On the Margulis constant for Kleinian groups, I

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Abstract

The *Margulis constant* for Kleinian groups is the smallest constant c such that for each discrete group G and each point x in the upper half space $\mathbf{H}^3$, the group generated by the elements in G which move x less than distance c is elementary. We take a first step towards determining this constant by proving that if $\langle f, g\rangle$ is nonelementary and discrete with f parabolic or elliptic of order $n \geq 3$, then every point x in $\mathbf{H}^3$ is moved at least distance c by f or g where $c = .1829\ldots$. This bound is sharp.

1 Introduction

Let $\mathbf{M}$ denote the group of all Möbius transformations of the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We associate with each Möbius transformation

$$f = \frac{az+b}{cz+d} \in \mathbf{M},\ ad - bc = 1,$$

the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})$$

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and set $\mathrm{tr}(f) = \mathrm{tr}(A)$ where $\mathrm{tr}(A)$ denotes the trace of A. Next for each f and g in $\mathbf{M}$ we we let $[f, g]$ denote the commutator $fgf^{-1}g^{-1}$. We call the three complex numbers

$$\beta(f) = \mathrm{tr}^2(f) - 4,\ \beta(g) = \mathrm{tr}^2(g) - 4,\ \gamma(f, g) = \mathrm{tr}([f, g]) - 2 \tag{1.1}$$

the parameters of the two generator group $\langle f, g\rangle$ and write

$$\mathrm{par}(\langle f, g\rangle) = (\gamma(f, g), \beta(f), \beta(g)).$$

These parameters are independent of the choice of representations for f and g and they determine $\langle f, g\rangle$ up to conjugacy whenever $\gamma(f, g) \neq 0$.

A Möbius transformation f may be regarded as a matrix A in $\mathrm{SL}(2, \mathbb{C})$, a conformal self map of $\overline{\mathbb{C}}$ or a hyperbolic isometry of $\mathbf{H}^3$. There are three different norms, corresponding to these three roles, which measure how much f differs from the identity [3]:

$$m(f) = \|A - A^{-1}\|, \tag{1.2}$$

$$d(f) = \sup\{q(f(z), z) : z \in \overline{\mathbb{C}}\}, \tag{1.3}$$

$$\rho(f) = h(f(j), j). \tag{1.4}$$

Here $\|B\|$ denotes the euclidean norm of the matrix $B \in \mathrm{GL}(2, \mathbb{C})$, q the chordal metric in $\overline{\mathbb{C}}$, j the point $(0, 0, 1) \in \mathbf{H}^3$ and h the hyperbolic metric with curvature -1 in $\mathbf{H}^3$. We will refer to $m(f)$, $d(f)$ and $\rho(f)$ as the *matrix*, *chordal* and *hyperbolic* norms of f. All three are invariant with respect to conjugation by chordal isometries.

A subgroup G of $\mathbf{M}$ is *discrete* if

$$\inf\{d(f) : f \in G, f \neq id\} > 0$$

or equivalently if

$$\inf\{m(f) : f \in G, f \neq id\} > 0;$$

G is *nonelementary* if it contains two elements with infinite order and no common fixed point and G is *Fuchsian* if $G(\mathbf{H}^2) = \mathbf{H}^2$ where $\mathbf{H}^2$ is the upper half plane in $\overline{\mathbb{C}}$.

The *Margulis constant* for Kleinian groups G in $\mathbf{M}$ acting on the upper half space $\mathbf{H}^3$ is the largest constant $c = c_K$ with the following property. For each discrete group G and each $x \in \mathbf{H}^3$, the group generated by

$$S = \{f \in G, h(f(x), x) < c\}$$

is elementary. The Margulis constant c_F for Fuchsian groups G in $\mathbf{M}$ acting on $\mathbf{H}^2$ is defined exactly as above with $\mathbf{H}^3$ replaced by $\mathbf{H}^2$. That such constants exist follows from [1], [10], [11].

The constant c_F was determined by Yamada who showed in [14] that

$$c_F = 2\text{arcsinh}\left(\sqrt{\frac{2\cos(2\pi/7)-1}{8\cos(\pi/7)+7}}\right) = .2629\ldots \tag{1.5}$$

by establishing the following result.

Theorem 1.6 *If $G = \langle f, g\rangle$ is discrete, nonelementary and Fuchsian, then*

$$\max\{h(f(x),x), h(g(x),x)\} \geq c_F$$

for $x \in \mathbf{H}^2$. Equality holds only for the case where G is the $(2,3,7)$ triangle group and f and g are elliptics of orders 3 and 2.

We shall establish in this paper the following partial analog of Theorem 1.6 for the case of Kleinian groups.

Theorem 1.7 *If $G = \langle f, g\rangle$ is discrete and nonelementary and if f is parabolic or elliptic of order $n \geq 3$, then*

$$\max\{h(f(x),x), h(g(x),x)\} \geq c \tag{1.8}$$

for $x \in \mathbf{H}^3$ where

$$c = \text{arccosh}\left(\frac{2\sqrt{8+2\sqrt{5}}-(1+\sqrt{5})}{6-\sqrt{5}}\right) = .1829\ldots. \tag{1.9}$$

Inequality (1.9) *is sharp and equality holds only if f and g are elliptics of orders 3 and 2.*

The following alternative formula

$$c = 2\text{arcsinh}\left(\sqrt{\frac{4\cos(2\pi/5)-1}{4\sqrt{8\cos(2\pi/5)+10}+14}}\right)$$

for c is similar to that for the constant c_F.

Let

$$c(3) = \operatorname{arccosh}\left(\frac{2\sqrt{8+2\sqrt{5}}-(1+\sqrt{5})}{6-\sqrt{5}}\right) = .1829\ldots, \tag{1.10}$$

$$c(4) = \operatorname{arccosh}\left(\frac{\sqrt{6+2\sqrt{3}}-\sqrt{3}}{3-\sqrt{3}}\right) = .3453\ldots, \tag{1.11}$$

$$c(5) = \operatorname{arccosh}\left(\frac{4(2+\sqrt{5}-\sqrt{9-\sqrt{5}})}{\sqrt{5}-1}\right) = .3401\ldots, \tag{1.12}$$

$$c(6) = \operatorname{arccosh}\left(\frac{17}{16}\right) = .3517\ldots, \tag{1.13}$$

$$c(n) = \operatorname{arccosh}\left(\frac{5-2\sin^2(\pi/n)}{4+2\sin^2(\pi/n)}\right) \geq .3343\ldots \tag{1.14}$$

for $n \geq 7$ and set

$$c(\infty) = \lim_{n\to\infty} c(n) = \operatorname{arccosh}(5/4) = .6931\ldots. \tag{1.15}$$

Then Theorem 1.7 is a consequence of the following two results.

Theorem 1.16 *If $G = \langle f, g\rangle$ is discrete and nonelementary and if f is elliptic of order $n \geq 3$, then*

$$\max\{h(f(x),x), h(g(x),x)\} \geq c(n) \tag{1.17}$$

for $x \in \mathbf{H}^3$. Inequality (1.17) *is sharp for each $n \geq 3$ and equality holds only if $\theta(f) = \pm 2\pi/n$ and f and g are elliptics of orders $n \neq 6$ and 2 or of orders 6 and 3.*

Theorem 1.18 *If $G = \langle f, g\rangle$ is discrete and nonelementary and if f is parabolic, then*

$$\max\{h(f(x),x), h(g(x),x)\} \geq c(\infty) \tag{1.19}$$

for $x \in \mathbf{H}^3$. Inequality (1.19) *is sharp and equality holds only if g is elliptic of order 2.*

Given $f, g \in \mathbf{M} \setminus \{id\}$, we let $\mathrm{fix}(f)$ denote the set of points in $\overline{\mathbb{C}}$ fixed by f. Next if f is nonparabolic, we let $\mathrm{ax}(f)$ denote the axis of f, i.e. the closed hyperbolic line in $\mathbf{H}^3$ with endpoints in $\mathrm{fix}(f)$. Finally if f and g are both nonparabolic, we let $\delta(f, g)$ the hyperbolic distance between $\mathrm{ax}(f)$ and $\mathrm{ax}(g)$ in $\mathbf{H}^3$. Then $\delta(f, g) > 0$ unless $\mathrm{ax}(f) \cap \mathrm{ax}(g) \neq \emptyset$.

We prove Theorem 1.16 by considering in §3, §4, §5, respectively, the three cases where

1. f is of order $n \geq 3$, $\mathrm{ax}(f) \cap \mathrm{ax}(gfg^{-1}) = \emptyset$,
2. f is of order $n \geq 3$, $\mathrm{ax}(f) \cap \mathrm{ax}(gfg^{-1}) \neq \emptyset$, $\mathrm{fix}(f) \cap \mathrm{fix}(gfg^{-1}) = \emptyset$,
3. f is of order $n \geq 3$, $\mathrm{fix}(f) \cap \mathrm{fix}(gfg^{-1}) \neq \emptyset$.

The proof depends on the estimates in [4] for the distance between axes of elliptics and on the diagrams in [7] for the possible values of the commutator parameter for a two generator group with an elliptic generator. Our argument shows also that the extremal groups for which (1.17) holds with equality for some $x \in \mathbf{H}^3$ are unique up to conjugacy.

The proof for Theorem 1.18 is given in §6.

2 Preliminary results

We derive here some formulas and inequalities which will be needed in what follows. First each nonparabolic Möbius transformation $f \neq id$ in $\mathbf{M}$ is conjugate to a transformation of the form ae^{ib} where $a > 0$ and $-\pi < b \leq \pi$. Then $\tau(f) = |\log(a)|$ and $\theta(f) = b$ are the *translation length* and *rotation angle* of f and it is easy to check that [6]

$$4\cosh(\tau(f)) = |\beta(f) + 4| + |\beta(f)|, \tag{2.1}$$

$$4\cos(\theta(f)) = |\beta(f) + 4| - |\beta(f)|. \tag{2.2}$$

The following result gives alternative formulas for the matrix and hyperbolic norms for a nonparabolic Möbius transformation f in terms of the trace parameter $\beta(f)$ and the *axial displacement*

$$\delta(f) = h(j, \mathrm{ax}(f)),$$

that is, the hyperbolic distance between $j = (0, 0, 1)$ and the axis of f.

Lemma 2.3 *If $f \in \mathbf{M} \setminus \{id\}$ is nonparabolic, then*

$$m(f)^2 = 2\cosh(2\delta(f))|\beta(f)|, \tag{2.4}$$

$$4\cosh(\rho(f)) = \cosh(2\delta(f))|\beta(f)| + |\beta(f) + 4|. \tag{2.5}$$

Proof. Let z_1, z_2 denote the fixed points of f. Then

$$m(f)^2 = 2\frac{8 - q(z_1, z_2)^2}{q(z_1, z_2)^2}|\beta(f)| = 2\cosh(2\delta(f))|\beta(f)|$$

by p. 37 and p. 48 in [3], and we obtain

$$8\cosh(\rho(f)) = m(f)^2 + 2|\mathrm{tr}(f)^2| = m(f)^2 + 2|\beta(f) + 4| \tag{2.6}$$

from p. 46 in [3]. □

Lemma 2.3 yields a formula for the hyperbolic displacement of a point $x \in \mathbf{H}^3$ under a Möbius transformation f.

Lemma 2.7 *If $f \in \mathbf{M} \setminus \{id\}$ is nonparabolic, then*

$$4\cosh(h(x, f(x)) = \cosh(2h(x, \mathrm{ax}(f)))|\beta(f)| + |\beta(f) + 4| \tag{2.8}$$

for each $x \in \mathbf{H}^3$.

Proof. Fix $x \in \mathbf{H}^3$ and let $g = \phi f \phi^{-1}$ where ϕ is a Möbius transformation which maps x onto j. Then $\beta(g) = \beta(f)$,

$$\delta(g) = h(j, \mathrm{ax}(g)) = h(\phi(x), \phi(\mathrm{ax}(f))) = h(x, \mathrm{ax}(f))$$

and

$$\rho(g) = h(g(j), j) = h(\phi(x), \phi(f(x))) = h(x, f(x)).$$

Then (2.8) follows from (2.4) and (2.5) applied to g. □

The proof of Theorem 1.16 for the first case in §3 depends on the following two upper bounds for the axial displacement $\delta(f)$.

Lemma 2.9 *If $f \in \mathbf{M}$ is elliptic of order $n \geq 3$, then*

$$\cosh(2\delta(f)) \leq \frac{\cosh(\rho(f)) - \cos^2(\pi/n)}{\sin^2(\pi/n)}, \tag{2.10}$$

$$\sinh^2(2\delta(f)) \leq \frac{(\cosh(\rho(f)) - 1)(\cosh(\rho(f)) + 1 - 2\cos^2(\pi/n))}{\sin^4(\pi/n)}. \tag{2.11}$$

There is equality in (2.10) *and* (2.11) *if and only if $\theta(f) = \pm 2\pi/n$.*

Proof. Suppose that $\theta(f) = 2m\pi/n$ where $|m| \leq n-1$. Then

$$\beta(f) = -4\sin^2(m\pi/n), \quad \beta(f) + 4 = 4\cos^2(m\pi/n)$$

and thus by (2.5)

$$\begin{aligned} \cosh(2\delta(f)) &= \frac{4\cosh(\rho(f)) - |\beta(f) + 4|}{|\beta(f)|} \\ &= \frac{\cosh(\rho(f)) - \cos^2(m\pi/n)}{\sin^2(m\pi/n)} \\ &\leq \frac{\cosh(\rho(f)) - \cos^2(\pi/n)}{\sin^2(\pi/n)}. \end{aligned}$$

Hence we obtain (2.10), which in turn implies (2.11), with equality in each case if and only if $|m| = 1$. □

Lemma 2.12 *If $g \in \mathbf{M} \setminus \{id\}$ is nonparabolic, then*

$$\cosh(2\delta(g)) \leq \frac{4\cosh(\rho(g))}{|\beta(g)|}, \tag{2.13}$$

$$\sinh(2\delta(g)) \leq \frac{4\sinh(\rho(g))}{|\beta(g)|}. \tag{2.14}$$

There is equality in (2.13) *if and only if g is elliptic of order 2 and equality in* (2.14) *if and only if either g is of order 2 or g is elliptic with $\delta(g) = 0$.*

Proof. For (2.13) we see by (2.5) that

$$\cosh(2\delta(g)) = \frac{4\cosh(\rho(g)) - |\beta(g)+4|}{|\beta(g)|} \leq \frac{4\cosh(\rho(f))}{|\beta(g)|}$$

with equality if and only if $\beta(g) = -4$, that is, if and only if g is of order 2. Next (2.5) implies that

$$\sinh^2(2\delta(g)) = \frac{N}{|\beta(g)|^2}$$

where

$$\begin{aligned} N &= (4\cosh(\rho(g)) - |\beta(g)+4|)^2 - |\beta(g)|^2 \\ &= 16\cosh^2(\rho(g)) - 8\cosh(\rho(g))|\beta(g)+4| + |\beta(g)+4|^2 - |\beta(g)|^2 \\ &\leq 16\cosh^2(\rho(g)) - 8\cosh(\tau(g))|\beta(g)+4| + |\beta(g)+4|^2 - |\beta(g)|^2 \\ &= 16\cosh^2(\rho(g)) - (|\beta(g)+4| + |\beta(g)|)^2 \\ &\leq 16\sinh^2(\rho(g)) \end{aligned}$$

by (2.1) and (2.5). This yields (2.14). Equality holds if and only if either g is of order 2 or g is elliptic with $\delta(g) = 0$. □

Finally we will use the following two lower bounds for the maximum of the hyperbolic norms $\rho(f)$ and $\rho(g)$ in the proof of Theorem 1.16 for the second and third cases in §4 and §5.

Lemma 2.15 *If $f, g \in \mathbf{M} \setminus \{id\}$ and if $\rho = \max\{\rho(f), \rho(g)\}$, then*

$$8\cosh(\rho) \geq M \tag{2.16}$$

where

$$M = |\beta(f)+4| + |\beta(g)+4| + \sqrt{m(f)^2 m(g)^2 + (|\beta(f)+4| - |\beta(g)+4|)^2}.$$

In addition,

$$m(f)^2 m(g)^2 \geq 2(|4\gamma(f,g) + \beta(f)\beta(g)| + |4\gamma(f,g)| + |\beta(f)\beta(g)|). \tag{2.17}$$

There is equality in (2.16) *if and only if $\rho(f) = \rho(g)$ and in* (2.17) *for non-parabolic f and g if and only if $\delta(f) = \delta(g) = \delta(f,g)/2$.*

Proof. Let $t = \cosh(\rho)$. Then

$$8\cosh(\rho(f)) - 2|\beta(f)+4| = m(f)^2, \quad 8\cosh(\rho(g)) - 2|\beta(g)+4| = m(g)^2$$

by (2.6). Hence

$$(8t - 2|\beta(f)+4|)(8t - 2|\beta(g)+4|) \geq m(f)^2 m(g)^2 \tag{2.18}$$

and we obtain

$$8t \geq |\beta(f)+4| + |\beta(g)+4| + \sqrt{m(f)^2 m(g)^2 + (|\beta(f)+4| - |\beta(g)+4|)^2}$$

with equality whenever $\rho(f) = \rho(g)$.

Next if f or g is parabolic, then $\beta(f)\beta(g) = 0$ and

$$\begin{aligned} m(f)^2 m(g)^2 &\geq 16|\gamma(f,g)| \\ &= 2(|4\gamma(f,g) + \beta(f)\beta(g)| + |4\gamma(f,g)| + |\beta(f)\beta(g)|) \end{aligned}$$

by Theorem 2.7 in [3]. Otherwise choose $x \in \mathrm{ax}(f)$ and $y \in \mathrm{ax}(g)$ so that $\delta(f) = h(x,j)$ and $\delta(g) = h(x,j)$. Then

$$\delta(f,g) \leq h(x,y) \leq h(x,j) + h(y,j) = \delta(f) + \delta(g)$$

and hence by (2.4) and Lemma 4.4 of [4],

$$\begin{aligned} m(f)^2 m(g)^2 &= 4\cosh(2\delta(f))\cosh(2\delta(g))|\beta(f)\beta(g)| \\ &\geq 4\cosh^2(\delta(f)+\delta(g))|\beta(f)\beta(g)| \\ &\geq 4\cosh^2(\delta(f,g))|\beta(f)\beta(g)| \\ &= 2(\cosh(2\delta(f,g)) + 1)|\beta(f)\beta(g)| \\ &= 2(|4\gamma(f,g) + \beta(f)\beta(g)| + |4\gamma(f,g)| + |\beta(f)\beta(g)|) \end{aligned}$$

with equality throughout if and only if $\delta(f) = \delta(g) = \delta(f,g)/2$. □

Lemma 2.19 *If $f, g \in \mathbf{M} \setminus \{id\}$ and if $\rho = \max\{\rho(f), \rho(g)\}$, then*

$$|\beta(g)+4| \leq 4\cosh(\rho) - \frac{4|\gamma(f,g)|}{4\cosh(\rho) - |\beta(f)+4|}. \tag{2.20}$$

Proof. Theorem 2.7 of [3] implies that

$$m(f)^2 m(g)^2 \geq 16|\gamma(f,g)|.$$

Hence

$$(4\cosh(\rho) - |\beta(f)+4|)(4\cosh(\rho) - |\beta(g)+4|) \geq 4|\gamma(f,g)|$$

by (2.18) and we obtain (2.20). □

3 Case where $\mathbf{ax}(f) \cap \mathbf{ax}(gfg^{-1}) = \emptyset$

We shall establish here in Theorem 3.2 a sharp version of Theorem 1.16 for the case where f is of order $n \geq 3$ with

$$\mathrm{ax}(f) \cap \mathrm{ax}(gfg^{-1}) = \emptyset.$$

In this case,

$$\delta = \delta(f, gfg^{-1}) > 0.$$

Then the fact f and gfg^{-1} are elliptic of order $n \geq 3$ allows us to combine Lemmas 2.9 and 2.12 with the sharp lower bound $b(n)$ for δ in [4] to obtain a lower bound for the maximal hyperbolic displacement of each point x in $\mathbf{H}^3$ under f and g.

For convenience of notation, for $n \geq 3$ we set

$$d(n) = \left\{ \begin{array}{ll} c(n) & \text{if } n \neq 6 \\ \mathrm{arccosh}(6 - \sqrt{24}) = .4457\ldots & \text{if } n = 6 \end{array} \right\} \tag{3.1}$$

since the lower bound in Theorem 3.2 is greater than that in Theorem 1.16 when $n = 6$.

Theorem 3.2 *If $\langle f, g\rangle$ is discrete, if f is elliptic of order $n \geq 3$ and if $\delta(f, gfg^{-1}) > 0$, then*

$$\max\{h(f(x), x), h(g(x), x)\} \geq d(n) \tag{3.3}$$

for $x \in \mathbf{H}^3$. Inequality (3.3) is sharp for each $n \geq 3$ and equality holds only if $\theta(f) = \pm 2\pi/n$ and g is elliptic of order 2.

Proof. Fix $x \in \mathbf{H}^3$ and let $f_1 = \phi f \phi^{-1}$ and $g_1 = \phi g \phi^{-1}$ where ϕ is a Möbius transformation which maps x onto j. Then $\langle f_1, g_1\rangle$ satisfies the hypotheses of Theorem 3.2,

$$\max\{h(f(x), x), h(g(x), x)\} = \max\{h(f_1(j), j), h(g_1(j), j)\}$$

and hence it suffices to establish (3.3) for the case where x is the point j.

Next choose $x \in \mathrm{ax}(f)$ and $y \in \mathrm{ax}(g)$ so that

$$\delta(f) = h(x, j), \qquad \delta(g) = h(y, j).$$

Then $g(x) \in \mathrm{ax}(gfg^{-1})$,

$$\delta = \delta(f, gfg^{-1}) \leq h(x, g(x)),$$

and thus

$$4\cosh(\delta) \leq 4\cosh(h(x, g(x)) \leq \cosh(2h(x, \mathrm{ax}(g)))|\beta(g)| + |\beta(g) + 4|.$$

by Lemma 2.7. Next

$$h(x, \mathrm{ax}(g)) \leq h(x, y) \leq h(x, j) + h(y, j) = \delta(f) + \delta(g)$$

by the triangle inequality and we obtain

$$4\cosh(\delta) \leq \cosh(2\delta(f) + 2\delta(g))|\beta(g)| + |\beta(g) + 4| = \mathrm{R}_1 + \mathrm{R}_2 \tag{3.4}$$

where

$$\begin{aligned} \mathrm{R}_1 &= \cosh(2\delta(f))\cosh(2\delta(g))|\beta(g)| + |\beta(g) + 4| \\ &\leq (\cosh(2\delta(g))|\beta(g)| + |\beta(g) + 4|)\cosh(2\delta(f)) \\ &= 4\cosh(\rho(g))\cosh(2\delta(f)) \end{aligned} \tag{3.5}$$

and

$$\mathrm{R}_2 = \sinh(2\delta(g))|\beta(g)|\sinh(2\delta(f)) \tag{3.6}$$

by (2.5) of Lemma 2.3.

Let

$$\rho = \max\{\rho(f), \rho(g)\}, \quad t = \cosh(\rho) > 1.$$

Then by Lemma 2.9,

$$\cosh(2\delta(f))\sin^2(\pi/n) \leq t - \cos^2(\pi/n), \tag{3.7}$$

$$\sinh(2\delta(f)\sin^2(\pi/n) \leq \sqrt{(t-1)(t+1-2\cos^2(\pi/n))} \tag{3.8}$$

with equality in (3.7) and (3.8) only if $\theta(f) = \pm 2\pi/n$. Similarly by Lemma 2.12,

$$\sinh(2\delta(g))|\beta(g)| \leq 4\sqrt{t^2 - 1} \tag{3.9}$$

with equality only if g is of order 2. We conclude from (3.4) through (3.9) that

$$\cosh(\delta)\sin^2(\pi/n) \leq \phi(t, n), \tag{3.10}$$

where

$$\phi(t,n) = t(t - \cos^2(\pi/n)) + (t-1)\sqrt{(t+1)(t+1-2\cos^2(\pi/n))}, \quad (3.11)$$

and that (3.10) holds with equality only if $\theta(f) = \pm 2\pi/n$ and g is of order 2.

For $n \geq 3$ let

$$\psi(n) = \cosh(b(n))\sin^2(\pi/n)$$

where $b(n)$ denotes the minimum distance between disjoint axes of elliptics of order n in a discrete group. Then

$$\psi(n) = \left\{ \begin{array}{ll} (1+\sqrt{5})/4 = .8090\ldots & \text{if } n = 3 \\ (1+\sqrt{3})/4 = .6803\ldots & \text{if } n = 4 \\ .5 & \text{if } n = 5 \\ .5 & \text{if } n = 6 \\ \cos(\pi/n)^2 - .5 \geq .3117\ldots & \text{if } n \geq 7 \end{array} \right\} \quad (3.12)$$

by Theorem 4.18 in [4]. Next $\phi(t,n)$ is increasing in t for $1 \leq t < \infty$,

$$\phi(1,n) = \sin(\pi/n)^2 < \psi(n) \leq \cosh(\delta)\sin^2(\pi/n) \leq \phi(t,n) \quad (3.13)$$

by (3.10) and we obtain

$$\rho = \operatorname{arccosh}(t) \geq \operatorname{arccosh}(t(n)) \quad (3.14)$$

where $t(n)$ is the unique root of the equation $\phi(t,n) = \psi(n)$. An elementary but technical calculation shows that

$$\operatorname{arccosh}(t(n)) = d(n)$$

and hence that

$$\max\{\rho(f), \rho(g)\} = \rho \geq d(n) \quad (3.15)$$

with equality only if $\theta(f) = \pm 2\pi/n$ and g is of order 2. This completes the proof for inequality (3.3).

To show that (3.3) is sharp, fix $n \geq 3$. Then by §8 in [4] we can choose choose elliptics f and g of orders n and 2 such that $\langle f, g\rangle$ is discrete with

$$2\delta(f,g) = \delta(f, gfg^{-1}) = b(n), \qquad \theta(f) = 2\pi/n.$$

Choose $x \in \mathrm{ax}(f)$ and $y \in \mathrm{ax}(g)$ so that $h(x,y) = \delta(f,g)$. By means of a preliminary conjugation we may assume that x and y lie on the j-axis in $\mathbf{H}^3$ with the point j situated so that

$$\cosh(2\delta(f))\sin^2(\pi/n) + \cos^2(\pi/n) = \cosh(2\delta(g)).$$

Then $\beta(f) = -4\sin^2(\pi/n)$, $\beta(g) = -4$ and

$$\begin{aligned} 4\cosh(\rho(f)) &= \cosh(2\delta(f))|\beta(f)| + |\beta(f)+4| \\ &= 4(\cosh(2\delta(f))\sin^2(\pi/n) + \cos^2(\pi/n))) \\ &= 4\cosh(2\delta(g)) \\ &= \cosh(2\delta(g))|\beta(f)| + |\beta(g)+4| \\ &= 4\cosh(\rho(g)). \end{aligned}$$

Hence if we set

$$t = \cosh(\rho(f)) = \cosh(\rho(g)),$$

then

$$\cosh(2\delta(f)) = \frac{t - \cos^2(\pi/n)}{\sin^2(\pi/n)}, \quad \cosh(2\delta(g)) = t$$

and we obtain

$$\begin{aligned} \psi(n) &= \cosh(b(n))\sin^2(\pi/n) \\ &= \cosh(2\delta(f,g))\sin^2(\pi/n) \\ &= \cosh(2\delta(f) + 2\delta(g))\sin^2(\pi/n) \\ &= t(t - \cos^2(\pi/n)) + (t-1)\sqrt{(t+1)(t+1-2\cos^2(\pi/n))} \\ &= \phi(t,n). \end{aligned}$$

Thus

$$\max\{\rho(f), \rho(g)\} = \mathrm{arccosh}(t) = \mathrm{arccosh}(t(n)) = d(n)$$

and (3.3) holds with equality. $\square$

4 Case where $\mathbf{ax}(f) \cap \mathbf{ax}(gfg^{-1}) \neq \emptyset$

We next prove Theorem 1.16 for the case where f is of order $n \geq 3$ with

$$\mathrm{ax}(f) \cap \mathrm{ax}(gfg^{-1}) \neq \emptyset, \quad \mathrm{fix}(f) \cap \mathrm{fix}(gfg^{-1}) = \emptyset.$$

Then $\langle f, gfg^{-1}\rangle$ is an elliptic group, that is, either the cyclic group C_n, the dihedral group D_n, the tetrahedral group A_4, the octahedral group S_4 or the iscosahedral group A_5. The hypothesis that f and gfg^{-1} have no common fixed point implies that $\langle f, gfg^{-1}\rangle \neq C_n$ while the fact that f and gfg^{-1} are both of order $n \geq 3$ implies that $\langle f, gfg^{-1}\rangle \neq D_n$. The remaining three cases are considered in the following result.

Theorem 4.1 *If $\langle f, g\rangle$ is discrete, if f is elliptic of order $n \geq 3$ and if $\langle f, gfg^{-1}\rangle$ is one of the three groups A_4, S_4, A_5, then either $\langle f, g\rangle$ is itself one these three groups or*

$$\max\{h(f(x), x), h(g(x), x)\} > c(n) \tag{4.2}$$

for $x \in \mathbf{H}^3$.

We will make use of the following results concerning the parameters of Möbius transformations in the proof for Theorem 4.1.

Lemma 4.3 *Suppose that $f, g \in \mathbf{M} \setminus \{id\}$. Then*

$$\gamma(f, g^2) = \gamma(f, g)(\beta(g) + 4), \qquad \beta(g^2) = \beta(g)(\beta(g) + 4) \tag{4.4}$$

and

$$\gamma(f, gfg^{-1}) = \gamma(f, g)(\gamma(f, g) - \beta(f)). \tag{4.5}$$

If f is of order 2, then

$$\beta(fg) = \gamma(f, g) - \beta(g) - 4. \tag{4.6}$$

If f and g have disjoint fixed points, then there exists $\tilde{f} \in \mathbf{M}$ of order 2 such that $\langle \tilde{f}, g\rangle$ is discrete whenever $\langle f, g\rangle$ is and such that

$$\gamma(\tilde{f}, g) = \beta(g) - \gamma(f, g). \tag{4.7}$$

Proof. The identities in (4.4) and (4.5) follow from direct calculation and from Lemma 2.1 in [4]. Next by the Fricke identity,

$$\gamma(f, g) = \mathrm{tr}([f, g]) - 2 = \mathrm{tr}(g)^2 + \mathrm{tr}(fg)^2 - 4 = \beta(g) + \beta(fg) + 4$$

and we obtain (4.5). Finally for (4.6) set $\tilde{f} = \phi f$ where ϕ is the Lie product of f and g which conjugates f and g to their inverses [8] and [9]. See also Lemma 2.29 of [4].

We shall also need the following list of possible parameters for the groups A_4, S_4, A_5 with conjugate elliptic generators.

Lemma 4.8 *Suppose that f and h are conjugate elliptics of order $n \geq 3$. If $\langle f, h\rangle = A_4$, then*

$$\text{par}(\langle f, h\rangle) = (-2, -3, -3). \tag{4.9}$$

If $\langle f, h\rangle = S_4$, then

$$\text{par}(\langle f, h\rangle) = (-1, -2, -2). \tag{4.10}$$

If $\langle f, h\rangle = A_5$, then

$$\text{par}(\langle f, h\rangle) = (-1, -3, -3) \tag{4.11}$$

or

$$\text{par}(\langle f, h\rangle) = (-.3819\ldots, -1.3819\ldots, -1.3819\ldots) \tag{4.12}$$

or

$$\text{par}(\langle f, h\rangle) = (-2.618\ldots, -3.618\ldots, -3.618\ldots). \tag{4.13}$$

Proof for Theorem 4.1. Suppose that $\langle f, g\rangle$ satisfies the hypotheses of Theorem 4.1 and that $\langle f, g\rangle$ is not any of the groups A_4, S_4, A_5. We must prove that (4.2) holds for each $x \in \mathbf{H}^3$. As in the proof of Theorem 3.2, it suffices to do this for the case where x is the point j. Let

$$\rho = \max\{\rho(f), \rho(g)\}, \qquad t = \cosh(\rho), \qquad \beta = \beta(g).$$

Then

$$|\beta + 4| + |\beta| = |\beta(g) + 4| + |\beta(g)| \leq 4\cosh(\rho(g)) \leq 4t \tag{4.14}$$

by (2.5). We will show that $\rho > c(n)$ by considering separately the three cases where $\langle f, gfg^{-1}\rangle$ is A_4, S_4 or A_5.

Case where $\langle f, gfg^{-1}\rangle = A_4$

By (4.9),

$$\mathrm{par}(\langle f, gfg^{-1}\rangle) = (-2,-3,-3), \qquad \mathrm{par}(\langle f, g\rangle) = (\gamma, -3, \beta)$$

where $\gamma(\gamma+3) = -2$ by (4.5). Hence $\gamma = -1$ or $\gamma = -2$.

Suppose that $\gamma = -1$. Then $\beta(f) = -3$ and

$$|\beta + 4| \leq 4t - \frac{4}{4t-1}. \tag{4.15}$$

by Lemma 2.19. Next by Lemma 4.3, $\langle f, g^2\rangle$ is discrete with commutator parameter

$$\tilde{\gamma} = \gamma(f, g^2) = \gamma(f,g)(\beta(g)+4) = -\beta - 4.$$

Since f is of order 3, it follows from §5.13 of [4] that

$$\tilde{\gamma} \in \{-3, -2.618\ldots, -2, -1, -.3819\ldots, 0\}$$

or that

$$|\tilde{\gamma} + 3| + |\tilde{\gamma}| \geq \sqrt{5} + 1.$$

In the first case

$$\beta \in \{-1, -1.3819\ldots, -2, -3, -3.618\ldots, -4\}$$

and $\langle f, g\rangle$ is S_4 or A_5 unless $\beta = -1$ in which case Lemma 2.15 implies that $\rho \geq .3418 > c(3)$. In the second case we have

$$|\beta + 4| + |\beta + 1| \geq \sqrt{5} + 1. \tag{4.16}$$

Then (4.14), (4.15) and (4.16) imply that $\rho \geq .2036 > c(3)$.

Suppose next that $\gamma = -2$. Then

$$|\beta + 4| \leq 4t - \frac{8}{4t-1} \tag{4.17}$$

by Lemma 2.19. By Lemma 4.3, there exists $\tilde{f}$ of order 2 such that $\langle \tilde{f}, g\rangle$ is discrete with

$$\gamma(\tilde{f}g, g) = \gamma(\tilde{f}, g) = \beta(g) - \gamma(f,g) = \beta + 2$$

and

$$\beta(\tilde{f}g) = \gamma(\tilde{f}, g) - \beta(g) - 4 = -2.$$

Hence $\tilde{\gamma} = \beta + 2$ is the commutator parameter of the discrete group $\langle \tilde{f}g, g\rangle$ with a generator of order 4 and by §5.9 of [4] either $\tilde{\gamma} \in \{-2, -1, 0\}$ or

$$|\tilde{\gamma} + 2| + |\tilde{\gamma}| \geq \sqrt{3} + 1.$$

In the first case, either $\beta \in \{-4, -3\}$ and $\langle f, g\rangle$ is A_4 or $\beta = -2$ and $\rho \geq .4281 > c(3)$ by Lemma 2.15. In the second case,

$$|\beta + 4| + |\beta + 2| \geq \sqrt{3} + 1 \tag{4.18}$$

and we obtain $\rho \geq .3899 > c(3)$ from (4.14), (4.17) and (4.18). □

Case where $\langle f, gfg^{-1}\rangle = S_4$

By (4.10)

$$\mathrm{par}(\langle f, gfg^{-1}\rangle) = (-1, -2, -2), \qquad \mathrm{par}(\langle f, g\rangle) = (\gamma, -2, \beta)$$

where $\gamma(\gamma + 2) = -1$ by (4.5). Hence $\gamma = -1$ and we obtain

$$|\beta + 4| \leq 4t - \frac{2}{2t - 1} \tag{4.19}$$

from Lemma 2.19. Next $\langle f, g^2\rangle$ is discrete with commutator parameter $\tilde{\gamma} = -\beta - 4$ and f is of order 4. Hence $\tilde{\gamma} \in \{-2, -1, 0\}$ or

$$|\tilde{\gamma} + 2| + |\tilde{\gamma}| \geq \sqrt{3} + 1.$$

by §5.9 of [4]. In the first case, $\beta \in \{-2, -3, -4\}$ and $\langle f, g\rangle$ is S_4. Otherwise

$$|\beta + 4| + |\beta + 2| \geq \sqrt{3} + 1 \tag{4.20}$$

Next by [7]

$$\tilde{\gamma} \in \{-2.618\ldots, -1.844\ldots \pm .7448i\ldots, -2.419\ldots \pm .6062i\ldots\}$$

or $|\tilde{\gamma} + 2| > 2$. In the first case either $\langle f, g\rangle$ is not discrete or $\rho \geq .4051 > c(4)$ by Lemma 2.15. Otherwise

$$|\beta + 2| > 2. \tag{4.21}$$

Finally if we combine (4.14), (4.19), (4.20) and (4.21), we obtain $\rho \geq .3526 > c(4)$. $\square$

Case where $\langle f, gfg^{-1} \rangle = A_5$

In this case we have the following three possibilities given in (4.11), (4.12) and (4.13). If (4.11) holds, then

$$\mathrm{par}(\langle f, gfg^{-1} \rangle) = (-1, -3, -3), \quad \mathrm{par}(\langle f, g \rangle) = (\gamma, -3, \beta)$$

where $\gamma(\gamma + 3) = -1$ by (4.5); hence $\gamma = -.3819\ldots$ or $\gamma = -2.618\ldots$.

If $\gamma = -.3819\ldots$, then $\langle f, g^2 \rangle$ is discrete with commutator parameter $\tilde{\gamma} = \gamma(\beta + 4)$ and a generator of order 3. Hence by Theorem 3.4 and Lemmas 2.26 and 6.1 of [4], either $\tilde{\gamma} \in \{-1, -.3819\ldots, 0\}$ whence $\beta \in \{-1.3819\ldots, -3, -4\}$ or

$$|\tilde{\gamma} + 1| \geq .618\ldots, \quad |\tilde{\gamma} + .3819\ldots| \geq .3819\ldots, \quad |\tilde{\gamma}| \geq .2469\ldots$$

whence

$$|\beta + 1.3819\ldots| \geq 1.618, \quad |\beta + 3| \geq 1, \quad |\beta| \geq .6466. \tag{4.22}$$

Since $\langle f, g \rangle$ is A_5 if $\beta \in \{-4, -1.3819\ldots\}$ and not discrete if $\beta = -3$, we obtain (4.22). This and (4.14) imply that $\rho \geq .4812 > c(3)$.

If $\gamma = -2.618\ldots$, then

$$|\beta + 4| \leq 4t - \frac{10.472}{4t - 1} \tag{4.23}$$

by Lemma 2.19. By Lemma 4.3 there exists $\tilde{f}$ of order 2 such that $\langle \tilde{f}g, g \rangle$ is discrete with commutator parameter $\tilde{\gamma} = \beta + 2.618\ldots$ and $\beta(\tilde{f}g) = -1.3819\ldots$. Since $\tilde{f}g$ is of order 5, $\tilde{\gamma} \in \{-1.3819\ldots, -1, -.3819\ldots, 0\}$ or

$$|\tilde{\gamma} + 1.3819\ldots| + |\tilde{\gamma}| \geq 2$$

by §5.4 of [4]. In the first case, $\beta \in \{-4, -3.618\ldots, -3, -2.618\ldots\}$ and either $\langle f, g \rangle$ is A_5 or not discrete or $\rho \geq .3418 > c(3)$ by Lemma 2.15. In the second case

$$|\beta + 4| + |\beta + 2.618\ldots| \geq 2 \tag{4.24}$$

and we obtain $\rho \geq .4073 > c(3)$ from (4.14), (4.23) and (4.24).

Next if (4.12) holds, then

$$\mathrm{par}(\langle f, gfg^{-1}\rangle) = (-.3819\ldots, -1.3819\ldots, -1.3819\ldots).$$

and

$$\mathrm{par}(\langle f, g\rangle) = (\gamma, -1.3819\ldots, \beta)$$

where $\gamma = -.3819\ldots$ or $\gamma = -1$.

If $\gamma = -.3819\ldots$, then

$$|\beta + 4| \leq 4t - \frac{1.5278}{4t - 2.618} \tag{4.25}$$

by Lemma 2.19 and $\langle f, g^2\rangle$ is discrete with commutator parameter $\tilde{\gamma} = \gamma(\beta + 4)$ and f of order 5. Then $\tilde{\gamma} \in \{-1.3819\ldots, -1, -.3819\ldots, 0\}$ or

$$|\tilde{\gamma} + 1.3819\ldots| + |\tilde{\gamma}| \geq 2.$$

In the first case $\beta \in \{-.3819\ldots, -1.3819\ldots, -3, -4\}$ and $\langle f, g\rangle$ is A_5 or not discrete. In the second case

$$|\beta + 4| + |\beta + 2.618\ldots| \geq \sqrt{5} + 3 \tag{4.26}$$

and (4.14), (4.25) and (4.26) imply that $\rho \geq .6893 > c(5)$.

When $\gamma = -1$,

$$|\beta + 4| \leq 4t - \frac{4}{4t - 2.618} \tag{4.27}$$

and $\tilde{\gamma} = -\beta - 4$ is the commutator parameter of $\langle f, g^2\rangle$ where f is of order 5. From §5.4 of [4] it follows that $\beta \in \{-2.618\ldots, -3, -3.618\ldots, -4\}$ or that

$$|\beta + 4| + |\beta + 2.618\ldots| \geq 2. \tag{4.28}$$

In the first case $\langle f, g\rangle$ is A_5 or not discrete; hence we obtain (4.28). Next from [7] it follows that either

$$\beta \in \{-.6909\ldots \pm .7722i\ldots, -1.5 \pm .6066i\ldots\}$$

or that

$$|\beta + 3| > 1. \tag{4.29}$$

In the first case $\langle f, g\rangle$ is A_5 or not discrete and we conclude from (4.14), (4.27), (4.28) and (4.29) that $\rho > .3615 > c(5)$.

Finally (4.13) implies that

$$\mathrm{par}(\langle f, g\rangle) = (\gamma, -3.618\ldots, \beta)$$

where $\gamma = -1$ or $\gamma = -2.618\ldots$.

If $\gamma = -1\ldots$, then

$$\mathrm{par}(\langle f^2, g\rangle) = (-.3819\ldots, -1.3819\ldots, \beta)$$

and we conclude that $2\rho \geq .6893 > 2c(5)$ from what was proved above.

If $\gamma = -2.618\ldots$, then

$$|\beta + 4| \leq 4t - \frac{10.742}{4t - .3819} \tag{4.30}$$

by Lemma 2.15. By Lemma 4.3 we can choose $\tilde{f}$ so that $\langle \tilde{f}g, g\rangle$ is discrete with commutator parameter $\tilde{\gamma} = \beta + 2.618\ldots$ and $\tilde{f}g$ of order 5. Then as above, either

$$\beta \in \{-2.618\ldots, -3, -3.618\ldots, -4\},$$

in which case $\langle f, g\rangle$ is A_5 or not discrete, or

$$|\beta + 4| + |\beta + 2.618\ldots| \geq 2. \tag{4.31}$$

Next by [7] either

$$\beta \in \{-.6909\ldots \pm 7722i\ldots, .1180\ldots \pm .6066\ldots\},$$

in which case $\langle f, g\rangle$ is not discrete or $\rho > .4452 > c(5)$ by Lemma 2.15, or

$$|\beta + 3| > 1. \tag{4.32}$$

Then (4.14), (4.30), 4.31) and (4.32) imply that $\rho \geq .426 > c(5)$. $\square$

5 Case where $\mathbf{fix}(f) \cap \mathbf{fix}(gfg^{-1}) \neq \emptyset$

We establish here Theorem 1.16 for the case where f is an elliptic of order $n \geq 3$ and

$$\mathrm{ax}(f) \cap \mathrm{ax}(gfg^{-1}) = \emptyset, \quad \mathrm{fix}(f) \cap \mathrm{fix}(gfg^{-1}) \neq \emptyset.$$

Then since $\langle f, g\rangle$ is nonelementary,

$$\mathrm{fix}(f) \cap \mathrm{fix}(g) = \emptyset \quad \text{and} \quad \gamma(f,g) \neq 0.$$

Hence $[f, g]$ is parabolic and $n = 3, 4, 6$ [2] or [12]. Next

$$0 = \gamma(f, gfg^{-1}) = \gamma(f,g)(\gamma(f,g) - \beta(f))$$

by (4.5) of Lemma 4.3 and thus

$$\mathrm{par}(\langle f, g\rangle) = (\beta(f), \beta(f), \beta(g)). \tag{5.1}$$

Theorem 1.16 follows for the case considered here from the following result.

Theorem 5.2 *If $\langle f, g\rangle$ is discrete and not dihedral, if f is elliptic of order $n \geq 3$ and if*

$$\mathrm{par}(\langle f, g\rangle) = (\beta(f), \beta(f), \beta(g)),$$

then

$$\max\{h(f(x), x), h(g(x), x)\} \geq c(n) \ldots \tag{5.3}$$

for $x \in \mathbf{H}^3$. Equality (5.3) is sharp only when $n = 6$ and g is of order 3.

Proof. Again it suffices to establish (5.3) for the case where $x = j$. Let

$$\rho = \max\{\rho(f), \rho(g)\}, \qquad t = \cosh(\rho), \qquad \beta = \beta(g).$$

From the above discussion we see that n is either 3, 4 or 6. We establish (5.3) by considering each of these cases separately.

Case n=3

In this case

$$\mathrm{par}(\langle f, g\rangle) = (-3, -3, \beta)$$

and

$$|\beta + 4| \leq 4t - \frac{12}{4t-1} \tag{5.4}$$

by Lemma 2.19. Next by Lemma 4.3 there exists $\tilde{f}$ such that $\langle \tilde{f}, g \rangle$ is discrete with

$$\tilde{\gamma} = \gamma(\tilde{f}, g) = \beta(g) - \gamma(f, g) = \beta + 3$$

and

$$\beta(\tilde{f}g) = \gamma(\tilde{f}, g) - \beta(g) - 4 = -1.$$

Thus $\langle \tilde{f}g, g \rangle$ has a generator of order 6 and either $\tilde{\gamma} \in \{-1, 0\}$ or

$$|\tilde{\gamma} + 1| + |\tilde{\gamma}| \geq 2$$

by §5.3 of [4]. In the first case, $\beta = -4$ and $\langle f, g \rangle$ is the dihedral group D_3 or $\beta = -3$ and $\rho \geq .4771 > c(3)$ by Lemma 2.15. Otherwise

$$|\beta + 4| + |\beta + 3| \geq 2 \tag{5.5}$$

and this together with (5.4) implies that $\rho \geq .5007 > c(3)$. □

Case n=4

Here

$$\text{par}(\langle f, g \rangle) = (-2, -2, \beta)$$

and

$$|\beta + 4| \leq 4t - \frac{8}{4t-2} \tag{5.6}$$

by Lemma 2.19. By Lemma 4.3 there exists $\tilde{f}$ such that $\langle \tilde{f}, g \rangle$ is discrete with

$$\tilde{\gamma} = \gamma(\tilde{f}, g) = \beta + 2, \quad \beta(\tilde{f}g) = -2.$$

Then $\tilde{\gamma}$ is the commutator parameter of a two generator group with a generator of order 4 and hence $\tilde{\gamma} \in \{-2, -1, 0\}$ or

$$|\tilde{\gamma} + 2| + |\tilde{\gamma}| \geq \sqrt{3} + 1$$

by §5.9 of [4]. In the first case $\beta = -4$ and $\langle f, g \rangle$ is D_4 or $\beta \in \{-3, -2\}$ and $\rho \geq .4281 > c(4)$ by Lemma 2.15. Otherwise

$$|\beta + 4| + |\beta + 2| \geq \sqrt{3} + 1 \tag{5.7}$$

which together with (5.6) implies that $\rho \geq .5026 > c(4)$. $\square$

Case n=6

Finally in this case

$$\mathrm{par}(\langle f, g\rangle) = (-1, -1, \beta)$$

while

$$|\beta + 4| \leq 4t - \frac{4}{4t-3} \tag{5.8}$$

by Lemma 2.19. Next $\langle f, g^2\rangle$ is discrete with a generator of order 6 and

$$\tilde{\gamma} = \gamma(f, g^2) = \gamma(f, g)(\beta(f) + 4) = -\beta - 4$$

by Lemma 4.3. Thus $\tilde{\gamma} = -1$, $\tilde{\gamma} = 0$ or

$$|\tilde{\gamma} + 1| + |\tilde{\gamma}| \geq 2$$

by §5.3 of [4]. In the second case $\beta = -4$ and $\langle f, g\rangle$ is D_6. In the third case

$$|\beta + 4| + |\beta + 3| \geq 2 \tag{5.9}$$

and this with (5.8) implies that $\rho \geq .3942$.

It remains to consider the first case where $\beta = -3$ and where $\langle f, g\rangle$ is discrete and nonelementary by [13]. Then

$$\rho = \max\{\cosh(2\delta(f)) + 3, 3\cosh(2\delta(g)) + 1\} \tag{5.10}$$

by (2.5). Next if we choose $x \in \mathrm{ax}(f)$ and $y \in \mathrm{ax}(g)$ so that $\delta(f) = h(x, j)$ and $\delta(g) = h(y, j)$, then

$$\delta(f, g) \leq h(x, y) \leq h(x, j) + h(y, j) = \delta(f) + \delta(g)$$

and hence

$$5/3 = \cosh(2\delta(f, g)) \leq \cosh(2\delta(f) + 2\delta(g)) \tag{5.11}$$

by Lemma 4.4 in [4]. It is then easy to verify from (5.10) and (5.11) that

$$\rho \geq \mathrm{arccosh}(17/16) = c(6)$$

with equality if x and y lie in the j-axis and are situated so that

$$\cosh(2\delta(f)) + 3 = 3\cosh(2\delta(g)) + 1. \quad \square$$

6 Case where f is parabolic

Finally we establish Theorem 1.18, and hence complete the proof of Theorem 1.7, by showing that if $\langle f, g\rangle$ is discrete and nonelementary and if f is parabolic, then

$$\max\{h(f(x), x), h(g(x), x)\} \geq \operatorname{arccosh}(5/4) = c(\infty) \tag{6.1}$$

for each $x \in \mathbf{H}^3$. As before it suffices to establish (6.1) for the case where $x = j$.

Let

$$\rho = \max\{\rho(f), \rho(g)\}, \quad t = \cosh(\rho).$$

Since j is fixed by chordal isometries, we may assume that $f(\infty) = \infty$ and hence that f and g can be represented by the matrices

$$A = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $ad = 1 + bc$. Then by Theorem 4.21 of [2],

$$|u|^2 + 2 = 2\cosh(\rho(f)) \leq 2t \tag{6.2}$$

and

$$\begin{aligned} 2 + (|c| - |b|)^2 &\leq 2|1 + bc| + |c|^2 + |b|^2 \\ &\leq |a|^2 + |d|^2 + |b|^2 + |c|^2 \\ &= 2\cosh(\rho(g)) \leq 2t. \end{aligned} \tag{6.3}$$

In addition,

$$|c|^2|u|^2 = |\gamma(f, g)|^2 \geq 1 \tag{6.4}$$

by the Shimizu-Leutbecher inequality; see II.C in [12].

Suppose that $t < 5/4$. Then (6.2), (6.4), (6.3) imply, respectively, that

$$|u| < 1/\sqrt{2}. \quad |c| > \sqrt{2}, \quad |b| > 1/\sqrt{2}$$

and hence that

$$5/2 < |c|^2 + |b|^2 \leq 2\cosh(\rho(g)) \leq 2t < 5/2,$$

a contradiction. Thus $t \geq 5/4$ and we obtain inequality (6.1).

Suppose next that $t = 5/4$. Then

$$|u| \leq 1/\sqrt{2}. \quad |c| \geq \sqrt{2}, \quad |b| \geq 1/\sqrt{2}$$

as above and

$$5/2 \leq |c|^2 + |b|^2 \leq |a|^2 + |d|^2 + |b|^2 + |c|^2 = 2\cosh(\rho(g)) \leq 5/2.$$

Hence in this case, $a = d = 0$ and we conclude that (6.1) holds with equality only if g is of order 2.

Finally if f and g are as above with $a = d = 0$, $u = b = 1/\sqrt{2}$ and $c = -\sqrt{2}$, then $\langle f, g \rangle$ is conjugate to the Modular group, and hence discrete and nonelementary, with

$$\cosh(\rho(f)) = \cosh(\rho(g)) = 5/4.$$

Thus inequality (6.1) is sharp. □

References

[1] B. N. Apanasov, *A universal property of Kleinian groups in the hyperbolic metric*, Dokl. Akad. Nauk SSSR **225** (1975) 1418-1421.

[2] A. F. Beardon, *The geometry of discrete groups*, Springer-Verlag 1983.

[3] F. W. Gehring and G. J. Martin, *Inequalities for Möbius transformations and discrete groups*, J. reine angew. Math. **418** (1991) 31-76.

[4] F. W. Gehring and G. J. Martin, *Commutators, collars and the geometry of Möbius groups*, J. d'Analyse Math. **63** (1994).

[5] F. W. Gehring and G. J. Martin, *On the minimal volume hyperbolic 3-orbifold*, Math. Res. Letters **1** (1994) 107-114.

[6] F. W. Gehring and G. J. Martin, *Chebyshev polynomials and discrete groups*, Proceedings of the International Conference on Complex Analysis at the Nankai Institute of Mathematics 1992, International Press (1994) 5-16.

[7] F. W. Gehring and G. J. Martin, *Commutator spectra for discrete groups with an elliptic generator* (in preparation).

[8] T. Jørgensen, *Compact 3-manifolds of constant negative curvature fibering over the circle*, Ann. Math. **106** (1977) 61–72.

[9] T. Jørgensen, *Comments on a discreteness condition for subgroups of* SL(2,$\mathbb{C}$), Can. J. Math. **31** (1979) 87–92.

[10] D. A. Každan and G. A. Margulis, *A proof of Selberg's conjecture*, Math. USSR–Sbornik **4** (1968) 147–152.

[11] A. Marden, *Universal properties of Fuchsian groups in the Poincaré metric*, Ann. Math. Studies **79** (1974) 315-339.

[12] B. Maskit, *Kleinian groups*, Springer-Verlag 1988.

[13] B. Maskit, *Some special 2-generators Kleinian groups*, Proc. Amer. Math. Soc. **106** (1989) 175–186,

[14] A. Yamada *On Marden's universal constant of Fuchsian groups*, Kodai Math. J. **4** (1981) 266-277.

[15] A. Yamada *On Marden's universal constant of Fuchsian groups, II*, J. d'Analyse Math. **41** (1982) 234-248.

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