DYNAMICS OF QUADRATIC POLYNOMIALS.
I. COMBINATORICS AND GEOMETRY OF THE YOCCOZ PUZZLE.

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Abstract. This work studies combinatorics and geometry of the Yoccoz puzzle for quadratic polynomials. It is proven that the moduli of the "principal nest" of annuli grow at linear rate. As a corollary we obtain complex a priori bounds and local connectivity of the Julia set for many infinitely renormalizable quadratics.

§1. Introduction

This is the beginning of a series of notes on dynamics of quadratic polynomials. The forthcoming notes will be concerned with parameter geometry, applications to the complex and real Rigidity Problems, and measurable dynamics. This first part is a detailed version of §§2,3 of [L4].

The goal of this part is to study combinatorics and geometry of the Yoccoz puzzle, with an application to the problem of local connectivity of Julia sets. The puzzle was introduced by Branner and Hubbard [BH] for cubic maps with one escaping critical point and by Yoccoz for quadratics (see [H], [M2]). The main geometric result of these works is the divergence property of moduli of a certain nest of annuli. This implies that the corresponding domains ("puzzle pieces") shrink to points, which yields local connectivity of the Julia set. The corresponding result in the parameter plane yields rigidity.

The geometric result of Branner-Hubbard and Yoccoz does not contain information on the *rate* at which the pieces shrink to points. In this work we tackle this problem. We consider a smaller nest $V^0 \supset V^1 \supset \dots$ called *principal*, and prove that the annuli moduli of this nest grow at linear rate over a certain combinatorially specified subsequence of levels (Theorem III). This implies that the first return maps $g_n : V^n \to V^{n-1}$ are becoming purely quadratic at exponential rate. We expect this fact to have many applications.

Theorem III also yields *a priori bounds* for infinitely renormalizable quadratics of "sufficiently big type" (Theorems IV and IV$'$). For real quadratics of "bounded type" the problem of complex bounds was resolved by Sullivan (see [S2], [MvS]). Our result for real quadratics is a kind of complement to Sullivan's, though there is still one special combinatorial situation, "saddle-node cascades", which is not covered by either of these results. In the forthcoming notes we will give an appropriate extension of Sullivan's bounds which pick saddle-node cascades [LY]: Local connectivity of the Julia sets for all real

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quadratics follows.** As to the non-real situation, to the best of our knowledge the problem of a priori bounds was not settled before for any one quadratic of bounded type (compare Rees [R]).

An immediate consequence of our a priori bounds is that the puzzle pieces of the "full principal nest" shrink to points, which yields local connectivity of the Julia set. This is a nice property, since such a Julia set has an explicit topological model (see Douady [D2]). Note that a large pool of locally connected examples was already known (Douady - Hubbard, Yoccoz (see [DH1], [H]), Hu - Jiang [HJ], [J], Petersen [P]). Unfortunately there are also counter-examples: Cremer and some infinitely renormalizable quadratics have non-locally connected Julia sets (see [L3], [M2]).

However, the main reason why complex a priori bounds are important is that they provide a key to the rigidity problem and the renormalization theory (compare Sullivan [S2] and McMullen [McM]). These applications will be the subject of the forthcoming notes.

Let us now describe the structure of the paper.

In the next section, §2, we overview the necessary preliminaries in holomorphic dynamics, particularly Douady-Hubbard renormalization and the Yoccoz puzzle.

In §3 we present our approach to combinatorics of the puzzle. The main concepts involved are the *principal nest* of puzzle pieces, *generalized renormalization* and *central cascades.* As we indicated above, the principal nest $V^0 \supset V^1 \supset \dots$ contains the key combinatorial and geometric information about the puzzle. We describe the combinatorics of this nest by means of *generalized renormalizations*, that is, appropriately restricted first return maps considered up to rescaling.

It may happen that the quadratic-like map $g_n : V^n \to V^{n-1}$ has "almost connected" Julia set. This phenomenon often requires a special treatment. Such a map generates a subnest of the principal nest called a *central cascade.* The number of central cascades in the principal nest is called the *height* $\chi(f)$ of a map f. In other words, $\chi(f)$ is the number of *different* quadratic-like maps among the g_n's (where "different" means: "with different Julia sets"). It will play an important role for our discussion.

In §4 we study the initial geometry of the puzzle. The main result of this section is the construction of an initial annulus $V^0 \backslash V^1$ with definite modulus, provided the hybrid class of a map is selected from a truncated secondary limb (Theorem I).

In §5 we prove the main geometric result of the paper saying that the principal moduli grow linearly with the number of central cascades. We introduce a new geometric parameter (worked out jointly with J. Kahn), *the asymmetric modulus*, and prove first that it is *monotonically non-decreasing* when we go down along the principal nest (Theorem II). This already provides us with lower bounds for the principal moduli $\mu_m = \operatorname{mod}(V^{m-1} \backslash V^m)$ (which, by the way, implies the Branner-Hubbard-Yoccoz divergence property), and upper bounds on the distortion. We reach these results by means of a purely combinatorial analysis plus the standard Grötzsch inequality.

However this analysis does not always yield the linear growth of moduli. In particular, it is not enough for the basic example called the *Fibonacci map.* The crucial part of the proof is to gain this extra growth for Fibonacci-like combinatorics. The argument based on the *Definite Grötzsch inequality* involves estimates of hyperbolic distances between puzzle

** This result with a different proof has been also announced by Levin and van Strien [LS].

pieces and analysis of their shapes. The key observation is that sufficiently pinched pieces make a definite extra contribution to the moduli growth (Theorem III).

At the end of this section we prove a priori bounds for infinitely renormalizable quadratics of sufficiently big type (Theorems IV and IV$'$). The meaning of this condition is that certain combinatorial parameters of the renormalized maps $R^n f$ are sufficiently big (depending on the truncated secondary limb to which the internal class of $R^n f$ belongs). The main such parameter is the above mentioned height, but there are also several others. These conditions together mean roughly that the *periods* of $R^n f$ are sufficiently big. The only difference is a possibility of long "parabolic or Siegel cascades".

In the next short section, §6, we show that the Julia sets of the quadratics under consideration are locally connected. This follows from shrinking of the puzzle-pieces.

In the last section, §7, we outline the content of the forthcoming notes.

Before finishing let us draw the reader's attention to the work of Graczyk and Swiatek [GS] containing a related result on the growth of moduli (within a certain nest of domains different from the principal nest) for *real* quadratics.

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§2. Preliminaries: Douady-Hubbard renormalization and Yoccoz puzzle.

2.1. General terminology and notations. Given two subsets V and W of the complex plane, we say that V is *strictly contained* in W, $V \subset\subset W$, if $\mathrm{cl}\, V \subset \mathrm{int}\, W$.

Let $B(z,\epsilon)$ denote the disk of radius ϵ centered at z. Given two sets A and B, let $\mathrm{dist}\,(A,B) = \inf\{\mathrm{dist}\,(z,\zeta) : z \in A, \zeta \in B\}$.

By $\mathrm{orb}\, z$ we denote the forward orbit $\{f^n z\}_{n=0}^{\infty}$ of z, and by $\omega(z)$ its ω-limit set. Let also $\mathrm{orb}_n z = \{f^m z\}_{m=0}^{n}$. By a *topological disk* we will mean a simply connected region in $\mathbf{C}$. By an *annulus* we mean a doubly connected region. A *horizontal curve* in an annulus A is a preimage of a circle centered at 0 by the Riemann mapping $A \to \{z : 1 < |z| < R\}$.

We assume that the reader is familiar with the basic theory of quasi-conformal maps (see [A]). Quasi-conformal and quasi-symmetric maps will be abbreviated as qc and qs correspondingly.

2.2. Polynomials. By now there are many surveys and books on holomorphic dynamics. The reader can consult [Be], [GC], [M1] for general reference, and [B], [DH1] for the quadratic case. Below we will state the main definitions and facts required for discussion. However we assume that the reader is familiar with the notions of periodic points, and their classification as attracting, neutral, parabolic and repelling.

Let $f:\mathbf{C}\to\mathbf{C}$ be a polynomial of degree $d\geq 2$. The *basin of* ∞ is the set of points escaping to ∞:

$$D_f(\infty)\equiv D(\infty)=\{z\in\mathbf{C}: f^n z\to\infty\}.$$

Its complement is called *the filled Julia set*:

$$K(f)=\mathbf{C}\backslash D(\infty).$$

The *Julia set* is the common boundary of $K(f)$ and $D(\infty)$:

$$J(f)=\partial K(f)=\partial D(\infty).$$

The Fatou set $F(f)$ is defined as $\mathbf{C}\backslash J(f)$. The Julia set (and the filled Julia set) is connected if and only if non of the critical points escape to ∞, that is, all of them belong to $K(f)$.

Given a polynomial f, there is a conformal map (the *Böttcher function*)

$$B_f: U_f\to\{z:|z|>r_f\geq 1\}$$

of a neighborhood U_f of infinity onto the exterior of a disk such that $B_f(fz)=(B_f z)^d$ and $B_f(z)\sim z$ as $z\to\infty$. There is an explicit dynamical formula for this map:

$$B_f(z)=\lim_{n\to\infty}(f^n z)^{1/d^n} \tag{2-1}$$

with an appropriate choice of the branch of the d^n th root.

If the Julia set $J(f)$ is connected then ∂U_f contains a critical point b of f. Otherwise B_f coincides with he Riemann mapping of the whole basin of infinity $D(\infty)$ onto $\{z:|z|>1\}$ (in this case $r_f=1$).

The *external rays* R_θ with angle θ and *equipotentials* E_h of level h are defined as the B_f-preimages of the straight rays $\{re^{i\theta}: r_f<r<\infty\}$ and the round circles $\{e^{i\theta}:0\leq\theta\leq 2\pi\}$. They form two orthogonal invariant foliations of $D(\infty)$.

Theorem 2.1 (see [M1], §18, or [H]). *If a is a repelling periodic point of f, then there is at least one but at most finitely many external rays landing at a.*

Sketch. Let us sketch the geometric construction of these rays (see the above references for the details). Let us linearize f^p near the periodic point a: Let $\psi:(U,0)\to(V,a)$ be a local conformal map normalized by $\psi'(0)=1$ and such that

$$\psi(\lambda z)=f^p(\psi z) \qquad (2\text{-}2),$$

where λ is the multiplier at a. By means of this functional equation ψ can be analytically extended to the whole complex plane. Thus we obtain an entire function $\psi:\mathbf{C}\to\mathbf{C}$ satisfying (2-2).

Let $\mathcal{D}_i$ be the components of $\psi^{-1}D(\infty)$. It is not hard to see that $\psi:\mathcal{D}_i\to D(\infty)$ is a universal covering map. Moreover, the function

$$h_i=\log B_f\circ\psi:\mathcal{D}_i\to\mathbf{H} \tag{2-3}$$

gives the Riemann map of $\mathcal{D}_i$ onto the upper half-plane $\mathbf{H}$.

The key point is that there are only finitely many components $\mathcal{D}_i$ which hence invariant under some iterate f^p. Then the map h_i conjugates $z\mapsto\lambda^p z$ on D_i to $z\mapsto d^p z$

on $\mathbf{H}$. Take now the hyperbolic geodesic γ joining 0 and ∞ in $\mathbf{H}$ (the vertical ray) and the corresponding hyperbolic geodesic $\hat{R}_i = h_i^{-1}\gamma$ in $\mathcal{D}_i$. It follows that $\hat{R}_i$ is invariant with respect to $z \mapsto \lambda^p z$, and hence lands at 0. Hence the external ray $R_i = \psi \hat{R}_i$ lands at a. □

Thus the external rays landing at a are organized in several cycles. The rotation number of these cycles is the same, and is called the *combinatorial rotation number* $\rho(a)$ of a. Let $\mathcal{R}(a)$ denote the union of the external rays landing at a, and

$$\mathcal{R}(\bar{a}) = \cup_{k=0}^{p-1} \mathcal{R}(f^k a)$$

(where p is the period of a). This configuration (with the external angles marked on the rays) is called the *rays portrait* of the cycle $\bar{a}$. The class of isotopic portraites is called the *abstract rays portrait.*

2.3. Quadratic family. Let now $f \equiv P_c : z \mapsto z^2 + c$ be the quadratic family. In this case the rays portraites of periodic cycles have quite specific combinatorial properties. The reader can consult [DH1], [At], [GM], [Sch], [M4] for the proofs of the results quated below.

Proposition 2.2 (see [M4]). *Let $\bar{a} = \{a_k\}_{k=0}^{p-1}$ be a repelling periodic cycle such that there are at least two rays landing at each point a_k.*

(i) Let S_1 be the components of $\mathbf{C}\backslash\mathcal{R}(\bar{a})$ containing the critical value c. Then S_1 is a sector bounded by two external rays.

(ii) Let S_0 be the component of $\mathbf{C}\backslash f^{-1}\mathcal{R}(\bar{a})$ containing the critical point 0. Then S_0 is bounded by four external rays: two of them lands at a periodic point a_k, and two others land at the symmetric point $-a_k$.

(iii) The rays of $\mathcal{R}(\bar{a})$ form either one or two cycles under iterates of f.

A particular situation of such kind is the following. Let $\bar{b} = \{b_k\}_{k=0}^{p-1}$ be an attracting cycle, $p > 1$. Let D_k be the components of its basin of attraction containing b_k. Then the boundaries of D_k are Jordan curves, and the restrictions $f^p|D_k$ are topologically conjugate to the doubling map $z \mapsto z^2$ of the unit circle. Hence there is a unique f^p-fixed point $a_k \in D_k$. Altogether these points form a repelling periodic cycle $\bar{a}$ (whose period may be smaller than p), with at least two rays landing at each a_k. The portrait $\mathcal{R}(\bar{a})$ will be also called the rays portrait associated to the attracting cycle $\bar{b}$.

A case of special interest for what follows is the fixed points portraites. There is always a fixed point called β which is the landing point of the invariant ray $\mathcal{R}_0$. Moreover, this is the only ray landing at this point, so that it is non-dividing: the set $K(f)\backslash\{\beta\}$ is connected.

If the second fixed point called α is also repelling, it is much more interesting. Namely, this point is dividing, so that there are several external rays landing at it. These rays are cyclically permuted by dynamics with some combinatorial rotation number q/p.

The *Mandelbrot set* M is defined as the set of $c \in \mathbf{C}$ for which $J(P_c)$ is connected, that is, 0 does not escape to ∞ under iterates of P_c. If $c \in \mathbf{C}\backslash M$, then $J(P_c)$ is a Cantor set.

The Mandelbrot set itself is connected (see [DH1], [CG]). This is proven by constructing explicitly the Riemann mapping $B_M : \mathbf{C} \setminus M \to \{z : |z| > 1\}$. Namely, let $D_c(\infty)$ be the basin of ∞ of P_c, and B_c be the Böttcher function (2-1) of P_c. Then

$$B_M(c) = B_c(c). \tag{2-4}$$

The meaning of this formula is that the "conformal position" of a parameter $c \in \mathbf{C} \setminus M$ coincides with the "conformal position" of the critical value c in the basin $D_c(\infty)$. This relation is the key to the similarity between dynamical and parameter planes.

Using the Riemann mapping B_M we can define the *parameter external rays* and *equipotentials* as the preimages of the straight rays going to ∞ and round circles centered at 0. This gives us two orthogonal foliations in the complement of the Mandelbrot set.

A quadratic polynomial P_c with $c \in M$ is called *hyperbolic* if it has an attracting cycle. The set of hyperbolic parameter values is the union of some components of $\operatorname{int} M$ called *hyperbolic components*. Conjectually all components of $\operatorname{int} M$ are hyperbolic. This Conjecture would follow from the MLC Conjecture (Douady & Hubbard [DH1]).

Let $H \subset \operatorname{int} M$ be hyperbolic component of the Mandelbrot set, and let $\bar{b}(c) = \{b_k(c)\}_{k=0}^{p-1}$ be the corresponding attracting cycle. On the boundary of H the cycle $\bar{b}$ becomes neutral, and there is a single point $c_H \in \partial H$ where $f^p(b_0) = 1$ [DH1]. This point is called the *root* of H.

Given a $c \in H$, there is the rays portrait $\mathcal{R}_c$ associated to the corresponding attracting periodic point. Let θ_1 and θ_2 be the external angles of the two rays bounding the sector S_1 of Proposition 2.2.

Theorem 2.3 (see [DH1], [M4], [Sch]). *The parameter rays $\mathcal{R}_1^H$ and $\mathcal{R}_2^H$ with angles θ_1 and θ_2 land at the root c_H. There are no other rays landing at c_H.*

The region W_H in the parameter plane bounded by the rays $\mathcal{R}_i^H$ and containing H is called the *wake* (originated from H). The part of the Mandelbrot set contained in the wake together with the root c_H is called the *limb* L_H of the Mandelbrot set originated at H. The root of H is also called the root of wake W_H and limb L_H.

Recall that for $c \in H$, $\bar{a}_c$ denotes the repelling cycle associated to the basin of attraction of the attracting cycle $\bar{b}_c$. The dynamical meaning of the wakes is reflected in the following statement.

Proposition 2.4 (see [GM]). *The repelling cycle $\bar{a}_c$ stays repelling throughout the wake W_H. The corresponding rays portrait $\mathcal{R}(a_c)$ preserves its isotopic type throughout this wake.*

The *main cardioid* of M is defined as the set of points c for which P_c has a neutral fixed point α_c, that is, $|P_c'(\alpha_c)| = 1$. It encloses the hyperbolic component where P_c has an attracting fixed point. In the exterior of the main cardioid both fixed points are repelling.

The limbs attached to the main cardioid are called *primary*. Let H be a hyperbolic component attached to the main cardioid. The limbs attached to such a component are called *secondary*. We refer to a *truncated limb* if we remove from it a neighborhood of its root (Figure 1).

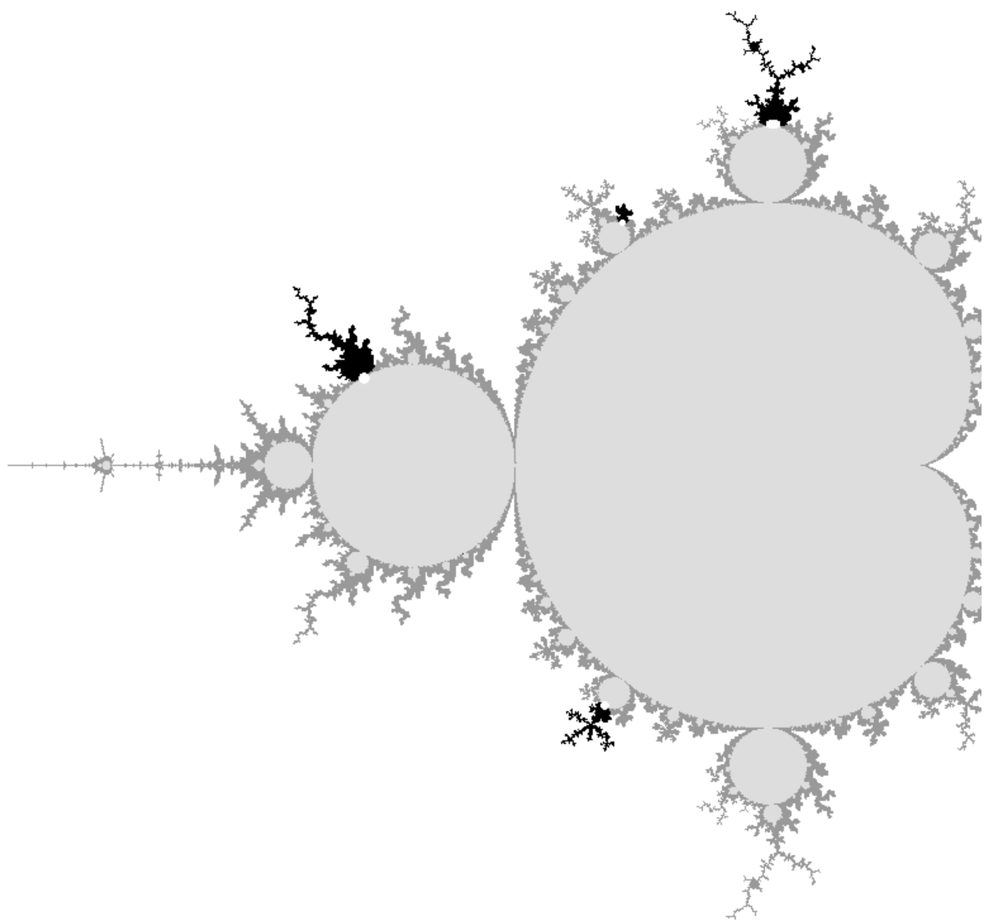

Figure 1: Truncated secondary limbs of the Mandelbrot set.

We refer to a *truncated limb* if we remove from it a neighborhood of its root (Figure 1).

2.4. Douady-Hubbard polynomial-like maps. The main reference for the following material is [DH2]. Let $U' \subset\subset U$ be two topological disks (with a non-degenerate annulus $U \backslash U'$). A branched covering $f : U' \to U$ is called a *DH polynomial-like map* (we will sometimes skip "DH" in case this does not cause confusion with "generalized" polynomial-like maps defined below). Every polynomial with connected Julia set can be viewed as a polynomial-like map after restricting it onto an appropriate neighborhood of the filled Julia set. Polynomial-like maps of degree 2 are called *(DH) quadratic-like*. We will always normalize quadratic-like maps so that the origin 0 will be its critical point.

One can naturally define the filled Julia set of f as the set of non-escaping points:

$$K(f) = \{z : f^n z \in U' : n = 0, 1, \ldots\}.$$

The Julia set is defined as $J(f) = \partial K(f)$. These sets are connected if and only if non of the critical points is escaping.

The choice of the domain U' and range U of a polynomial-like map is not canonical. Two polynomial-like maps $f : U' \to U$ and $g : V' \to V$ are considered to be the same if $K(f) = K(g)$ and f coincides with g on a neighborhood of $K(f)$.

Given a polynomial-like map $f : U' \to U$, we can consider a *fundamental annulus* $A = U' \backslash U$. It is certainly not a canonical object but rather depending on the choice of U' and U. Let

$$\operatorname{mod}(f) = \sup \operatorname{mod} A,$$

where A runs over all fundamental annuli of f.

Two polynomial-like maps f and g are called *topologically (quasi-conformally, conformally) conjugate* if there is a homeomorphism (qc map, conformal isomorphism correspondingly) h from a neighborhood of $K(f)$ to a neighborhood of $K(g)$ such that $h \circ f = g \circ h$.

If there is a qc conjugacy h between f and g with $\bar\partial h = 0$ almost everywhere on the filled Julia set $K(f)$, then f and g are called *hybrid* or *internally* equivalent. A *hybrid class* $\mathcal{H}(f)$ is the space of DH polynomial-like maps hybrid equivalent to f modulo conformal equivalence. According to Sullivan [S1], a hybrid class of polynomial-like maps should be viewed as as infinitely dimensional Teichmüller space. In contrast with the classical Teichmüller theory this space has a preferred point:

Straightening Theorem [DH2]. *Any hybrid class $\mathcal{H}(f)$ of DH polynomial-like maps with connected Julia set contains a unique (up to affine conjugacy) polynomial.*

In particular, any hybrid class of quadratic-like maps with connected Julia set contains a unique quadratic polynomial $z \mapsto z^2 + c$ with $c = c(f) \in M$. So the hybrid classes of quadratic-like maps are labeled by the points of the Mandelbrot set. In what follows we will freely identify a quadratic hybrid class with its label $c \in M$.

Sullivan supplied any hybrid class with the following *Teichmüller metric* [S1]:

$$\operatorname{dist}_T(f, g) = \inf \log K_h,$$

where h runs over all hybrid conjugacies between f and g, and K_h denotes the qc dilatation of h. The Teichmüller distance from f to the quadratic $P_{c(f)} : z \mapsto z^2 + c(f)$ in its hybrid class is controlled by the modulus of f:

Proposition 2.5. *If $\operatorname{mod}(f) \geq \mu > 0$ then $\operatorname{dist}_T(f, P_{c(f)}) \leq C$ with a $C = C(\mu)$ depending only on μ. Moreover, $C(\mu) \to 0$ as $\mu \to \infty$.*

This is the reason why the control of moduli of polynomial-like maps is crucial for the renormalization theory (see [S2], [McM]).

Let us finish with the following remark: Given a polynomial-like map with connected Julia set, we can define external rays and equipotentials near the filled Julia set by conjugating it to a polynomial. This definition is certainly not canonical but rather depends on the choice of conjugacy. We will specify the particular choice whenever it matters.

2.5. Douady-Hubbard renormalization. The reverse procedure under the name of *tuning* is discussed in [DH2], [D] and [M3]. A more general point of view (but which is equivalent to the tuning, after all) is presented in [McM].

Let $f : U' \to U$ be a quadratic-like map. Let $\bar{a}$ be a dividing repelling cycle, so that there are at least two rays landing at each point of $\bar{a}$. Let $\mathcal{R} \equiv \mathcal{R}(\bar{a})$ denote the configuration of rays landing at $\bar{a}$, and let $\mathcal{R}' = -\mathcal{R}$ be the symmetic configuration. Let us aslo consider an arbitrary equipotential E. Let now Ω be the component of $\mathbf{C}\backslash(E \cup \mathcal{R} \cup \mathcal{R}')$ containing the critical point 0. By Proposition 2.2, it is bounded by four arcs γ_i of external rays ant two pieces of the equipotential E.

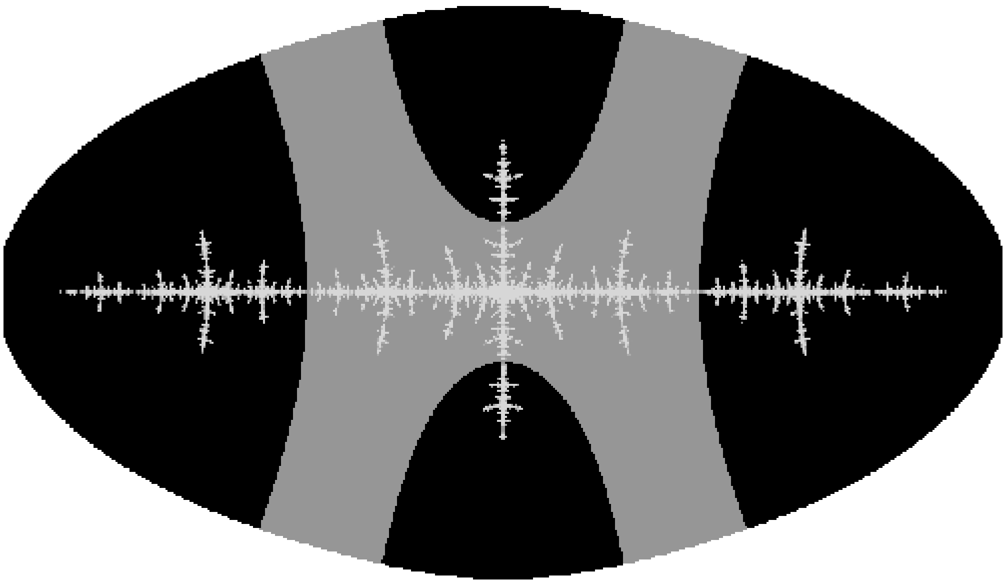

Figure 2: A renormalization domain for the Feigenbaum polynomial.

Let p be the period of the above rays, and $a \in \partial\Omega$ be the periodic point of cycle

$\bar{a}$. Let us consider a domain $\Omega' \subset \Omega$, the component of $f^{-p}\Omega$ attached to a (see Figure 2). If $\Omega' \ni 0$ then $f^p : \Omega' \to \Omega$ is a double covering map. A quadratic-like map f is called *DH-renormalizable* if there is a repelling cycle $\bar{a}$ as above such that $\Omega' \ni 0$, and 0 does not escape Ω' under iterates of f^p. We will also say that this renormalization is associated to the periodic point a. We call f *immediately DH renormalizable* if a can be selected as the dividing fixed point α of f.

Note that the disks $\Omega', f\Omega', \dots, f^{p-1}\Omega'$ have disjoint interiors. Indeed, otherwise $f^k\Omega'$ would be inside Ω for some $k < p$. But this is impossible since the external rays which bound $f^k\Omega'$ are outside of Ω.

In the DH-renormalizable case by means of a "thickening procedure" (see [DH1] or [M2]) one can extract a polynomial-like map $f^p : V' \to V$. Namely, let us consider a little bit bigger domain $V \supset \Omega$ bounded by arcs of four external rays close to γ_i, two arcs of circles going around the points a and $a' = -a$, and two arcs of E. Pulling V back by f^p, we obtain a domain $V' \subset\subset V$ such that the map $f^p : V' \to V$ is quadratic-like. This map considered up to conformal conjugacy is called the *DH renormalization* of f.

Let now $f : z \mapsto z^2 + c_0$ be a quadratic polynomial, $c_0 \in M$. If it is renormalizable then there is a homeomorphic copy $M_0 \ni c_0$ of the Mandelbrot set with the following properties (see [DH2, [D]). For $z \in M_0' = M_0 \backslash \{\textit{one point}\}$ the polynomial $P_c : z \mapsto z^2 + c$ is renormalizable. Moreover, there is the parameter analytic extension a_c of the periodic point a to a neighborhood of M_0' such that the above renormalization of P_c is associated to a_c. At the parameter value b removed from M_0 the periodic point a_c is becoming parabolic with multiplier one. This parameter value is called the *root* of M_0. The component H of M_0 corresponding to the component of M enclosed by the main cardioid "gives origin" for the copy M_0. Vice versa, any hyperbolic component H of the Mandelbrot set gives origin to a copy of M. In particular, the copies corresponding to the immediate renormalization are attached to the main cardioid.

We will see below that among all renormalizations there is the *first* one, which we denote Rf (see §3.4). This renormalization corresponds to a *maximal* copy of the Mandelbrot (that is a copy, which is not contained in any bigger copies except M itself). Let $\mathcal{M}$ denote the family of maximal Mandelbrot copies.

Given any sequence $M_0, M_1, \dots$ of maximal copies of M, there is an infinitely renormalizable quadratic polynomial P_b such that $c(R^m P_b) \in M_m$, $m = 0, 1, \dots$. Indeed, the sets

$$C_N = \{b : c(R^m P_b) \in M_m,\ m = 0, 1, \dots, N\}$$

form a nest of copies of M whose intersection consists of the desired parameter values.

We say that these infinitely renormalizable quadratics have the same combinatorics determined by the sequence $M_0, M_1, \dots$. The MLC problem for these parameter values is equivalent to the assertion that there is only one quadratic with a given combinatorics. In other words, the above copies C_N shrink to a point as $N \to \infty$.

2.6. Yoccoz puzzle. Let $f : U' \to U$ be a quadratic-like map with both fixed points α and β repelling. As usual, α denotes the dividing fixed point with rotation number $\rho(\alpha) = q/p$, $p > 1$. Let E be an equipotential sufficiently close to $K(f)$ (so that both E and fE are closed curves). Let $\mathcal{R}_\alpha$ denote the union of external rays landing at α. These rays cut the domain bounded by E into p closed topological disks $Y_i^{(0)}$, $i=0,\dots,p-1$, called *puzzle pieces of zero depth* (Figure 3). The main property of this partition is that $f\partial(\cup Y_i^{(0)})$ is outside of $\operatorname{int}(\cup Y_i^{(0)})$.

Let us now define *puzzle-pieces* $Y_i^{(n)}$ *of depth* n as the connected components of $f^{-n}Y_k^{(0)}$. They form a finite partition of the the neighborhood of $K(f)$ bounded by $f^{-n}E$. If the critical orbit does not land at α, then for every depth there is a single puzzle-piece containing the critical point. It is called *critical* and is labeled as $Y^{(n)} \equiv Y_0^{(n)}$.

Let $\mathcal{M}(f)$ denote the family of all puzzle pieces of f of all levels. It is *Markov* in the following sense:

(i) Any two puzzle pieces are either nested or have disjoint interiors. In the former case the puzzle-piece of bigger depth is contained in the one of smaller depth.

(ii) The image of any puzzle-piece $Y_i^{(n)}$ of depth $n>0$ is a puzzle piece $Y_k^{(n-1)}$ of the previous depth. Moreover, $f: Y_i^{(n)} \to Y_k^{(n-1)}$ is a two-to-one branched covering or a conformal isomorphism depending on whether $Y_i^{(n)}$ is critical or not.

We say that $f^kY_i^{(n)}$ *l-to-one covers* a union of pieces $\cup_{m,j}Y_j^{(m)}$ if $f^k|\operatorname{int} Y_i^{(n)}$ is l-to-one covering map onto its image, and

$$f^k|Y_i^{(n)} \cap J(f) = \bigcup_{m,j} Y_j^{(m)} \cap J(f).$$

In this case $\cup Y_j^{(m)}$ is obtained from $f^k|Y_i^{(n)}$ by cutting with appropriate equipotentials.

On depth 1 we have $2p-1$ puzzle pieces: one central $Y^{(1)}$, $p-1$ non-central $Y_i^{(1)}$ attached to the fixed point α (cuts of $Y_i^{(0)}$ by the equipotential $f^{-1}E$), and $p-1$ symmetric ones Z_i^1 attached to α'. Moreover, $fY^{(1)}$ two-to-one covers $Y_1^{(1)}$, $Y_i^{(1)}$ univalently covers $Y_{i+1}^{(1)}$, $i=1,\dots,p-2$, and $fY_{p-1}^{(1)}$ univalently covers $Y^{(1)} \cup_i Z_i^{(1)}$. Thus $f^pY^{(1)}$ cut by $f^{-1}E$ is the union of $Y^{(1)}$ and $Z_i^{(1)}$ (Figure 3).

Theorem 2.6. *(Yoccoz, 1990). Assume that both fixed points of a polynomial-like map f are repelling, and that f is DH non-renormalizable. Then the following divergence property holds:*

$$\sum_{n=0}^{\infty} \operatorname{mod}(Y^{(n)} \backslash Y^{(n+1)}) = \infty.$$

Hence $\operatorname{diam} Y^{(n)} \to 0$ *as* $n \to \infty$.

Corollary 2.7. *Under the circumstances of the above theorem the Julia set $J(f)$ is locally connected.*

The reader can consult [H], [M] or [L2] for a proof (or read §5 of this paper).

The Yoccoz puzzle provides us with the Markov family of puzzle pieces to play with. There are several different ways to do this: by means of the Branner-Hubbard tableaux [BH], or by means of the Yoccoz τ-function (unpublished), or by means of the principal nest and generalized renormalization, as will be described below (compare [L2], [L3]).

§3. Principal nest and generalized renormalization.

In what follows we will always assume that both fixed points of the DH quadratic-like maps under consideration are repelling.

3.1 Principal nest. Given a set $W = \mathrm{cl}\,(\mathrm{int}\, W)$ and a point z such that $f^l z \in \mathrm{int}\, W$, let us define the *pull-back* of W along the $\mathrm{orb}_l z$ as the chain of sets $W_0 = W, W_{-1} \ni f^{n-1}z, \dots, W_{-l} \ni z$ such that W_{-k} is the closure of the component of $f^{-1}(\mathrm{int}\, W_{-k+1})$ containing $f^{l-k}z$. In particular if $z \in \mathrm{int}\, W$ and $l > 0$ is the moment of first return of $\mathrm{orb}\, z$ back to $\mathrm{int}\, W$ we will refer to the *pull-backs corresponding to the first return of* $\mathrm{orb}\, z$ to $\mathrm{int}\, W$.

Let us consider the puzzle pieces of depth 1 as described above: $Y^{(1)}, Y_i^{(1)}$ and $Z_i^{(1)}$, $i = 1, \dots p-1$ (Figure 3). If $z \in Y^{(1)}$ then $f^p z$ is either in $Y^{(1)}$ or in one of $Z_i^{(1)}$. Hence either $f^{pk}0 \in Y^{(1)}$ for all $k = 0, 1, \dots$, or there is the smallest $t > 0$ and a ν such that $f^{tp}0 \in Z_\nu^{(1)}$. Thus either f is immediately DH renormalizable, or the critical point escapes through one of the non-critical pieces, attached to α'.

In the immediately renormalizable case the principal nest of puzzle pieces consists of just single puzzle piece Y^0 (which is not too informative). In the escaping case we will construct *the principal nest*

$$Y^0 \supset V^1 \supset V^2 \supset \dots \tag{3-0}$$

in the following way. Let t be the first moment when $f^{tp}0 \in Z_i^{(1)}$. Then let $V^0 \ni 0$ be the pull-back of $Z_i^{(1)}$ along the $\mathrm{orb}_{tp}0$. Further, let us define V^{n+1} as the pull-back of V^n corresponding to the first return of the critical point 0 back to $\mathrm{int}\, V^n$. Of course it may happen that the critical point never returns back to $\mathrm{int}\, V^n$. Then we stop, and the principal nest turns out to be finite. This case is called *combinatorially non-recurrent.* If the critical point is *recurrent* in the usual sense, that is $\omega(0) \ni 0$, it is also combinatorially recurrent, and the principal nest is infinite.

Let $l = l(n)$ be the first return time of the critical point back to $\mathrm{int}\, V^{n-1}$. Then the map $g_n = f^{l(n)} : V^n \to V^{n-1}$ is two-to-one branched covering. Indeed, by the Markov property of the puzzle, $f^k V^n \cap \mathrm{int}\, V^{n-1} = \emptyset$ for $k = 1, \dots, l-1$, so that the maps $f : V^k \to V^{k-1}$ are univalent for those k's.

Let us call return to level $n-1$ *central* if $g_n 0 \in V^n$. In other words $l(n) = l(n+1)$. Let us say that a sequence $n, n+1, n+N-1$ (with $N \geq 1$) of levels (or corresponding puzzle pieces) of the principal nest form a *(central) cascade* if the returns to all levels $n, \dots, n+N-2$ are central, while the return to level $n+N-1$ is non-central. In this case

$$g_{n+k}|V^{n+k} = g_{n+1}|V^{n+k}, \quad k = 1, \dots, N.$$

and $g_{n+1}0 \in V^{n+N-1} \backslash V^{n+N}$. Thus all maps $g_{n+1}, \dots, g_{n+N}$ are the same as quadratic-like. We call the number N of levels in the cascade its *length.* Note that a cascade of length 1 consists of a single non-central level. Let us call the cascade *maximal* if the return to level $n-1$ is non-central. Clearly the whole principal nest is the union of disjoint maximal cascades. The number of such cascades is called the *height* $\chi(f)$ of f. In other words, $\chi(f)$ is the number of different quadratic-like maps among the g_n's. (If f is immediately renormalizable set $\chi(f) = -1$.)

Remark. Given a quadratic polynomial $f : z \mapsto z^2 + c$, the principal nest determines a specific way to approximate c by superattracting parameter values. Namely, one should perturb c in such a way that the critical point becomes fixed under g_n, while the combinatorics on the preceeding levels keeps unchanged. The number of points in this approximating sequence is equal to the height $\chi(f)$. This resembles "internal addresses" of Dierk Schleicher [Sch] but turns out to be different.

3.2. Initial Markov partition. Let P_i be a finite or countable family of topological disc with disjoint interiors, and $g : \cup P_i \to \mathbf{C}$ be a map such that the restrictions $g|P_i$ are branched coverings onto their images. This map is called *Markov* if $gP_i \supset P_j$ whenever $\operatorname{int} gP_i \cap \operatorname{int} P_j \neq \emptyset$. Let us call it a *standard Markov map* if all restrictions $g|P_i$ are one-to-one onto their images.

A Markov map is called *Bernoulli (with range D)* if $gP_i \supset D \supset \cup P_j$ for all i. Similarly we can define a standard Bernoulli map.

We know that $f^p Y^{(1)}$ two-to-one covers $Y^{(1)}$ itself and the puzzle pieces $Z_i^{(1)}$ attached to α'. If $f^p 0 \in Y^{(1)}$ (central return) then the pull back of $Y^{(1)}$ by this map is the critical piece $Y^{(1+p)}$, while each $Z_i^{(1)}$ has two univalent pull-backs $Z_j^{(1+p)}$ (we count them by j in an arbitrary way) (see Figure 3).

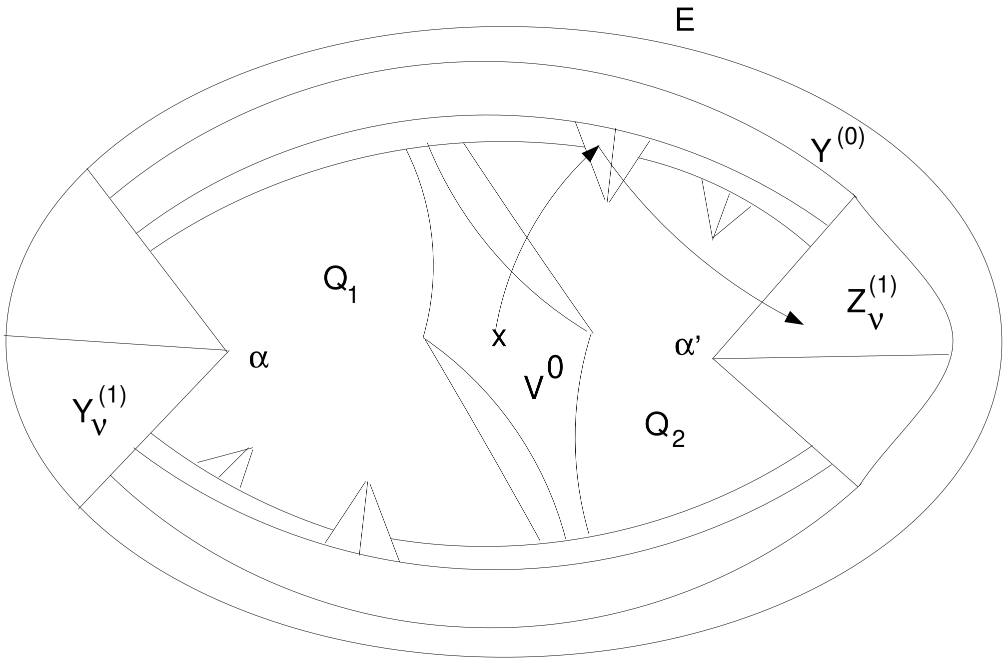

Figure 3: Initial partition ($p = 3$, $t = 2$).

Now, $f^p Y^{(1+p)}$ two-to-one covers all these puzzle pieces. If we again have a central return, that is $f^p 0 \in Y^{(1+p)}$, then the pull-back will give us one critical piece $Y^{(1+2p)}$, and $4(p-1)$ off-critical $Z_j^{(1+2p)}$.

Repeating this procedure t times (where $f^{tp} 0 \in Z_i^{(1)}$), we obtain the initial central nest

$$Y^{(1)} \supset Y^{(1+p)} \supset \ldots \supset Y^{(1+tp)}, \tag{3-1}$$

and a family of non-critical puzzle pieces $Z_j^{(1+sp)}$, $0 \le s \le k-1$. Moreover

$$f^p 0 \in Z_\nu^{(1+tp)} \tag{3-2}$$

for some ν.

Let us say that a set D is *partitioned by pieces* W_i rel $F(f)$ if the $\operatorname{int} W_i$ are disjoint, and $D \cap J(f) = \cup W_i \cap J(f)$.

Thus we have partitioned $Y^{(0)}$ rel $F(f)$ by the pieces $Z_i^{(1+sp)}$, $0 \le s \le t$, and $Y^{(1+tp)}$. Let us look closer at this last piece. Its image under f^p two-to-one covers all above puzzle pieces of depth $1+tp$. The pull-back of $Z_\nu^{(1+tp)}$ from (3-2) gives us exactly $V^0 \ni 0$, the first puzzle-piece in the principal nest. The pull-backs of other pieces $Z_j^{(1+tp)}$ give some non-critical pieces $Z_j^{(1+(t+1)p)}$. Finally, we have two univalent pull-backs Q_1 and Q_2 of $Y^{(1+tp)}$. Altogether these pieces give a partition of $Y^{(1+tp)}$ rel $F(f)$.

To understand how the critical point returns back to V^0 we need to partition further $Q_1 \cup Q_2$. To this end let us iterate the standard Bernoulli map $f^p|Q_1 \cup Q_2$ with range $Q_1 \cup Q_2 \cup V^0 \cup Z_j^{(1+(t+1)p)}$. So take a point $z \in Q_1 \cup Q_2$ and push it forward by iterated f^p until it escapes $Q_1 \cup Q_2$ (or iterate forever if it does not escape). It can escape through the piece V^0 or through a piece $Z_j^{(1+(t+1)p)}$. In any case pull the corresponding piece back to this point. In such a way we will obtain the partition

$$Q_1 \cup Q_2 = \bigcup_{k>0} \bigcup_i X_i^{kp} \bigcup_{k>t+1} \bigcup_j Z_j^{(1+kp)} \cup K \ \text{ rel } F(f),$$

where X_i^k denote the pull-backs of V^0 under f^{kp} (for $k=0$ we have just one piece $X_0^0 \equiv V^0$), Z_j^{1+kp} denote the pull-backs of $Z_j^{(1+(t+1)p)}$ under $f^{(k-t-1)p}$, and K denote the residual set of non-escaping points.

Altogether we have constructed the *initial Markov partition*:

$$Y^{(0)} \backslash K = V^0 \bigcup_{k>0} \bigcup_i X_i^{kp} \bigcup_{k>0} \bigcup_j Z_j^{(1+kp)} \text{ rel } F(f). \tag{3-3}$$

It is convenient (to restrict the number of iterates in the following considerations) to pass here to the following Markov map:

$$G : \bigcup_{k,i} X_i^k \bigcup_{k,j} Z_j^{(1+kp)} \to \mathbf{C} \tag{3-4}.$$

Observe that for any j there is an i such that $f^{sp} Z_j^{(1+sp)}$ univalently covers $Z_i^{(1)}$. Moreover $f^{p-i} Z_i^{(1)}$ univalently covers $Y^{(0)}$. Let us set $G|Z_j^{1+sp} = f^{ps+(p-i)}$. The image of each piece $Z_j^{(1+sp)}$ under this map univalently covers $Y^{(0)}$. Similarly let us set $G|V^0 = f^{tp+p-i}$, where $f^{tp}0 \in Z_i^{(1)}$. The image of this piece two-to-one covers $Y^{(0)}$. Finally $G|W_i^k = f^{kp}$ for $k>0$. These pieces are univalently mapped onto V^0.

3.3. Non-degenerate annulus. Yoccoz has shown that if f is non-renormalizable then in the nest $Y^{(1)} \supset Y^{(2)} \dots$ there is a non-degenerate annulus $Y^{(n)} \setminus Y^{(n+1)}$. However the modulus of this annulus is not under control. We will construct a different non-degenerate annulus whose modulus we will be able to control.

Let $A^n = V^n \setminus V^{n+1}$.

Proposition 3.1. *Let f be a DH quadratic-like map which is not immediately DH renormalizable. Then all annuli A^n are non-degenerate.*

Proof. Observe first that V^0 is strictly inside $Y^{(0)}$, that is, the annulus $Y^{(0)} \setminus V^0$ is non-degenerate. Indeed, V^0 is the pull-back of $Z_i^{(1)}$ which is strictly inside $Y^{(0)}$. As the iterates of $\partial Y^{(0)}$ stay outside $\operatorname{int} Y^{(0)}$, V^0 may not touch $\partial Y^{(0)}$.

For the same reason all other pieces $Z_j^{(1+kp)}$ and X_i^k of the initial Markov partition are strictly inside $Y^{(0)}$ as well.

Let us consider the orbit of the critical point 0 under iterates of the map G from (3-4) until it returns back to V^0. It first goes through the Z-pieces of the initial Markov partition, then at some moment $l \geq 1$ lands at a X_i^s, and the latest at the next moment returns back to V^0.

Since the map $G : V^0 \cup Z_j^{(1+kp)} \to \mathbf{C}$ is Bernoulli with range $Y^{(0)}$, there is a piece $P \subset V^0$, such that $G^l P$ two-to-one covers $Y^{(0)}$. Clearly V^1 is the pull-back of X_i^s by G^l. Since $X_i^s \subset\subset Y^{(0)}$, we conclude that $V^1 \subset\subset V^0$.

Now it is easy to see that all annuli A^n are non-degenerate as well. Indeed, it follows that the orbit of ∂V^1 stays away from V^1. Hence V^2 cannot touch ∂V^1, for otherwise there would be a point on ∂V^1 which returns back to V^1. So A^2 is no-degenerate. Now we can proceed inductively. $\square$

3.4. DH renormalization and central cascades. In this section quadratic-like maps and renormalization are understood in the sense of Douady and Hubbard.

Proposition 3.2. *A quadratic-like map is renormalizable if and only if it is either immediately renormalizable, or the principal nest $V^0 \supset V^1 \supset \dots$ ends with an infinite cascade of central returns. Thus the height $\chi(f)$ is finite if and only if f is renormalizable.*

Proof. Let the principle nest ends with an infinite central cascade $V^{m-1} \supset V^m \supset \dots$ Then $l_m = l_{m+1} = \dots = l$, and $g \equiv g_m = f^l | V^m$. Hence $\cap V^k$ consists of all points which never escape V^m under iterates of g, that is, $\cap V^k = K(g)$. Since $0 \in \cap V^k$, $K(g)$ is connected, and $g : V^m \to V^{m-1}$ is quadratic-like (take into account that $V^m \subset\subset V^{m-1}$ by Proposition 3.1).

Take now the non-dividing fixed point b of g. Let us show that b is dividing for the big Julia set $K(f)$. To this end let us consider the configuration of the *full* external rays whose segments bound $V \equiv V^m$. They divide the plane into the central component containing V, and the family $\mathcal{S} = \{S_i\}_{i=1}^t$ of sectors. As g maps V onto a bigger piece, $g(\partial S_i) \subset S_{\sigma(i)}$ for any sector $S_i \in \mathcal{S}$.

Let $i_1, \dots, i_r$ be a set of indeces which are cyclically permuted by the map $\sigma : \{1, \dots, t\} \to \{1, \dots, t\}$. Then there is a sequence of sectors $S_{i_1} \equiv T_1, \dots, T_r, \dots$ with the following properties:

- Each T_k is bounded by two external rays, and does not contain 0;

- The angle of each T_k at infinity is smaller than π ;
- $T_k \supset S_{i_l}$ with $l \equiv k \bmod r$;
- $g(\partial T_{k+1}) = \partial T_k$, $k = 1, 2, \dots$
- $T_k \subset T_{k+r}$, while $T_{k+1}, \dots, T_{k+r}$ are pairwise disjoint.

Then for any k we have a nest of sectors, $T_k \subset T_{k+r} \subset T_{k+2r} \dots$, converging to a sector $\tilde{T}_k$. These sectors also satisfy all the above properties, and moreover $\tilde{T}_k = \tilde{T}_{k+r}$. Hence g cyclically permut the vertices of these sectors, which therefore form a cycle of period r . Moreover, the points of this cycle are clearly dividing.

Let us finally note that actually r must be equal to 1, so that this cycle actually coincides with the fixed point b . Indeed, the opposite situation would contradict Proposition 2.2 (ii).

So we have a dividing periodic point b . Let $\Omega' \subset \Omega$ be the corresponding domains constructed in §2.5. Then $\Omega' \supset K(g)$ and hence $\Omega' \ni 0$. It follows that $g : \Omega' \to \Omega$ is a double covering. Moreover, $g^n 0 \in K(g) \subset \Omega'$, $n = 0, 1, \dots$. Thus f is renormalizable.

Assume now that f is renormalizable. Let a be the corresponding periodic point, and Ω' be the region bounded by rays landing at a and arcs of an equipotential, such that $Rf = f^l : \Omega' \to \Omega$ is a double covering.

Then the fixed point α may not lie in $\operatorname{int} \Omega'$, for otherwise $\operatorname{int} f\Omega'$ would intersect $\operatorname{int} \Omega'$. Hence α does not cut the filled Julia set $K(Rf)$. But then the preimages of α don't cut $K(Rf)$ either. Hence given a puzzle-piece $Y_i^{(n)}$, either $K(Rf)$ is contained in $Y_i^{(n)}$, or $K(Rf) \cap \operatorname{int} Y_i^{(n)} = \emptyset$. In particular $V^m \supset K(Rf)$. But then $f^l 0 \in V^m$ for all m , so that the first return times to V^m are uniformly bounded. But then this nest must end up with a central cascade. □

The above discussion shows that there is a well-defined first renormalization Rf with the biggest Julia set, and it can be constructed in the following way. If f is immediately renormalizable, then Rf is obtained by thickening $Y^{(1)} \to Y^{(0)}$. Otherwise the principal nest ends up with the infinite central cascade $V^{m-1} \supset V^m \supset \dots$. Then $Rf = g_m : V^m \to V^{m-1}$.

The internal class $c(Rf)$ of the first renormalization belongs to a maximal copy M_0 of the Mandelbrot set, that is, a copy which is not contained in any other one except the whole M .

3.5. Return maps and Koebe space. Let f be a DH quadratic-like map, and let $V \in \mathcal{M}(f)$ be a puzzle piece.

Lemma 3.3. *Let z be a point whose orbit passes through $\operatorname{int} V$. Let l be the first positive moment of time for which $f^l z \in \operatorname{int} V$. Let $U \ni z$ be the puzzle piece mapped onto V by f^l . Then $f^l : U \to V$ is either a univalent map or two-to-one branched covering depending on whether U is non-critical or otherwise.*

Proof. Let $U_k = f^k U$, $k = 0, 1, \dots l$. Since $f^k z \notin \operatorname{int} V$ for $0 < k < l$, by the Markov property of the puzzle, $U_k \cap \operatorname{int} V = \emptyset$ for those k 's. Hence $f : U_k \to U_{k+1}$ is univalent for $k = 1, \dots, l-1$, and the conclusion follows. □

Let $z \in \operatorname{int} V$ be a point which returns back to $\operatorname{int} V$, and let $l > 0$ be the first return time. Then there is a puzzle piece $V(z) \subset V$ containing z such that $f^l V(z) = V$.

It follows that the first return map $A_V f$ to $\operatorname{int} V$ is defined on the union of disjoint open puzzle pieces $\operatorname{int} V_i$. Moreover, if

$$f^m \partial V \cap V = \emptyset, \ m = 1, 2, \dots \tag{3-4}$$

then it is easy to see that the closed pieces V_i are pairwise disjoint and are contained in $\operatorname{int} V$. Indeed, otherwise there would be a boundary point $\zeta \in \partial V$ whose orbit would return back to V, despite (3-4).

Somewhat non-rigorously, we will call the map

$$A_V f : \bigcup V_i \to V \tag{3-5}$$

the first return map to V. (Note: it perhaps may happen that a point $z \in \partial V$ returns back to V but does not belong to $\cup V_i$.) Let V_0 denote the critical ("central") puzzle piece (provided the critical point returns back to V). Now Lemma 3.3 immediately implies:

Lemma 3.4. *The first return map f_V univalently maps all non-critical pieces V_i onto V, and maps the critical piece V_0 onto V as a double branched covering.*

Thus the first return map $A_V f$ is Bernoulli, and is standard Bernoulli on $\bigcup_{i \neq 0} V_i$.

Let us now improve Lemma 3.3.

Lemma 3.5. *Let z be a point whose orbit passes through the central domain $\operatorname{int} V_0$ of the first retun map (3-5), and $l \geq 0$ be the first moment when $f^l z \in V_0$. Then there is a puzzle piece $\Omega \ni z$ mapped univalently by f^l onto V.*

Proof. Let s be the first moment when $f^s z \in V$. Then $f^l z = f_V^k(f^s z)$ for some $k \geq 0$. Moreover, $(A_V f)^r(f^s z) \notin \operatorname{int} V_0$ for $r < k$.

Since the return map is standard Bernoulli outside of the central piece, there is a piece $X \subset V$ containing $f^s z$ which is univalently mapped by $(A_V f)^k$ onto V. On the other hand, by Lemma 3.3 there is a domain $D \ni z$ which is univalently mapped by f^s onto V. Hence the domain $D \cap f^{-s} X$ is univalently mapped by f^l onto V. □

Let us now consider the principal nest of f:

$$Y^0 \supset V^0 \supset V^1 \supset \dots .$$

Let

$$g_n : \cup V_i^n \to V^{n-1} \tag{3-6}$$

be the first return map to V^{n-1}, where $V_0^n \equiv V^n \ni 0$. We will call them the *principal return maps.* We will also let $g_0 \equiv f$.

Let $\Phi : z \mapsto z^2$ be the quadratic map. Since $f : U' \to U$ is a double covering with the critical point at 0, it can be decomposed as $h \circ \Phi$ where h is a univalent map with range U.

Corollary 3.6. *For $n \geq 1$ the pieces V_i^n are pairwise disjoint, and the annuli $V^{n-1} \backslash V_i^n$ are non-degenerate. For $n \geq 2$ the map $g_n | V_i^n$ can be decomposed as $h_{n,i} \circ \Phi$ where $h_{n,i}$ is a univalent map with range V^{n-2}.*

Proof. As $V^n \subset \operatorname{int} V^{n-1}$ (Proposition 3.1), and

$$f^m(\partial V^n) \cap \operatorname{int} V^{n-1} = \emptyset,$$

condition (3-4) is satisfied here, and the first statement follows.

Take a piece V_i^n, and let $g_n|V_i^n = f^l$. By Lemma 3.5 the map $f^{l-1} : (fV_i^n) \to V^{n-1}$ can be extended to a univalent map with range V^{n-2}, and the second statement also follows. □

3.6. Bernoulli scheme associated to a central cascade. In the case of a central cascade we need a more precise analysis of the Koebe space. Let us consider a central cascade $\mathcal{C} \equiv \mathcal{C}^{m+N}$:

$$V^m \supset V^{m+1} \supset \ldots \supset V^{m+N-1} \supset V^{m+N}, \tag{3-7}$$

where $g_{m+1}0 \in V^{m+N-1} \backslash V^{m+N}$. Set $g = g_{m+1}|V^{m+1}$. Then $g : V^k \to V^{k-1}$ is a double branched covering, $k = m+1, \ldots m+N$.

Let us consider the first return map $g_{m+1} : V_i^{m+1} \to V^m$ as in (3-6). Let us pull the pieces V_i^{m+1} back to the annuli $A^k = V^{k-1} \backslash V^k$ by iterates of g, $k = m+N, \ldots, m+1$. We obtain a family $\mathcal{W}(\mathcal{C})$ of pieces W_j^k. By construction, $W_j^k \subset A^k$ and g^{k-m+1} univalently maps W_j^k onto some $V_i^{m+1} \equiv W_i^{m+1}$.

Let us define a standard Bernoulli map $G \equiv G^{m+N}$:

$$G : \bigcup_{\mathcal{W}(\mathcal{C})} W_j^k \to V^m \tag{3-8}$$

as follows: $G|W_j^k = f_{m+1} \circ g^{k-m+1}$ (see Figure 4).

Remark. This Bernoulli map is similar to the initial Bernoulli map constructed in §3.5. Actually in the initial construction we deal with the central cascade (3-1) with degenerate annuli.

Lemma 3.7. *Let us consider the central cascade (3-7). Let z be a point whose orbit passes through* $\operatorname{int} V^{m+N}$*, and l be the first moment for which $f^l z \in V^{m+N}$. Then there is a piece $\Omega \ni z$ which is univalently mapped by f^l onto V^m.*

Proof. Let s be the first moment for which $f^s z \in V^m$. Then $f^l z = G^k(f^s z)$ where G is the Bernoulli map (3-8). Now repeat the argument of Lemma 3.5 just using G instead of the first return map. □

Corollary 3.8. *Let us consider the central cascade (3-7). Then the map $g_{m+N+1} : V^{m+N+1} \to V^{m+N}$ can be represented as $h_{m+N+1}(z^2)$ where h_{m+N+1} is a univalent map with range V^m.*

Proof. Repeat the argument for Corollary 3.6 using Lemma 3.7 instead of 3.5. □

3.7. Generalized polynomial-like maps and renormalization. Let $\{U_i\}$ be a finite or countable family of topological discs with disjoint interiors strictly contained in a topological disk U. We call a map $g : \cup U_i \to U$ a *(generalized) polynomial-like map* if $g : U_i \to U$ is a branched covering of finite degree which is univalent on all but finitely many U_i.

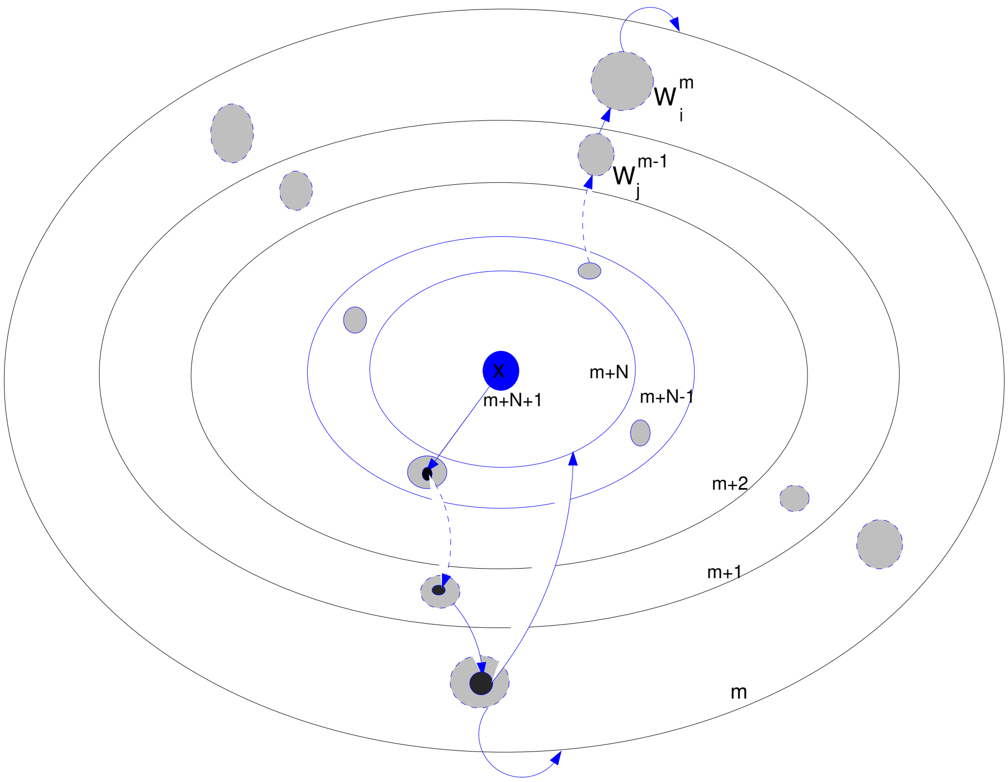

Figure 4: Bernoulli scheme associated to a central cascade.

Then we can define the filled Julia set $K(g)$ as the set of all non-escaping points, and the Julia set $J(g)$ as its boundary. The DH polynomial-like maps correspond to the case of a single disk U_0.

Let us say that a polynomial-like map g is *of finite type* if its domain consists of finitely many disks U_i.

Generalized Straightening Theorem. *Any generalized polynomial-like map of finite type is qc conjugate to a polynomial with the same number of non-escaping critical points.*

Lemma 3.9. *A generalized polynomial-like map with non-escaping critical point has a connected Julia set/(filled Julia set) if and only if it is DH polynomial-like.*

Let us call a (generalized) polynomial-like map a *(generalized) quadratic-like* if it has a single (and non-degenerate) critical point. In such a case we will always assume that 0 is the critical point, and count the discs U_i in such a way that $U_0 \ni 0$. In what follows we will deal exclusively with quadratic-like maps, namely with the principal sequence g_n of the first return maps (3-6).

Given a V_j^{n+1}, let l be its first return time back to V^n, that is, $g_{n+1}|V_j^{n+1} = g_n^l$. Then

$$g_n^k V_j^{n+1} \subset V_{i(k)}^n,\ k = 0, 1, \ldots, l,$$

with $i(0) = i(s) = 0$. Moreover, $g_n^k V_j \subset\subset V_{i(k)}^n$ for $k < s$. The sequence $0 =$

$i(0), i(1), \ldots, i(s) = 0$ is called the *itinerary* of V_j^{n+1} through the domains of previous level.

Philosophically the dynamical renormalization is the first return map to an appropriate piece of the phase space considered up to rescaling. In our setting let us define the n-fold *generalized renormalization* $T^n f$ of f as the first return map g_n restricted to the union of puzzle pieces V_i^n meeting the critical set $\omega(0)$, and considered up to rescaling. In the most interesting situations these maps are of finite type (compare [L3]):

Lemma 3.10. *If f is a DH renormalizable quadratic-like map, then all maps $T^n f$ are of finite type.*

Proof. In the tail of the principal nest the maps g_n are DH quadratic-like, and their domains consist just of one component. So we should take care only of the initial piece of the cascade.

Let us take the renormalization $Rf = f^l : V^{t+1} \to V^t$ with $t \geq n$. Since 0 is non-escaping under iterates of Rf, we have the following property: the first return time of any point $f^k 0$ back to V^{t+1} is at most l. All the more, the return time to any bigger domain $V^n \supset V^t$ is bounded by l. Hence the components of $f^{-t} V^n$, $t = 0, 1, \ldots, l-1$, cover the whole postcritical set. For sure there is only finitely many such components. But the domain of $T^n f$ is the union of some of these components (which are inside V^n). □

3.8. Return graph. Let $\mathcal{I}^n$ be the family of puzzle pieces V_i^n intersecting $\omega(0)$, that is, the pieces in the domain of the generalized renormalization

$$T^n f : \bigcup_{\mathcal{I}^n} V_i^n \to V^{n-1}.$$

Let us consider a graded graph Υ_f whose vertices of level n are the pieces $V_j^n \in \mathcal{I}^n$, $n = 0, 1, \ldots$. Let us take a vertex $V_j^n \in \mathcal{I}^n$, and let $i(1), \ldots, i(t) = 0$ be its itinerary through the pieces of the previous level under the iterates of g_n (see the previous section). Then for $n \geq 1$ join V_j^{n+1} with V_i^n by k edges, provided the symbol i appears in the above itinerary k times. This means that the piece V_j^{n+1} under iterates of g_n passes through V_i^n k times before the first return back to V^{n-1}. As to the top level, let us join each V_j^1 with V^0 by the number of edges equal to the first return time of V_j^1 back to V^0 under iterates of $f = g_0$.

Remark. A similar graph in the real one-dimensional setting was introduced by Marco Martens [Ma]. The above graph is not exactly the same as Marten's graph, as the latter is related to the iterates of f itself rather than the renormalized maps.

Note that for any vertex V_j^{n+1} with $n > 0$ there is exactly one edge joining it to the critical vertex V_0^n of the previous level. Note also that by Lemma 3.10 in the DH renormalizable case the number of vertices on a given level is finite. In any case there are clearly only finitely many edges leading from a V_j^{n+1} to the previous level.

By a *path* in the graph Υ_f we mean a connected sequence of edges such that no two of them join the same two levels, up to reversing the order of the sequence. So we don't endow the paths with orientation, and can go along them either strictly upwards or strictly downwards.

Diverse combinatorial data can be easily read off this graph. For example, given $n \geq m$, the number of paths joining V_j^{n+1} to V_i^m is equal to the number of times which the g_m-orbit of V_j^{n+1} passes through V_i^m before the first return back to V^n. Hence the return time of V_j^{n+1} back to V^n under iterates of g_m is equal to the total number of paths in Υ_f leading from V_j^{n+1} up to level m. Let $g_m - time(V_j^{n+1})$ denote this return time.

Let now the map f be DH renormalizable, and t be a renormalization level in the principal nest, that is, $g_{t+1} : V^{t+1} \to V^t$ is a quadratic-like map with non-escaping critical point. Then there is a single vertex V^{t+1} at level $t+1$, and below it the return graph is just the "vertical path" through the critical vertices. Let $per(f) \equiv g_0 - time(V^{t+1})$ denote *the renormalization period*, that is, the return time of V^{t+1} back to V^t under iterates of the original map $f = g_0$.

By the above discussion, $per(f)$ is equal to the total number of paths in the graph Υ_f (joining the top vertex V^0 to the bottom vertex V^{t+1}). It follows that the $per(f)$ is bounded if and only if the DH-level t is bounded, and all return times $g_m - time(V_i^{m+1})$ are bounded for $1 \leq m \leq t$ and any i. For example, the "if" statement means: If $t \leq T$ and $g_m - time(V_i^{m+1}) \leq R$, then $per(f) \leq P(T, R)$. Indeed, in this case the total number of paths in the graph is bounded by R^T.

Let us now take a closer look at the central cascades. Let us consider a central cascade (3-7) and a non-pre-critical piece V_j^{m+N+1}, that is $g_{m+N} V_j^{m+N+1} \neq V_0^{m+N}$. Then let us denote by $G_{m+N} - time(V_j^{m+N+1})$ the first return time of $g_{m+N} V_i^{m+N+1}$ back to V^{m+N} under the iterates of the map G_{m+N} from (3-8).

This time can be expressed in terms of the following *reduced graph* Υ_f^r. This graph is obtained from the Υ_f by removing from all central cascades (3-7) the edges leading from the non-critical pieces V_j^{k+1}, $j \neq 0$, to the critical piece V_0^k, $k = m+N-1, \ldots, m$. Then $G_{m+N} - time(V_j^{m+N+1})$ is equal to the number of paths in the reduced graph leading from V_j^{m+N+1} up to level m.

Let us define the *reduced period* $per_r(f)$ as the total number of paths in the reduced graph. This means that we don't count the moments of time when the orbit goes through the intermediate levels of central cascades. Clearly *the reduced period of f is bounded if and only if the height $\chi(f)$ and all $G - times$ are bounded.*

Let us finally define one more combinatorial notion, the rank. Let V_j^{n+1} and V_i^n be two Υ_f-adjacent puzzle pieces, and let γ denote an edge joining them. Let $D^n \supset V_i^n$ be a puzzle piece contained in V^{n-1}. Consider the first moment t for which $g_n^t V_j^{n+1} \subset V_i^n$. The piece $D^{n+1} \supset V_j^{n+1}$ in V^n such that $g_n^t D^{n+1} = D^n$ will be called the *pull-back of D^n along the edge γ*. More generally, let us define the *pull-back of D^n along a path γ* leading from $V_i^n \subset D^n$ downwards by composing the pull-backs along the edges.

Given a piece V_k^n, let $rank V_k^n$ be the number of non-central levels in the shortest path leading from V_k^n down to a critical piece. For a piece $D^n \subset V^{n-1}$ as above, let $rank(D^n)$ denote the minimum of ranks of puzzle pieces V_k^n contained in D^n. Note that $rank(D^n) = 0$ iff D^n is critical.

By a path through D^n we will mean a path through a piece $V_i^n \subset D^n$.

Lemma 3.11. *Let γ be the shortest path leading from D^n down to a central piece*

V_0^{n+t}, *and let* D^{n+t} *be the pull-back of* D^n *along this path. Then*

$$V^{n+t} \subset D^{n+t} \subset V^{n+t-1},$$

and the map $D^{n+t} \to D^n$ *is a double branched covering.*

Proof. Follows from the definitions. □

3.9. Full principal nest. Let f be a DH renormalizable, but not immediately, quadratic-like map. Then its principal nest

$$Y^{(0,0)} \supset V^{0,0} \supset V^{0,1} \supset \ldots \supset V^{0,t(0)} \supset V^{0,t(0)+1} \supset \ldots$$

ends up with an infinite cascade of central returns (we mark this nest with two labels for the reason which will become clear in a moment). Let us select a level $t(0)$ of this cascade, so that the return map $Rf = g_{0,t(0)+1} : V^{0,t(0)+1} \to V^{0,t(0)}$ is DH quadratic-like. We will call such a level DH. (The particular choice of DH renormalizable levels in what follows will depend on the geometry).

If Rf is not immediately DH renormalizable, let us cut the puzzle piece $V^{0,t(0)+1}$ by the external rays, and construct its short principal nest:

$$Y^{(1,0)} \supset V^{1,0} \supset V^{1,1} \supset \ldots \supset V^{1,t(1)} \supset V^{1,t(1)+1} \supset \ldots$$

If Rf is DH renormalizable, then this nest also ends up with an infinite central cascade. Then select a DH level $t(1)+1$, and pass to the next short nest.

If f is infinitely DH renormalizable but non of the renormalizations are immediate, then in such a way we construct the *full principal nest*

$$\begin{gathered} Y^{(0,0)} \supset V^{0,0} \supset V^{0,1} \supset \ldots \supset V^{0,t(0)} \supset V^{0,t(0)+1} \supset \\ Y^{(1,0)} \supset V^{1,0} \supset V^{1,1} \supset \ldots \supset V^{1,t(1)} \supset V^{1,t(1)+1} \supset \ldots . \\ Y^{(m,0)} \supset V^{m,0} \supset V^{m,1} \supset \ldots \supset V^{m,t(m)} \supset V^{m,t(m)+1} \supset \ldots \end{gathered} \tag{3-9}$$

Here $Y^{(m,0)}$ is the first critical Yoccoz puzzle piece for the m-fold DH renormalization $R^m f$, while the pieces $V^{m,n}$ form the corresponding short principal nest. Moreover, for $m > 1$, $Y^{(m,0)}$ is obtained by cutting $V^{m-1,t(m-1)+1}$ with the external rays of $R^m f : V^{m-1,t(m-1)+1} \to V^{m-1,t(m-1)}$.

The annuli $A^{m,n} = V^{m,n-1} \setminus V^{m,n}$ will be called the *principal annuli.*

3.10. Big type: special families of Mandelbrot copies. Assume that we associated to any quadratic-like map its "combinatorial type" $\tau(f)$, which depends only on the hybrid class $c(f)$ and is constant over any maximal copy of the Mandelbrot set. (Keep in mind the height function $\chi(f)$ or the period $per(f)$.) Thus we can use the notation $\tau(M')$.

Let $\mathcal{S} \subset \mathcal{M}$ be a family of maximal copies of the Mandelbrot set. Let us call it τ*-special* if it satisfies the following property: For any truncated secondary limb L there is a τ_L such that $\mathcal{M}$ contains all copies $M' \subset L$ of the Mandelbrot set with $\tau(M') \geq \tau_L$.

Let f be an infinitely DH renormalizable quadratic-like map. Let us say that it is of $\mathcal{S}$-type if all internal classes $c(R^n f)$ belong to copies M' from $\mathcal{S}$.

§4. Initial geometry.

4.1. Geometry of rays. Let $L \equiv L_b$ be a limb of the Mandelbrot set with root at b. By $L^{tr} = L_b^{tr}$ we denote a truncated limb, that is, L with a neighborhood of the root removed.

We say that a pre-periodic point is *repelling* if the corresponding periodic point is. Given a repelling periodic or pre-periodic point a of P_c, let $\mathcal{R}(a)$ be the union of the segments of rays landing at a up to the equipotential level 1 together with point a. We will use the Hausdorff topology on the space of configurations of curves. We will often suppress the label c unless it may cause confusion. Let B_c denote the Böttcher function (2-1) of P_c.

Lemma 4.1 (see [GM]). *Let a_c be a periodic point of P_c continuously depending on $c \in L_b$ which stays repelling on the unrooted limb $L_b \backslash \{b\}$. Then the rays configuration $\mathcal{R}(a_c)$ depends continuously on $c \in L_b \backslash \{b\}$.*

Proof. By [GM] the external angles of the rays landing at a_c are the same through the limb. So $\mathcal{R}(a_c)$ is the B^{-1}-image of the fixed configuration of segments in the annulus $\{z : 1 < |z| < e\}$ with the point a_c added.

Let us go back to the construction of the external rays in §2.2. It is easy to see from the explicit formulas for the Rimann map and the linearizing coordinate that the map (2-3) $h_i^{-1} : \mathbf{H} \to \mathcal{D}_i$ depends continuously (in the compact-open topology) on c ranging over the unrooted limb. Let $I = \{iy : 1 \leq y \leq 2^p\}$ be a fundamental segment of the vertical geodesic in $\mathbf{H}$. It follows that the curves $\hat{J} = h_i^{-1} I$ depend smoothly on c. Hence the curves

$$\hat{R}_i = \bigcup_{n=0}^{\infty} \lambda^{-n} \hat{J}$$

depend continuously on c in the Hausdorff topology. Then the rays $R_i = \psi(\hat{R}_i)$ depend continuously on c as well. □

Given a configuration $\mathcal{C}_0$ of finitely many parametrized curves and point in $\mathbf{C}$, let us consider the space Teich $(\mathcal{C}_0)$ of all configurations qc equivalent to $\mathcal{C}_0$ modulo conformal equivalence. There is a natural Teicmüller distance on this space:

$$\operatorname{dist}_T(\mathcal{C}_1, \mathcal{C}_2) = \inf \log K_h,$$

where h runs over all qc equivalences between C_1 and C_2.

We say that configurations of some family have *bounded geometry* if they stay bounded Teichmüller distance from a reference configuration $\mathcal{C}_0$ whose curves are smooth and intersect transversally.

Let us consider a family of two topological nested disks $D_1 \subset D_2$ with $\Gamma_i = \partial D_i$, and $A = D_2 \backslash D_1$. The statement that $\operatorname{mod}(A) > \epsilon$ with an $\epsilon > 0$ uniform over the family will be freely expressed in the following ways: "The annulus A has a definite modulus", or "D_1 is well inside D_2", or "There is a definite space in between Γ_1 and Γ_2."

A statement like "If f has a definite modulus then a certain quantity is bounded" means: "If $\operatorname{mod}(f) > \epsilon$ then there is a bound on that quantity depending only on ϵ".

Lemma 4.2. *Under the circumstances of Lemma 4.1 the configuration $\mathcal{R}(a_c)$ has a bounded geometry when c ranges over the truncated limb L_b^{tr} of the Mandelbrot set.*

Proof. Let us consider the configuration $\mathcal{I}$ of intervals obtained by rotating the interval $[0,1]$ by the cyclic group Z_p of order p. We will show that the configurations $\mathcal{R}(\alpha_c)$ of rays stay bounded Teichmüller distance from $\mathcal{I}$.

Let us use the notations from the proof of the previous lemma. Since the configuration $\hat{\mathcal{R}}$ of rays continuously depends on c ranging over the truncated limb L^{tr}, we can find a smooth Jordan curve γ enclosing 0 and smoothly depending on c, which intersects each infinite ray $\hat{\mathcal{R}}_i^\infty$ at only one point, namely the endpoint of $\hat{\mathcal{R}}_i$, and this intersection is transversal. The curves $\gamma_{-N} = \lambda^{-N}\gamma$ clearly satisfy the same property. But for sufficiently big N (uniform over the truncated limb) γ_{-N} lies strictly inside γ with a definite space in between.

Let us consider the annulus bounded by γ and γ_{-N} with the segments of the rays R_i in between. It follows from the above that this configuration $\mathcal{C}$ stays bounded Teichmüller distance from the standard one $\mathcal{C}_0$: the round annulus $\{re^{i\theta} : 1/2 \le r \le 1\}$ with p equally spaced straight intervals inside.

So there a K-qc map $h : \mathcal{C} \to \mathcal{C}_0$ with dilatation K depending only on the truncated limb, which conjugates $z \mapsto \lambda^N z$ to $z \mapsto 2z$ on the inner boundaries of the configurations. Pulling this map back by linear dynamics, we obtain a desired K-qc equivalence between the configuration $\hat{\mathcal{R}}$ and $\mathcal{I}$.

Finally, it is easy to see that the linearizing map ψ is univalent in a neighborhood of 0 whose size is uniform over the truncated limb. Hence there is a uniform l such that ψ is univalent on $\mathcal{R}_{-l} \equiv \lambda^{-l}\hat{\mathcal{R}}$. Hence the configuration $\mathcal{R} = f^l\psi(\mathcal{R}_{-l})$ has bounded geometry. □

Corollary 4.3. *Let us consider a limb L_b of M and a finite set A_c of periodic or preperiodic points which stay repelling through the unrooted limb L_b^{tr}. Let $\mathcal{C}_c$ be the union of configuration $\mathcal{R}(A_c)$ of rays landing at points of A_c cut at level 1, together with A_c and several equipotentials (whose levels don't depend on c). Then the configuration $\mathcal{C}_c$ has bounded geometry when c ranges over a truncated limb L_b^{tr}*

Proof. It follows from Lemma 4.2 that near A_c the configuration $\mathcal{C}_c$ has bounded geometry (where "near" is uniform over the truncated limb). On the other hand, it clearly has bounded geometry outside a uniform neighborhood of A_c, since ϕ_c^{-1} is a normal family of maps. □

Let $\mathcal{Y}_c^{(n)} \equiv \mathcal{Y}_c^{(n)}(f)$ denote the configuration of cutting curves for the Yoccoz puzzle of depth n for a map f. Now we immediately conclude:

Corollary 4.4. *For any given n the Yoccoz configuration $\mathcal{Y}^{(n)}(P_c)$ has bounded geometry when c ranges over a truncated primary limb L_b.*

If we consider a DH quadratic-like map f with $\mathrm{mod}\,(f) > \epsilon$, then we can $K(\epsilon)$-qc conjugate it to a quadratic polynomial, and transfer the net of rays and equipotentials from this quadratic. In what follows we always assume that the choice of the net is made in this way.

Corollary 4.5. *If f is a quadratic-like map with a definite modulus and internal class $c(f)$ ranging over a primary truncated limb L_b of M, then Yoccoz configuration $\mathcal{Y}^{(n)}(f)$ has a bounded geometry for any given n.*

4.2. Fundamental domain near the fixed point. The goal of this subsection is to construct a combinatorially defined fundamental domain with bounded geometry near the fixed point α. It is where the secondary limbs condition comes into the scene.

Let f be a DH quadratic-like map whose α-fixed point has rotation number q/p. This map has a single periodic point $\gamma \in \operatorname{int} Y^{(1)}$ of period p. Let γ' be the symmetric point. Consider the family $\mathcal{R}(\gamma')$ of rays landing at γ'. Let $D = D(f)$ be the component of $Y^{(1)} \backslash \mathcal{R}(\gamma')$ attached to the fixed point α (see Figure 5). Then $f^p D$ univalently covers the component of $Y^{(0)} \backslash \mathcal{R}(\gamma)$ attached to α. Note that $\partial D \cap \partial(fD)$ is contained in the union of two rays landing at α.

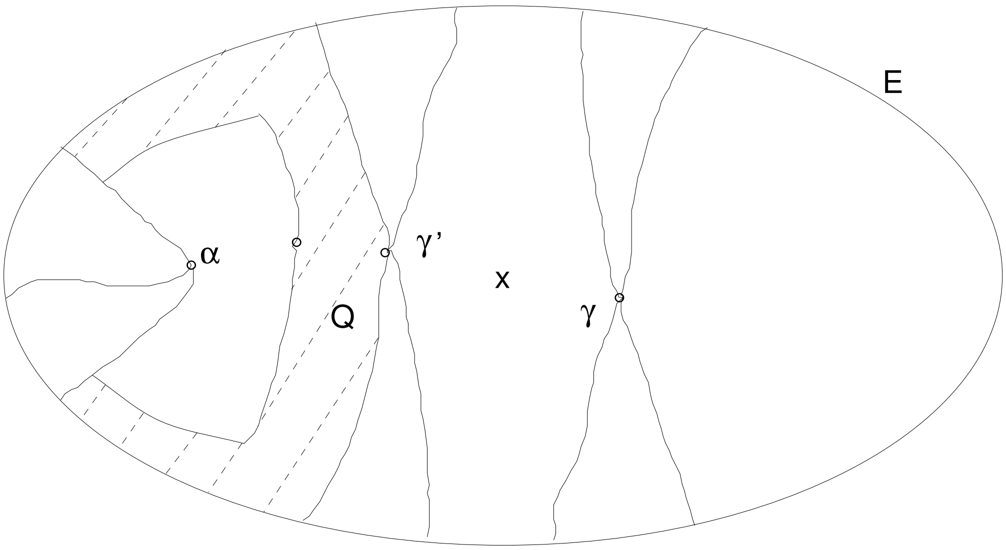

Figure 5: Fundamental domain near the fixed point.

Hence there is a univalent branch of f^{-p} which fixes α and maps D inside itself. It in now easy to see that $f^{-pn} D$ shrink to α as $n \to \infty$. So we can select $Q = Q(f) = D \backslash f^{-p} D$ as a fundamental domain for f^p near α: any trajectory which starts near α must pass through $Q = Q(f)$. Now Corollary 4.3 yields:

Lemma 4.6. *Geometry of the fundamental domain $Q(f)$ is bounded if $c(f)$ ranges over a truncated secondary limb and f has a definite modulus.*

4.5. Modulus of the first annulus.

Lemma 4.7. *Let P_c be a quadratic polynomial with c outside the main cardioid but not immediately renormalizable. If c ranges over a truncated secondary limb L_b, then all pieces X of the initial Markov partition (3-3) are well inside $Y^{(0)}$: $\operatorname{mod}(Y^{(0)} \backslash X) > \nu(L_b) > 0$.*

Proof. Take a little $\epsilon > 0$. Then find an N and $\delta \in (0, \epsilon)$ such that the equipotential $E_{1/2^N}$ does not intersect the δ-neighborhood of $\partial Y^{(0)} \backslash B(\alpha, \epsilon)$ (for all c in the truncated limb).

It follows from Proposition 3.1 and Corollary 4.4 that the statement is true for all pieces of depth $\leq N$.

Any other piece X is enclosed by the equipotential $E_{1/2^N}$ (where $E \equiv E_1$ is the outer-most equipotential of external radius 1). Hence if $\operatorname{dist}(X, \{\alpha\}) > \epsilon$ then $\operatorname{dist}(X, \partial Y^{(0)}) \geq \delta$. As $\operatorname{diam} X$ is uniformly bounded, we conclude that X is well inside $Y^{(0)}$.

Assume now that $\operatorname{dist}(X, \{\alpha\}) < \epsilon$. Then X intersects the domain $D = D(f)$. Since $\partial D \cap \partial X = \emptyset$, $X \subset D$. Let us consider the iterates $f^{pk}X$, $k = 0, 1, \ldots$ until the last moment l such that $f^{pl}X \subset D$. At this moment $f^{pl}X$ must intersect the fundamental domain Q. Since their boundaries don't intersect, we conclude that $f^{pl}X \subset Q$.

Let us consider domain $Q' \subset Q$ obtained by truncating Q with the equipotential $f^{-p}E$. This domain has a bounded geometry since the fundamental domain Q does (Lemma 4.6). Hence Q' is well inside f^pD. Moreover, $f^{pl}X \subset Q'$ since all puzzle pieces inside Y^1 are enclosed by the equipotential $f^{-p}E$ (see Figure 3). Hence $f^{pl}X$ is well inside fD as well.

On the other hand, if part of the puzzle-piece $f^{pl}X$ lies in $Q \backslash Q'$, it belongs to one of the pieces of depth $n \leq 2p$, of the initial partition (3-2). As we have noted above, this piece is well inside fD. Hence $f^{pl}X$ is also well inside fD.

We conclude that there is always a definite space around $f^{pl}X$ in f^pD. Pulling this space back by iterates of the univalent branch $f^{-p}f^pD \to D$, we obtain a definite space around X in D. $\square$

Remember that $A^n = V^{n-1} \backslash V^n$ are the the principal annuli.

Theorem I. *Let f be a quadratic-like map with internal class $c(f)$ ranging over a truncated secondary limb L_b^{tr}. If $\operatorname{mod}(f) \geq R > 0$ then*

$$\operatorname{mod}(A^1) \geq C(R)\, \nu(L_b^{tr}) > 0$$

where $C(R) > 0$ monotonically depends on $R > 0$ and $C(R) \nearrow 1$ as $\mu \nearrow \infty$.

Proof. Let us go through the proof of Proposition 3.1. We found an l and a puzzle piece $P \subset V^0$ such that G^lP two-to-one covers $Y^{(0)}$, where G is the Markov map (3-4). Let $G^l 0 \in X_i^s$ where X_i^s is a puzzle-piece of the initial partition (3-3). Then V^1 is the pull-back of X_i^s by $G^l|P$. But by Lemma 4.7 X_i^s is well inside $Y^{(0)}$. Hence V^1 is well inside V^0. $\square$

§5 Increasing of moduli.

5.1. Statement of the results. Let f be a Douady-Hubbard quadratic-like map. Let us consider its principal nest $Y^0 \supset V^0 \supset V^1 \supset \ldots$, and the corresponding nest of annuli $A^n = V^n \backslash V^{n-1}$. Let us call their moduli $\mu_n = \operatorname{mod}(A^n)$ the *principal moduli* of g_1.

In this section we will prove the central result of the paper:

Theorem III. *Let $n(k)$ counts the non-central levels in the principal nest $\{V^n\}$. Then*

$$\operatorname{mod} A^{n(k)+1} \geq Bk,$$

where the constant B depends only on the first modulus $\mu_1 = \operatorname{mod} A_1$.

On the way to this result we prove a priori moduli and distortion bounds along the principal nest (Theorem II). Note that already this result yields the divergence property of the Yoccoz Theorem (see §2).

Theorem III will also imply a priori bounds for infinitely renormalizable quadratics of sufficiently big height:

Theorem IV. *There is a χ-special family $\mathcal{S}$ of the Mandelbrot copies with the following property. Let f be an infinitely renormalizable quadratic of $\mathcal{S}$-type then, and $A^{n,m}$ be its principal annuli. Then there is an $Q = Q(\chi) \to \infty$ as $\chi \to \infty$, such that* $\operatorname{mod}(R^n f) \geq \operatorname{mod} A^{n,1} \geq Q$, $n = 0, 1, \ldots$.

At the end of this section we will describe other combinatorial factors which yield big space. This is summarized in Theorem IV$'$ which loosely says that if the periods of $R^n f$ are sufficiently big and there are no "parabolic" or "Siegel cascades" in the principal nests then there are a priori bounds.

Recall that $g_n : \cup V_i^n \to V^{n-1}$ denotes the principal sequence of return maps (3-6), and $\mathcal{M} \equiv \mathcal{M}(f)$ denotes the full family of all puzzle pieces (§2.6). Let $\mathcal{V}^n \subset \mathcal{M}$ denote the family of all pieces V_i^n of level n.

5.2. First estimates. Let us start with a lemma which partly explains the importance of the principal nest: the principal moduli control the distortion of the first return maps (see the Appendix for the definition of the distortion). Let us consider the decomposition:

$$g_n|V^n = h_n \circ \Phi, \tag{5-0}$$

where Φ is a purely quadratic map and h_n is a diffeomorphism of ΦV^n onto V^{n-1}.

Lemma 5.1. *Let $D \in \mathcal{M}$ be a puzzle piece such that $f^l D = V^n$, while $f^k D \cap V^n = \emptyset$, $k = 0, \ldots, l-1$. If $\mu_n \leq \bar{\mu}$ then the distortion of f^l on D is $O(\exp(-\mu_{n-1}))$ with a constant depending on $\bar{\mu}$. Hence the distortion of h_n is $O(\exp(-\mu_{n-2}))$.*

Proof. This follows from Lemma 3.5, Corollary 3.6 and the Koebe Theorem (see the Appendix). □

Let us fix a level $n > 0$, denote $V^{n-1} = \Delta$, $V_i = V_i^n$, $g = g_n$, $A = A^n = \Delta \backslash V_0$, $\mu = \mu_n$, and mark the objects of the next level $n+1$ with prime. Thus $\Delta' \equiv V \equiv V_0$, and $g' : \cup V_i' \to \Delta'$. (We restore the index n whenever we need it).

Lemma 5.2. *Let $D' \subset \Delta'$ be a puzzle piece such that $g^{i(k)} D' \subset V_{i(k)}$, $k = 0, 1, \ldots, l$ with $i(k) \neq 0$ for $0 < k < l$. Then*

$$\operatorname{mod}(\Delta' \backslash D') \geq \frac{1}{2} \sum_{k=1}^{l} \operatorname{mod}(\Delta \backslash V_{i(k)}).$$

Proof. Let us consider the following nest of topological disks:

$$\Delta' = W_1 \supset \ldots \supset W_l \supset W_{l+1} \equiv D',$$

where W_k is the pullback of Δ under g^k, $k = 1, \ldots l$ (which has itinerary $0 = i(0), i(1), \ldots, i(k-1)$ through the pieces of level n). Then g^k is a two-to-one branched covering of the annulus $W_k \backslash W_{k+1}$ over the annulus $\Delta \backslash V_{i(k)}$. Hence

$$\operatorname{mod}(W_k \backslash W_{k+1}) = \frac{1}{2} \operatorname{mod}(\Delta \backslash V_{i(k)}), \quad (1 \le k \le l).$$

But by the Grötzsch inequality

$$\operatorname{mod}(\Delta' \backslash D') \ge \sum_{k=1}^{l} \operatorname{mod}(W_k \backslash W_{k+1}),$$

and the desired estimate follows. □

This lemma immediately yields:

Corollary 5.3. *Given a puzzle piece V'_j , we have*

$$\operatorname{mod}(\Delta' \backslash V'_j) \ge \frac{1}{2}\mu.$$

Moreover, if the return to level n is non-central, that is $g0 \in V_i$ with an $i \ne 0$, then

$$\operatorname{mod}(\Delta' \backslash V'_j) \ge \frac{1}{2}(\mu + \operatorname{mod}(\Delta \backslash V_i)).$$

So, a definite principal modulus on some level produces a definite space on the next level.

5.3. Isles and asymmetric moduli. Let $\{V_i\}_{i \in \mathcal{I}} \subset \mathcal{V}^n$ be a finite family of disjoint puzzle pieces consisting of at least two pieces (that is $|\mathcal{I}| \ge 2$) and containing a critical puzzle piece V_0 . Let us call such a family *admissible.* We will freely identify the label set $\mathcal{I}$ with the family itself.

Given a puzzle piece D , let $\mathcal{I}|D$ denote the family of puzzle pieces of $\mathcal{I}$ contained in D . Let D be a puzzle piece containing at least two pieces of family $\mathcal{I}$. For $V_i \subset D$ set

$$R_i \equiv R_i(\mathcal{I}|D) \subset D \backslash \bigcup_{j \in \mathcal{I}|D} V_j$$

be an annulus of maximal modulus enclosing W_i but not enclosing other pieces of the family $\mathcal{I}$. Such an annulus exists by the Montel Theorem (see Figure 6). We will briefly call it the *maximal annulus* enclosing V_i in D (rel the family $\mathcal{I}$).

Let us define *the asymmetric modulus of the family $\mathcal{I}$ in D* as

$$\sigma(\mathcal{I}|D) = \sum_{i \in \mathcal{I}} \frac{1}{2^{1-\delta_{i0}}} \operatorname{mod} R_i(R(\mathcal{I}|D), \tag{5-1}$$

where δ_{ji} is the Kronecker symbol. So the critical modulus is supplied with weight 1, while the non-critical moduli are supplied with weights 1/2 (if D is a non-critical island then all weights are actually 1/2).

Remark. A real analogue of this parameter, "the asymmetic Poincaré length", appeared in [L3]. Its complex counterpart for the Fibonacci map was suggested by Jeremy Kahn. A general notion involving admissible familes and isles is given by the author.

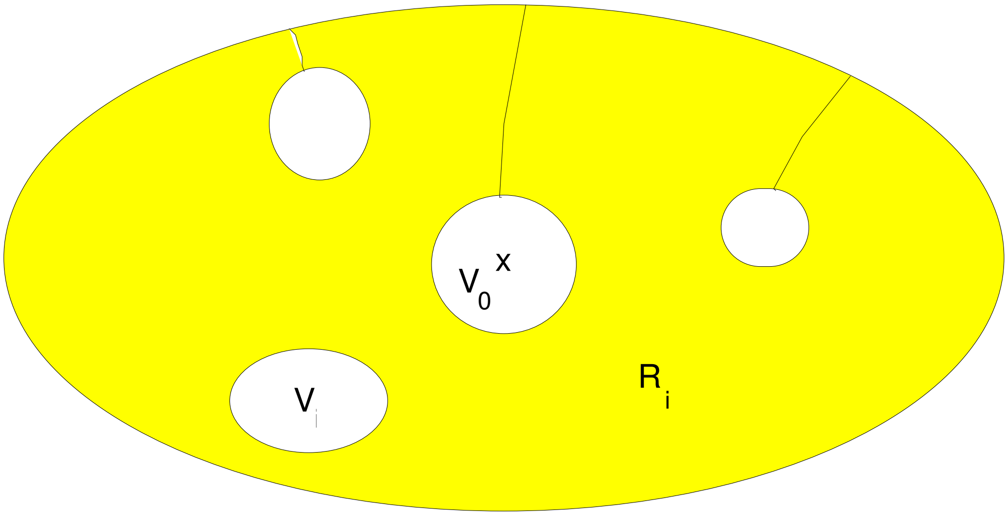

Figure 6: Annulus R_i .

Take a level $n-1$ *which is not DH renormalizable*, that is, the family $\mathcal{V}^n$ consists of more than one piece. Let $D = V^{n-1}$, and let $\{V_i^n\}_{i\in\mathcal{I}}$ be an admissible subfamily of $\mathcal{V}^n$. Then set $\sigma_n(\mathcal{I}) \equiv \sigma(\mathcal{I}|V^{n-1})$ and

$$\sigma_n = \min_{\mathcal{I}} \sigma_n(\mathcal{I}), \tag{5-2}$$

where $\mathcal{I}$ runs over all admissible subfamilies of $\mathcal{V}^n$.

The principal moduli μ_n and the asymmetric moduli σ_n are the main geometric parameters of the renormalized maps g_n . Again, in what follows the label n will be suppressed as long as the level is not changed.

Let $\{V_i'\}_{i\in\mathcal{I}'}$ be an admissible subfamily of $\mathcal{V}'$. Let us organize the pieces of this family in *isles* in the following way. A puzzle piece $D' \subset \Delta'$ is called an *island* (for family I') if

- D' contains at least two puzzle pieces of family $\mathcal{I}'$;
- There is a $t \geq 1$ such that $g^k D' \subset V_{i(k)}$, $k = 1, \dots t-1$, with $i(k) \neq 0$, while $g^t D = \Delta$.

Given an island D' , let $\phi_{D'} = g^t : D' \to \Delta$. This map is either a double covering or a biholomorphic isomorphism depending on whether D' is critical or not. In the former case, $D' \supset V_0'$ (for otherwise $D' \subset V_0'$ contradicting the first part of the definition of isles).

We call a puzzle piece $V_j' \subset D'$ $\phi_{D'}$*-pre-critical* if $\phi_{D'}(V_j') = V_0$. There are at most two pre-critical pieces in any D' . If there are actually two of them, then they are non-critical and symmetric with respect to the critical point 0. Thus in this case D' contains also the critical puzzle piece V_0' .

Let $\mathcal{D}' = \mathcal{D}(\mathcal{I}')$ be the family of isles associated with $\mathcal{I}'$. Let us consider the asymmetric moduli $\sigma(\mathcal{I}'|D')$ as a function on this family. This function is clearly monotone:

$$\sigma(\mathcal{I}'|D') \geq \sigma(\mathcal{I}'|D_1') \quad \text{if} \quad D' \supset D_1', \tag{5-3}$$

and superadditive:

$$\sigma(\mathcal{I}'|D') \geq \sigma(\mathcal{I}'|D'_1) + \sigma(\mathcal{I}'|D'_2),$$

provided D'_i are disjoint subisles in D'.

Let us call an island D' *innermost* if it does not contain any other isles of the family $\mathcal{D}(\mathcal{I}')$. As this family is finite, innermost isles exist.

5.4. Non-decreasing of moduli.

Lemma 5.4. *Let $\mathcal{I}'$ be an admissible family of puzzle pieces. Let D' be an innermost island associated to the family $\mathcal{I}'$, and let $\mathcal{J}' = \mathcal{I}'|D$. Let $i(j)$ is defined for $j \in \mathcal{J}'$ by the property $\phi_{D'}(V'_j) \subset V_{i(j)}$, and let $\mathcal{I} = \{i(j) : j \in \mathcal{J}'\} \cup \{0\}$. Then $\{V_i\}_{i\in\mathcal{I}}$ is an admissible family of puzzle pieces, and*

$$\sigma(\mathcal{I}'|D') \geq \frac{1}{2}\left((|\mathcal{J}'| - s)\mu + s \operatorname{mod} R_0 + \sum_{j\in\mathcal{J}',\, i(j)\neq 0} \operatorname{mod} R_{i(j)}\right), \tag{5-4}$$

where $s = \#\{j : i(j) = 0\}$ is the number of $\phi_{D'}$-pre-critical pieces, and R_i are the maximal annuli enclosing V_i in Δ rel family $\mathcal{I}$.

Proof. Let $\phi \equiv \phi_{D'}$. Let us show first that the family $\mathcal{I}$ is admissible. This family is finite since $\mathcal{J}' \subset \mathcal{I}'$ is finite. The critical puzzle piece belongs to $\mathcal{I}$ by definition. So the only property to check is that $|\mathcal{I}| \geq 2$. But otherwise $\mathcal{J}'$ would consist of two pre-critical puzzle pieces. But then D' would be critical, and thus should have also contained the critical piece V'_0, which is a contradiction.

Let us observe next that

$$\operatorname{mod}(V_{i(j)} \setminus \phi V'_j) \geq \mu \quad \text{if} \quad i(j) \neq 0. \tag{5-5}$$

Indeed, in this case $g^m(\phi V'_j) = V_0$ for some $m > 0$. Let $W \subset V_{i(j)}$ be the pull-back of Δ under g^m. Then the annulus $W\setminus\phi V'_j$ is univalently mapped by g^m onto the annulus $\Delta\setminus V_0$. Hence $\operatorname{mod}(W\setminus\phi V'_j) = \operatorname{mod}(\Delta\setminus V_0) = \mu$, and (5-5) follows.

Given a $i \in I$, let us consider a topological disk $Q_i = R_i \cup V_i \subset \Delta$ ("filled annulus R_i"). By the Grötzsch inequality and (5-5),

$$\operatorname{mod}(Q_{i(j)}\setminus\phi V_j) \geq \operatorname{mod} R_{i(j)} + (1 - \delta_{0,i(j)})\mu. \tag{5-6}$$

For a $j \in J'$, let us consider an annulus $B_j \subset D'$, the component of $\phi^{-1}R_{i(j)}$ enclosing V'_j. This annulus does not enclose any other pieces $V'_k \in \mathcal{J}'$, $k \neq j$. Indeed, otherwise the inner component of $\mathbf{C}\setminus B'_j$ would be an island contained in D', despite the assumption that D' is innermost.

Let us now consider a topological disk P'_j obtained by filling the annulus B'_j. As it contains a single puzzle piece V'_j of family J', the annulus $P'_j\setminus V'_j$ does not go around any other puzzle piece $V'_k \in \mathcal{J}'$, $k \neq j$. Hence

$$\operatorname{mod} R'_j \geq \operatorname{mod}(P'_j\setminus V'_j), \tag{5-7}$$

where $R'_j \subset D'$ is the maximal annulus enclosing V'_j rel $\mathcal{J}'$. Moreover $\phi : P'_j \to Q_{i(j)}$ is univalent or double covering depending on whether $j \neq 0$ or $j = 0$. Hence

$$\mathrm{mod}\,(P'_j\backslash V'_j) \geq \frac{1}{2^{\delta_{j0}}}\mathrm{mod}\,(Q_{i(j)}\backslash \phi V_j). \tag{5-8}$$

Inequalities (5-6)-(5-8) yield

$$\mathrm{mod}\,R'_j \geq \frac{1}{2^{\delta_{j0}}}(\mathrm{mod}\,R_{i(j)} + (1-\delta_{0,i(j)})\mu). \tag{5-9}$$

Summing up estimastes (5-9) over $\mathcal{J}'$ with weights $1/2^{1-\delta_{j0}}$, we obtain the desired inequality (5-5). □

Corollary 5.5. *For any island D' of the family $\mathcal{D}'$ the following estimates hold:*

$$\sigma(\mathcal{I}'|D') \geq \frac{1}{2}\mu \quad \text{and} \quad \sigma(\mathcal{I}'|D') \geq \sigma(\mathcal{I}) \geq \sigma.$$

Proof. By monotonicity (5-3), it is enough to check the case of an innermost island D'. Let us use the notations of the previous lemma. Since the family $\mathcal{I}$ is admissible, it contains a non-critical piece. Hence $|\mathcal{J}'|$ is always strictly greater than the number s of pre-critical pieces in D', and (5-4) implies the first of the above inequality.

Furthermore, as $\mu \geq \mathrm{mod}(R_0)$ and $|\mathcal{J}'| \geq 2$, the right-hand side in (5-4) is bounded from below by

$$\frac{1}{2}\left(|\mathcal{J}'|\,\mathrm{mod}(R_0) + \sum_{i\in\mathcal{I}, i\neq 0} \mathrm{mod}(R_i)\right) \geq \sigma(\mathcal{I}).$$

(Note that $\sigma(\mathcal{I})$ makes sense since $\mathcal{I}$ is admissible). Finally $\sigma(\mathcal{I}) \geq \sigma$, and the second inequality follows. □

Let us fix a "big" integer quantifier $N_* > 0$. We say that a level n is in the "tail of a cascade" if all levels $n-1, n-N_*$ belong to a cascade (note that level $n-1$ itself may be non-central). Cascades of length at least N_* we call "long".

Theorem II. *Given a generalized quadratic-like map g_1, we have the following bounds of the geometric parameters within its principal nest:*

- *The asymmetric moduli σ_n grow monotonically and hence stay away from 0 on all levels (until the first DH renormalizable level): $\sigma_n \geq \bar{\sigma} > 0$.*
- *The principal moduli μ_n stays away from 0 (that is, $\mu_n \geq \bar{\mu} > 0$) everywhere except for the case when $n-1$ is in the tail of a long cascade (the bound $\bar{\mu}$ depends on the choice of N_*).*
- *The non-critical puzzle-pieces V_i^n are well inside V^{n-1} (that is, $\mathrm{mod}\,(V^{n-1}\backslash V_i^n \geq \bar{\mu} > 0$) except for the case when V_i^n is pre-critical and $n-2$ is the last level of a long cascade.*
- *The distortion of h_n from (5-0) is uniformly bounded on all levels by a constant $\bar{K}$.*

All bounds depend only on the first principal modulus μ_1 and (as $\bar{\mu}$ is concerned) on the choice of N_*).

Proof. The first assertion follows from the second inequality of Corollary 5.5. Together with Corollary 5.3 it implies the second one (note that the second inequality of this corollary implies that $\mu' \geq \sigma/2$ in the non-central case). One more application of Corollary 5.3 yields the next assertion.

Let us check the last statement. If $n-2$ is not in the tail of a central cascade, then $\mu_{n-1} \geq \bar{\mu}$ by the second statement, and the desired follows from Lemma 5.1.

Let $n-2$ be in the tail of a central cascade $V^m \supset \ldots \supset V^{n-2} \supset \ldots$. If this is not the last level of this cascade then $g_n|V^n = g_{m+2}|V^n$, so that h_n is just a restriction of the map h_{m+2} with bounded distortion.

Finally, if $n-2$ is the *last* level of a central cascade, then by Corollary 3.8 h_n can be extended to a univalent map with range V^m, and the Koebe Theorem implies the distortion bound. □

5.5. Linear growth of moduli. Our goal is to prove that $\sigma' \geq \sigma + a$ with a definite $a > 0$ (that is, dependent only on $\operatorname{mod} A_0$) at least on every other level, except for the tails of long cascades and a couple of the following levels. (Theorem II shows the reason why these tails play a special role: In the tails the principal moduli become tiny which slows down the growth rate of asymmetric moduli.)

Clearly it is enough to show that for any innermost island D'

$$\sigma(\mathcal{I}'|D') \geq \sigma + a \tag{5-10}$$

with a definite $a > 0$. The analysis will be split into a tree of cases.

Case I. An island with at least three puzzle pieces.

Proposition 5.6. *If an innermost island D' contains at least three puzzle-pieces V_j', $j \in \mathcal{J}'$, then*

$$\sigma(\mathcal{J}'|D') \geq \sigma(I) + \frac{1}{2}\mu.$$

Proof. Let us split off $(1/2)\mu$ in (5-4) and estimate all other μ's by $\operatorname{mod}(R_0)$. This estimates the right-hand side by

$$\frac{1}{2}\mu + \frac{|\mathcal{J}|-1}{2}\operatorname{mod}(R_0) + \frac{1}{2}\sum_{i\in\mathcal{I}, i\neq 0}\operatorname{mod}(R_i),$$

which immediately yields what is claimed. □

Hence under the circumstances of Proposition 5.6 we observe a definite growth of the asymmetric modulus provided level $n-1$ is not in the tail of a long cascade. Indeed then by Theorem II μ is bounded away from 0, and (5-10) follows.

Case II. An island with two puzzle pieces. The further analysis needs some preparation in the geometric function theory summarized in the Appendix.

Suppose we have an innermost island D' containing two puzzle-pieces V_j', $j \in \mathcal{J}'$. Let $\phi \equiv \phi_{D'}$ and let $\phi V_j' \subset V_i$ with $i = i(j)$. Fix a quantifier $L_* > 0$. When we say that something is "big", this means that it is at least $C(L_*)$ where $C(L_*) \to \infty$ as $L_* \to \infty$. Similarly "small" means an upper bound by $\epsilon(L_*) \to 0$ as $L_* \to \infty$. The sign $\approx$ will mean an equality up to an small (in the above sense) error, while the sign $\succ$ will mean the inequality up to a small error.

Subcase (i). *Assume that there is a non-critical puzzle-piece* $V_{i(j)}$ *whose Poincaré distance in* Δ *from the critical point is less than* L_* . Then by Lemma A.1

$$\mu \geq \operatorname{mod}(R_0) + \alpha \tag{5-11}$$

with a definite $\alpha = \alpha(L_*) > 0$. But observe that when we passed from Lemma 5.4 to Corollary 5.5 we estimated μ by mod (R_0) . Using the better estimate (5.10), we obtain a definite increase of σ .

Subcase (ii). *Assume now that the hyperbolic distance in* Δ *from any non-critical puzzle piece* $V_{i(j)}$ *, from the critical point is at least* L_* . Let also $n-2$ don't belong to the tail of a long cascade (for the sake of linear growth it is enough to prove definite growth on such levels). Then V_0 may not belong to any non-trivial island together with some non-critical piece $V_{i(j)}$. Indeed, by Theorem II all puzzle pieces of level $n-1$ are well inside V^{n-2} . But then by Lemma 5.2 all non-trivial isles of level n are well inside of $V^{n-1} \equiv \Delta$. (The quantifier L_* should be chosen bigger than the a priori bound on the hyperbolic diameters of the isles).

Subcase (ii-a). *Assume that both* $V_{i(j)}$ *are non-critical.* Then by Corollary 5.5 $\sigma(\mathcal{J}'|D')$ is estimated by $\sigma_n(\mathcal{I})$ where the family $\mathcal{I}$ consists of three puzzle pieces: two pieces $V_{i(j)}$ and the central puzzle piece V_0 .

If puzzle pieces $V_{i(j)}$, $j \in \mathcal{J}'$, don't belong to the same non-trivial island, then by Proposition 5.6 $\sigma(\mathcal{I}) \geq \sigma_{n-1} + a$ with a definite $a > 0$, and we are done.

Otherwise the puzzle pieces $V_{i(j)}$ belong to an island W . Since by Lemma 5.2 W is well inside of Δ , it stays on the big Poincaré distance from the critical point (namely, on distance $L_* - O(1)$). Hence mod $(R_0) \approx \mu$, and

$$\sigma(\mathcal{I}) \geq \sigma(\mathcal{I}|W) + \operatorname{mod}(R_0) \succ \sigma_{n-1} + \mu$$

where $\mu \equiv \mu_n$ is bounded away from 0, since level $n-1$ is not in the tail of a long cascade. So we have gained some extra growth, and can pass to the next case.

Below we will restore labels n and $n+1$ since many levels will be involved in the consideration.

Subcase (ii-b). *Let one of the puzzle pieces* V_i^n *be critical.* So we have the family $\mathcal{I}^n$ of two puzzle-pieces V_0^n and V_1^n . Remember that we also assume that *the hyperbolic distance between these pieces is at least* L . Hence, V^{n-1} is the only island containing both of them, so that $g_{n-1}V_0^n$ and $g_{n-1}V_1^n$ belong to different puzzle-pieces of level $n-1$. For the same reason we can assume that one of these puzzle-pieces is critical. Denote them by V_0^{n-1} and V_1^{n-1} . Then one of the following two possibilities on level $n-2$ can occur:

1) *Fibonacci return* when $g_{n-1}V_0^n \subset V_1^{n-1}$ and $g_{n-1}V_1^n = V_0^{n-1}$ (see Figure 7);

2) *Central return* when $g_{n-1}V_0^n = V_0^{n-1}$ and $g_{n-1}V_1^n \subset V_1^{n-1}$.

We can assume that one of these schemes occur on several previous levels $n-3, n-4, ...$ as well (otherwise we gain an extra growth by the previous considerations). To fix the idea, let us first consider the following particular case, which plays the key role for the whole theorem.

Fibonacci cascade. Assume that on both levels $n-1$ and $n-2$ the Fibonacci returns

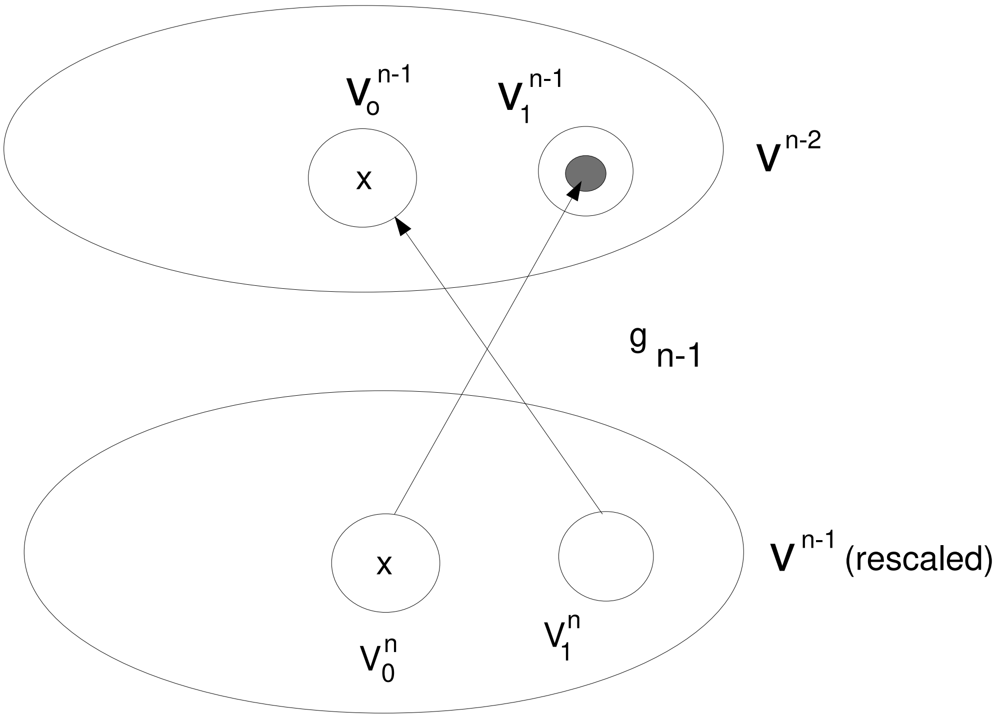

Figure 7: Fibonacci scheme.

occur. Let us look more carefully at the estimates of Lemma 4. In the Fibonacci case we just have:

$$\operatorname{mod}(R_1^n) \geq \operatorname{mod}(R_0^{n-1}), \tag{5-12}$$

$$\operatorname{mod}(R_0^n) \geq \frac{1}{2}\operatorname{mod}(Q_1^{n-1} \backslash g_{n-1} V_0^n), \tag{5-13}$$

where $Q_i^n = V_i^n \cup R_i^n$. Applying g_{n-2} we see that

$$\operatorname{mod}(Q_1^{n-1} \backslash g_{n-1} V_0^n) \geq \operatorname{mod}(Q_0^{n-2} \backslash V_0^{n-1}). \tag{5-14}$$

But since V_1^{n-2} is hyperbolically far away from the critical point,

$$\operatorname{mod}(Q_0^{n-2} \backslash V_0^{n-1}) \approx \operatorname{mod}(V_0^{n-3} \backslash V_0^{n-1}). \tag{5-15}$$

By the Grötzsch Inequality there is an $a \geq 0$ such that

$$\operatorname{mod}(V_0^{n-3} \backslash V_0^{n-1}) = \mu_{n-1} + \mu_{n-2} + a. \tag{5-16}$$

Clearly

$$\mu_{n-1} \geq \operatorname{mod}(R_0^{n-1}). \tag{5-17}$$

Furthermore, let $P_1^{n-1} \subset V^{n-2}$ be the pull-back of Q_0^{n-2} by g_{n-2}. Since ∂P_1^{n-1} is hyperbolically far away from V_1^{n-1} , we have:

$$\mu_{n-2} \geq \operatorname{mod}(R_0^{n-2}) = \operatorname{mod}(P_1^{n-1} \backslash V_1^{n-1}) \approx \operatorname{mod}(V^{n-2} \backslash V_1^{n-1}) \geq \operatorname{mod}(R_1^{n-1}). \tag{1-18}$$

Combining estimates (5-13) through (5-18) we get

$$\mathrm{mod}(R_0^n) \succ \frac{1}{2}(\mathrm{mod}(R_0^{n-1}) + \mathrm{mod}(R_1^{n-1}) + a). \tag{5-19}$$

We see from (5-12) and (5-19) that we need to check that the constant a in (5-16) is definitely positive. Assume that this is not the case, that is, for any $\delta > 0$ we can find a level n in the Fibonacci cascade as above such that $a < \delta$. Set $\Gamma_n = \partial V^n$. Then by the Definite Grötzsch Inequality (see the Appendix), the width (Γ_{n-2}) in the annulus $T = V^{n-3} \backslash V^{n-1}$ is at most $\xi(\delta)$ with $\xi(\delta) \to 0$ as $\delta \to 0$. Since Γ_{n-2} is well inside of T, we conclude by the Koebe Distortion Theorem that Γ_{n-2} is contained in a narrow neighborhood of a curve γ with a bounded geometry. Hence there is a $k = k(\delta) \to 0$ as $\delta \to 0$ and and $\epsilon = \epsilon(\delta, k) > 0$ such that the curve Γ_{n-2} is not (k, ϵ)-pinched.

On the other hand, the hyperbolic distance from the puzzle piece V_1^{n-1} to the critical point 0 in V^{n-2} is at least L_*. Hence by Lemma A.4 it must be located Euclideanly very close to Γ_{n-2} relatively the Euclidean distance to the critical point (that is, the relative distance is at most $\beta(\delta) \to 0$ as $\delta \to 0$). Hence the critical value $g_{n-1}0$ is also very close to Γ_{n-2} relatively the distance to the critical point, that is

$$\frac{\mathrm{dist}\,(g_n 0, \Gamma_{n-2})}{\mathrm{dist}\,(g_n 0, 0)} \le \epsilon(L_*),$$

where $\epsilon(L_*) \to 0$ as $L_* \to \infty$.

By the last statement of Theorem II, g_{n-1} is a quadratic map up to a bounded distortion. Hence the curve Γ_{n-1} which is the pull-back of Γ_{n-2} by g_{n-1} must have a huge eccentricity around the critical point. But then by Lemma 7.2 the width of Γ_{n-1} in $V^{n-2} \backslash V^n$ is also big, which by the above considerations gives a definite linear growth on the next level.

Remark. The actual shape of a deep level puzzle-piece for the Fibonacci cascade is shown on Figure 8. There is a good reason why it resembles the filled Julia set for $z \mapsto z^2 - 1$ (see [L5]). As the geodesic in V_0^{n-1} joining the puzzle-pieces V_0^n and V_1^n goes through the pinched region, the Poincaré distance between these puzzle-pieces is, in fact, big.

Now it is the time to look closer at central cascades.

Central cascades. Let $N \ge 2$, $n = m + N$, and let us consider a nest $\mathcal{C}^{m+N}$ of puzzle pieces

$$V^m \supset V^{m+1} \supset \ldots \supset V^{m+N-1} \supset V^{m+N} \supset D^{m+N} \tag{5-20}$$

satisfying the following properties (see Figure 8):

- The return on level $m-1$ is non-central: $g_m 0 \notin V_0^m$;
- Central returns occur on levels $m, m+1, \ldots m+N-2$, that is $g_{m+1}0 \in V^{m+N-1}$;
- D^{m+N} is an island with a family $\mathcal{I}^{m+N+1}$ of two puzzle pieces inside. Let $\phi_{m+N} \equiv \phi_{D^{m+N}}$ denote the corresponding double covering $D^{m+N} \to V^{m+N-1}$;
- One of the puzzle pieces $\phi_{m+N} V_0^{m+1}$, $\phi_{m+N} V_1^{m+1}$ is critical.

Though the logic of our argument so far allowed to assume that $N \le N_*$, the following argument will require consideration of arbitrary big N. So let N be arbitrary integer ≥ 2.

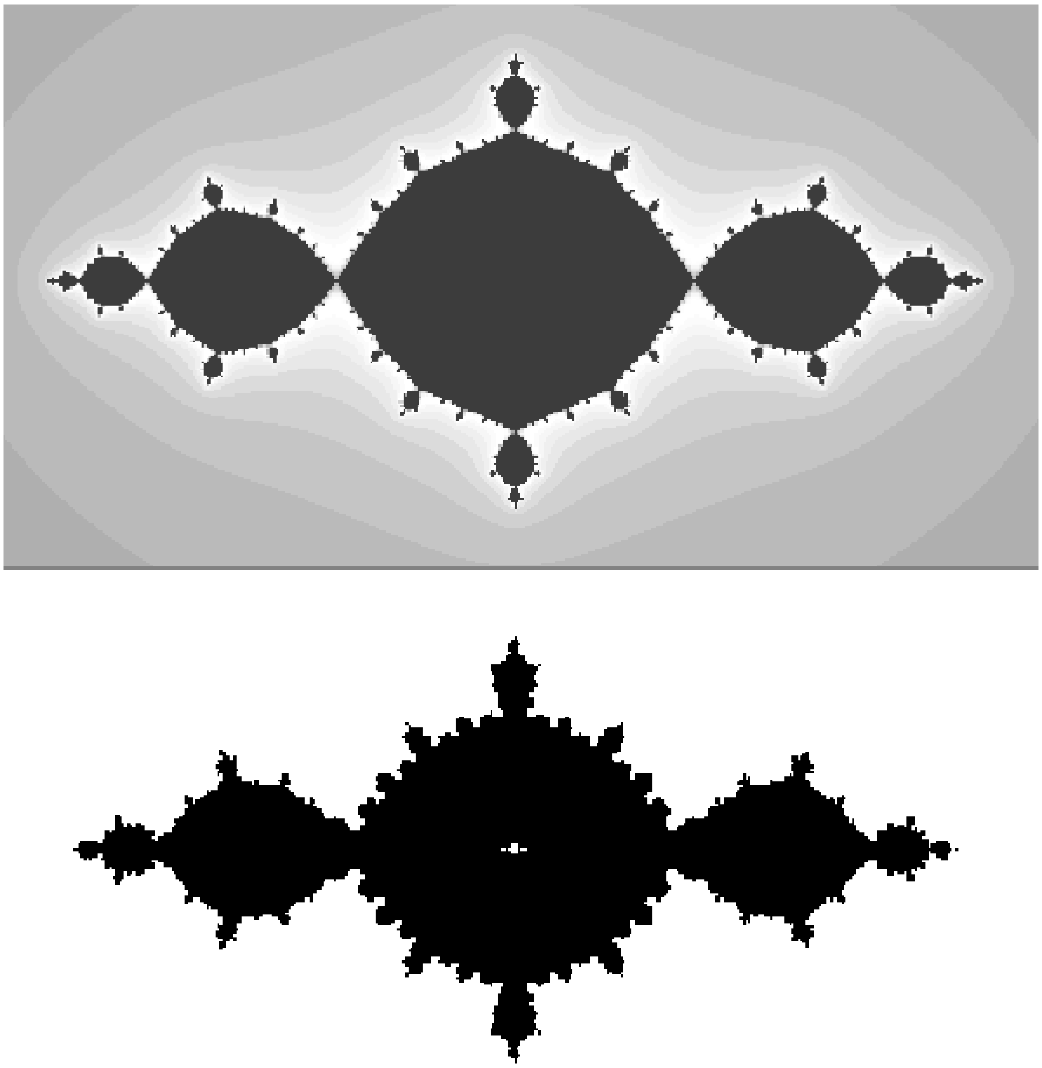

Figure 8: Fibonacci puzzle piece (below) vs the Julia set of $z \mapsto z^2 - 1$ (above) .

We would like to analyze when

$$\sigma(I^{m+N+1}|D^{m+N}) \geq \sigma_{m+1} + a \tag{5-21}$$

with a definite $a > 0$. To this end we need to pass from level $m+N$ all way up to level m.

Let us consider the family $\mathcal{W}(\mathcal{C}^{m+N})$ of puzzle pieces W_i^k, $k = m+1, ..., m+N$, $i \neq 0$, the pull-backs of the $V_i^{m+1} \equiv W_i^{m+1}$ to the annuli A^k (compare with the initial Markov partition in §3.2). Given a $W_i^k \subset A^k$, there is a $V_j^{m+1} \subset A^{m+1}$, $j \neq 0$, such that $g_{m+1}^{m+1-k} W_i^k = V_j^{m+1}$. Hence $g_m \circ g_{m+1}^{m+1-k} W_i^k = V^m$. So we can define a Bernoulli map

$$G : \cup W_i^k \to V^m \tag{5-22}$$

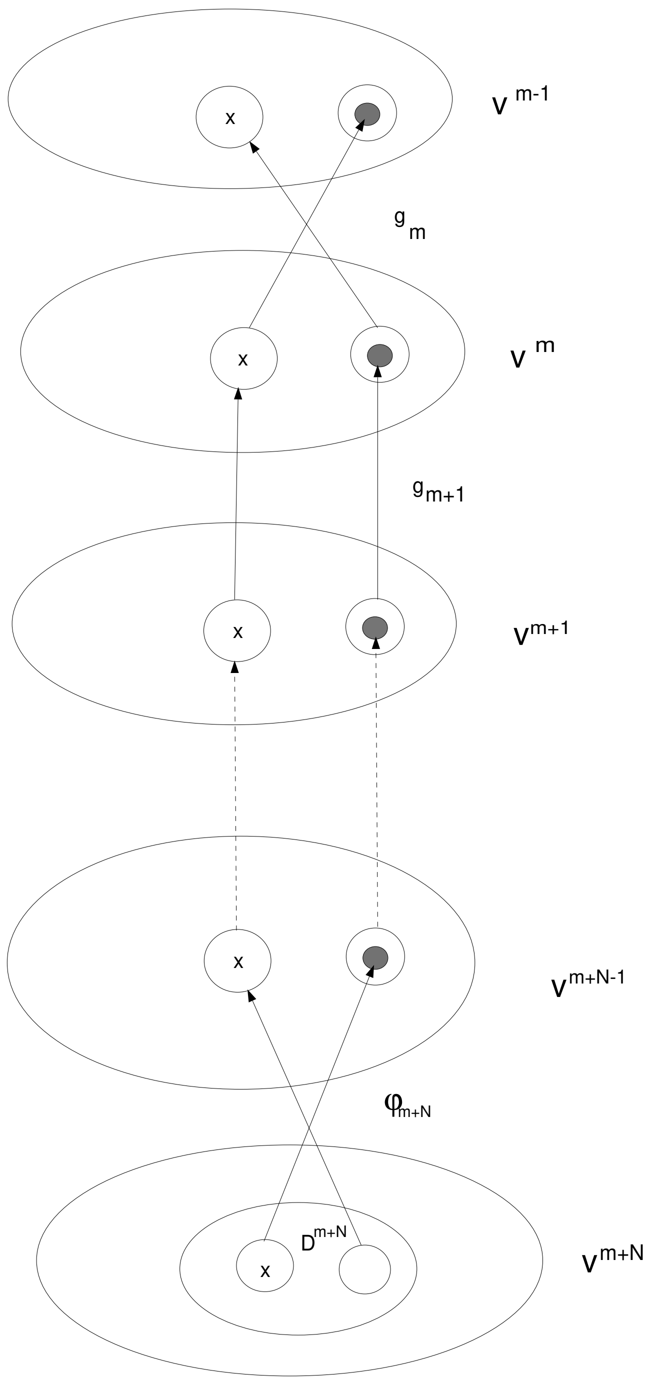

Figure 9: Central cascade
(with Fibonacci returns on top and bottom)

by letting $G|W_i^k = g_m \circ g_{m+1}^{m+1-k}|W_i^k$.

Let $V_*^{m+N+1} \subset D^{m+N}$ be a non-pre-critical piece of the family $\mathcal{I}^{m+N+1}$, and

$$\phi V_*^{m+N+1} \subset V_1^{m+N} \subset W_i^{m+N}$$

for some $i \neq 0$. Then the return map $g_{m+N+1} : V_*^{m+N+1} \to V^{m+N}$ can be decomposed as $G^l \circ \phi$ for an appropriate $l \geq 1$. Since the map G is Bernoulli with range V^m,

$$\mathrm{mod}\,(W_i^{m+N} \backslash \phi V_*^{m+N+1}) \geq \mathrm{mod}\,(V^m \backslash V^{m+N}) \qquad (5\text{-}23).$$

Let $\Gamma^k = \partial V^k$, and

$$w_k = \mathrm{width}\,(\Gamma^k | V^{k-1} \backslash V^{k+1}). \qquad (5\text{-}24)$$

For $k \in [m+1, m+N]$ let V_1^k denote the puzzle piece of level k containing $g_{m+1}^{\circ(k-m-N)} \phi V_*^{m+N+1}$, and $\mathcal{I}^k$ denote the family of two puzzle pieces: V_0^k and V_1^k. Moreover, let $R_i^k \subset V^{k-1}$ denote an annulus of maximal modulus going around V_i^k but not going around the other piece of family $\mathcal{I}^k$, $i = 0, 1$.

By the Definite Grötzsch Inequality and the second part of Theorem II, there is an $a = a(w_{m+1})$ such that

$$\begin{aligned} \mathrm{mod}\,(V^m \backslash V^{m+N}) &\geq \sum_{k=m+N}^{m+1} \mathrm{mod}\, A^k + a = \\ \sum_{k=0}^{N-1} \frac{1}{2^k} \mathrm{mod}(A^{m+1}) + a &\geq (2 - \frac{1}{2^{N-1}}) \mathrm{mod}\, R_0^{m+1} + a. \end{aligned} \qquad (5\text{-}25)$$

Let S_0^{m+N} and S_1^{m+N} denote the pull-backs of the annuli R_0^{m+1} and R_1^{m+1} by the map $g_{m+1}^{\circ(N-1)} : V^{m+N-1} \to V^m$. Then

$$\mathrm{mod}\, S_0^{m+N} \geq \frac{1}{2^{N-1}} \mathrm{mod}\, R_0^{m+1} \quad \text{and} \quad \mathrm{mod}\, S_1^{m+N} \geq \mathrm{mod}\, R_1^{m+1}. \qquad (5\text{-}26)$$

Note that the inner boundary of S_1^{m+N} coincides with the outer boundary of $W_i^{m+N} \backslash \phi V_*^{m+N+1}$. Let Q_1^{m+N} denote the union of these two annuli. This annulus goes around ϕV_*^{m+N+1} but not around V_0^{m+N}. Now estimates (5-23), (5-25), (5-26) yield

$$\mathrm{mod}\, S_0^{m+N} + \mathrm{mod}\, Q_1^{m+N} \geq 2\mathrm{mod}\, R_0^{m+1} + \mathrm{mod}\, R_1^{m+1} + a \geq 2\sigma(\mathcal{I}^{m+1}) + a.$$

Finally, pulling S_0^{m+N} and Q_1^{m+N} back by ϕ_{m+N} to the island D^{m+N} we obtain:

$$\sigma(\mathcal{I}^{m+N+1} | D^{m+N}) \geq \frac{1}{2}(\mathrm{mod}\, S_0^{m+N} + \mathrm{mod}\, Q_1^{m+N}) \geq \sigma(\mathcal{I}^{m+1}) + a/2. \qquad (5\text{-}27)$$

So we come up with the following statement:

Statement 5.7. *There is an increasing function $a : \mathbf{R}_+ \to \mathbf{R}_+$, $a(0) = 0$, such that for the cascade $\mathcal{C}^{m+N}$ estimate (5-21) holds with $a = a(w_{m+1})$, where $w_{m+1} =$* $\mathrm{width}\,(\Gamma^{m+1} | V^m \backslash V^{m+2})$.

Let us fix a quantifier w_* which distinguishes "small width" w from a "definite" one. For further analysis let us go several levels up. Let $m-1-l$ be the highest non-central level preceding $m-1$, $l \geq 1$. We are going to study when

$$\sigma(\mathcal{I}^{m+N+1} | D^{m+N}) \geq \sigma_{m-l} + a \qquad (5\text{-}28)$$

with a definite $a > 0$. We cannot now assume that l is bounded, so we face a possibility of a long cascade $\mathcal{C}^{m-1}:\ V^{m-l} \supset \ldots \supset V^{m-1}$. Set $g = g_{m-l+1}$; then $g0 \in V^{m-1}$.

Assume first that $m-2$ is not the last piece of a long cascade (in particular, this is the case when central return occurs on level $m-2$, that is, $l \geq 2$). Then by the third part of Theorem II all non-central pieces of level m are well inside V^{m-1}: $\operatorname{mod}(V^{m-1} \setminus V_j^m) \geq \bar{\mu}$. Hence $g_m V_0^{m+1}$ and $g_m V_1^{m+1}$ belong to different pieces of level m. Indeed, otherwise the hyperbolic distance between V_0^{m+1} and V_1^{m+1} in V^m would be bounded by a constant $L(\bar{\mu})$. But according to our assumption this distance is at least L_*. So this situation is impossible if L_* was a priori selected bigger than $L(\bar{\mu})$.

For the same reason the pieces $g^k \circ g_m V_i^{m+1}$, $i = 0, 1$, also belong to different pieces V_j^{m-k} for $0 \leq k \leq l-3$. Indeed, assume they belong to the same piece V_j^{m-k}. Clearly this piece is non-central, that is $j \neq 0$. Then it is contained in a piece W_j^{m-k} of the Bernoulli family $\mathcal{W}(\mathcal{C}^{m-1})$ associated to the central cascade $\mathcal{C}^{m-1}$. Hence $g_m V_0^{m+1}$ and $g_m V_1^{m+1}$ belong to W_j^m, the pull-back of W_j^{m-k} by g^k. As $\operatorname{mod}(W_j^m \setminus g_m V_i^{m+1}) \geq \bar{\mu}$, the hyperbolic distance between V_0^{m+1} and V_1^{m+1} in V^m is at most $L(\bar{\mu})$ contradicting our assumptions.

Let us show now that (5-28) holds if both $g_m V_i^{m+1}$ are non-central. Indeed let us then consider the family $\mathcal{I}^m$ of three pieces: two pieces of level m containing $g_m V_i^{m+1}$ and the central piece V^m. Let $\mathcal{I}^{m-k}$ denote the family of puzzle pieces of level $m-k$ containing the pieces of $g^k \mathcal{I}^m$. By the previous two paragraphs, $\mathcal{I}^{m-k}$ consists of three puzzle pieces. Then by Corollaries 5.5 and 5.7,

$$\sigma(\mathcal{I}^{m+1}) \geq \sigma(\mathcal{I}^m) \geq \ldots \geq \sigma(\mathcal{I}^{m-l+2}) \geq \sigma(\mathcal{I}^{m-l+1}) + \frac{1}{2}\bar{\mu},$$

and we are done.

Thus let us assume that the Fibonacci return occurs on level $m-1$. In this case let $\mathcal{I}^{m-k}$ denote the family of two puzzle pieces V_0^{m-k} and V_1^{m-k} containing $g^k \circ g_m V_i^{m+1}$, $i = 1, 2$, $k \leq l-1$.

Note that in order to have (5-28) it is enough to to have a definite increase of the $\sigma(\mathcal{I}^{m-k})$ in the beginning of the cascade $\mathcal{C}^{m-1}$. By Statement 5.7 applied to this cascade this is the case if $\operatorname{width}(\Gamma^{m-l+1} | V^{m-l} \setminus V^{m-l+2}) \geq w_*$. So assume that the opposite inequality holds. Similarly, because of Subcase (i), we can assume that the hyperbolic distance from V_1^{m-l+2} to 0 in V^{m-l+1} is at least L_*.

It follows from Lemma A.4 from the Appendix that the piece V_1^{m-l+2} stays Euclidean distance at most $\epsilon \operatorname{diam} \Gamma^{m-l+1}$ from Γ^{m-l+1} where $\epsilon = \epsilon_{\bar{\mu}}(w_*, L_*) \to 0$ as $w_* \to 0$, $L_* \to \infty$ (for a fixed $\bar{\mu} > 0$). It follows that the Euclidean distance from V_1^{m-l+2} to Γ^{m-l+1} is relatively small as compared with its distance from Γ^{m-l} and Γ^{m-l+2}. More precisely, there is a $\delta = \delta_{\bar{\mu}}(w_*, L_*)$ with the same properties as ϵ above such that for any $z \in V_1^{m-l+2}$,

$$\operatorname{dist}(z, \Gamma^{m-l+1}) \leq \delta \operatorname{dist}(z, \partial(V^{m-l} \setminus V^{m-l+2})) \tag{5-29}$$

Take $z_0 \in V_1^{m-l}$, and let $r = \operatorname{dist}(z_0, \partial(V^{m-l} \setminus V^{m-l+2})$. Note that the disk $B(z_0, r)$ can be univalently pulled by g^{l-3} to the annulus $V^{m-3} \setminus V^{m-1}$. By the Koebe Distortion

Theorem and (5-29), for any $\zeta \in V_1^{m-3}$

$$\operatorname{dist}(\zeta, \Gamma^{m-2}) \le C\delta \operatorname{dist}(\zeta, \partial(V^{m-3} \setminus V^{m-1})) \le C\delta \operatorname{dist}(\zeta, 0)$$

with an absolute constant C. All the more,

$$\operatorname{dist}(\zeta, \Gamma^{m-2}) \le C\delta \operatorname{diam} V^{m-2},$$

so that Γ^{m-2} has a big eccentricity about V_1^{m-2} (that is, this eccentricity is at least $e(w_*, L_*)$, where $e(w_*, L_*) \to \infty$ as $w_* \to 0$, $L_* \to \infty$).

Pulling Γ^{m-2} back by $g_{m+1} \circ g_m \circ g_{m-1}$, we conclude that Γ^{m+1} has a big eccentricity about 0. Hence it has big width in the annulus $V^m \setminus V^{m+2}$, and Statement 5.7 yields the desired.

Let us summarize the information which will be useful in what follows:

Statement 5.8. *If the width w_{m-l+1} is at most w_* and the Poincaré distance from V_1^{m-l+2} to 0 in V^{m-l+1} is at least L_*, then the eccentricity Γ^m about the origin is at least $e(w_*, L_*)$, where $e(w_*, L_*) \to \infty$ as $w_* \to 0$ and $L_* \to \infty$.*

Let us assume now that $m-2$ is the last piece of a long cascade

$$\mathcal{C}^{m-2} : V^{m-2-t} \supset \dots V^{m-2}, \quad t \ge N_*.$$

Then non-central return occurs on level $m-2$. We will show that

$$\sigma_(\mathcal{I}^{m+N+1} | D^{m+N}) \ge \sigma_{m-2-t} + a \tag{5-30}$$

with a definite $a > 0$.

Let $D^m \subset V^m$ be the island containing V_0^m and V_1^m, and $\phi_m : D_m \to V^{m-1}$ be the corresponding two-to-one map. Note that in the case under consideration this island may be non-trivial and still the Poincaré distance between V_0^{m+1} and V_1^{m+1} be big (since the pre-critical puzzle pieces in V^{m-1} are not well inside V^{m-1}). Moreover the map ϕ_m is not necessarily bounded perturbation of the quadratic map. These are the circumstances which make this case special.

As $m-2$ is a non-central level, $\mu_{m+1} \le \bar{\mu}$, and by the previous considerations we are done unless

- The return on level $m-1$ is Fibonacci, that is $\phi_m V_1^{m+1} = V_0^m$ and $\phi_m V_0^{m+1} \subset V_1^m$ for some puzzle piece V_1^m;
- The hyperbolic distance between the puzzle pieces V_0^m and V_1^m is at least L_*.
- The return on level $m-2$ is also Fibonacci: $g_{m-1} V_1^m = V_0^{m-1}$ and $g_{m-1} V_0^m = V_1^{m-1}$ for some puzzle piece V_1^{m-1}.

Let V_0^k and V_1^k be the pieces containing the corresponding push forwads of V_0^{m-1} and V_1^{m-1} along the cascade $\mathcal{C}^{m-1}$, $m-1 \le k \le m-t-1$. Then (5-30) follows unless

- The width w_{m-t-3} is at most w_*, and the distance between V_0^{m-t-4} and V_1^{m-t-4} in V^{m-t-3} is at least L_*.

But then by Statement 5.8 applied to the cascade $\mathcal{C}^{m-2}$ the eccentricity of Γ^{m-1} about 0 is at least $e = e(w_*, L_*)$. As g_m is a bounded perturbation of the quadratic map, by Lemma A.5 the curve Γ^m is $(0.1, \epsilon)$-pinched, where $\epsilon = \epsilon_{\bar{\mu}}(e) \to 0$ as $e \to \infty$. (Note that the pinched region is not necessarily around V_1^m, since ϕ_m may differ from

g_m). Applying Lemma A.5 again, we conclude that the curve Γ^{m+1} is $((10C)^{-1}, C\sqrt{\epsilon})$-pinched. By Lemma A.3 Γ^{m+1} has a definite width inside $V^m \backslash V^{m+2}$. Now Statement 5.7 yields (5-30).

5.6. Other factors yielding big space. Theorem III ensures that after many central cascades we will observe a big principal modulus. However, there are other combinatorial factors which yield the same effect. Altogether they are quite close to a "big renormalization period", except that "parabolic or Siegel cascades" may interfere.

Big return time implies big modulus. This section will rely on combinatorial considerations of §3.8. Recall that $\mathcal{I}^n$ denotes the family of puzzle pieces V_i^n intersecting $\omega(0)$. Given two Υ_f-adjacent puzzle pieces $V_j^{n+1} \in \mathcal{I}^{n-1}$ and $V_i^n \in \mathcal{I}^n$, let t be the first return time of V_j^{n+1} back to V_i^n under iterates of g_n. Then we will use the notation $\mathrm{mod}\,(V_j^{n+1} \to V_i^n)$ for $\mathrm{mod}\,(V_i^n \backslash g_n^t V_j^{n+1})$. If $i \neq 0$ then

$$\mathrm{mod}\,(V_j^{n+1} \to V_i^n) \geq \mu_n. \tag{5-31}$$

Lemma 5.9. *Let $D^n \supset V^{n-1}$ be a puzzle piece containing at least one piece of $\mathcal{I}^n$. Let γ be the shortest path leading from D^n down to some central piece V_0^{n+t}, and let D^{n+t} be the pull-back of D^n along this path. Then*

$$\mathrm{mod}\,(D^{n+t} \backslash V^{n+t}) \geq \frac{\bar{\mu}}{2} rank(D^n).$$

Proof. Let $D^{n+k} \subset V_{j(k)}^{n+k}$, $k = 0, 1, \ldots, t,$ be the pull-back of D^n along the path γ. By Lemma 3.11, all edges of this pull-back except the last one are univalent, and the last one is a double covering. Hence

$$\mathrm{mod}\,(D^{n+t} \backslash V^{n+t}) \geq \frac{1}{2} \sum_{k=1}^{t} \mathrm{mod}\,(V_{j(k)}^{n+k} \to V_{j(k-1)}^{n+k-1}),$$

and by (5-31) and Theorem II the right-hand side of this inequality is estimated from below by the right-hand side of the desired inequality. □

Lemma 5.10. *Let $n-1$ be not in the tail of a central cascade. Assume that for a puzzle piece $V_j^{n+1} \in \mathcal{I}^{n+1}$, $g_n-time(V_j^{n+1}) \geq r$. Then there is a level m such that $\mu_m \geq L(r)$, where $L(r) \to \infty$ as $r \to \infty$.*

Proof. Let $M > 0$. We need to find a level m with $\mu_m \geq M$, provided r is sufficiently big. If $rank(V_j^{n+1}) > M/\bar{\mu}$, then Lemma 5.9 yields the desired. So let us assume that

$$rank(V_j^{n+1}) \leq N \equiv M/\bar{\mu} + 1. \tag{5-32}$$

Let $0 = i(0), i(1), \ldots, i(r) = 0$ be the itinerary of V_j^{n+1} through the pieces of the previous level. Let us consider the nest of puzzle pieces

$$D_{r-1} \supset \ldots \supset D_0 \equiv V_i^n, \tag{5-33}$$

where $g_n^{r-k} D_k = V_{i(r-k)}^n$. Then

$$D_{k+1} \backslash D_k \geq \bar{\mu}/2, \quad k = 1, \ldots r-1. \tag{5-34}$$

Let us now pull the pieces D_k down along the shortest path γ joining V_j^{n+1} with a critical vertex V^{n+t}. Denote the corresponding pull-backs by D_k^{n+l}. If this pull-back turns out to be univalent then by (5-34)

$$\mu_{n+t} \geq \operatorname{mod}(D_r \backslash V_j^{n+1}) \geq r\bar{\mu}/2,$$

which is greater than M for sufficiently big r. Otherwise let us consider the first level $n+s$ where D_{r-1}^{n+s} hits the critical point. Let us find such an l that $0 \in D_{l+1}^{n+s} \backslash D_l^{n+s}$.

If $l > A \equiv 2M/\bar{\mu}$ then it follows from (5-33) that $\mu_{n+s} \geq M$, and we are done. Otherwise by (5-34) $\operatorname{mod}(D_l \backslash V_j^{n+1}) \geq (r-A)\bar{\mu}$. Let us now repeat the same procedure with D_l^{n+s} instead of D_{r-1}. Note that $rank(D_l^{n+s}) \leq rank(D_{r-1}) - 1$, since the pull-back of D_{r-1} along the first central cascade is univalent. Hence this procedure can be repeated at most N times, and the principal modulus at the end will be at least M, provided $(r - NA)\bar{\mu} \geq 2M$. □

Let us improve this lemma in the case of central cascades. Below we will use notions from §3.8.

Lemma 5.11. *Let us consider a central cascade (3-7) such that $m-1$ is not in the tail of a central cascade. Assume that $G_{m+N} - time(V_j^{m+N+1}) \geq r$ for some non-pre-critical puzzle piece V_j^{m+N+1}. Then there is a level m such that $\mu_m \geq M(r)$, where $M(r) \to \infty$ as $r \to \infty$.*

Proof. The argument is the same as for the previous lemma except one modification: To construct a nest (5-33), use the Bernoulli map G_{m+N} instead of g_{m+N}. □

Note that if $V^{m-1} \supset V^m \ldots \supset V^{m+N}$ is a central cascade then the first condition is satisfied for the sub-cascade $V^m \ldots \supset V^{m+N}$.

Parabolic and Siegel cascades. We will show that we usually will observe a big principal modulus after just one long central cascade. Let us consider a central cascade:

$$V^m \supset \ldots \supset V^{m+N-1} \supset V^{m+N}, \tag{5-35}$$

where $g_{m+1}0 \in V^{m+N-1} \backslash V^{m+N}$. The double covering $g_{m+1} : V^{m+1} \to V^m$ can be viewed as a small perturbation of a quadratic-like map g_* with a definite modulus and with non-escaping critical point.

To make this precise, let us consider the space $\mathcal{Q}$ of double coverings $g : U' \to U$, with $0 \in U' \subset\subset U$ and $g'(0) = 0$, modulo affine conjugacy. Let us supply it with the *Catathéodory topology* (see [McM]). Convergence in this topology means Carathéodory convergence of the domains and the ranges, and uniform convergence of the maps on compact subsets.

Given a $\mu > 0$, let $\mathcal{Q}(\mu)$ denote the set of double coverings $g \in \mathcal{Q}$ with $\operatorname{mod}(g) \geq \mu$. By Theorem II, the return maps $g_{m+1} : V^{m+1} \to V^m$ of the principal nest belong to $\mathcal{Q}(\bar{\mu})$.

Compactness Lemma (see [McM]). *The set $\mathcal{Q}(\mu)$ is Carathéodory compact.*

Let $\mathcal{Q}_N(\mu)$ denote the space of double coverings $g : U' \to U$ from $\mathcal{Q}(\mu)$ such that $g^n 0 \in U, n = 0, 1, \ldots N$. Note that $\mathcal{Q}_\infty(\mu)$ is the space of DH quadratic-like maps with modulus at least μ.

As $\bigcap_N \mathcal{Q}_N(\mu) = \mathcal{Q}_\infty(\mu)$, for any neighborhood $\mathcal{U} \supset \mathcal{Q}_\infty(\mu)$, there is an N such that $\mathcal{Q}_N(\mu) \subset \mathcal{U}$. In this sense any double map $g \in \mathcal{Q}_N(\mu)$ is close to some quadratic-like map g_*. In particular, this concerns the above return map g_{m+1} generating the cascade (5-35) of big length N. Moreover, since g_{m+1} has an escaping fixed point, the neighborhood of g_* containing g_{m+1} also contains a quadratic-like map with hybrid class $c(g_*) \in \partial M$.

If we have a sequence of maps $f_n \in \mathcal{Q}$ converging to a map $g_* \in \partial M$, we also say that the f_n*-central cascades converge to the* g_**-cascade.*

Let us say that the principal nest is minor modified if a piece V^m is replaced by a piece $\tilde{V}^n \subset V^n$ such that $\operatorname{cl} V_i^{n+1} \subset \tilde{V}^n$ for all pieces $V_i^{n+1} \in \mathcal{I}^{n+1}$.

Lemma 5.12. *Let g_* be a DH quadratic-like map with $c(g_*) \in \partial M$ which does not have neither parabolic points, nor Siegel disks. Let g_{m+1} be the return map of the principal nest generating cascade (5-35). Take an arbitrary big $M > 0$. If g_{m+1} is sufficiently close to g_* (depending on a priori bound $\bar{\mu}$ from Theorem II) then the principal nest can be minor modified in such a way that $\tilde{\mu}_n \geq M$ for some $n > m + N$.*

Proof. Take a big number $e > 0$.

By the above assumptions, the Julia set $J(g_*)$ has empty interior. If g_{m+1} is sufficiently close to g_* then $\Gamma^{m+N-1} = \partial V^{m+N-1}$ is close in the Hausdorff metric to the Julia set $J(g_*)$. Hence Γ^{m+N-1} has an eccentricity at least e with respect to any point $z \in V^{m+N-1}$.

As the g_m are quadratic maps up to bounded distortion (Theorem II), the curves Γ_{m+N}, Γ_{m+N+1} and Γ_{m+N+2} also have big eccentricity with respect to any enclosed point. Moreover, by the same theorem, there is a definite space in between these two curves. Hence by Lemma A.2, $\operatorname{mod}(V^{m+N+1} \setminus V^{m+N+3})$ is at least $M(e)$ where $M(e) \to \infty$ as $e \to \infty$.

Let us assume that non-central return occurs on level $m+N+1$: $g_{m+N+2} 0 \in V_i^{m+N+2}$ with $i \neq 0$. As the map $g_{m+N+2} : V_i^{m+N+2} \to V^{m+N+1}$ is quadratic up to bounded distortion, the curve $\Gamma_i^{m+N+2} = \partial V_i^{m+N+2}$ has a big eccentricity e' about any enclosed point (that is, e' can be made arbitrary big by a sufficiently big choice of e, depending on a priori bound $\bar{\mu}$). By Lemma A.2

$$\operatorname{mod}(V^{m+N+1} \setminus g_{m+N+2} V_0^{m+N+3}) \geq M(e),$$

where $M(e) \to \infty$ as $e \to \infty$. Hence $\operatorname{mod} A^{m+N+3} \geq M(e)/2$, and we are done.

Let the central return occurs on level $m+N+1$ but this is not yet a DH-renormalizable level. Then the corresponding central cascade is finite. Let $m + N + T$ be the last level of this cascade. Then by Statement 5.8 and Lemma A.2, $\mu_{m+N+T+2} \geq M(e)$, where $M(e) \to \infty$ as $e \to \infty$.

Assume finally that $m + N + 1$ is a DH-renormalizable level. Then let us take a horizontal curve $\Gamma \subset A^{m+N+2}$ which divides this annulus into two subannuli of moduli at least $\bar{\mu}/2$. Let $\Gamma' \subset A^{m+N+3}$ be its pull-back by g_{m+N+3}, and $\tilde{A}$ be the annulus bounded by Γ and Γ'. Then by Lemma A.2 $\operatorname{mod}(\tilde{A}) \geq M(e)$ with $M(e)$ as above. As this is a minor modification of the nest, we are done. □

5.7. Proof of Theorem IV. Let us fix a $Q > 0$. Take a truncated secondary limb $L \equiv L_b^{tr}$, and find $q = C(Q)\,\nu(L)$ from Theorem I. Note that $q(L, Q) \geq \nu(L)/2$ for sufficiently big Q (independently of L). Let us now select all copies M' of the Mandelbrot set with the height $\chi(M') \geq Q/B$, where $B = B(q)$ is the constant from Theorem III. Taking the union of all these copies over all truncated limbs, we obtain a desired special family $\mathcal{M}$.

Let us now consider an infinitely renormalizable DH quadraticlike map f of $\mathcal{S}$-type with $\operatorname{mod}(f) \geq Q$ (to start with, take a quadratic polynomial). Then by Theorem I, $\operatorname{mod}(A^1) \geq q$. Hence by Theorem III, $\operatorname{mod}(Rf) \geq B\,\chi(f) \geq Q$.

By induction, $\operatorname{mod}(R^n f) \geq R$ for all n. □

5.8. Variations. Let us now improve Theorem IV by taking into account not only the height but also the other factors yielding big space.

Theorem IV$'$. *Let f be an infinitely renormalizable quadratic polynomial, and let $P_m : z \mapsto z^2 + c_m$ be the straightened $R^m f$. Assume that*

- *All c_m are selected from a finite number of truncated secondary limbs L_i, $i = i, \dots s$;*
- *The set $\mathcal{A} \subset \mathcal{Q}$ of accumulation points of the central cascades of P_m (of lengths growing to ∞) does not contain parabolic or Siegel maps;*
- *$per(R^m f) \geq p$.*

Then $\liminf_{n\to\infty} \operatorname{mod}(R^n f) \geq Q(p)$*, where the function $Q(p)$ depends on the choice of the limbs and the accumulation set $\mathcal{A}$, and $Q(p) \to \infty$ as $p \to \infty$.*

Proof. By Theorem II the top modulus of the central cascades of P_m is bounded from below by some $\bar{\mu}$. Hence the set $\mathcal{A} \subset \mathcal{Q}(\bar{\mu})$ is compact. By Lemma 5.12, for any Q there is a neighborhood $\mathcal{U} \supset \mathcal{A}$ such that: If $f \in \mathcal{U}$ is renormalizable then $\operatorname{mod} Rf > Q$.

As $\mathcal{A}$ is the accumulation set for the central cascades of the P_m, there is an N such that all but finitely many of these cascades of length $\geq N$ belong to $\mathcal{U}$. Hence if the principal nest of P_m contains a cascade of length $\geq N$ then $\operatorname{mod}(R(P_m)) \geq Q$ (for sufficiently big m).

Further, by Theorems I and III, there is a χ such that if the height $\chi(P_m) \geq \chi$ then $\operatorname{mod}(R(P_m)) \geq Q$. Let us also find a T such that if for some cascade the return time from Lemma 5.11 is at least T, then $\operatorname{mod}(R(P_m)) \geq Q$.

It is easy to see that there is a p such that: If $per(P_m) \geq p$ then either P_m has a central cascade of length at least N, or $\chi(P_m) \geq \chi$, or one of the above return times is at least T. In any case $\operatorname{mod}(R(P_m)) \geq Q$.

Now the same argument as for Theorem IV yields a priori bounds. □

§6. Local connectivity of the Julia sets.

In this section we will show that the Julia sets of quadratic polynomials from Theorems IV and IV$'$ are locally connected. This follows from the moduli bounds in the full principal nest, which make puzzle pieces shrink to points (compare Yoccoz (see [H]), and Hu-Jiang [HJ], [J]). I thank J. Kahn and C. McMullen for useful discussions of this issue.

Theorem V. *Let $\mathcal{S}$ be a special family of Mandelbrot copies from Theorem V. Let f be an infinitely renormalizable quadratic of $\mathcal{S}$-type. Then the Julia set $J(f)$ is locally connected.*

Proof. Let us consider the full principal nest (3-9). Let $f_m \equiv R^m f$ and $J_m \equiv J(f_m)$. It follows from Theorem IV and the Grötcsz inequality that

$$\operatorname{mod}(Y^{0,0} \backslash J_m) \geq \sum_{k=0}^{m-1} \operatorname{mod} A^{m,0} \geq \epsilon m \to \infty \quad as \quad m \to \infty.$$

Hence the "little" Julia sets J_m shrink down to the critical point. Let us take an $\delta > 0$, and find an m such that J_m is contained in the $B(0, \delta)$.

Let us now inscribe into $B(0, \delta)$ a domain bounded by equipotentials and external rays of the original map f (compare Hu and Jiang [HJ], [J]). Let α_m denote the dividing fixed point of the Julia set J_m, and $\alpha'_m = -\alpha_m$ be the symmetric point. Let us consider a puzzle piece $P_m^0 \ni 0$ bounded by any equipotential and four external rays of the *original map* f landing at α_m and α'_m. This is a "degenerate" domain of the renormalized map f_m (see §2.5). By definition of the renormalized Julia set, the preimages $P_m^k \equiv f_m^{-k} P_m^0$ shrink down to J_m. Hence there is a puzzle piece P_m^l contained in the δ-neighborhood of the critical point. As $J(f) \cap P_m^l$ is clearly connected, the Julia set $J(f)$ is locally connected at the critical point.

Let us now prove local connectivity at any other point $z \in J(f)$. This is done by a standard spreading of the local information near the critical point around the whole dynamical plane. Let $f_m : U'_m \to U_m$, where $U_m \equiv V^{m,t(m)}$, $U'_m \equiv V^{m,t(m)+1}$ are domains from the principal nest (3-9). Let us consider two cases.

Case (i). Let the orbit of z accumulates on all Julia sets J_m. Find an $l = l(m)$ such that $P_m^l \subset U'_m$, and then take the first moment $k = k(m) \geq 0$ such that $f^k z \in P_m^l$. Let us consider the pull-back $V_m \supset Q_m^l \ni z$ of $U_m \supset P_m^l$ along the orbit $\operatorname{orb}_k(z) = \{z, ..., f^k z\}$. By Lemma 3.3, the pull-back of the puzzle piece P_m^l is univalent. Moreover, by construction of the principal nest, $U_m \backslash J_m$ does not intersect the critical set $\omega(0)$. Hence the pull-back of U_m along the $\operatorname{orb}_k(z)$ is also univalent.

Let $\Gamma_m \subset U_m \backslash U'_m$ be a horizontal curve in the annulus $U_m \backslash U'_m$ which divides it into two sub-annuli of modulus at least $\epsilon/2$. By the Koebe Theorem, it has a bounded eccentricity about 0 (with a bound depending on ϵ). Applying Koebe again, we conclude that its pull-back γ_m along $\operatorname{orb}_k(z)$ has a bounded eccentricity about z. Since the inner radius of this curve about z tends to 0 as $m \to \infty$ (follows from the fact that the sufficiently high iterates of any disk in $J(f)$ cover the whole $J(f)$), the $\operatorname{diam} \gamma_m \to 0$ as well. All the more, the $\operatorname{diam} Q_m^l \to 0$ as $m \to \infty$. As $Q_m^l \cap J(f)$ are connected, the Julia set is locally connected at z.

Case (ii). Assume now that the orbit of z does not accumulate on some J_m. Hence it accumulates on some point $a \notin \omega(0)$. Let us consider the puzzle associated with the periodic point α_m (so that the initial configuration consists of a certain equipotential and the external rays landing at α_m). Since the critical puzzle pieces shrink to J_m, the puzzle pieces Y_i^l of sufficiently big depth l containing a are disjoint from $\omega(0)$ (there are several such pieces if a is a preimage of α_m). Take such an l, and let X be the union of these puzzle pieces. It is a closed topological disk disjoint from $\omega(0)$ whose interior contains a. Hence there is a simply connected neighborhood $V \supset X$ still disjoint from $\omega(0)$. Consider now the moments $k_i \to \infty$ when the orbit of z lands at $\operatorname{int} X$, and pull $V \supset X$ back to z. As the pull-backs of V are univalent, the pull-backs of X shrink to

z (by the Koebe argument as above). It follows that $J(f)$ is locally connected at z. $\square$

§7. Forthcoming notes.

Let us briefly outline the content of the forthcoming notes. They will be mostly based on already existing preprints:

• Rigidity of quadratics of sufficiently big height (§4 of [L4]).

• Parapuzzle geometry: the moduli in the principal parameter nest grow at the same rate as in the dynamical nest. This also yields the rigidity of the corresponding quadratics. The Fibonacci case has been worked out by LeRoy Wenstrom (in preparation).

• Geometry of real quadratics ([L3] and [L4], §5). We give a criterion when the geometry of a real quadratic polynomial is "essentially bounded": It happens if and only if its "essential period" is bounded. On each level with sufficiently high essential period the renormalized map has a big modulus.

• We prove local connectivity of the Mandelbrot set at all real infinitely renormalizable points with sufficiently high essential period on all levels (following §5 of [L4]). Rigidity for all real quadratics follows (compare Swiatek [Sw]).

• Extension of Sullivan's complex a priori bounds onto infinitely renormalizable quadratics of essentially bounded type (joint work with Michael Yampolsky [LY]). Together with [L4] this yields complex bounds (and hence local connectivity of the Julia sets) for all real quadratics.

• Teichmüller metric on the space of quasi-quadratic maps following [L5], [L6].

• Applications to complex and real measurable dynamics.

Appendix: Conformal maps and geometry of curves.

A.1. Poincaré metric and distortion. A domain $D \subset \mathbf{C}$ is called *hyperbolic* if its universal covering space is conformally equivalent to the unit disk. This happens if and only if $\mathbf{C} \backslash D$ consists of at least two point. Hyperbolic domains possess the *hyperbolic (or Poincaré) metric* ρ_D of constant negative curveture. This metric is obtained by pushing down the Poincaré metric $d\rho_{\mathbf{D}} = |dz/(1-z^2)|$ from the unit disk $\mathbf{D}$.

In the case of a simply connected hyperbolic domain D ("conformal disk"), $d\rho_D = p_D(z)|dz|$ is the pull-back of the $\rho_{\mathbf{D}}$ by the Riemann mapping $D \to \mathbf{D}$. In this case its density $p_D(z)$ is comparable with $1/\mathrm{dist}\,(z, \partial D)$:

$$(1/4)\mathrm{dist}\,(z, \partial D)^{-1} \le p(z) \le \mathrm{dist}\,(z, \partial D)^{-1} \qquad \text{(A-1).}$$

In the simply connected case, a set $K \subset D$ has a bounded hyperbolic diameter $\mathrm{diam}_D K$ if and only if there is an annulus $A \subset D$ of definite modulus surrounding K. More precisely, let $\mu_{min}(R)$ and $\mu_{max}(R)$ denote the minimal and maximal possible modulus of an annulus $A \subset D$ surrounding K, where K runs over all subsets of hyperbolic diameter R. Then $0 < \mu_{min} < \mu_{max} < \infty$ (all estimates are clearly independent. of D). This can be readily seen by passing to the disk model and moving one point of K to the origin. The extremal moduli correspond to the cases of a pair of points and hyperbolic disk of radius R. Moreover, both minimal an maximal moduli behave as $\log(1/R)+0(1)$ (see [A], Ch. III).

Given a univalent holomorphic function $f: D \to \mathbf{C}$, the *distortion* of f on K is defined as

$$\sup_{z,\zeta\in K} \log\left|\frac{f'(z)}{f'(\zeta)}\right|.$$

Koebe Distortion Theorem. *Let D be a conformal disk, $K \subset D$, $r = \operatorname{diam}_D K$ be the Poincaré diameter of K in D. Then the distortion of any univalent function f on K is bounded by a constant $C_D(r)$ independent of a particular choice of K. Moreover $C_D(r) = O(r)$ as $r \to 0$.*

A.2. Moduli defect and capacity. Let D be a topological disk, $\Gamma = \partial D$, $a \in D$, and $\psi : (D,a) \to (\mathbf{D}_r, 0)$ be the Riemann map onto a round disk of radius r with $\psi'(a) = 1$. Then $r \equiv r_a(\Gamma)$ is called the *conformal radius* of Γ about a. The *capacity* of Γ rel a is defined as

$$\operatorname{cap}_a(\Gamma) = \log r_a(\Gamma).$$

Lemma A.1. *Let $D_0 \supset D_1 \supset K$, where D_i are topological disks and K is a connected compact. Assume that the hyperbolic diameter of K in D_0 and the hyperbolic dist $(K, \partial D_1)$ are both bounded by a L. Then there is an $\alpha(L) > 0$ such that*

$$\operatorname{mod}(D_1\backslash K) \le \operatorname{mod}(D_0\backslash K) - \alpha(L).$$

Proof. Let us take a point $z \in \partial D_1$ whose hyperbolic distance to K is at most L. Then there is an annulus of a definite modulus contained in D_0 and enclosing both K and z.

Let us uniformize $D_0\backslash K$ by a round annulus $A_r = \{\zeta : r < |\zeta| < 1\}$, and let $\tilde z$ correspond to z under this uniformization. Then $\tilde z$ stays a definite Euclidian distance d from the unit circle.

If $R \subset A_r$ is any annulus enclosing the inner boundary of A_r but not enclosing $\tilde z$ then by the normality argument $\operatorname{mod}(R) < \operatorname{mod}(A_r) - \alpha_r(d)$ with an $\alpha_r(d) > 0$. (Actually, the extremal annulus is just A_r slit along the radius from $\tilde z$ to the unit circle).

We have to check that $\alpha_r(d)$ is not vanishing as $r \to 0$. Let us fix an outer boundary Γ of B (the unit circle + the slit in the extremal case). We may certainly assume that the inner boundary coincides with the r-circle. Then the defect $\operatorname{mod}(R) - \log(1/r)$ monotonically increases to the $\operatorname{cap}_0(\Gamma)$. By normality this capacity is bounded above by an $-\alpha(d) < 0$, and we are done. $\square$

Let A be a standard cylinder of finite modulus, $K \subset A$. Define the $\operatorname{width}(K) \equiv \operatorname{width}(K|A)$) as the modulus of the smallest concentric sub-cylinder $A' \subset A$ containing K.

Definite Grötzsch Inequality. *Let A_1 and A_2 be homotopically non-trivial disjoint topological annuli in A. Let K be the set of points in their complement which are separated by $A_1 \cup A_2$ from the boundary of A. Then there is a function $\beta(x) > 0$ ($x > 0$) such that*

$$\operatorname{mod}(A) \ge \operatorname{mod}(A_1) + \operatorname{mod}(A_2) + \beta(\operatorname{width}(K)).$$

Proof. For a given cylinder this follows from the usual Grötzsch Inequality and the normality argument. Let us fix a K, and let $\mathrm{mod}\,(A) \to \infty$. We can assume that A_i are lower and upper components of $A \backslash K$ correspondingly. Then the modulus defect

$$\mathrm{mod}(A) - \mathrm{mod}(A_1) - \mathrm{mod}(A_2)$$

decreases by the usual Grötzsch inequality. At the limit the cylinder becomes the punctured plane, and the modulus defect converges to -(cap $_0(K)$+ cap $_\infty(K)$).

It follows from the area inequality that this sum of capacities is negative, unless K is a circle centered at the origin. Moreover the estimate depends only on $\mathrm{width}\,(K)$. Indeed, let D_0 and D_∞ be the components of $\mathbf{C} \setminus K$ containing 0 and ∞ correspondingly. Let $\phi_0 : B(0, R_0) \to D_0$ and $\phi_\infty : \mathbf{C}\backslash B(0, R_\infty) \to D_\infty$ be the Riemann mappings normalized by: $\phi_0(z) \sim z$ as $z \to 0$, and $\phi_\infty(z) \sim z$ as $z \to \infty$. Then $\mathrm{cap}_0 K = \log R_0$ and $\mathrm{cap}_\infty K = \log(1/R_\infty)$.

As scaling does not change $\mathrm{cap}_0(K) + \mathrm{cap}_\infty(K)$, we can assume that $R_\infty(K) = 1$. Let

$$\phi_\infty(z) = z - \sum_{k=1}^{\infty} \frac{a_k}{z^k}.$$

Then:

$$\mathrm{area}\,(\mathbf{C}\backslash D_\infty) = \frac{i}{2} \int_{|z|=1} \phi_\infty d\bar{\phi}_\infty = \pi(1 - \sum_{k=1}^{\infty} k|a_k|^2) \leq \pi,$$

with equality only in the case when $\phi_\infty = \mathrm{id}$. Hence $\mathrm{area}\,(D_0) \leq \pi$ with equality only in the case when K is the unit circle S^1.

As $|\phi_0|^2$ is a subharmonic function,

$$1 = |\phi_0(0)|^2 \leq \frac{\mathrm{area}\,(D_0)}{\pi R_0^2} \leq \frac{1}{R_0^2},$$

with equality only on the case when $K = S^1$. Hence $\mathrm{cap}_0(K) < 0$ unless $K = S^1$. Moreover, by a normality argument $\mathrm{cap}_0(K) \leq c(\mathrm{width}\,(K)) < 0$. (Indeed, otherwise there would be a sequence of domains D_0^m as above converging to a domain Ω different from the unit disk, with $\mathrm{cap}_0(\Omega) = 0$.) The lemma is proved. □

A.3. Eccentricity and pinching.

Let Γ be a Jordan curve surrounding a point a. Let $d_a(\Gamma)$ and $\rho_a(\Gamma)$ be the Euclidian radii of the inscribed and circumscribed circles about Γ centered at a. Then let us define the eccentricity of Γ about a as

$$e_a(\Gamma) = \log \frac{\rho_a(\Gamma)}{d_a(\Gamma)}.$$

Lemma A.2. *Let $A \subset \mathbf{C}\backslash\{0\}$ be an annulus homotopically non-trivially embedded in the punctured plane, $\Gamma \subset A$ be a homotopically non-trivial Jordan curve, and A_i be the components of $A\backslash\Gamma$. Assume that mod $(A_i) \geq \mu > 0$. If $e_0(K) \geq e$ then* $\mathrm{width}\,(\Gamma|A) \geq w(e)$*, where $w(e) \to \infty$ as $e \to \infty$.*

Proof. Assume that there is a sequence of annuli A^n and curves $\Gamma^m \subset A^m$ satisfying the assumptions of the lemma, such that $\mathrm{width}\,(\Gamma^m|A^m) \leq w$, while $e_0(\Gamma^m) \to \infty$. Let us consider the uniformization $\phi_m : \bar{A}^m \to A^m$ of the A^m by round annuli centered at 0. Let $\bar{\Gamma}^m = \phi^{-1}\Gamma^m$. Then $\bar{\Gamma}^m$ is contained in a round annulus $\bar{R}^m$ of modulus $\leq w$ cocentric with $\bar{A}^m$. Let $R^m = \phi\bar{R}^m$.

Let us normalize the annuli A^m and $\bar{A}^m$ (by scaling and rotation) so that the inner radii of R^m and $\bar{R}^m$ are equal to 1, and $\phi_m(1) = 1$. Passing to a subsequence (without change of notations) we can find cocentric annuli $\bar{R} \subset \bar{A}$ such that the inner radius of $\bar{R}$ is equal to 1, $\operatorname{mod}\bar{R} \leq w$, the both components of $\bar{A}\setminus\bar{R}$ have moduli at least $\alpha(\mu, w)$, and $\bar{R}^m \subset \bar{R}$, $\bar{A}^m \supset \bar{A}$.

By the Koebe Theorem, the family of functions ϕ_m is normal in A. Hence these functions are uniformly bounded on $\bar{R}$ contradicting the assumption that the eccentricities of Γ^m about 0 go to ∞. □

"Pinching" of a Jordan curve means creating of a narrow region which in limit makes the curve non-simple. Below we will quantify this process.

Let us take a number $0 < k < 1$ called the "pinching parameter". Let us define the k*-pinching* of a Jordan curve Γ as $\xi_k(\Gamma) = \inf \operatorname{dist}(z_1, z_2)$, where the infimum is taken over all pairs of points $z_i \in \Gamma$ such that both components Γ_1 and Γ_2 of $\Gamma\setminus\{z_1, z_2\}$ have diameter at least $k \operatorname{diam}\Gamma$.

We say that a curve Γ is (k, ϵ)*-pinched* if $\xi_k(\Gamma) < \epsilon$. Note that if the curve is symmetric about 0 and $e_0(\Gamma) \geq e$, then it is $(0.5 - e^{-1}, e^{-1})$-pinched.

The following lemma shows that a sufficiently pinched curve has a definite width:

Lemma A.3. *Let Γ, A and A_i be the same objects as in Lemma A.2. Let also mod $(A_i) \geq \mu > 0$. If Γ is (k, ϵ)-pinched, then there exists a $w = w(\mu, k) > 0$ such that* $\operatorname{width}(\Gamma|A) \geq w > 0$ *for all sufficiently small $\epsilon > 0$.*

Proof. Otherwise we can find a sequence A^m of annuli as above with $\operatorname{width}(\Gamma^m|A^m) \to 0$ as $m \to \infty$, and Γ^m is $(k, 1/m)$-pinched. As in the previous lemma, let $\phi_m : \bar{A}^m \to A^m$ be the uniformizations by round annuli normalized in such a way that Γ and $\bar{\Gamma} = \phi^{-1}\Gamma$ pass through 1.

Then the curves $\bar{\Gamma}^m$ should converge to the unit circle in the Hausdorff metric. Moreover, the family ϕ_m is well-defined and normal on a cocentric annulus $\bar{A}$ of modulus, say, $\sqrt{\mu}$. Hence any Hausdorff limit of the famlily of curves Γ^m is an analytic Jordan curve. On the other hand, these curves should be $(k, 0)$-pinched (that is, they are non-simple). Contradiction. □

Lemma A.4. *Let Γ, A and A_i be the same objects as in Lemma A.2, and mod $(A_i) \geq \mu > 0$. Let D be a topological disk bounded by Γ, and $b \in D$. Then there is a function $\delta(L) \to 0$ as $L \to \infty$ such that*

$$\operatorname{dist}(b, \partial\Gamma) \leq \delta(L) \operatorname{diam}\Gamma,$$

provided $\rho_D(0, b) \geq L$.

(Thus, if b is hyperbolically far away from 0 then it is Euclideanly close to the ∂D, in the scale of D.)

Proof. Otherwise there is a sequence of the curves $\Gamma^m = \partial D^m$ as above, and points

$b^m \in D^m$ such that $\operatorname{diam} D^m = 1$,

$$\rho_{D^m}(0, b_m) \to \infty, \tag{A-2),}$$

and

$$\operatorname{dist}(b^m, \Gamma^m) \geq \delta. \tag{A-3}$$

Passing to a Caratheodory limit along some subsequence (without change of notations), we have: $(D^m, 0, b^m) \to (D, 0, b)$, where D is a topological disk (note that $b \in D$ due to (A-3)). But then $\rho_{D^m}(0, b_m) \to \rho_D(0, b) < \infty$, contradicting (A-2). □

Lemma A.5. *Let Γ be a Jordan curve which does not pass through 0. If it is (k, ϵ)-pinched then its pull-back under the quadratic map $\Phi : z \mapsto z^2$ is $(C^{-1}k, C\sqrt{\epsilon})$-pinched, where $C > 1$ is an absolute constant.*

Proof. Let z_1 and z_2 be two points on Γ such that $\operatorname{dist}(z_1, z_2) < \epsilon$, while $\operatorname{diam} \Gamma_i > k$, where Γ_i are complementary components of $\Gamma \backslash \{z_1, z_2\}$. Let us mark the Φ-preimages of the corresponding objects with twilde (select the closest preimages of the points z_i).

We can assume that $\operatorname{diam} \Gamma = 1/4$. If $\operatorname{dist}(\Gamma, 0) > 1/4$ then the distortion of the quadratic map on Γ is bounded by an absolute constant, and the conclusion follows. Otherwise Γ is contained in the unit disk. Hence $1/2 \geq \operatorname{diam} \tilde{\Gamma} \geq 1/8$ and $\operatorname{diam} \tilde{\Gamma}_i \geq (1/2) \operatorname{diam} \Gamma_i \geq k/8$. Moreover, $\operatorname{dist}(\tilde{z}_1, \tilde{z}_2) \leq \sqrt{\epsilon}$, and we are done. □

References.

[A] L. Ahlfors. Lectures on quasi-conformal maps. Van Nostrand Co, 1966.

[At] P. Atela. Bifurcations of dynamical rays in complex polynomials of degree two. Ergod. Th. & Dynam. Sys., v. 12 (1991), 401-423.

[Be] A. Beardon. Iteration of rational functions. Graduate texts in mathematics. Springer-Verlag, 1991.

[B] B.Branner. The Mandelbrot set. In: "Chaos and fractals", Proceedings of Symposia in Applied Mathematics, v. 39, 75-106.

[BH] B.Branner & J.H.Hubbard. The iteration of cubic polynomials, Part II. Acta Math. v. 169 (1992), 229-325.

[CG] L. Carleson & W. Gamelin. Complex dynamics. Springer-Verlag, 1993.

[D1] A. Douady. Shirurgie sur les applications holomorphes. In: "Proc. Internat. Congress Math. Berkeley", 1986, v. 1, pp. 724-738.

[D2] A. Douady. Description of compact sets in **C**. In: "Topological Methods in Modern Mathematics, A Symposium in Honor of John Milnor's 60th Birthday", Publish or Perish, 1993.

[DH1] A.Douady & J.H.Hubbard. Étude dynamique des polynômes complexes. Publication Mathematiques d'Orsay, 84-02 and 85-04.

[DH2] A. Douady & J.H. Hubbard. On the dynamics of polynomial-like maps. Ann. Sc. Éc. Norm. Sup., v. 18 (1985), 287-343.

[GS] J. Graczyk & G. Swiatek. Induced expansion for quadratic polynomials. Stony Brook IMS Preprint 1993/8.

[GM] L. Goldberg & J. Milnor. Part II: Fixed point portraits. Ann. Sc. Éc. Norm. Sup.,

[H] J.H. Hubbard. Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz. In: "Topological Methods in Modern Mathematics, A Symposium in Honor of John Milnor's 60th Birthday", Publish or Perish, 1993.

[HJ] J. Hu & Y. Jiang. The Julia set of the Feigenbaum quadratic polynomial is locally connected. Preprint 1993.

[J] Y. Jiang. Infinitely renormalizable quadratic Julia sets. Preprint, 1993.

[LS] G. Levin & S. van Strien. Local connectivity of the Julia sets of real polynomials. Preprint 1995.

[L1] M. Lyubich. The dynamics of rational transforms: the topological picture. Russian Math. Surveys, v. 41 (1986), # 4, 43-117.

[L2] M. Lyubich. Milnor's attractors, persistent recurrence and renormalization, "Topological Methods in Modern Mathematics, A Symposium in Honor of John Milnor's 60th Birthday", Publish or Perish, 1993.

[L3] M. Lyubich. Combinatorics, geometry and attractors of quasi-quadratic maps. Preprint IMS at Stony Brook #1992/18, Annals of Math., v. 140 (1994), 347-404.

[L4] M. Lyubich. Geometry of quadratic polynomials: moduli, rigidity and local connectivity. Preprint IMS at Stony Brook #1993/9.

[L5] M. Lyubich. Teichmüller space of Fibonacci maps. Preprint IMS at Stony Brook, #1993/12.

[L6] M. Lyubich. Teichmüller space of quasi-quadratic maps. Manuscript in preparation.

[LY] M. Lyubich & M. Yampolsky. Dynamics of quadratic polynomials: Complex bounds for real maps, Manuscript 1995.

[LM] M. Lyubich & J.Milnor. The unimodal Fibonacci map. Journal of AMS, v. 6 (1993), 425-457.

[M1] J. Milnor. Dynamics in one complex variable: Introductory lectures, Stony Brook IMS Preprint # 1990/5.

[M2] J. Milnor. Local connectivity of Julia sets: expository lectures. Preprint IMS Stony Brook, #1992/11.

[M3] J. Milnor. Self-similarity and hairiness in the Mandelbrot set, pp. 211-257 of "Computers in geometry and topology", Lect. Notes in Pure Appl Math, v. 114, Dekker 1989.

[M4] J. Milnor. Periodic orbits, external rays and the Mandelbrot set: An expository account. Manuscript.

[McM] C. McMullen. Complex dynamics and renormalization. Preprint, 1993.

[Ma] M. Martens. Rotation numbers for minimal Cantor sets. Manuscript in preparation.

[MS] W. de Melo & S. van Strien. One dimensional dynamics. Springer-Verlag, 1993.

[P] C. L. Petersen. Local connectivity of some Julia sets containing a circle with an irrational rotation, Preprint IHES/M/94/26.

[R] M. Rees. A possible approach to a complex renormalization problem. In: "Linear and Complex Analysis Problem Book 3, part II", p. 437-440. Lecture Notes in Math., v. 1574.

[Sch] D. Schleicher. Internal addresses in the Mandelbrot set and irreducibility of polynomials. Thesis, 1994.

[Sw] G. Swiatek. Hyperbolicity is dense in the real quadratic family. Preprint IMS at Stony Brook, #1992/10.

[S1] D.Sullivan. Quasiconformal homeomorphisms in dynamics, topology and geometry. Proceedings of the ICM, Berkeley, 1986, v. 2, 1216.

[S2] D.Sullivan. Bounds, quadratic differentials, and renormalization conjectures. AMS Centennial Publications. **2**: Mathematics into Twenty-first Century (1992).

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