_{\epsilon k}} \mathcal{H}_{X_k}$ is nef.

The proof goes verbatim to the one for the genuine Sarkisov program, taking the common resolution (W, B_W) which Sarkisov-relates (X_k, B_{X_k}) and $(X', B_{X'})$ into consideration.

Flowchart for Log Sarkisov Program with KLT Singularities

The flowchart for the log Sarkisov program with klt singularities goes almost parallel to the one for the genuine Sarkisov program replacing K with $K + B$, constructing inductively a log pair (W, B_W) which dominates all the intermediate log Mori fiber spaces in the process of untwisting.

Let

$$\begin{array}{ccc} (X, B_X) & \overset{\Phi}{\dashrightarrow} & (X', B_{X'}) \\ \phi \downarrow & & \downarrow \phi' \\ S & & S' \end{array}$$

be a birational map between two log Mori fiber spaces with only klt singularities, which are Sarkisov related. We fix a rational number $0 \leq \epsilon < 1$ such that all the coefficients in B_X or $B_{X'}$ are $\leq \epsilon$ and that all the discrepancies of (X, B_X) or $(X', B_{X'})$ are $> -\epsilon$.

We remark that for all the relevant log pairs (U, B_U) that appear in the course of the log Sarkisov program (including the auxiliary resolutions we take) the boundaries B_U are always taken to be of the form

$$B_U = D_U(B_X, B_{X'}) + \Sigma_{E_j \text{ not appearing as a divisor either on } X \text{ or on } X'} \epsilon E_j.$$

Before we start the flowchart, we note that by Proposition 3.5 there exists a log variety (W_0, B_{W_0}) consisting of a nonsingular projective variety W_0 and an S.N.C. divisor B_{W_0} as a boundary such that $(X, B_X) = (X_0, B_{X_0})$ (resp. $(X', B_{X'})$) is dominated by a birational morphism $p = p_{0,0} : (W_0, B_{W_0}) \to (X_0, B_{X_0})$ (resp.

$q = q_0 : (W_0, B_{W_0}) \to (X', B_{X'}))$ and an end result of $K + B$-MMP over X_0 (resp. over X') starting from (W_0, B_{W_0}) and that

$$B_{W_0} = D_{W_0}(B_X, B_{X'}) + \Sigma_{E_j \text{not appearing as a divisor on either } X \text{ or on } X'} \epsilon E_j.$$

Suppose we have untwisted the birational map up to the k-th stage and constructed a log pair (W_k, B_{W_k}) consisting of a nonsingular projective variety W_k and an S.N.C. divisor B_{W_k} as a boundary such that each log Mori fiber space (X_m, B_{X_m}) $m = 0, 1, \cdots, k$ (and $(X', B_{X'})$) is dominated by a birational morphism $p_{k,m} : (W_k, B_{W_k}) \to (X_m, B_{X_m})$ (and $q_k : (W_k, B_{W_k}) \to (X', B_{X'})$) and an end result of $K + B$-MMP over X_m (and over X') starting from (W_k, B_{W_k}) and that

$$B_{W_k} = D_{W_k}(B_X, B_{X'}) + \Sigma_{E_j \text{not appearing as a divisor on either } X \text{ or on } X'} \epsilon E_j.$$

The first question then to ask as in the genuine Sarkisov program is:

$$\lambda_{\epsilon k} > \mu_k?$$

$$\boxed{\text{Case:} \lambda_{\epsilon k} \leq \mu_k}$$

In this case, thanks to the Nöther-Fano inequality for klt singularities, the program works completely parallel to the genuine Sarkisov program. If $(K_{X_k} + B_{X_k}) + \frac{1}{\lambda_{\epsilon k}} \mathcal{H}_{X_k}$ is nef, then the program comes to an end. If not, then after untwisting the birational map either by a link of type (III) or (IV), the quasi-effective threshold strictly drops

$$\mu_{k+1} < \mu_k$$

as before.

We also take a nonsingular projective variety W_{k+1} by blowing up W_k further so that it dominates each log Mori fiber space by a birational morphism $p_{k+1,m}$: $(W_{k+1}, B_{W_{k+1}}) \to (X_m, B_{X_m})$ $m = 0, 1, \cdots, k, k+1$ (and $q_{k+1} : (W_{k+1}, B_{W_{k+1}}) \to (X', B_{X'})$) and that

$$B_{W_{k+1}} := D_{W_{k+1}}(B_X, B_{X'}) + \Sigma_{E_j \text{not appearing as a divisor on either } X \text{ or on } X'} \epsilon E_j$$

is an S.N.C. divisor. Then by Lemma 3.6 (W_k, B_{W_k}) is an end result of $K + B$-MMP over W_k starting from $(W_{k+1}, B_{W_{k+1}})$ and thus each log Mori fiber space $(X_m, B_{X_m}) m = 0, 1, \cdots, k$ (and $(X', B_{X'})$) is an end result of $K + B$-MMP starting from $(W_{k+1}, B_{W_{k+1}})$ over X_m. As for $(X_{k+1}, B_{X_{k+1}})$ which is an end result of $K + B + \frac{1}{\lambda_{\epsilon k}} \mathcal{H}$-MMP starting from (X_k, B_{X_k}) (over T), all the coeffiecients of

$B_{X_{k+1}}$ are $\leq \epsilon$ by construction. Moreover, for any valuation E whose center on X_{k+1} has codimension at least 2

$$a(E, X_{k+1}, B_{X_{k+1}}) = a(E, X_k, B_{X_k}) < -\epsilon$$

if the birational map $(X_k, B_{X_k}) \dashrightarrow (X_{k+1}, B_{X_{k+1}})$ is isomorphic at the center of the valuation E, and

$$\begin{aligned} a(E, X_{k+1}, B_{X_{k+1}}) &\leq a(E, X_{k+1}, B_{X_{k+1}} + \frac{1}{\lambda_{\epsilon k}} \mathcal{H}_{X_{k+1}}) \\ &< a(E, X_k, B_{X_k} + \frac{1}{\lambda_{\epsilon k}} \mathcal{H}_{X_k}) \leq -\epsilon. \end{aligned}$$

if the birational map $(X_k, B_{X_k}) \dashrightarrow (X_{k+1}, B_{X_{k+1}})$ is not isomorphic at the center of the valuation E.

Thus again by Lemma 3.6 $(X_{k+1}, B_{X_{k+1}})$ is an end result of $K+B$-MMP starting from $(W_{k+1}, B_{W_{k+1}})$ over X_{k+1}. Therefore, $(W_{k+1}, B_{W_{k+1}})$ satisfies the desired inductive property.

$$\boxed{\text{Case:} \lambda_{\epsilon k} > \mu_k}$$

In this case we

Take a maximal divisorial blow up $d_k : (Z_k, B_{Z_k}) \to (X_k, B_{X_k})$

with respect to $(K_{X_k} + B_{X_k}) + \frac{1}{\lambda_{\epsilon k}} \mathcal{H}_{X_k}$, and in the log Sarkisov program we also require (Z_k, B_{Z_k}) is obtained through $K + B + \frac{1}{\lambda_{\epsilon k}} \mathcal{H}$-MMP possibly followed by $K + B$-MMP starting from (W_k, B_{W_k}) over X_k, i.e.,

o) in the proof of Proposition 1.5 instead of carrying out a $K + \frac{1}{\lambda}\mathcal{H}$-MMP over X starting from Y to get a minimal model $(Z', \frac{1}{\lambda}\mathcal{H})$ and then running a K-MMP over X to obtain a maximal divisorial blow up, we carry out a $K+B+\frac{1}{\lambda_{\epsilon k}}\mathcal{H}$-MMP over X_k starting from (W_k, B_{W_k}) to get a minimal model $(Z'_k + B_{Z'_k} + \frac{1}{\lambda_{\epsilon k}} \mathcal{H}_{Z'_k})$ and then run a $K + B$-MMP over X_k to obtain the maximal divisorial blow up $d_k : (Z_k, B_{Z_k}) \to (X_k, B_{X_k})$,

i) $\rho(Z_k/X_k) = 1$,

ii) the exceptional locus of d_k is a prime divisor E_k, and

iii) d_k is $K + B + \frac{1}{\lambda_{\epsilon k}}\mathcal{H}$-crepant, i.e.,

$$(K_Z + B_Z) + \frac{1}{\lambda_{\epsilon k}} \mathcal{H}_Z = d_k^*((K_{X_k} + B_{X_k}) + \frac{1}{\lambda_{\epsilon k}} \mathcal{H}_{X_k}).$$

Then the rest goes parallel to the genuine Sarkisov program, untwisting the birational map further by a link of either type (II) or (I). After untwisting, we have

$$\mu_{k+1} \leq \mu_k$$

and if $\mu_{k+1} = \mu_k$ and $\dim S_k = \dim S_{k+1}$, then ψ_k is a square. Moreover,

$$\lambda_{\epsilon,k+1} \leq \lambda_{\epsilon k}$$

and if $\lambda_{\epsilon,k+1} = \lambda_{\epsilon k}$ then $e_{\epsilon,k+1} < e_{\epsilon k}$.

We also take a nonsingular projective variety W_{k+1} by blowing up W_k further so that it dominates each log Mori fiber space by a birational morphism $p_{k+1,m} : (W_{k+1}, B_{W_{k+1}}) \to (X_m, B_{X_m})$ $m = 0, 1, \cdots, k, k+1$ (and $q_{k+1} : (W_{k+1}, B_{W_{k+1}}) \to (X', B_{X'})$) and that

$$B_{W_{k+1}} := D_{W_{k+1}}(B_X, B_{X'}) + \Sigma_{E_j \text{not appearing as a divisor on either } X \text{ or on } X'} \epsilon E_j$$

is an S.N.C. divisor. Then by Lemma 3.6 (W_k, B_{W_k}) is an end result of $K + B$-MMP over W_k starting from $(W_{k+1}, B_{W_{k+1}})$ and thus each log Mori fiber space (X_m, B_{X_m}) $m = 0, 1, \cdots, k$ (and $(X', B_{X'})$) is an end result of $K + B$-MMP starting from $(W_{k+1}, B_{W_{k+1}})$ over X_m.

As for $(X_{k+1}, B_{X_{k+1}})$, note that it is an end result of $K + B + \frac{1}{\lambda_{\epsilon k}}\mathcal{H}$-MMP starting from (Z_k, B_{Z_k}) (over S_k), which itself is an end result of $K + B + \frac{1}{\lambda_{\epsilon k}}\mathcal{H}$-MMP possibly followed by $K + B$-MMP over X_k starting from (W_k, B_{W_k}). All the coeffiecients of $B_{X_{k+1}}$ are $\leq \epsilon$ by construction.

Moreover, for any valuation E whose center on ${Z'}_k$ has at least codimension 2

$$a(E, {Z'}_k, B_{{Z'}_k}) = a(E, W_k, B_{W_k}) < -\epsilon$$

if the birational map $(W_k, B_{W_k}) \dashrightarrow ({Z'}_k, B_{{Z'}_k})$ is isomorphic at the center of the valuation E, and

$$\begin{aligned} a(E, {Z'}_k, B_{{Z'}_k}) &\leq a(E, {Z'}_k, B_{{Z'}_k} + \frac{1}{\lambda_{\epsilon k}}\mathcal{H}_{{Z'}_k}) \\ &< a(E, W_k, B_{W_k} + \frac{1}{\lambda_{\epsilon k}}\mathcal{H}_{W_k}) \leq -\epsilon \end{aligned}$$

if the birational map $(W_k, B_{W_k}) \dashrightarrow ({Z'}_k, B_{{Z'}_k})$ is not isomorphic at the center of the valuation E.

For any valuation E whose center on Z_k has at least codimension 2

$$a(E, Z_k, B_{Z_k}) = a(E, {Z'}_k, B_{{Z'}_k}) < -\epsilon$$

if the birational map $({Z'}_k, B_{{Z'}_k}) \dashrightarrow (Z_k, B_{Z_k})$ is isomorphic at the center of the valuation E, and

$$a(E, Z_k, B_{Z_k}) < a(E, {Z'}_k, B_{{Z'}_k}) \leq -\epsilon$$

if the birational map $({Z'}_k, B_{{Z'}_k}) \dashrightarrow (Z_k, B_{Z_k})$ is not isomorphic at the center of the valuation E.

Finally for any valuation E whose center on X_{k+1} has at least codimension 2

$$a(E, X_{k+1}, B_{X_{k+1}}) = a(E, Z_k, B_{Z_k}) < -\epsilon$$

if the birational map $(Z_k, B_{Z_k}) \dashrightarrow (X_{k+1}, B_{X_{k+1}})$ is isomorphic at the center of the valuation E, and

$$\begin{aligned} a(E, X_{k+1}, B_{X_{k+1}}) &\leq a(E, X_{k+1}, B_{X_{k+1}} + \frac{1}{\lambda_{\epsilon k}}\mathcal{H}_{X_{k+1}}) \\ &< a(E, Z_k, B_{Z_k} + \frac{1}{\lambda_{\epsilon k}}\mathcal{H}_{Z_k}) \leq -\epsilon \end{aligned}$$

if the birational map $(Z_k, B_{Z_k}) \dashrightarrow (X_{k+1}, B_{X_{k+1}})$ is not isomorphic at the center of the valuation E.

Thus again by Lemma 3.6 $(X_{k+1}, B_{X_{k+1}})$ is an end result of $K+B$-MMP starting from $(W_{k+1}, B_{W_{k+1}})$ over X_{k+1}. Therefore, $(W_{k+1}, B_{W_{k+1}})$ satisfies the desired inductive property.

Termination of Flowchart for Log Sarkisov Program with KLT Singularities

One of the key points to show the terminatin of of the flowchart for the genuine Sarkisov program is the discreteness of the quasi-effective thresholds, which follows from the boundedness of $\mathbb{Q}$-Fano varieties which are fibers of the Mori fiber spaces (a nonsingular rational curve, Del Pezzo surfaces and $\mathbb{Q}$-Fano 3-folds for the genuine Sarkisov program in dimension 3). In the log Sarkisov program, we rely on the Borisov conjecture, which is a theorem in dimension 2 thanks to Nikulin[26][Alexeev[2], to show the discreteness of the quasi-effective thresholds.

Claim 3.10. *Let $\phi_k : (X_k, B_{X_k}) \to S_k$ be a log Mori fiber space in dimension 3 with only klt singularities in the process of the log Sarkisov program untwisting a birational map*

$$\begin{tikzcd} (X, B_X) \arrow[r, dashed, "\Phi"] \arrow[d, "\phi"'] & (X', B_{X'}) \arrow[d, "\phi'"] \\ S & S' \end{tikzcd}$$

between two log Mori fiber spaces in dimension 3 with only klt singularities which are Sarkisov related, fixing ϵ as before.

If $\dim S_k \geq 1$, then the denominator of the quasi-effective threshold μ_k is universally bounded by a fixed constant depending only on ϵ and the coefficients of B_X and $B_{X'}$.

proof.

Let d be the l.c.m. of the denominator of ϵ and those of coefficients of B_X and $B_{X'}$.

When $\dim S_k = 2$, l being a nonsingular rational curve which is a general fiber of ϕ_k,

$$\begin{aligned} -2 \leq (K_{X_k} + B_{X_k}) \cdot l < 0 \\ (K_{X_k} + B_{X_k}) \cdot l \in \frac{1}{d}\mathbb{Z}_{<0} \\ \{\mu_k(K_{X_k} + B_{X_k}) + \mathcal{H}_{X_k}\} \cdot l = 0 \end{aligned}$$

imply

$$\mu_k \in \frac{1}{(2d)!}\mathbb{N}.$$

When $\dim S_k = 1$, a general fiber F_k is a log Del Pezzo surface (a normal projective surface with only quotient singularities having an ample anti-canonical divisor) whose discrepancies are all $> -\epsilon$. Therefore, by Nikulin[26][Alexeev[2] we conclude that the family of such log Del Pezzo surfaces is bounded. Now q being the universal $\mathbb{Q}$-factorial index for such surfaces and r the index for the canonical divisors, we have

$$\mu_k \in \frac{1}{(r \cdot 2 \cdot \dim X)!q}\mathbb{N}.$$

In arbitrary dimension, we have to use the Borisov conjecture for boundedness of log $\mathbb{Q}$-Fano d-folds for $d \leq n-1$ in general to derive this claim.

Thanks to Claim 3.10, the argument for termination goes parallel replacing K with $K+B$ for Claims 2.1, 2.2 and the first case in Claim 2.3 to that of termination of the genuine Sarkisov program. (We note that in Step 3 of the proof of Claim 2.2 we repalce the local canonical threshold with the local version of $\frac{1}{\lambda_\epsilon}$.) We have to use a conjecture of Borisov to establish the last step of the second case in Claim 2.3.

Conjecture 3.11 (cf.Borisov[3]). *Fix a rational number $0 \leq \epsilon < 1$. Then the family of log $\mathbb{Q}$-Fano 3-folds (normal projective 3-folds X with only $\mathbb{Q}$-factorial log terminal (equivalently, klt) singularities s.t. the anti-canonical divisors $-K_X$ are ample) with Picard number 1 and whose discrepancies are all $> -\epsilon$, is bounded.*

In general, to establish Claim 2.3 in the second case we need the conjecture above for log $\mathbb{Q}$-Fano n-folds.

This completes the discussion of termination of log Sarkisov program with klt singularities.

§4. Log Sarkisov Program with WKLT Singularities.

In this section, we establsih the log Sarkisov program with weakly kawamata log terminal singularities in dimension 2, and then discuss the problems one has to face attempting to establish the log Sarkisov program with weakly kawamata log terminal singularities in higher dimension. We also prove that Sarkisov related log minimal 3-folds with klt or wklt singularities are connected by a sequence of log flops.

Definition 4.1 (Local) (cf.Kollár et al[16]). *Let (X, B_X) be a germ (with respect to Zariski topology) around a point $P \in X$. (X, B_X) has only weakly kawamata log terminal singularities if there exists a Zariski open set $P \in U \subset X$ s.t. there exists a log resolution $f : V \to U$ such that all the log discrepancies of the exceptional divisors with center on U are positive and we have an f-anti-ample effective divisor whose support coincides with that of the exceptional locus of f. (Note that B_X may have components with coefficient 1.)*

The relation between local and global properties of wklt singularities was clarified by the following result of Szabó[32].

Proposition 4.2 (Global) (cf.Szabó[32]). *Let (X, B_X) be a projective log variety which has locally only weakly kawamata log terminal singularities. Then there exists a log resolution (global) $f : Y \to X$ such that all the log discrepancies of the exceptional divisors with center on X are positive and we have an f-anti-ample effective divisor whose support coincides with that of the exceptional locus of f.*

Corollary 4.3 (Characterization of a log Mori fiber space (or a log minimal model) with wklt singularities). *A log Mori fiber space $\phi : (X, B_X) \to S$ (resp. a log minimal model (X, B_X)) in dimension n (≤ 3) with only $\mathbb{Q}$-factorial wklt singularities is an end result of a $K + B$-MMP starting from (Y, B_Y) where Y is a nonsingular projective n-fold and $B_Y = \Sigma b_i B_i$ is an S.N.C. divisor with $0 \leq b_i \leq 1$, and the converse holds, i.e., any end result of fibering type (resp. of minimal model type) of a $K + B$-MMP starting from (Y, B_Y) as above is a log Mori fiber space (resp. a log minimal model) with only $\mathbb{Q}$-factorial wklt singularities. (Once we have the log-MMP in dimension n, the same statement holds in dimension n.)*

We prove the well-behavior of the Sarkisov relation for log minimal models with wklt singularities as follows, thanks to the fact that the nef log canonical divisors of the Sarkisov related log minimal models are all essentially the same and uniquely characterized as the nef part of the Zariski decomposition of the log canonical divisor of an arbitrary log resolution. However, we fail to prove the well-behavior of the Sarkisov relation for log Mori fiber spaces with wklt singularities in dimension

> 2, mainly because the lack of the Zariski decomposition and the failure for the statement (ii) of Proposition 3.5 to hold.

Proposition 4.4 = Proposition 3.5 for log minimal models with wklt singularities. *Let*

$$(X_0, B_{X_o}), (X_1, B_{X_1}), \cdots, (X_k, B_{X_k}), \cdots, (X_l, B_{X_l})$$

be log minimal models with only $\mathbb{Q}$-factorial wklt singularities. Then (i) (ii) and (iii) as in Proposition 3.5 (replacing the assumption of klt singularities with that of wklt singularities and allowing the possibility $\epsilon = 1$) are equivalent.

Proof.

The implications $(iii) \Rightarrow (ii) \Rightarrow (i)$ are obvious (rgardless whether they are log minimal models or log Mori fiber spaces). We only have to prove $(i) \Rightarrow (iii)$.

Take a log pair (W, B_W) as in (i) and let

$$p_k : (W, B_W) \dashrightarrow (X_k, B_{X_k})$$

be a birational map which is a $K + B$-MMP over *Speck*. We denote by C_k the closed set in W so that

$$p_k : (W - C_k, B_W|_{W - C_k}) \overset{\sim}{\to} (X_k - I_k, B_{X_k}|_{X_k - I_k}),$$

where I_k is the indeterminacy of the birational map ${p_k}^{-1}$. We can take a blowup $\sigma : W' \to W$ whose centers are all over $\cup_k C_k$ such that each X_k is dominated by a birational morphism ${p_k}' : W' \to X_k$ and that $\sigma_*^{-1}(B_W) \cup E(\sigma)$ is an S.N.C. divisor.

Set

$$\hat{B}_{W'} := \sigma_*^{-1}(B_W) + \Sigma_{E_j \text{not appearing as a divisor on } W} \epsilon E_j.$$

For each k we have the ramification formulae

$$K_{W'} + \hat{B}_{W'} = \sigma^*(K_W + B_W) + R_\sigma$$

where R_σ is an effective ramification divisor (whose support may not coincide with $E(\sigma)$), and

$$\sigma^*(K_W + B_W) = \sigma^*\{p_k^*(K_{X_k} + B_{X_k}) + R_{p_k}\}$$

where a $p_k \circ \sigma$-exceptional divisor E has a strictly positive coefficient in $\sigma^* R_{p_k}$ iff the center of E on W is contained in C_k.

Now an easy application of the Negativity Lemma (cf.Kollár[15],Lemma 4.3) shows that

$$(p_0 \circ \sigma)^*(K_{X_0} + B_{X_0}) = \cdots = (p_k \circ \sigma)^*(K_{X_k} + B_{X_k}) = \cdots = (p_l \circ \sigma)^*(K_{X_l} + B_{X_l})$$

giving the nef part of the Zariski decomposition of $K_{W'} + \hat{B}_{W'}$, and hence

$$R_\sigma + \sigma^* R_{p_0} = \cdots = R_\sigma + \sigma^* R_{p_k} = \cdots = R_\sigma + \sigma^* R_{p_l}.$$

Now any k, a $p_k \circ \sigma$-exceptional divisor E has the center on W contained in $\cup_i I_i$ and thus has a strictly positive coefficient in $\sigma^* R_{p_i}$ for some i. Therefore, the above equality implies it has a strictly positive coefficient in $R_\sigma + \sigma^* R_{p_k}$. Thus

$$K_{W'} + \hat{B}_{W'} = (p_k \circ \sigma)^*(K_{X_k} + B_{X_k}) + \hat{R}_{p_k \circ \sigma}$$

where

$$\text{supp } \hat{R}_{p_k \circ \sigma} = \text{supp } E(p_k \circ \sigma).$$

Therefore, finally by setting

$$B_{W'} = D_W(B_{X_0}, B_{X_1}, \cdots, B_{X_k}, \cdots, B_{X_l}) + \Sigma_{E_j \text{not appearing as a divisor on any of } X_k} \epsilon E_j$$

we have

$$K_{W'} + B_{W'} = (p_k \circ \sigma)^*(K_{X_k} + B_{X_k}) + R_{p_k \circ \sigma}$$

with

$$\text{supp } R_{p_k \circ \sigma} = \text{supp } E(p_k \circ \sigma).$$

Hence by Lemma 3.6 (i) each (X_k, B_{X_k}) is an end result of $K + B$-MMP over X_k starting from $(W', B_{W'})$. This completes the proof.

Theorem 4.5. *Let (X, B_X) and $(X', B_{X'})$ be log minimal models with only $\mathbb{Q}$-factorial wklt singularities in dimension 3. Suppose they are Sarkisov related. Then they are connected by a sequence of log flops*

$$(X, B_X) \dashrightarrow (X_1, B_{X_1}) \dashrightarrow \because (X_k, B_{X_k}) \dashrightarrow \cdots (X', B_{X'})$$

where all the (X_k, B_{X_k}) are Sarkisov related.

proof.

An easy application of the Negativity Lemma shows (cf.Kollár[15],Lemma 4.3) that (X, B_X) and $(X', B_{X'})$ are isomorphic in codimension 1, since they are Sarkisov related. Let $\mathcal{H}_{X'}$ be a very ample divisor on X' and $\mathcal{H}_X$ its strict transform on X. We take a log pair (W, B_W) dominating both log minimal models by birational morphisms $p : (W, B_W) \to (X, B_X)$ and $q : (W, B_W) \to (X', B_{X'})$ as in (iii) of Proposition 3.5, whose existence is guaranteed by Proposition 4.4. We claim that $K_X + B_X + \eta\mathcal{H}_X$ is wklt for $0 < \eta << 1$, since

$$Bs(\mathcal{H}_X) \subset p(R_p)$$

where

$$K_W + B_W = p^*(K_X + B_X) + R_p.$$

If $K_X + B_X + \eta\mathcal{H}_X$ is nef, then by Log Abundance (cf.KeMaMc[12]) it is semi-ample. Since it is the strict transform of an ample divisor $K_{X'} + B_{X'} + \eta\mathcal{H}_{X'}$ and since (X, B_X) and $(X', B_{X'})$ are isomorphic in codimension one and $\mathbb{Q}$-factorial, this implies $(X, B_X) \overset{\sim}{\to} (X', B_{X'})$.

If $K_X + B_X + \eta\mathcal{H}_X$ is not nef, then there is an $K_X + B_X + \eta\mathcal{H}_X$-negative extremal ray, which must be $K_X + B_X$-trivial and of flopping type (cf.Kollár[15],Lemma 4.4). We flop this extremal ray to get another log minimal model $(X, B_X) \dashrightarrow (X_1, B_{X_1})$ with only $\mathbb{Q}$-factorial wklt singularities. By construction it is easy to see that $(X, B_X), (X_1, B_{X_1})$ and $(X', B_{X'})$ are all Sarkisov related and that $K_{X_1} + B_{X_1} + \eta\mathcal{H}_{X_1}$ is wklt. We proceed inductively and this procedure has to come to an end, since any sequence of log flops has to terminate (cf.Shokurov[31]Kollár et al[16]). Thus we obtain the desired connecting sequence of log flops between (X, B_X) and $(X', B_{X'})$.

We go back to the discussion of the log Sarkisov program with wklt singularities.

In the following we prove the well-behavior of the Sarkisov relation for log Mori fiber spaces with wklt singularities in dimension 2.

Lemma 4.6 = Proposition 3.5 for log Mori fiber spaces with wklt singularities in dimension 2. *Let*

$$(X_0, B_{X_o}), (X_1, B_{X_1}), \cdots, (X_k, B_{X_k}), \cdots, (X_l, B_{X_l})$$

be log Mori fiber spaces with only $\mathbb{Q}$-factorial wklt singularities in dimension 2. Then (i) (ii) and (iii) as in Proposition 3.5 (replacing the assumption of klt singularities with that of wklt singularities and allowing the possibility $\epsilon = 1$) are equivalent.

Proof.

Again we only have to show the implication $(i) \Rightarrow (iii)$.

Take a log pair (W, B_W) as in (i) and let

$$p_k : (W, B_W) \dashrightarrow (X_k, B_{X_k})$$

be a birational map which is a $K + B$-MMP over *Speck*. First observe that in dimension 2 all the p_k are birational morphisms and that

$$\cup_k {p_k^{-1}}_*(B_{X_k}) \subset B_W$$

is an S.N.C. divisor. We take a blowup $\sigma : W' \to W$ with centers over

$$\cup_{E_l \text{not appearing as a divisor on any of } X_k} E_l$$

until

$$\cup_{E_m \text{ not appearing as a divisor on any of } X_k} E_m \cup \cup_k {p_k^{-1}}_* (B_{X_k})$$

is an S.N.C. divisor. Then by setting

$$B_{W'} = D_W(B_{X_0}, \cdots, B_{X_k}, \cdots, B_{X_l}) + \Sigma_{E_j \text{ not appearing as a divisor on any of } X_k} E_j,$$

we have

$$K_{W'} + B_{W'} = \sigma^*(K_W + B_W) + R_\sigma$$

where R_σ is an effective ramification divisor (whose support may not coincide with the exceptional locus $E(\sigma)$).

On the other hand, since for each k

$$K_W + B_W = {p_k}^*(K_{X_k} + B_{X_k}) + R_{p_k}$$

where the effective ramification divisor R_{p_k} has the support which coincides with that of the exceptional locus $E(p_k)$, and since the blowup has all the centers over

$$\cup_{E_l \text{ not appearing as a divisor on any of } X_k} E_l \subset \cap_i E(p_i),$$

we conclude

$$K_{W'} + B_{W'} = (p_k \circ \sigma)^*(K_{X_k} + B_{X_k}) + R_{p_k \circ \sigma}$$

with

$$\text{supp } R_{p_k \circ \sigma} = \text{supp } E(p_k \circ \sigma).$$

Thus by Lemma 3.6 (i) each (X_k, B_{X_k}) is an end result of $K + B$-MMP over X_k starting from $(W', B_{W'})$. This completes the proof.

In dimension 2, any MMP (log or genuine) is a succession of contractions of divisors without any flip and thus the resulting Mori fiber space is dominated by the starting variety through a birational morphism. Also any $K + B$-MMP over *Spec k* is a process of $K + B$-MMP over any variety T which is dominated by relevant log pairs. These easy observations unique to dimension 2 make the flowchart for the log Sarkisov with wklt singularities rather straightforward in dimension 2, compared to higher dimensional case, where the Sarkisov relation seems more subtle.

The log Sarkisov degree $(\mu_k, \lambda_{\epsilon k}, e_{\epsilon k})$ is defined in the following way.

The quasi-effective threshold μ_k is as before defined to be the positive rational number s.t.

$$\mu_k(K_{X_k} + B_{X_k}) + \mathcal{H}_{X_k} \equiv 0 \text{ over } S_k.$$

$\lambda_{\epsilon k}$ is defined in the exactly same way as in the case with klt singularities setting $\epsilon = 1$.

We pay extra attention to how we define $e_{\epsilon k}$. If we try to define it in the same way as in the case with klt singularities setting $\epsilon = 1$, then it would not be well defined, since we may have infinitely many crepant divisors. This is one of the difficulties one has to face once we hit the critical value $\epsilon = 1$ creating the neccessity to deal with the log canonical locus.

In the flowchart below we show all the divisorial blow ups and intermediate log Mori fiber spaces are dominated by (W, B_W) that we fix from the beginning as above satisfying the conditions in (iii) in Proposition 3.5 .

We define $e_{\epsilon k}$ of the intermediate log Mori fiber space that appear in the due course of the flowchart to be the number of $K + B + \frac{1}{\lambda_{\epsilon k}}\mathcal{H}$-crepant divisors **ON W**.

Flowchart for Log Sarkisov Program in dimenion 2

with WKLT Singularities and its Termination

$$\boxed{\text{Case} : \lambda_{\epsilon k} \leq \mu_k}$$

The flowchart for the log Sarkisov program with WKLT singularities in this case goes parallel to the one for the genuine Sarkisov program. After untwisting the birational map by a link of type (III) or (IV), the quasi-effective threshold strictly decreases.

Moreover, since $\phi_{k+1} : (X_{k+1}, B_{X_{k+1}}) \to S_{k+1}$ is obtained as an end result of $K + B + \frac{1}{\mu_k}\mathcal{H}$-MMP (over T), it follows immediately that $(X_{k+1}, B_{X_{k+1}})$ is an end result of $K + B$-MMP starting from (W, B_W) (over X_{k+1}) and thus dominated by (W, B_W) through a birational morphism.

The claim that there is no infinite number of untwisting (successiove or unsuccessive) by the links unedr the case $\lambda_{\epsilon k} \leq \mu_k$ can be shown similarly, proving the discreteness of the quasi-effective thresholds noting that a general fiber of ϕ_k is $\mathbb{P}^1$.

$$\boxed{\text{Case} : \lambda_{\epsilon k} > \mu_k}$$

We take a maximal divisorial blow up constructed starting from (W, B_W), which dominates (X_k, B_{X_k}) by inductive assumption.

Then just as in the genuine Sarkisov program after untwisting the birational map by a link of type (II) or (I), the quasi-effective threshold does not increase

$$\mu_{k+1} \leq \mu_k$$

with equality holding only if

$$\begin{aligned} &\text{either } \dim S_{k+1} > \dim S_k \\ &\quad \text{or } \dim S_{k+1} = \dim S_k \text{ and } \psi_k \text{ is square.} \end{aligned}$$

Also it follows similarly that

$$\lambda_{\epsilon,k+1} \leq \lambda_{\epsilon k}.$$

Note that $\phi_{k+1} : (X_{k+1}, B_{X_{k+1}}) \to S_{k+1}$ is an end result of $K + B + \frac{1}{\lambda_{\epsilon k}}\mathcal{H}$-MMP (over S_k) starting from the maximal divisorial blowup (Z_k, B_{Z_k}), which in turn is an end result of $K + B + \frac{1}{\lambda_{\epsilon k}}\mathcal{H}$-MMP possibly followed by $K + B$-MMP (over X_k) starting from (W, B_W). Therefore, it is easy to see that $(X_{k+1}, B_{X_{k+1}})$ is an end result of $K + B$-MMP (over X_{k+1}) starting from (W, B_W). Thus $e_{\epsilon,k+1}$ is well -defined and in the above inequality

$$\text{if } \lambda_{\epsilon,k+1} = \lambda_{\epsilon k} \text{ then } e_{\epsilon,k+1} \leq e_{\epsilon k} - 1 < e_{\epsilon k}.$$

This immediately proves the claim that there is no infinite (successive) sequence of untwisting by the links under the case $\lambda_{\epsilon k} > \mu_k$ with stationary $\lambda_{\epsilon k}$.

Now we take a closer look at the proof of the claim that there is no infinite (successive) sequence of untwisting by the links under the case $\lambda_{\epsilon k} > \mu_k$ with stationary quasi-effective threshold μ_k.

The proof goes parallel for Steps 1 and 2 (cf. Claim 2.2) replacing K with $K+B$. Step 3 becomes meaningless and irrelevant in the case with wklt singularities and we disregard it, i.e., we don't use Step 3 in our argument below. Instead we conclude the argument as follows. First we remark that the valuations of $k(X)$ corresponding to the unique exceptional divisors E_k of the maximal divisorial blowups are all distinct. Moreover, all the E_k are divisors on W. On the other hand,

$$a(E_k, X_1, B_{X_1} + \alpha\mathcal{H}_{X_1}) \leq a(E_k, X_k, B_{X_k} + \alpha\mathcal{H}_{X_k}) < 0,$$

but there are only finitely many divisors on W with negative discrepancies w.r.t. $B_{X_1} + \alpha\mathcal{H}_{X_1}$, a contradiction!

We finish the proof of termination by showing the claim that there is no infinite (successive) sequence of untwisting by the links under the case $\lambda_{\epsilon k} > \mu_k$ with nonstationary quasi-effective threshold.

For the case where $\dim S_{k_0} \geq 1$ for some k_0 (and thus for $\forall k \geq k_0$), we show the discreteness of the quasi-effective thresholds again noting that a general fiber of $\phi_k (k \geq k_0)$ is $\mathbb{P}^1$.

For the case $\forall k, \dim S_k = 0$, we note that the X_k are all log Del Pezzo surfaces (normal projective surfaces with only quotient singularities having ample anti-canonical divisors) which are dominated by one fixed nonsingular projective surface W. Therefore, it is easy to see that the X_k belong to a bounded family, from which fact the discreteness of the quasi-effective thresholds follows just as before.

This completes the discussion of the flowchart and its termination for the log Sarkisov program with wklt singularities in dimension 2.

Fianlly we discuss briefly the problems we face when we try to establish the log Sarkisov program with wklt singularities in higher dimension.

Problem 1.

Does the Sarkisov relation behave well with wklt singularities, i.e., do we have the equivalence of (i) (ii) and (iii) in proposition 3.5 replacing klt singularities with wklt singularities and allowing $\epsilon = 1$?

Problem 2.

If the answer to Problem 1 is affirmative, then we can construct a maximal divisorial blowup of (X_k, B_{X_k}) with respect to $\mathcal{H}_{X_k}$ in the case $\lambda_{\epsilon k} \leq \mu_k$ from a good log pair (W, B_W) as in (iii) of Proposition 3.5. After $K + B + \frac{1}{\lambda_{\epsilon k}}\mathcal{H}$-MMP we reach $(X_{k+1}, B_{X_{k+1}})$. In the case $\lambda_{\epsilon k} \leq \mu_k$, after $K + B + \frac{1}{\mu_k}\mathcal{H}$-MMP we reach $(X_{k+1}, B_{X_{k+1}})$.

Show in both cases that the (X_m, B_{X_m}) $m = 0, 1, \cdots, k, k+1$ are all Sarkisov related establishing the inductive procedure.

We note that the Nöther-Fano criterion remains valid as long as we know (X, B_X), $(X', B_{X'})$ and (X_k, B_{X_k}) are Sarkisov related. (This is not a trivial remark as at one point of the proof (See Corti[4],Theorem 4.2.) the positivity of some coefficient in the ramification divisor does not follow without the assumption of being Sarkisov related in the case of wklt singularities.)

Problem 3.

Show the discreteness of the quasi-effective thresholds under the case $\lambda_{\epsilon k} \leq \mu_k$, which follows from the (conjectural) boundedness of the fibers of the ϕ_k.

We remark that this is not a straight consequence of $S_d(Global)$ ($d \leq n-1$ in dimension n), since we do not know that the number of components in the boundary or the coefficient $\frac{1}{\mu_k}$ to be bounded.

Problem 4.

Show that there is no infinite (successive) sequence of untwisting by the links under the case $\lambda_{\epsilon k} > \mu_k$ in the following manner:

i) Show that $\lambda_{\epsilon k}$ cannot be stationary, by adopting an appropriate definition of $e_{\epsilon k}$ as demonstrated in the case of dimension 2. This should be relatively easy.

ii) Show that μ_k cannot be stationary. Steps 1 and 2 of Calim 2.2 go without change, while Step 3 is irrelevant. We should conclude the argument by looking at $S_d(Global)$ $d \leq n-1$ on the exceptional divisors of the maximal divisorial blowups that appear with coefficient 1. (This line of argument was suggested to us by A. Corti. In fact we could argue this way in the proof of termination in dimension $n = 2$, though it becomes substantially lengthier than the one we give.)

Problem 5.

Finally show the discreteness of the quasi-effective thresholds under the case $\lambda_{\epsilon k} > \mu_k$, which follows again from the boundedness of the fibers of the ϕ_k. This seems to be the most difficult part.

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