GENERALIZED KUMMER CONGRUENCES AND p-ADIC FAMILIES OF MOTIVES

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Abstract. We describe some new general constructions of p-adic L-functions attached to certain arithmetically defined complex L-functions coming from motives over $\mathbf{Q}$ with coefficiens in a number field T, $[T:\mathbf{Q}]<\infty$. These constructions are equivalent to proving some generalized Kummer congruences for critical special values of these complex L-functions.

The paper is based on some talks given by the auther during his visit to MSRI in February and March 1994. The purpose of this paper is to describe some new general constructions of p-adic L-functions attached to certain arithmetically defined complex L-functions coming from motives over $\mathbf{Q}$ with coefficiens in a number field T, $[T:\mathbf{Q}]<\infty$. These constructions are equivalent to proving some generalized Kummer congruences for critical special values of these complex L-functions.

The starting point in the theory of L-functions is the expansion of the Riemann zeta-function $\zeta(s)$ into the Euler product:

$$\zeta(s)=\prod_p (1-p^{-s})^{-1} \quad =\sum_{n=1}^{\infty} n^{-s} \qquad (\mathrm{Re(s)}>1).$$

The set of arguments s for which $\zeta(s)$ is defined can be extended to all $s\in\mathbf{C}, s\neq 1$, and we may regard $\mathbf{C}$ as the group of all continuous quasicharacters

$$\mathbf{C}=\ \mathrm{Hom}(\mathbf{R}_+^\times,\mathbf{C}^\times), \quad y\mapsto y^s$$

of $\mathbf{R}_+^\times$. The special values $\zeta(1-k)$ at negative integers are rational numbers:

$$\zeta(1-k)=-\frac{B_k}{k}\ ,$$

where B_k are Bernoulli numbers, which are defined by the formal power series equality

$$e^{Bt}=\sum_{n=0}^{\infty}\frac{B_n t^n}{n!}=\frac{t\,e^t}{e^t-1}\ ,$$

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and we know (by the classical Sylvester–Lipschitz theorem) that

$$c \in \mathbf{Z} \text{ implies } c^k(c^k-1)\frac{B_k}{k} \in \mathbf{Z}.$$

The theory of non-Archimedean zeta-functions originates in the work of Kubota and Leopoldt containing p-adic interpolation of these special values. Their construction turns out to be equivalent to classical Kummer congruences for the Bernoulli numbers, which we recall here in the following form. Let p be a fixed prime number, $c > 1$ an integer prime to p. Put

$$\zeta_{(p)}^{(c)}(-k) = (1-p^k)(1-c^{k+1})\zeta(-k)$$

and let $h(x) = \sum_{i=0}^{n} \alpha_i\, x^i \in \mathbf{Z}_p[x]$ be a polynomial over the ring $\mathbf{Z}_p$ of p-adic integers such that s

$$x \in \mathbf{Z}_p \Longrightarrow h(x) \in p^m \mathbf{Z}_p.$$

Then we have that

$$\sum_{i=0}^{n} \alpha_i\, \zeta_{(p)}^{(c)}(-i) \in p^m \mathbf{Z}_p.$$

This property expresses the fact that the numbers $\zeta_{(p)}^{(c)}(-k)$ depend continuously on k in the p-adic sense; it can be deduced from the known formula for the sum of k-th powers:

$$S_k(N) = \sum_{n=1}^{N-1} n^k \quad = \quad \frac{1}{k+1}[B_{k+1}(N) - B_{k+1}]$$

in which $B_k(x) = (x+B)^k = \sum_{i=0}^{k} \binom{k}{i} B_i x^{k-i}$ denotes the Bernoulli polynomial. Indeed, all summands in $S_k(N)$ depend p-adic analytically on k, if we restrict ourselves to numbers n, prime to p, so that the desired congruence follows if we express the numbers $\zeta_{(p)}^{(c)}(-k)$ in terms of Bernoulli numbers.

The domain of definition of p-adic zeta functions is the p-adic analytic Lie group

$$X_p = \mathrm{Hom}_{\mathrm{contin}}(\mathbf{Z}_p^\times, \mathbf{C}_p^\times)$$

of all continuous p-adic characters of the profinite group $\mathbf{Z}_p^\times$, where $\mathbf{C}_p = \widehat{\overline{\mathbf{Q}}}_p$ denotes the Tate field (completion of an algebraic closure of the p-adic field $\mathbf{Q}_p$), so that all integers k can be regarded as the characters $x_p^k: \; y \mapsto y^k$. The construction of Kubota and Leopoldt is equivalent to existence a p-adic analytic function $\zeta_p: \; X_p \to \mathbf{C}_p$ with a single pole at the point $x = x_p^{-1}$, which becomes a bounded holomorphic function on X_p after multiplication by the elementary factor $(x_p\, x - 1)$ $(x \in X_p)$, and is *uniquely determined* by the condition

$$\zeta_p({x_p}^k) = (1-p^k)\zeta(-k) \qquad (k \geq 1).$$

This result has a very natural interpretation in framework of the theory of non-Archimedean integration (due to Mazur): there exists a p-adic measure $\mu^{(c)}$ on $\mathbf{Z}_p^\times$ with values in $\mathbf{Z}_p$ such that $\int_{\mathbf{Z}_p^\times} x_p^k\, \mu^{(c)} \;=\; \zeta_{(p)}^{(c)}(-k)$. Indeed, if we integrate $h(x)$ over $\mathbf{Z}_p^\times$ we exactly get the above congruence. On the other hand, in order to define a measure $\mu^{(c)}$ satisfying the above condition it suffices for any continuous function $\phi : \mathbf{Z}_p^\times \to \mathbf{Z}_p$ to define its integral $\int_{\mathbf{Z}_p^\times} \phi(x)\, \mu^{(c)}$. For this purpose we approximate $\phi(x)$ by a polynomial (for which the integral is already defined), and then pass to the limit.

The important feature of the construction is that it equally works for primitive Dirichlet characters χ modulo a power of p: if we fix an embedding $i_p : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$ then the character $\chi : (\mathbf{Z}/\mathbf{Z}_{p^N})^\times \to (\overline{\mathbf{Q}})^\times$ becomes an element of the torsion subgroup $X_p^{\mathrm{tors}} \subset X_p$ and the above equality also holds for the special values $L(-k, \chi)$ of the Dirichlet L-series

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n)\, n^{-s} = \prod_p (1 - \chi(p) p^{-s})^{-1},$$

so that we have

$$\zeta_p(\chi x_p^k) = i_p[(1-\chi(p)p^k)L(-k,\chi)] \quad (k \geq 1,\ k \in \mathbf{Z},\ \chi \in X_p^{\mathrm{tors}}).$$

The original construction of Kubota and Leopoldt was successesfully used by Iwasawa for the description of the class groups of cyclotomic fields. Since then the class of functions admitting p-adic analogues has gradually extended.

L-functions (of complex variable) can be attached as certain Euler products to various objects such as diophantine equations, representations of Galois groups, modular forms etc., and they play a crucial role in modern number theory. Deep interrelations between these objects discovered in last decades are based on identities for the corresponding L-functions which presumably all fit into a general concept of the Langlands of L-functions associated with automorphic representations of a reductive group G over a number field K. ¿From this point of view the study of arithmetic properties of these zeta function is becoming especially important.

The major sources of such L-functions are:

1) *Galois representations* of $G_K = \mathrm{Gal}(\overline{K}/K)$ for algebraic number fields K, $r : G_K \to \mathrm{GL}(V)$, V a finite dimensional vector space, and one can attach to r an Euler product due to Artin.

2) *Algebraic varieties* X defned over an algebraic number field K. In this case one can attach to X/K its Hasse–Weil zeta function.

3) *Automorphic forms and automorphic representations.* In the classical case one associates to a modular form $f(z) = \sum_{n=0}^{\infty} a_n \exp(2\pi i n z)$ its Mellin transform $L(s,f) = \sum_{n=1}^{\infty} a_n n^{-s}$. In general an automorphic form generates an automorphic representations in the space of smooth functions over an adelic reductive group, and one can attach an Euler product to it using a decomposition of such a representation into a tensor product indexed by prime numbers p and ∞.

Conjecturally, all the three type of L-functions can be related to each other using a general theory of motives over $\mathbf{Q}$ with coefficiens in a number field T, $[T : \mathbf{Q}] < \infty$ (this field coincides with the field $\mathbf{Q}(\{a_n\}_{n\geq 1})$ generated by the coefficients of the corresponding L-function $L(M,s) = \sum_{n=1}^{\infty} a_n n^{-s}$.) For a fixed prime number p one can also attach in many cases to the above complex L-function a p-adic L-function. These

p-adic L-functions are certain analytic functions in a p-adic domain obtained by an interpolation procedure of certain special values of the corresponding complex analytic L-functions. Their existence is equivalent to certain generalized Kummer congruences for the special values.

We describe a general conjecture on the existence of a p-adic family M_P of motives, parametrised by some dense subset of algebraic characters P of a p-adic commutative algebraic group (which we call group of Hida). This group can be regarded as (a maximal torus of) the p-adic part $G_{M,p}$ of the motivic Galois group G_M of M (the Tannakian group for the tensor category generated by M). The important condition of motives M_P of the above family is that they have the same fixed p-invariant $h = h_P$ (generalized slope), which is defined as the difference between the Newton polygon and the Hodge polygon of a motive at certain point d^+ (the dimension of the subspace M^+ of the Betti realization M_B of M). The corresponding p-adic L-functions of this family can be unbounded in the "cyclotomic direction" (of Amice–Vélu type [Am-Ve]) but they form a family which is conjectually bounded in the "weight direction", that is for P parametrized by algebraic characters of $G_{M,p}$.

More precisely, the values of the function $P \mapsto L(M_P, 0)$ satisfy generalised Kummer congruences in the followinfg sense: for any finite linear combination $\sum_P b_P \cdot P$ with $b_P \in \mathbf{C}_p$ which has the property $\sum_P b_P \cdot P \equiv 0(\mathrm{mod} p^N)$ we have that for some constant $C \neq 0$ the corresponding linear combination of the normalized L-values

$$C \sum_P b_P c_p(M_P) \cdot \frac{L_{(p,\infty)}(M_P, 0)}{c_\infty(M_P)} \equiv 0(\mathrm{mod} p^N).$$

Here $c_p(M_P)$ and $c_\infty(M_P)$ denote a p-adic and a complex period of M_P so that the ratio "$\dfrac{c_p(M_P)}{c_\infty(M_P)}$" is uniquely defined, and $L_{(p,\infty)}(M_P, s)$ denotes the above L-function $L(M_P, s)$ normalized by multiplying by a certain canonicaly defined Deligne's p-factor corresponding to a choice of inverse roots $\alpha^{(1)}(p)$, …, $\alpha^{(d^+)}(p) \in \mathbf{C}_p$ of p-local polynomial of M such that

$$\mathrm{ord}_p(\alpha^{(1)}(p)) \leq \mathrm{ord}_p(\alpha^{(2)}(p)) \leq \ldots \mathrm{ord}_p(\alpha^{(d)}(p)),$$

d being the common rank of the family M_P, d^+ the T-dimension of the Deligne'a subspace M^+ of M_B (the fixed subspaces of the canonical involution ρ of M over T).

Recent examples of such families related to modular forms were constructed by R.Coleman [CoPBa] who proved the following

Theorem. *Suppose $\alpha \in \mathbf{Q}$ and $\varepsilon : (\mathbf{Z}/pZ)^\times \to \mathbf{C}_p^\times$ is a character. Then there exists a number n_0 which depends on p, N and ε, and α with the following property: If $k \in \mathbf{Z}$, $k > \alpha + 1$ and there is a unique normalized cusp form F on $X_1(Np)$ of weight k, character $\varepsilon\omega^{-k}$ and slope α and if $k' > \alpha + 1$ is an integer congruent tok modulo p^{n+n_0}, for any positive integer n, then there exists a unique normalized cusp form F' on $X_1(Np)$ of weight k', character $\varepsilon\omega^{-k'}$ and slope α (ω denotes the Teichmü ller character). Moreover his form satisfies the congruence*

$$F'(q) \equiv F'(q)(\mathrm{mod} p^{n+1}).$$

This result can be regarded as a generalization of the work of Hida [HiGal] who considered the case $\alpha = 0$ and constructed interesting families of Galois representations of the type

$$\rho_p : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_p[[T]]), \quad G_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}),$$

which are non ramified outside p. These representations have the following property: if we consider the homomorphisms

$$\mathbf{Z}_p[[T]] \xrightarrow{s_k} \mathbf{Z}_p, \quad 1+T \ \mapsto (1+p)^{k-1},$$

then we obtain a family of Galois representations

$$\rho_p^{(k)} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_p),$$

which is parametrized by $k \in \mathbf{Z}$, and for $k = 2, 3, \cdots$, these representations are equivalent over $\mathbf{Q}_p$ to the p-adic representations of Deligne, attached to modular forms of weight k. This means that the representations of Hida are obtained by the p-adic interpolation of Deligne's representations. A geometric interpretation of Hida's representations was given by Mazur and Wiles [Maz–W], cf. [Maz]. For example, for the modular form Δ of weight 12 Hida has constructed a representation

$$\rho_{p,\Delta} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_p[[T]]),$$

as an example of his general theory, where the prime number p have the property $\tau(p) \not\equiv 0(\mathrm{mod} p)$ (e.g. $p < 2041, p \neq 2, 3, 5$ and 7).

The boundednes property is the subject of a research by G.Stevens and of a forthcoming paper of B.Mazur and F.Q.Gouvêa.

Note that other examples may include Rankin products, Garrett triple products of elliptic and Hilbert modular forms and standard L-functions of Siegel modular forms.

To describe this conjecture more precisely, let M is a motive over $\mathbf{Q}$ of with coefficients in T i.e.

$$M_B, \ M_{DR}, \ M_\lambda, \ I_\infty, \ I_\lambda,$$

where M_B is the Betti realization of M which is a vector space over T of dimension d endowed with a T-rational involution ρ ; $M_B = M^+ \oplus M^-$ denotes the corresponding decomposition into the sum of $(+1)$-eigenspaces and (-1)-eigenspaces of ρ.

M_{DR} is the de Rham realization of M, a free T-module of rank d, endowed with a decreasing filtration $\{F^i_{DR}(M) \subset M_{DR} \mid i \in \mathbf{Z}\}$ of T-modules;
M_λ is the λ-adic realization of M at a finite place λ of the coefficient field T (a T_λ-vector space of degree d over T_λ, a completion of T at λ) which is a Galois module over $G_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ so that we a have a compatible system of λ-adic representations denoted by

$$r_{M,\lambda} = r_\lambda : G_{\mathbf{Q}} \to GL(M_\lambda).$$

Also,

$$I_\infty : M_B \otimes_T \mathbf{C} \to M_{DR} \otimes_T \mathbf{C}$$

is the complex comparison isomorhism of complex vector spaces

$$I_\lambda : M_B \otimes_T T_\lambda \to M_\lambda$$

is the λ-adic comparison isomorphism of T_λ-vector spaces. It is assumed in the notation that the complex vector space $M_B \otimes_{\mathbf{Q}} \mathbf{C}$ is decomposed in the Hodge bigraduation

$$M_B \otimes_T \mathbf{C} = \oplus_{i,j} M^{i,j}$$

in which $\rho(M^{i,j}) \subset M^{j,i}$ and

$$h(i,j) = h(i,j,M) = \dim_{\mathbf{C}} M^{i,j}$$

are the Hodge numbers. Moreover,

$$I_\infty(\oplus_{i' \geq i} M^{i',j}) = F^i_{DR}(M) \otimes \mathbf{C}.$$

Also, I_λ takes ρ to the r_λ-image of the Galois automorphism which corresponds to the complex conjugation of $\mathbf{C}$. We assume that M is pure of weight w (i.e. $i + j = w$).

The L-function $L(M,s)$ of M is defined as the following Euler product:

$$L(M,s) = \prod_p L_p(M, p^{-s}),$$

extended over all primes p and where

$$\begin{aligned}&L_p(M,X)^{-1} = \det\left(1 - X \cdot r_\lambda(Fr_p^{-1}) \mid M_\lambda^{I_p}\right) = \\ &(1 - \alpha^{(1)}(p)X) \cdot (1 - \alpha^{(2)}(p)X) \cdot \ldots \cdot (1 - \alpha^{(d)}(p)X) = \\ &1 + A_1(p)X + \ldots + A_d(p)X^d;\end{aligned}$$

here $Fr_p \in G_{\mathbf{Q}}$ is the Frobenius element at p, defined modulo conjugation and modulo the inertia subgroup $I_p \subset G_p \subset G_{\mathbf{Q}}$ of the decomposition group G_p (of any extension of p to $\overline{\mathbf{Q}}$). We make the standard hypothesis that the coefficients of $L_p(M,X)^{-1}$ belong to T, and that they are independent of λ coprime to p. Therefore we can and we shall regard this polynomial both over $\mathbf{C}$ and over $\mathbf{C}_p$. We shall need the following twist operation: for an arbitrary motive M over $\mathbf{Q}$ with coefficients in T an integer m and a Hecke character χ of finite order one can define the twist $N = M(m)(\chi)$ which is again a motive over $\mathbf{Q}$ with the coefficient field $T(\chi)$ of the same rank d and weight w so that we have

$$L(N,s) = \prod_p L_p(M, \chi(p)p^{-s-n}).$$

The group of Hida and the algebra of Iwasawa–Hida. Now let us fix a motive M with coefficients in $T = \mathbf{Q}(\langle a(n)_n \rangle)$ of rang d and of weight w, and let $\mathrm{End}_T M$ denote the endomorphism algebra of M (i.e. the algebra of T-linear endomorphisms of any M_B, which commute with the Galois action under the comparison isomorphisms). Let

$$\mathrm{Gal}_p = \mathrm{Gal}(\mathbf{Q}_{p,\infty}^{ab}/\mathbf{Q})$$

denotes the Galois group of the maximal abelian extension $\mathbf{Q}_{p,\infty}^{ab}$ of $\mathbf{Q}$ unramified outside p and ∞. Define $\mathcal{O}_{T,p} = \mathcal{O}_T \otimes \mathbf{Z}_p$.

Definition. *The group of Hida* $GH_M = GH_{M,p}$ *is the following product*

$$GH_M = (\mathrm{End}_T M)^\times(\mathcal{O}_{T,p}) \times \mathrm{Gal}_p,$$

where $(\mathrm{End}_T M)^\times$ *denotes (a maximal torus of) the algebraic* T*-group of invertible elements of* $\mathrm{End}_T M$ *(it is implicitely supposed that the group* $\mathrm{End}_T M^\times$ *posesses an* $\mathcal{O}_T$*-integral structure given by an appropriate choice of an* $\mathcal{O}_T$*-lattice).*

Consider next the $\mathbf{C}_p$-analytic Lie group

$$\mathcal{X}_{M,p} = \mathrm{Hom}_{contin}(GH_M,\ \mathbf{C}_p^\times)$$

consisting of all continuous characters of the Hida group GH_M, which contains the $\mathbf{C}_p$-analytic Lie group

$$\mathcal{X}_p = \mathrm{Hom}_{contin}(\mathrm{Gal}_p,\ \mathbf{C}_p^\times)$$

consisting of all continuous characters of the Galois group Gal_p (via the projection of GH_M onto Gal_p.

The group $\mathcal{X}_{M,p}$ contains the discrete subgroup $\mathcal{A}$ of arithmetical characters of the type

$$\chi \cdot \eta \cdot x_p^m = (\chi, \eta, m),$$

where

$$\chi \in \mathcal{X}_{M,p}^{tors}$$

is a character of finite order of GH_M, η is a T-algebraic character of $(\mathrm{End}_T M)^\times(\mathcal{O}_{T,p})$, $m \in \mathbf{Z}$, and x_p denotes the following natural homomorphism

$$x_p : \mathrm{Gal}_p = \mathrm{Gal}(\mathbf{Q}_{p,\infty}^{ab}/\mathbf{Q}) \cong \mathbf{Z}_p^\times \to \mathbf{C}_p^\times,\ x_p \in \mathcal{X}_p.$$

Definition. *The algebra of Iwasawa–Hida* $I_M = I_{M,p}$ *of* M *at* p *is the completed group ring* $\mathcal{O}_p[[GH_M]]$*, where* $\mathcal{O}_p$ *denotes the ring of integers of the Tate field* $\mathbf{C}_p$*.*

Note that this definition is completely analogous to the usual definition of the Iwasawa algebra Λ as the completed group ring $\mathbf{Z}_p[[\mathbf{Z}_p]]$ if we take into account that $\mathbf{Z}_p$ coincides with the factor group of $\mathbf{Z}_p^\times$ modulo its torsion subgroup.

Now for each arithmetic point $P = (\chi, \eta, m) \in \mathcal{A}$ we have a homomorphism

$$\nu_P : I_{M,p} \to \mathcal{O}_p$$

which is defined by the corresponding group homomorphism

$$P : GH_M \to \mathcal{O}_p^\times \subset \mathbf{C}_p^\times.$$

For a I_M-module N and $P \in \mathcal{A}$ we put

$$N_P = N \otimes_{I_M, \nu_P} \mathcal{O}_p$$

("reduction of N modulo P", or a fiber of N at P).

Therefore, for a Galois representation

$$r_N : G_{\mathbf{Q}} \to \mathrm{GL}(N)$$

of the above type its reduction $r_{N_P} = r \bmod P$ is defined as the natural composition:

$$G_{\mathbf{Q}} \to \mathrm{GL}(N) \to \mathrm{GL}(N_P).$$

Remark. In his very recent work [HiGen] Hida gives another version of the above definition, but he starts from a Galois representation $\varphi : \mathrm{Gal}(\overline{F}/F) \longrightarrow GL_n(\mathbf{I})$, where $\mathbf{I} = \mathcal{O}_K[[T_n(\mathbf{Z}_p]]$ and T_n the maximal split torus of $\mathrm{Res}_{\mathcal{O}_F/\mathbf{Z}} GL(n)$ for the integer ring $\mathcal{O}_F$ of F, and for the integer ring $\mathcal{O}_K$ of a suffficiently large finite extension K of $\mathbf{Q}_p$. He is interested in representations φ satisfying the following condition:

> *There are arithmetic points P "densly populated" in* $\mathrm{Spec}(\mathbf{I}(K)$ *such that the Galois representation $\varphi_P = P \circ \varphi$ is the p-adic étale realization of a rank n pure motive M_P of weight w defined over F with coefficients in a number field E_P in $\overline{Q}$.*

We are trying to resolve an inverse problem and to include a given motive M in a maximal possible p-adic family M_P parametrized by arithmetic characters of a certain group which we suppose to consist of an "algebraic part" $(\mathrm{End}_T M)^\times(\mathcal{O}_{T,p})$ and of a "Galois part" Gal_p.

A conjecture on the existence of p-adic families of Galois representations attached to motives. Note first that the fixed embeddings $T \hookrightarrow \mathbf{C}$,

$$i_\infty : \overline{\mathbf{Q}} \to \mathbf{C}, \quad i_p : \overline{\mathbf{Q}} \to \mathbf{C}_p$$

define a place $\lambda(p)$ of T attached to the corresponding composition

$$T \hookrightarrow \overline{\mathbf{Q}} \overset{i_p}{\to} \mathbf{C}_p.$$

Conjecture I. *For every M of rang d with coefficients in T there esists a free I_M-module M_I of the same rang d, a Galois representation*

$$r_I : G_F \to \mathrm{GL}(M_I),$$

a dense subset $\mathcal{A}' \subset \mathcal{A}$ of characters, and a distinguished point $P_0 \in \mathcal{A}$ such that
(a) the reduced Galois representation

$$r_{I,P_0} : G_F \to \mathrm{GL}(M_{I,P_0})$$

is equivalent over $\mathbf{C}_p$ to the $\lambda(p)$-adic representation $r_{M,\lambda(p)}$ of M at the distinguished place $\lambda(p)$;

(b) for every $P \in \mathcal{A}'$ there exists a motive M_P over $\mathbf{Q}$ of the same rang d such that its $\lambda(p)$-adic Galois representation is equivalent over $\mathbf{C}_p$ to the reduction

$$r_{I,P} : G_{\mathbf{Q}} \to \mathrm{GL}(M_{I,P}).$$

We call the module M_I the *realization of Iwasawa* of M.

A generalization of the Hasse invariant for a motive. We define the generalized *Hasse invariant* of a motive in terms of the *Newton polygons* and the *Hodge polygons* of a motive. Properties of these polygons are closely related to the notions of a p-ordinary and a p-admissible motive.

Now we are going to define the Newton polygon $P_{Newton}(u) = P_{Newton}(u, M)$ and the Hodge polygon $P_{Hodge}(u) = P_{Hodge}(u, M)$ attached to M. First we consider (using i_∞) the local p-polynomial

$$L_p(M, X)^{-1} = 1 + A_1(p)X + \cdots + A_d(p)X^d$$

$$= (1 - \alpha^{(1)}(p)X) \cdot (1 - \alpha^{(2)}(p)X) \cdot \ldots \cdot (1 - \alpha^{(d)}(p)X),$$

and we assume that its inverse roots are indexed in such a way that

$$\mathrm{ord}_p\alpha^{(1)}(p) \leq \ \mathrm{ord}_p\alpha^{(2)}(p) \leq \cdots \leq \ \mathrm{ord}_p\alpha^{(d)}(p)$$

Definition. *The Newton polygon $P_{Newton}(u)(0 \leq u \leq d)$ of M at p is the convex hull of the points $(i, \mathrm{ord}_p A_i(p))$ $(i = 0, 1, \cdots, d)$.*

The important property of the Newton polygon is that the length the horizontal segment of slope $i \in \mathbf{Q}$ is equal to the number of the inverse roots $\alpha^{(j)}(p)$ such that $\mathrm{ord}_p\alpha^{(j)}(p) = i$ (note that i may not necessarily be integer but this will be the case for the p-ordinary motives below).

The *Hodge polygon* $P_{Hodge}(u)$ $(0 \leq u \leq d)$ of M is defined using the Hodge decomposition of the d-dimensional $\mathbf{C}$-vector space

$$M_B = M_B \otimes_T \mathbf{C} = \oplus_{i,j} M^{i,j}$$

where $M^{i,j}$ as a $\mathbf{C}$-subspace.

Definition. *The Hodge polygon $P_{Hodge}(u)$ is a function $[0, d] \to \mathbf{R}$ whose graph consists of segments passing through the points*

$$(0, 0), \ \ldots, \ (\sum_{i' \leq i} h(i', j), \ \sum_{i' \leq i} i' h(i', j)),$$

so that the length of the horizontal segment of the slope $i \in \mathbf{Z}$ is equal to the dimension $h(i, j)$.

Now we recall the definition of a p-ordinary motive (see [Co], [Co–PeRi]). We assume that M is pure of weight w and of rank d. Let G_p be the decomposition group (of the place $\lambda(p)$ in T over p) and

$$\psi_p : G_p \to \mathbf{Z}_p^\times$$

be the cyclotomic character of G_p. Then M is called p ordinary at p if the following conditions are satisfied:
(i) The inertia group $I_p \subset G_p$ acts trivially on each of the l-adic realizations M_l for $l \neq p$;
(ii) There exists a decreasing filtration $F_p^i V$ on $V = M_p = M_B \otimes \mathbf{Q}_p$ of $\mathbf{Q}_p$-subspaces which are stable under the action of G_p such that for all $i \in \mathbf{Z}$ the group G_p acts on $F_p^i V / F_p^{i+1} V$ via some power of the cyclotomic character, say $\psi_p^{-e_i}$. Then

$$e_1(M) \geq \cdots \geq e_t(M)$$

and the following properties take place:

(a)
$$\dim_{\mathbf{Q}_p} F_p^i V / F_p^{i+1} V = h(e_i, w - e_i);$$

(b) The Hodge polygon and the Newton polygon of M coincide:

$$P_{Newton}(u) = P_{Hodge}(u).$$

If furthermore M is critical at $s = 0$ then it is easy to verify that the number d_p of the inverse roots $\alpha^{(j)}(p)$ with

$$\mathrm{ord}_p \alpha^{(j)}(p) < 0 \text{ is equal to } d^+ = d^+(M) \text{ of } M_B^+.$$

However, it turns out that the notion of a p-ordinary motive is too restrictive, and we have introduced the following weaker version of it.

Definition. *The motive M over F with coefficients in T is called admissible at p if*

$$P_{Newton}(d^+) = P_{Hodge}(d^+)$$

here $d^+ = d^+(M)$ is the dimension of the subspace $M^+ \subset M_B$. In the general case we use the following quantity (a "generalized slope") $h = h_p$ which is defined as the difference between the Newton polygon and the Hodge polygon of M:

$$h_p = P_{Newton}(d^+) - P_{Hodge}(d^+).$$

of M at p. Note the following important properties of h:
(i) $h = h(M)$ does not change if we replace M by its Tate twist.
(ii) $h = h(M)$ does not change if we replace M by its twist $M = M(\chi)$ with a Dirichlet character χ of finite order whose conductor is prime to p.
(iii) $h = h(M)$ does not change if we replace M by its dual $M^\vee$

In the next section we state in terms of this quantity a general conjecture on p-adic L-functions.

A conjecture on the existence of certain families of p-adic L-functions. We are going to describe families of p-adic L-functions as certain analytic functions on the total analytic space, the $\mathbf{C}_p$-analytic Lie group

$$\mathcal{X}_{M,p} = \mathrm{Hom}_{contin}(GH_M,\ \mathbf{C}_p^\times),$$

which contain the $\mathbf{C}_p$-analytic Lie subgroup (the cyclotomic line) $\mathcal{X}_p \subset \mathcal{X}_{M,p}$:

$$\mathcal{X}_p = \mathrm{Hom}_{contin}(\mathrm{Gal}_p,\ \mathbf{C}_p^\times).$$

In order to do this we need a modified L-function of a motive. Following J. Coates this modified L-function has a form appropriate for further use in the p-adic construction. First we multiply $L(M,s)$ by an appropriate factor at infinity and define

$$\Lambda_{(\infty)}(M,\ s) = E_\infty(M,\ s)L(M,s)$$

where $E_\infty(M,s) = E_\infty(\tau, R_{F/\mathbf{Q}}M, \rho, s)$ is the modified Γ-factor at infinity which actually does not depend on the fixed embedding τ of T into $\mathbf{C}$. Also we put

$$c^\nu(M) = (c^\nu(M)^{(\tau)}) = c^\nu(RM)(2\pi i)^{r(RM)} \in (T \otimes \mathbf{C})^\times$$

where

$$\nu = (-1)^m, r(M) = \sum_{j<0} jh(i,j) = \sum_{j<0} jh(i,j),$$

$c^\nu(M)$ is the period of M. Note that the quantity $r(M)$ has a natural geometric interpretation as the minimum of the Hodge polygon $P_{Hodge}(M)$.

We define

$$\Lambda_{(p,\infty)}(M(m)(\chi),s) = \\ G(\chi)^{-d^{\varepsilon_0}(M(m)(\chi))} \prod_{p|p} A_p(M(m)(\chi),s) \cdot \Lambda_{(\infty)}(M(m)(\chi),s),$$

where

$$A_p(M(\chi),s) = \begin{cases} \prod_{i=d^++1}^{d}(1-\chi(p)\alpha^{(i)}(p)p^{-s}) \prod_{i=1}^{d^+}(1-\chi^{-1}(p)\alpha^{(i)}(p)^{-1}p^{s-1}) \\ \qquad\qquad\qquad\qquad \text{for } p \not| \mathfrak{c}(\chi) \\ \prod_{i=1}^{d^+} \left(\dfrac{p^s}{\alpha^{(i)}(p)}\right)^{\mathrm{ord}_p \mathfrak{c}(\chi)}, \text{otherwise.} \end{cases}$$

Let $\mathcal{A}$ be the discrete subgroup $\mathcal{A}$ of arithmetical characters,

$$\chi \cdot \eta \cdot x_p^m = (\chi, \eta, m) \in \mathcal{A},$$

$\mathcal{A}' \subset \mathcal{A}$ a certain "dense" subset of characters, $P_0 \in \mathcal{A}$ a distinguished point of conjecture I. Let $\mathcal{A}'' \subset \mathcal{A}'$ be the subset of critical elements, which consists of those P, for which the corresponding motives M_P are critical (at $s=0$). Now we are ready to formulate the following

Conjecture II. *There exists a certain choice of complex periods $\Omega_\infty(P) \in \mathbf{C}^\times$ and p-adic periods $\Omega_p(P) \in \mathbf{C}_p^\times$ for all $P \in \mathcal{A}''$ such that "the ratio" $\Omega_p(P)/\Omega_\infty(P)$ is canonicaly defined, and there exists a $\mathbf{C}_p$-meromorphic function*

$$\mathcal{L}_M : \mathcal{X}_{M,p} \to \mathbf{C}_p$$

with the properties:

(i)

$$\mathcal{L}_M(P) = \Omega_p(M)\frac{\Lambda_{p,(\infty)}(M(m)(\chi),0)}{\Omega_\infty(P)}$$

for almost all $P \in \mathcal{A}''$;

(ii) For arithmetic points of type

$$P = (\chi, \eta, m) \in \mathcal{A}''$$

with η fixed there exists a finite set $\Xi \subset \mathcal{X}_{M,p}$ of p-adic characters and positive integers $n(\xi)$ (for $\xi \in \Xi$) such that for any $g_0 \in \mathrm{Gal}_p$ we have that the function

$$\prod_{\xi \in \Xi}(x(g_0) - \xi(g_0))^{n(\xi)}\mathcal{L}_M(x \cdot P)$$

is holomorphic on $\mathcal{X}_p$;

(iii) For arithmetic points of type

$$P = (\chi, \eta, m) \in \mathcal{A}''$$

with η fixed the function in (ii) is bounded if and only if the invariant $h(P) = h(M_P)$ vanishes;

(iv) In the general case the function $\mathcal{L}_M(P \cdot x)$ of $x \in \mathcal{X}_p$ is of logarithmic growth type $o(\log(\cdot)^h{}_0)$ with

$$h_0 = [h] + 1.$$

(v) For arithmetic points of type

$$P = (\chi, \eta, m) \in \mathcal{A}''$$

with χ and m fixed the function in (ii) is always bounded if the Hasse invariant $h(P) = h(M_P)$ does not depend on η.

Note that the assertion (v) means in particulary that the values of the function

$$P \mapsto \Omega_p(M)\frac{\Lambda_{(p,\infty)}(M_P,0)}{\Omega_\infty(P)}$$

satisfy generalised Kummer congruences in the followinfg sense: for any finite linear combination $\sum_P b_P \cdot P$ with $b_P \in \mathbf{C}_p$ which has the property $\sum_P b_P \cdot P \equiv 0(\mathrm{mod} p^N)$ we have that for some constant $C \neq 0$ the corresponding linear combination of the normalized L-values

$$C\sum_P b_P\Omega_p(M_P)\cdot\frac{\Lambda_{(p,\infty)}(M_P,0)}{\Omega_\infty(P)} \equiv 0(\mathrm{mod} p^N).$$

In the case of families of supersingular modular forms studied by R.Coleman [CoPBa] the invariant $h(P)$ reduces to the slope of a modular form in such a family.

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