RECURRENT RANDOM WALKS, LIOUVILLE'S THEOREM, AND CIRCLE PACKINGS

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Abstract. It has been shown that univalent circle packings filling in the complex plane $\mathbf{C}$ are unique up to similarities of $\mathbf{C}$. Here we prove that bounded degree branched circle packings properly covering $\mathbf{C}$ are uniquely determined, up to similarities of $\mathbf{C}$, by their branch sets. In particular, when branch sets of the packings considered are empty we obtain the earlier result.

We also establish a circle packing analogue of Liouville's theorem: if f is a circle packing map whose domain packing is infinite, univalent, and has recurrent tangency graph, then the ratio map associated with f is either unbounded or constant.

1 Introduction

Over the period of last several years various results linking circle packing with other, classical, fields of mathematics have been established ([Bo],[D1],[HSc],[RS],[St1],[Th2]). The most prominent connection is the one with analytic function theory, which was originally suggested by Thurston in [Th1]. However, recent developments ([BeSc],[HSc],[Mc]) point to an equally interesting relation between the theory of circle packing and graph theory. This relationship was first observed by Stephenson ([St1],[St2]) who used Markov processes and electric network-type arguments to prove Thurston's conjecture (cf. [RS],[Th1]). The "analytical" side of circle packing theory has predominantly contained, until very recently, results about univalent circle packings. These include the existence and uniqueness statements, and the finite Riemann mapping theorem. On the other hand, the "graph-theoretical" side gives, for example, an answer to the type problem for graphs in the language of circle packing.

In [D1] and [D2] the author expanded "analytic" circle packing theory to the non-univalent case by proving results that have well-known parallels in the classical setting. Specifically, notions of discrete Blaschke products and discrete complex polynomials, and associated with them branched circle packings, have been introduced. As the new concepts have arrived, new questions have been posed. In particular, since it has been shown ([RS],[Sc],[St3]) that univalent circle packings which cover the complex plane $\mathbf{C}$ are unique up to similarities of $\mathbf{C}$, it is natural to ask whether the analogous fact holds for branched circle packings. More precisely,

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is it true that circle packings which are proper branched coverings of $\mathbf{C}$ are uniquely determined, up to similarities of $\mathbf{C}$, by their branch sets? This paper gives the positive answer to the above question in the case of circle packings of bounded degree; the exact statement is contained in Theorem 4.2. The proof of Theorem 4.2 uses both analytical developments of [D2] and graph-theoretical developments of [HSc] and [Mc]. With the help of [D2] we show that two proper branched circle packing coverings of $\mathbf{C}$ with identical branch sets must have comparable radius functions; using [HSc]/[Mc] and random walk techniques we prove that there are no nontrivial bounded perturbations of a circle packing whose combinatorial pattern of tangencies is a recurrent planar graph (see Corollary 3.2). These two results yield the uniqueness of branched circle packings. Particularly, we obtain the uniqueness of univalent packings.

Although this paper was motivated by studies of circle packings, it is worth mentioning that Corollary 3.2 is actually a special case of Theorem 3.1 when labelled complexes of the theorem are replaced by circle packings. Roughly speaking, Theorem 3.1 says that if two labels ρ_1 and ρ_2, of a 2-complex $\mathbb{K}$ with recurrent 1-skeleton, are such that the curvature of the labelled complex $\mathbb{K}(\rho_1)$ is no larger then that of $\mathbb{K}(\rho_2)$ then either $\rho_1 \equiv c\rho_2$ for some constant $c > 0$ or ρ_1 is not comparable with ρ_2. To prove the theorem we use variational methods (cf. [CdV]) and results about random walks. Also, as a special case of Theorem 3.1 we obtain a circle packing version of Liouville's theorem (Corollary 3.3).

We finish our introduction with the following remark. As graphs can be divided into two types of classes, recurrent and transient, similarly circle packings can be divided into two types of classes, parabolic and hyperbolic (see [BSt1]); in the case of packings of bounded degree, a circle packing is of parabolic/hyperbolic-type if and only if its combinatorial pattern of tangencies is a recurrent/transient graph, respectively. Thus, this paper deals with circle packings of parabolic-type. We would like to mention that the existence and uniqueness questions for hyperbolic-type branched circle packings are addressed in [D4]. Techniques used in [D4] are quite different from the ones used here as they reflect the distinction between hyperbolic-type packings and parabolic-type packings.

2 Vital Facts

In this section we will introduce essential terminology and definitions. We will also recall several facts regarding labelled complexes, circle packings, and random walks.

We begin with labelled complexes and circle packings; the interested reader should see [Bo], [BSt2], [D1], and [D2] for more details. Let $\mathbb{K}$ be a simplicial 2-complex given by a triangulation of a domain in $\mathbf{C}$. We will assume, throughout this paper, that $\mathbb{K}$ is either *infinite without boundary* or *finite*, and has an orientation induced from $\mathbf{C}$. Denote by $\mathbb{K}^{(0)}$, $\operatorname{int}\mathbb{K}^{(0)}$, $\operatorname{bd}\mathbb{K}^{(0)}$, $\mathbb{K}^{(1)}$, and $\mathbb{K}^{(2)}$ the sets of vertices, interior vertices, boundary vertices, edges, and faces of $\mathbb{K}$, respectively.

A function $\rho : \mathbb{K}^{(0)} \to (0, \infty)$ will be called a *label* for $\mathbb{K}$, and $\mathbb{K}(\rho)$ a *labelled complex* with label ρ (cf. [Bo],[BSt2],[Th2]). If $\mathbb{L}$ is a subcomplex of $\mathbb{K}$ and ρ is a label for $\mathbb{K}$ then $\mathbb{L}(\rho)$ will denote $\mathbb{L}$ with the label $\rho|_{\mathbb{L}^{(0)}}$. For a face $\triangle = \langle u, v, w\rangle$

in $\mathbb{K}$ let $\alpha_\rho(v, \triangle)$ be given by

$$\alpha_\rho(v,\triangle) := \arccos\left[\frac{\big(\rho(v)+\rho(u)\big)^2 + \big(\rho(v)+\rho(w)\big)^2 - \big(\rho(u)+\rho(w)\big)^2}{2\big(\rho(v)+\rho(u)\big)\big(\rho(v)+\rho(w)\big)}\right]. \tag{2.1}$$

Figure 1 shows the geometric interpretation of (2.1).

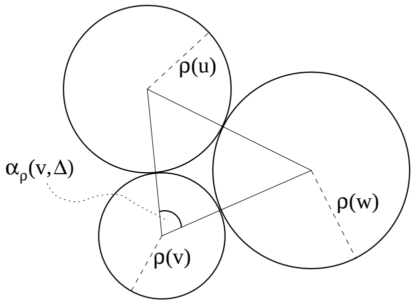

FIGURE 1. The angle $\alpha_\rho(v, \triangle)$.

If $v \in \operatorname{int} \mathbb{K}^{(0)}$, the quantity $\Theta_\rho(v) := \sum_{\triangle \in \mathbb{K}^{(2)}} \alpha_\rho(v, \triangle)$ will be termed the *angle sum* of $\mathbb{K}(\rho)$ at v, and $\Theta_\rho : \operatorname{int} \mathbb{K}^{(0)} \to (0, \infty)$ the angle sum function.

Labelled complexes can be regarded as generalizations of more rigid structures called circle packings. We say that a collection $\mathcal{P}$ of circles in $\mathbf{C}$ is a *circle packing* for $\mathbb{K}$ if there exists 1-to-1 relationship between $\mathbb{K}^{(0)}$ and $\mathcal{P}$ ($\mathbb{K}^{(0)} \ni v \leftrightarrow C_{\mathcal{P}}(v) \in \mathcal{P}$) such that $\langle C_{\mathcal{P}}(u), C_{\mathcal{P}}(v), C_{\mathcal{P}}(w)\rangle$ is a positively oriented triple of mutually externally tangent circles whenever $\langle u, v, w\rangle \in \mathbb{K}^{(2)}$ is a positively oriented triple.

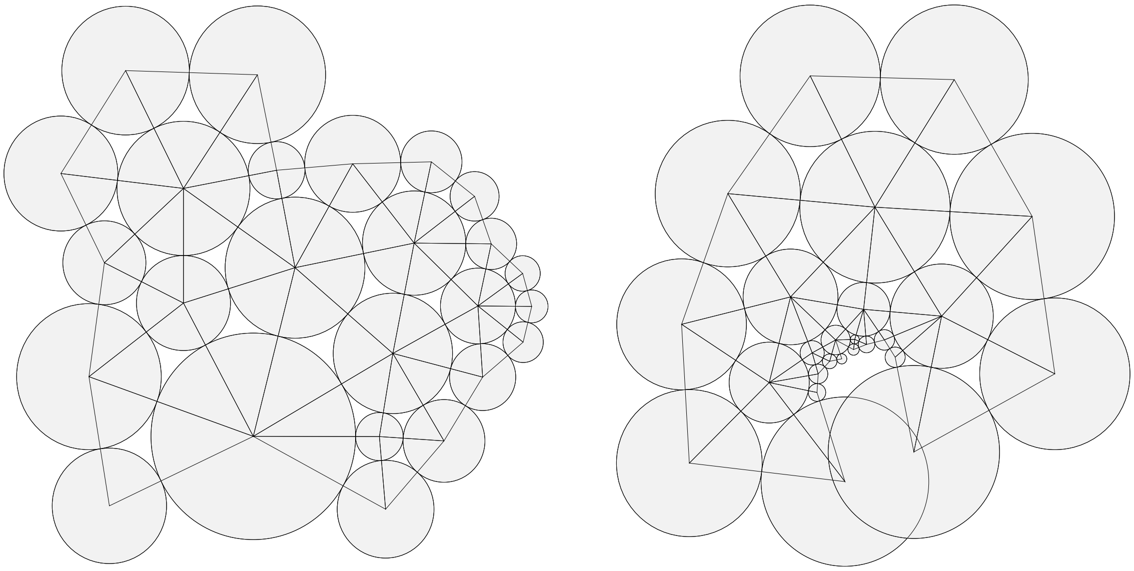

FIGURE 2*. Different circle packings (and their carriers) for the same 2-complex.

*This figure was created with the help of `CirclePack` by the kind permission of Ken Stephenson

Suppose $\mathcal{P}$ is a circle packing for $\mathbb{K}$. A function $r_{\mathcal{P}} : \mathbb{K}^{(0)} \to (0,\infty)$ with its value at $v \in \mathbb{K}^{(0)}$ equal to the radius of $C_{\mathcal{P}}(v)$ will be called the *radius function* of $\mathcal{P}$. The *carrier* of $\mathcal{P}$, $\mathrm{carr}(\mathcal{P})$, is the underlying geometric complex of $\mathcal{P}$ which is isomorphic to $\mathbb{K}$ (Figure 2). The isomorphism is defined as the function $S_{\mathcal{P}} : \mathbb{K} \to \mathbf{C}$ which maps each $v \in \mathbb{K}^{(0)}$ to the center of $C_{\mathcal{P}}(v)$ and then extends affinely to $\mathbb{K}^{(1)}$ and $\mathbb{K}^{(2)}$. Hence, $\mathrm{carr}(\mathcal{P}) := S_{\mathcal{P}}(\mathbb{K})$.

If $\triangle \in \mathbb{K}^{(2)}$ and v is a vertex of $\triangle$ then we define $\alpha_{\mathcal{P}}(v,\triangle) := \alpha_{r_{\mathcal{P}}}(v,\triangle)$. From the earlier geometric interpretation it follows that $\alpha_{\mathcal{P}}(v,\triangle)$ is the angle at the vertex $S_{\mathcal{P}}(v)$ in the euclidean triangle $S_{\mathcal{P}}(\triangle)$. For $v \in \mathbb{K}^{(0)}$, $\Theta_{\mathcal{P}}(v) := \sum_{\triangle \in \mathbb{K}^{(2)}} \alpha_{\mathcal{P}}(v,\triangle)$ is said to be the *angle sum* of $\mathcal{P}$ at v. It follows from our definition of the circle packing that for each $v \in \mathrm{int}\,\mathbb{K}^{(0)}$ there exists a non-negative integer $n_{\mathcal{P}}(v)$ such that $\Theta_{\mathcal{P}}(v) = 2(n_{\mathcal{P}}(v)+1)\pi$. If $n_{\mathcal{P}}(v) > 0$ then v will be called a *branch point* of $\mathcal{P}$ of *order* $n_{\mathcal{P}}(v)$; equivalently we can say that the chain of circles in $\mathcal{P}$ associated with the adjacent vertices of v in $\mathbb{K}^{(0)}$ winds $(n_{\mathcal{P}}(v)+1)$-times around $C_{\mathcal{P}}(v)$. The set $\mathrm{br}(\mathcal{P}) \subset \mathrm{int}\,\mathbb{K}^{(0)} \times \mathbf{N}$ will denote the *branch set* of $\mathcal{P}$, i.e. $\mathrm{br}(\mathcal{P}) = \big\{(v_1, n_{\mathcal{P}}(v_1)), \ldots, (v_m, n_{\mathcal{P}}(v_m))\big\}$, where the v_i's, $i = 1,\ldots,m$, are all the branch points of $\mathcal{P}$.

It is important to realize (see [BSt2]) that if all values of the angle function Θ_ρ of a labelled complex $\mathbb{K}(\rho)$ are positive integer multiples of 2π, then $\rho \equiv r_{\mathcal{P}}$ and $\Theta_\rho \equiv \Theta_{\mathcal{P}}$ for some circle packing $\mathcal{P}$ for $\mathbb{K}$.

When working with labelled complexes it is very helpful to have the notion of subpackings (see [BSt2]). Given a function $\Theta : \mathrm{int}\,\mathbb{K}^{(0)} \to (0,\infty)$ we say that a labelled complex $\mathbb{K}(\rho)$ is a *subpacking* or *packing* for Θ if $\Theta_\rho \geq \Theta$ or $\Theta_\rho \equiv \Theta$, respectively. In particular, a circle packing $\mathcal{P}$ for $\mathbb{K}$ is a subpacking or packing for Θ if the labelled complex $\mathbb{K}(r_{\mathcal{P}})$ is a subpacking or packing for Θ, respectively. It turns out that subpackings have a nice monotonicity property: if $\mathbb{K}(\rho_1)$ and $\mathbb{K}(\rho_2)$ are subpackings for Θ then $\mathbb{K}(\hat\rho)$ is also a subpacking for Θ, where $\hat\rho(v) := \max\{\rho_1(v), \rho_2(v)\}$. The following simple observation is known as Maximum Principle (cf. [D1]).

Fact 2.1 (Maximum Principle). *Let $\mathbb{K}(\rho_1)$ and $\mathbb{K}(\rho_2)$ be labelled complexes for $\mathbb{K}$. If $\mathbb{K}(\rho_2)$ is a subpacking for Θ_{ρ_1} then either $\frac{\rho_2(v)}{\rho_1(v)} < \sup_{v\in\mathbb{K}^{(0)}} \frac{\rho_2(v)}{\rho_1(v)}$ for all $v \in \mathrm{int}\,\mathbb{K}^{(0)}$ or there exists $c > 0$ such that $\rho_2 \equiv c\rho_1$. In particular, if $\Theta_{\rho_1} \equiv \Theta_{\rho_2}$ and $\mathbb{K}$ is finite then $\min_{v\in\mathrm{bd}\,\mathbb{K}^{(0)}} \frac{\rho_2(v)}{\rho_1(v)} \leq \frac{\rho_2(v)}{\rho_1(v)} \leq \max_{v\in\mathrm{bd}\,\mathbb{K}^{(0)}} \frac{\rho_2(v)}{\rho_1(v)}$ for all $v \in \mathbb{K}^{(0)}$.*

The next result deals with a boundary value problem for labelled complexes. Its first part can be easily verified using Maximum Principle and Perron's methods introduced in [BSt2] (cf. [Bo],[D1],[G]); the second part can be found in [D1] (cf. [Bo]).

Theorem 2.2. *Let $\Theta : \mathrm{int}\,\mathbb{K}^{(0)} \to (0,\infty)$ be a function. Suppose a labelled complex $\mathbb{K}(\rho)$ is a packing for Θ. Then*

(i) *For any function $\lambda : \mathrm{bd}\,\mathbb{K}^{(0)} \to (0,\infty)$ there exists a unique label $\rho_\lambda : \mathbb{K}^{(0)} \to (0,\infty)$ such that $\mathbb{K}(\rho_\lambda)$ is a packing for Θ and $\rho_\lambda|_{\mathrm{bd}\,\mathbb{K}^{(0)}} \equiv \lambda$.*

(ii) *If $\mathbb{K}$ is a finite triangulation of a disc and $\Theta(\mathrm{int}\,\mathbb{K}^{(0)}) \subset \{2n\pi\}_{n>0}$ then there exists a circle packing $\mathcal{P}$ for $\mathbb{K}$ contained in the unit disc such that $\Theta_{\mathcal{P}} \equiv \Theta$ and all boundary circles of $\mathcal{P}$ are internally tangent to the unit circle. Moreover, $\mathcal{P}$, called Bl-type packing for $\mathbb{K}$, is unique up to Möbius transformations preserving the unit disc.*

Let $\mathcal{P}$ and $\mathcal{Q}$ be circle packings for $\mathbb{K}$. A function $f : \mathrm{carr}(\mathcal{P}) \to \mathbf{C}$ is said to be the *circle packing map* (shortly, cp-map) from $\mathcal{P}$ to $\mathcal{Q}$ if f maps the center of $C_{\mathcal{P}}(v)$ to the center of $C_{\mathcal{Q}}(v)$ and extends affinely to faces of $\mathrm{carr}(\mathcal{P})$. In such a case we call $\mathcal{P}$ ($\mathcal{Q}$) the *domain* (*range*) packing of f. If $\mathcal{P}$ is *univalent*, i.e. all circles in $\mathcal{P}$ have mutually disjoint interiors (in particular, $\mathrm{br}(\mathcal{P}) = \emptyset$), then $\mathrm{carr}(\mathcal{P})$ can naturally be regarded as a subset of $\mathbf{C}$ and the map f as a function defined in such a subset. If $\mathcal{P}$ is not univalent then f should be thought of as a function from the underlying surface of $\mathcal{P}$ (=$\mathrm{carr}(\mathcal{P})$).

With the pair of packings $\mathcal{P}$ and $\mathcal{Q}$ we associate another map $f^{\#} : \mathrm{carr}(\mathcal{P}) \to (0, \infty)$ called the *ratio map* from $\mathcal{P}$ to $\mathcal{Q}$. It is defined on $\mathbb{K}^{(0)}$ by $f^{\#}(S_{\mathcal{P}}(v)) := \frac{r_{\mathcal{Q}}(v)}{r_{\mathcal{P}}(v)}$ and then extended affinely to $\mathbb{K}^{(1)}$ and $\mathbb{K}^{(2)}$. As before, depending on whether $\mathcal{P}$ is univalent or not, $f^{\#}$ can be regarded as a function from a subset of $\mathbf{C}$ or from the underlying surface of $\mathcal{P}$. Due to various approximation results (see [D1],[D2],[RS],[St1]), $f^{\#}$ should be thought of as a discrete analogue of the absolute value of the complex derivative of the cp-map f.

Before we state the next result we need the following three definitions. We will say that a complex $\mathbb{K}$ is of *bounded degree* d if every vertex in $\mathbb{K}$ has at most d neighboring vertices. A function $f : \Omega \to \mathbf{C}$, $\Omega \subset \mathbf{C}$, is of *valence* M if M is the least upper bound on the number of elements in $f^{-1}(z)$ for every point $z \in \mathbf{C}$. Finally, the *valence* of a circle packing $\mathcal{P}$ is defined as the valence of the cp-map from a univalent circle packing to the packing $\mathcal{P}$.

Theorem 2.3. *Let $\mathcal{P}$ be a univalent circle packing for $\mathbb{K}$ such that $\mathrm{carr}(\mathcal{P}) = \mathbf{C}$ and $\mathbb{K}$ is of bounded degree d. Let M be a positive integer. There exists a constant $\kappa = \kappa(M, d)$, depending only on M and d, with the following property: if $\mathcal{Q}$ is a finite valence circle packing for $\mathbb{K}$ with finite (possible empty) branch set, say $\{(v_1, n_1), \ldots, (v_m, n_m)\}$, satisfying $\sum_{i=1}^m n_i \leq M$ then:*

1. $\frac{r_{\mathcal{Q}}(v)}{r_{\mathcal{Q}}(w)} \in (1/\kappa, \kappa)$ *for all pairs of adjacent vertices $v, w \in \mathbb{K}^{(0)}$,*
2. *the cp-map f from $\mathcal{P}$ to $\mathcal{Q}$ is κ-quasiregular with a decomposition $f = \psi \circ h$, where $h : \mathbf{C} \to \mathbf{C}$ is κ-quasiconformal and ψ is a complex polynomial with the branch set $\big\{\big(h(S_{\mathcal{P}}(v_1)), n_1\big), \ldots, \big(h(S_{\mathcal{P}}(v_m)), n_m\big)\big\}$.*

The above fact is a result of Corollary 4.9 and Lemma 5.2 of [D2]. Notice that if $\mathrm{br}(\mathcal{Q}) = \emptyset$ then $\mathcal{Q}$ is univalent and f is κ-quasiconformal.

We now shift our attention to the subject of random walks. We will establish basic terminology and notation; the reader should refer to [So] and [W] for more information. If $\mathbb{K}$ is an infinite simplicial 2-complex given by a triangulation of a domain in $\mathbf{C}$ then its 1-skeleton, $\mathbb{K}^{(1)}$, is a graph. If $\mathbb{K}$ is of bounded degree then so is the graph $\mathbb{K}^{(1)}$. A function $p : \mathbb{K}^{(0)} \times \mathbb{K}^{(0)} \to [0, \infty)$ such that $\sum_w p(v, w) = 1$ for each $v \in \mathbb{K}^{(0)}$ will be called the *transition probability function* and $p(v, w)$ the *transition probability* from v to w. A stochastic process on $\mathbb{K}^{(1)}$ given by p will be called the *random walk* on $\mathbb{K}^{(1)}$ with the transition probability p, and denoted $(\mathbb{K}^{(1)}, p)$. For the purposes of this paper we assume that

1. $p(v, v) = 0$ for all $v \in \mathbb{K}^{(0)}$,
2. $p(v, w) = 0$ if v and w are not neighbors,
3. $p(v, w) > 0$ whenever v and w are adjacent.

An example of a random walk on $\mathbb{K}^{(1)}$ is the *simple random walk* on $\mathbb{K}^{(1)}$ given by

$$p(v,w) = \begin{cases} \frac{1}{\delta(v)} & \text{if } w \text{ is a neighbor of } v \\ 0 & \text{otherwise,} \end{cases}$$

where $\delta(v)$ denotes the number of adjacent vertices of v in $\mathbb{K}^{(1)}$.

A random walk $(\mathbb{K}^{(1)}, p)$ is called *reversible* if there exists a function $c : \mathbb{K}^{(0)} \to (0,\infty)$ such that $c(v)p(v,w) = c(w)p(w,v)$ for all $v, w \in \mathbb{K}^{(0)}$. Notice that $c(v,w)$, called the *conductence* between v and w, given by $c(v,w) := c(v)p(v,w)$ is symetric. Conversely, a symetric function $c : \mathbb{K}^{(0)} \times \mathbb{K}^{(0)} \to (0,\infty)$ such that $\sum_w c(v,w) \in (0,\infty)$ for all $v \in \mathbb{K}^{(0)}$ defines the reversible random walk $(\mathbb{K}^{(1)}, p)$ by $p(v,w) = c(v,w)/c(v)$. Note that in the case of the simple random walk, $c(v,w) = 1$ for all neighboring vertices v, w. If $f : \mathbb{K}^{(0)} \to \mathbf{R}$ is a function and $(\mathbb{K}^{(1)}, p)$ is a random walk then f is said to be *superharmonic* with respect to p if for every $v \in \mathbb{K}^{(0)}$

$$\sum_w p(v,w) f(w) \le f(v).$$

A random walk is called *recurrent* if the probability of visiting every vertex infinitely many times is equal to 1. If a random walk is not recurrent then it is called *transient*. If the simple random walk on a graph is recurrent (transient) then the graph is called recurrent (transient). The following fact is a consequence of Theorem 4.1 and Corollary 4.14 of [W].

Theorem 2.4.

(1) *A random walk $(\mathbb{K}^{(1)}, p)$ is recurrent if and only if there are no non-constant, non-negative superharmonic functions for p.*
(2) *Let $(\mathbb{K}^{(1)}, p)$ and $(\mathbb{K}^{(1)}, \tilde{p})$ be reversible random walks with conductances c and $\tilde{c}$, respectively. If $\tilde{c}(v,w) \le c(v,w)$ for all $v, w \in \mathbb{K}^{(0)}$ then the recurrence of $(\mathbb{K}^{(1)}, p)$ implies the recurrence of $(\mathbb{K}^{(1)}, \tilde{p})$.*

We finish this section with a result that determines a connection between the type of a graph and the type of a univalent circle packing associated with it (see [Mc], [HSc]).

Theorem 2.5. *Let $\mathbb{K}$ be a triangulation of an open disc. If $\mathbb{K}^{(1)}$ is recurrent then there exists an infinite univalent circle packing for $\mathbb{K}$ whose carrier is* $\mathbf{C}$. *Conversely, if there exists a circle packing for $\mathbb{K}$ whose carrier is* $\mathbf{C}$ *and $\mathbb{K}$ is of bounded degree then $\mathbb{K}^{(1)}$ is recurrent.*

3 Variations of labelled complexes

Our goal in this section is to show that there are no non-trivial, angle-sum-preserving, bounded perturbations of a circle packing for an infinite 2-complex with recurrent 1-skeleton. The precise statement is Corollary 3.2 to the following theorem.

Theorem 3.1. *Let $\mathbb{K}$ be a triangulation of an open domain in* $\mathbf{C}$ *with recurrent $\mathbb{K}^{(1)}$. Let $\mathbb{K}(\rho)$ and $\mathbb{K}(\tilde{\rho})$ be labelled complexes for $\mathbb{K}$. Suppose $\mathbb{K}(\tilde{\rho})$ is a subpacking* *for Θ_ρ. Then the function $A : \mathbb{K}^{(0)} \to (0,\infty)$, $A(v) := \frac{\tilde{\rho}(v)}{\rho(v)}$, is bounded if and only if it is constant.*

A proof of the above result will be given shortly, first we draw some corollaries.

Corollary 3.2. *Let $\mathbb{K}$ be as in Theorem 3.1. If $\mathcal{P}$ and $\mathcal{Q}$ are circle packings for $\mathbb{K}$ with $\Theta_\mathcal{P} \equiv \Theta_\mathcal{Q}$ then either both ratio functions from $\mathcal{P}$ to $\mathcal{Q}$ and from $\mathcal{Q}$ to $\mathcal{P}$ are unbounded or are constant.*

The next result is a discrete analogue of Liouville's theorem for ratio functions.

Corollary 3.3 (Liouville's Theorem)**.** *Let $\mathcal{P}$ be a univalent circle packing with* $\mathrm{carr}(\mathcal{P}) = \mathbf{C}$. *Suppose f is a cp-map with the domain packing $\mathcal{P}$, and $f^\#$ is the associated ratio function. If the 1-skeleton of the complex of $\mathcal{P}$ is recurrent then either $f^\#$ is unbounded or is constant.*

From Theorem 2.5 we get

Corollary 3.4. *Let $\mathcal{P}$ be a univalent circle packing whose complex is of bounded degree and* $\mathrm{carr}(\mathcal{P}) = \mathbf{C}$. *Then a bounded ratio function with the domain packing $\mathcal{P}$ is constant.*

To prove Theorem 3.1 we will need the following definition and lemma. We use C^l to denote the class of functions with continuous l-th derivative.

Definition 3.5. *A collection $\{\rho(t)\}_{t\in I}$, $I = [0,1]$, is called a C^l-family of labels for $\mathbb{K}$ if the following are satisfied:*

(i) *for each $t \in I$, $\rho(t)$ is a label for $\mathbb{K}$,*
(ii) *for every $v \in \mathbb{K}^{(0)}$, the map $\rho_v : I \to (0,\infty)$ given by $\rho_v(t) := \rho(t)(v)$ is C^l.*

Lemma 3.6. *Let $\mathbb{L}$ be a star with m boundary vertices, i.e. a triangulation of a closed disc as in Figure 3. Let $\{\rho(t)\}_{t\in I}$ be a C^l-family of labels for $\mathbb{L}$. Denote by $\Theta(t)$ the angle sum at v_0 given by $\rho(t)$. If $\Theta : I \to (0,\infty)$ is non-decreasing then for each $t \in I$*

$$0 \le \sum_{i=1}^{m} \frac{1}{\rho_0(t)+\rho_i(t)} \left[\frac{\sqrt{\rho_0(t)\rho_i(t)\rho_{i-1}(t)}}{\sqrt{\rho_0(t)+\rho_i(t)+\rho_{i-1}(t)}} + \frac{\sqrt{\rho_0(t)\rho_i(t)\rho_{i+1}(t)}}{\sqrt{\rho_0(t)+\rho_i(t)+\rho_{i+1}(t)}} \right] \left(\frac{\rho_i'(t)}{\rho_i(t)} - \frac{\rho_0'(t)}{\rho_0(t)} \right),$$
$$(\rho_{m+1} \equiv \rho_1),$$

where $\rho_i : I \to (0,\infty)$ and $\rho_i(t) = \rho(t)(v_i)$, $i = 0, 1, \ldots, m$.

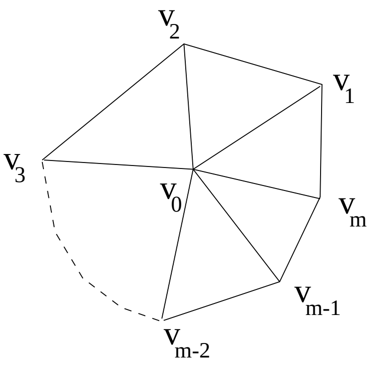

FIGURE 3. A star with m boundary vertices.

Proof. By the hypothesis, $0 \le \Theta'(t)$ for all $t \in I$. Notice that

$$\Theta(t) = \sum_{i=1}^{m} \arccos\left[1 - \frac{2\rho_i(t)\rho_{i+1}(t)}{(\rho_0(t)+\rho_i(t))(\rho_0(t)+\rho_{i+1}(t))}\right].$$

Therefore

$$\begin{aligned}
\Theta' &= \sum_{i=1}^{m} \frac{1}{\sqrt{1-\left(1-\frac{2\rho_i\rho_{i+1}}{(\rho_0+\rho_i)(\rho_0+\rho_{i+1})}\right)^2}} \left(\frac{2\rho_i\rho_{i+1}}{(\rho_0+\rho_i)(\rho_0+\rho_{i+1})}\right)' \\
&= \sum_{i=1}^{m} \frac{1}{\sqrt{\frac{(\rho_0+\rho_i)(\rho_0+\rho_{i+1})}{\rho_i\rho_{i+1}}-1}} \frac{\rho_i'\rho_0\rho_{i+1}(\rho_0+\rho_{i+1})+\rho_{i+1}'\rho_0\rho_i(\rho_0+\rho_i)-\rho_0'\rho_i\rho_{i+1}(2\rho_0+\rho_i+\rho_{i+1})}{\rho_i\rho_{i+1}(\rho_0+\rho_i)(\rho_0+\rho_{i+1})} \\
&= \sum_{i=1}^{m} \frac{\rho_0}{\sqrt{\frac{(\rho_0+\rho_i)(\rho_0+\rho_{i+1})}{\rho_i\rho_{i+1}}-1}} \left[\frac{\rho_i'}{\rho_i}\frac{1}{\rho_0+\rho_i} + \frac{\rho_{i+1}'}{\rho_{i+1}}\frac{1}{\rho_0+\rho_{i+1}} - \frac{\rho_0'}{\rho_0}\left(\frac{1}{\rho_0+\rho_i}+\frac{1}{\rho_0+\rho_{i+1}}\right)\right] \\
&= \sum_{i=1}^{m} \left[\frac{1}{\sqrt{\frac{(\rho_0+\rho_i)(\rho_0+\rho_{i-1})}{\rho_i\rho_{i-1}}-1}} + \frac{1}{\sqrt{\frac{(\rho_0+\rho_i)(\rho_0+\rho_{i+1})}{\rho_i\rho_{i+1}}-1}}\right] \frac{\rho_0}{\rho_0+\rho_i}\left(\frac{\rho_i'}{\rho_i}-\frac{\rho_0'}{\rho_0}\right) \\
&= \sum_{i=1}^{m} \frac{1}{\rho_0+\rho_i}\left[\frac{\sqrt{\rho_0\rho_i\rho_{i-1}}}{\sqrt{\rho_0+\rho_i+\rho_{i-1}}} + \frac{\sqrt{\rho_0\rho_i\rho_{i+1}}}{\sqrt{\rho_0+\rho_i+\rho_{i+1}}}\right]\left(\frac{\rho_i'}{\rho_i}-\frac{\rho_0'}{\rho_0}\right) \ge 0
\end{aligned}$$

□

We can now establish Theorem 3.1. The proof will proceed by contradiction; using variational methods we will generate a random walk which, by the hypothesis of the theorem, cannot exist. The reader might be interested to learn that Colin de Verdière have also applied variational techniques in [CdV] to show the existence of circle packings.

Proof of Theorem 3.1. Suppose that the assertion is false. Then there exist two vertices $w_0, w_1 \in \mathbb{K}^{(0)}$ such that $\frac{\tilde\rho(w_0)}{\rho(w_0)} \ne \frac{\tilde\rho(w_1)}{\rho(w_1)}$ and $M := \sup_{v\in\mathbb{K}^{(0)}} A(v) < \infty$. We may assume without loss of generality that $\tilde\rho(w_0) = \rho(w_0)$ and $\tilde\rho(w_1) > \rho(w_1)$. Define a family $\mathbf{F}$ of labels for $\mathbb{K}$ by

$$\left\{\varrho\colon \mathbb{K}(\varrho) \text{ is a subpacking for } \Theta\rho,\ \varrho(w_0) = \rho(w_0), \text{ and } \tfrac{\varrho(v)}{\rho(v)} \le M \text{ for all } v \in \mathbb{K}^{(0)}\right\}.$$

Notice that $\mathbf{F}$ is not empty as $\tilde\rho, \rho \in \mathbf{F}$. It follows from a monotonicity property of subpackings that the label $\hat\rho$ given by $\hat\rho(v) := \sup_{\varrho\in\mathbf{F}}\{\varrho(v)\}$ is a subpacking for $\Theta\rho$. From definitions of $\mathbf{F}$ and $\hat\rho$ we obtain that $\hat\rho \in \mathbf{F}$, and $\hat\rho \ge \rho$ with $\hat\rho(w_1) > \rho(w_1)$. Hence $\hat\rho$ is not a constant multiple of ρ. Moreover, $\sup_{v\in\mathbb{K}^{(0)}} \frac{\hat\rho(v)}{\rho(v)} = M$. From Maximum Principle, $\frac{\hat\rho(v)}{\rho(v)} < M$ for all $v \in \mathbb{K}^{(0)}$. This implies that $\Theta_{\hat\rho}(v) = \Theta_\rho(v)$ for all $v \ne w_0$, and $\Theta_{\hat\rho}(w_0) > \Theta_\rho(w_0)$ (for if $\Theta_{\hat\rho}(v_0) > \Theta_\rho(v_0)$ for some $v_0 \ne w_0$ then there is $\epsilon > 0$ such that $\bar\rho : \mathbb{K}^{(0)} \to (0,\infty)$, defined by $\bar\rho|_{\mathbb{K}^{(0)}\setminus\{v_0\}} \equiv \hat\rho|_{\mathbb{K}^{(0)}\setminus\{v_0\}}$ and $\bar\rho(v_0) = \hat\rho(v_0) + \epsilon$, belongs to $\mathbf{F}$).

Let $\{\mathbb{K}_n\}$ be a sequence of finite, connected 2-subcomplexes of $\mathbb{K}$ such that $\mathbb{K}_n \subset \mathbb{K}_{n+1}$, $\bigcup_{n=1}^{\infty} \mathbb{K}_n = \mathbb{K}$, and $w_0, w_1 \in \operatorname{int} \mathbb{K}_1^{(0)}$. It follows from Theorem 2.2(i) (and Implicite Function Theorem) that for each n there is a unique C^1-family of labels $\{\rho_n(t)\}_{t\in I}$ for $\mathbb{K}_n$ such that

(1) $\rho_n(t)(v) = (1-t)\rho(v) + t\hat{\rho}(v)$ for every $v \in \operatorname{bd} \mathbb{K}_n^{(0)}$,
(2) $\rho_n(t)(w_0) = \rho(w_0)$ for all t,
(3) $\Theta_{\rho_n(t)}(v) = \Theta_\rho(v)$ for every $v \in \operatorname{int} \mathbb{K}_n^{(0)} \setminus \{w_0\}$ and all t.

Observe that the uniqueness in Theorem 2.2(i) gives $\rho_n(0)(v) = \rho(v)$ and $\rho_n(1)(v) = \hat{\rho}(v)$ for all $v \in \mathbb{K}_n^{(0)}$. Since for each $v \in \operatorname{bd} \mathbb{K}_n^{(0)}$ the function $\rho_{v,n} : I \to (0,\infty)$, $\rho_{v,n}(t) := \rho_n(t)(v)$, is non-decreasing, Maximum Principle implies that for every $v \in \mathbb{K}_n^{(0)}$ the function $\rho_{v,n}$ is non-decreasing. Thus the function $\Theta_{w_0,n} : I \to (0,\infty)$, $\Theta_{v,n}(t) := \Theta_{\rho_n(t)}(v)$, is non-decreasing. Moreover, from Maximum Principle we get for all $t \in I$

$$1 = \min_{v\in\{w_0\}\cup \operatorname{bd}\mathbb{K}_n^{(0)}} \frac{\rho_{v,n}(t)}{\rho(v)} = \min_{v\in\mathbb{K}_n^{(0)}} \frac{\rho_{v,n}(t)}{\rho(v)} \le$$
$$\max_{v\in\mathbb{K}_n^{(0)}} \frac{\rho_{v,n}(t)}{\rho(v)} = \max_{v\in \operatorname{bd}\mathbb{K}_n^{(0)}} \frac{\rho_{v,n}(t)}{\rho(v)} \le \max_{v\in \operatorname{bd}\mathbb{K}_n^{(0)}} \frac{\hat{\rho}(v)}{\rho(v)} < M.$$

Hence

$$\rho_{v,n}(t) \in [\rho(v), M\rho(v)] \qquad \text{for every } n \text{ and all } t \in I. \tag{3.1}$$

Notice that using Maximum Principle we also obtain the following inequalities for small $h > 0$

$$0 \le \rho_{v,n}(t+h) - \rho_{v,n}(t) = \rho_{v,n}(t)\left(\tfrac{\rho_{v,n}(t+h)}{\rho_{v,n}(t)} - 1\right) \le$$
$$\rho_{v,n}(t) \max_{v\in \operatorname{bd}\mathbb{K}_n^{(0)}} \left(\tfrac{\rho_{v,n}(t+h)}{\rho_{v,n}(t)} - 1\right) = \rho_{v,n}(t) \max_{v\in \operatorname{bd}\mathbb{K}_n^{(0)}} \left(\tfrac{\rho_{v,n}(t+h)-\rho_{v,n}(t)}{\rho_{v,n}(t)}\right).$$

Thus, by dividing both sides in the above inequalities by h and then taking the limit as $h \to 0$ we get

$$0 \le \frac{\rho'_{v,n}(t)}{\rho_{v,n}(t)} \le \max_{v\in \operatorname{bd}\mathbb{K}_n^{(0)}} \frac{\rho'_{v,n}(t)}{\rho_{v,n}(t)} \qquad \text{for all } v \in \mathbb{K}_n^{(0)},\ t \in I, \text{ and every } n.$$

In particular,

$$0 \le \frac{\rho'_{v,n}(t)}{\rho_{v,n}(t)} \le M \qquad \text{for all } v \in \mathbb{K}_n^{(0)},\ t \in I, \text{ and every } n. \tag{3.2}$$

Denote by $\mu := \log \frac{\hat{\rho}(w_1)}{\rho(w_1)}$. Since $\mu > 0$ and

$$\mu = \int_0^1 (\log \rho_{w_1,n}(t))'\, dt = \int_0^1 \frac{\rho'_{w_1,n}(t)}{\rho_{w_1,n}(t)} dt,$$

for each n there exists $t_n \in I$ such that

$$0 < \mu \le \frac{\rho'_{w_1,n}(t_n)}{\rho_{w_1,n}(t_n)}. \tag{3.3}$$

Suppose v is an interior vertex in $\mathbb{K}_n^{(0)}$ and $w \in \mathbb{K}_n^{(0)}$ is a neighbor of v. Let w' and w'' be the other two vertices in $\mathbb{K}_n^{(0)}$ adjacent to both v and w. We define

$$c_n(v,w;t) := \frac{1}{\rho_{v,n}(t)+\rho_{w,n}(t)}\left[\sqrt{\frac{\rho_{v,n}(t)\rho_{w,n}(t)\rho_{w',n}(t)}{\rho_{v,n}(t)+\rho_{w,n}(t)+\rho_{w',n}(t)}}+\sqrt{\frac{\rho_{v,n}(t)\rho_{w,n}(t)\rho_{w'',n}(t)}{\rho_{v,n}(t)+\rho_{w,n}(t)+\rho_{w'',n}(t)}}\right]. \tag{3.4}$$

Notice that if w is also an interior vertex in $\mathbb{K}_n^{(0)}$ then

$$c_n(v,w;t) = c_n(w,v;t). \tag{3.5}$$

Moreover

$$c_n(v,w;t) \le 2\frac{\sqrt{\rho_{v,n}(t)\rho_{w,n}(t)}}{\rho_{v,n}(t)+\rho_{w,n}(t)} \le 1. \tag{3.6}$$

From Lemma 3.6 we get for every n and t

$$\begin{aligned} 0 &\le \sum_w c_n(w_0,w;t)\left(\frac{\rho'_{w,n}(t)}{\rho_{w,n}(t)} - \frac{\rho'_{w_0,n}(t)}{\rho_{w_0,n}(t)}\right) \qquad \text{and} \\ 0 &= \sum_w c_n(v,w;t)\left(\frac{\rho'_{w,n}(t)}{\rho_{w,n}(t)} - \frac{\rho'_{v,n}(t)}{\rho_{v,n}(t)}\right) \qquad \text{for } v \in \operatorname{int}\mathbb{K}_n^{(0)} \setminus \{w_0\}. \end{aligned} \tag{3.7}$$

We now make the following observation. Suppose $\{s_n\}$ is a sequence of functions $s_n : \mathbb{K}^{(0)} \times I \to \mathbf{R}$ such that for each $v \in \mathbb{K}^{(0)}$, the $s_n(v,\cdot) : I \to \mathbf{R}$, $n = 1,2,\ldots$, have uniformly bounded ranges. Then for any sequence $\{a_n\}$ of points in I there are subsequences $\{s_{n(k)}\}$ and $\{a_{n(k)}\}$ such that $\lim_{k\to\infty} s_{n(k)}\big(v, a_{n(k)}\big)$ exists for every $v \in \mathbb{K}^{(0)}$. This is due to the following diagonalization argument:

> We begin by enumerating all vertices in $\mathbb{K}^{(0)}$ so that $\{v_i\}_{i=1}^{\infty} = \mathbb{K}^{(0)}$. Since the maps $s_n(v_1,\cdot)$ have uniformly bounded ranges there exist subsequences $\{s_{n_1(k)}\}$ and $\{a_{n_1(k)}\}$ of $\{s_n\}$ and $\{a_n\}$, respectively, such that $\{s_{n_1(k)}\big(v_1, a_{n_1(k)}\big)\}_{k=1}^{\infty}$ is convergent. Next, we repeat the above construction for v_2 with $\{s_{n_1(k)}\}$ and $\{a_{n_1(k)}\}$ instead of $\{s_n\}$ and $\{a_n\}$, respectively. This yields two subsequences $\{s_{n_2(k)}\}$ and $\{a_{n_2(k)}\}$ of $\{s_{n_1(k)}\}$ and $\{a_{n_1(k)}\}$, respectively, such that $\{s_{n_2(k)}\big(v_2, a_{n_2(k)}\big)\}_{k=1}^{\infty}$ is convergent and, of course, $\lim_{k\to\infty} s_{n_2(k)}\big(v_1, a_{n_2(k)}\big) = \lim_{k\to\infty} s_{n_1(k)}\big(v_1, a_{n_1(k)}\big)$. We continue this process for all v_i's. Then by setting $n(k) := n_k(k)$ we obtain desired sequences $\{s_{n(k)}\}$ and $\{a_{n(k)}\}$.

We finish our observation by noting that analogous diagonalization argument is valid when $\mathbb{K}^{(0)}$ in the domains of the functions s_n is replaced by $\mathbb{K}^{(0)} \times \mathbb{K}^{(0)}$.

We now return to our sequence $\{t_n\}$ satisfying (3.3). We can assume, by using the inequalities (3.2) and (3.6) and by applying diagonalization argument if necessary, that

$$f(v) := \lim_{n\to\infty} \frac{\rho'_{v,n}(t_n)}{\rho_{v,n}(t_n)} \quad \text{and} \quad c(v,w) := \lim_{n\to\infty} c_n(v,w;t_n) \quad \text{exist for all } v,w \in \mathbb{K}^{(0)}.$$

From (3.1), (3.4), (3.5), and (3.6) it follows that

$$(3.8) \qquad 0 < c(v,w) = c(w,v) \leq 1 \quad \text{for all adjacent vertices } v \text{ and } w \text{ in } \mathbb{K}^{(0)}.$$

Also, (3.2) and (3.3) imply

$$(3.9) \qquad 0 \leq f(v) \leq M \quad \text{for all } v \in \mathbb{K}^{(0)}, \text{ and} \quad 0 = f(w_0) < \mu \leq f(w_1).$$

Moreover, from (3.7) we get

$$0 \leq \sum_w p(v,w)\,(f(w) - f(v)) \quad \text{for every } v \in \mathbb{K}^{(0)},$$

where $p(v,w) = c(v,w)/\sum_w c(v,w)$. Thus the function $\tilde{f} : \mathbb{K}^{(0)} \to \mathbf{R}$, $\tilde{f}(v) := M - f(v)$, is superharmonic for the reversible random walk $(\mathbb{K}^{(1)}, p)$ with conductance between v and w equal to $c(v,w)$. From Theorem 2.4(1) and (3.9) we conclude that $(\mathbb{K}^{(1)}, p)$ is transient. On the other hand, since by the hypothesis $\mathbb{K}^{(1)}$ is recurrent, Theorem 2.4(2) and (3.8) (recall that the conductances for a simple random walk are all equal to 1) imply that $(\mathbb{K}^{(1)}, p)$ is recurrent. This is a contradiction. Hence $\frac{\tilde{\rho}(w)}{\rho(w)} = \frac{\tilde{\rho}(v)}{\rho(v)}$ for all $v, w \in \mathbb{K}^{(0)}$, and the proof is complete. $\square$

4 Uniqueness of branched packings

We begin this section with the following definition which was originally introduced in [D2].

Definition 4.1. *Let $\mathbb{K}$ be a triangulation of an open disc. A cp-map f is said to be a discrete complex polynomial for $\mathbb{K}$ if the carrier of the domain packing of f is equal to $\mathbf{C}$ and the range packing is a proper, finite, branched covering of $\mathbf{C}$, i.e. f has a decomposition $f = \psi \circ h$, where h is a self-homeomorphism of $\mathbf{C}$ and ψ is a complex polynomial.*

It was shown in [D2] that discrete complex polynomials exist. The range packings of such maps form a natural expansion of the class of univalent circle packings that cover $\mathbf{C}$. However, the uniqueness result for branched packings similar to the one for univalent packings (see [RS],[Sc],[St3]) was proved so far only in the case of packings for regular hexagonal lattice ([D2]). Here we will show the uniqueness of branched circle packings of bounded valence for bounded degree triangulations whose 1-skeletons are recurrent; the case when 1-skeletons are transient is treated in [D4]. In particular, we obtain that range packings of discrete complex polynomials are unique, and furthermore, when branch sets of such packings are empty we get the earlier results of [RS], [Sc], and [St3], which are crucial for approximation schemes (cf. [D1],[D2],[H],[RS],[St1]).

Theorem 4.2. *Let $\mathbb{K}$ be a bounded degree triangulation of an open disc with recurrent $\mathbb{K}^{(1)}$. Let $\mathbf{F}_{\mathfrak{B}}$ denote the set of all bounded valence circle packings for $\mathbb{K}$ with identical finite branch sets equal to $\mathfrak{B}$. If $\mathbf{F}_{\mathfrak{B}} \neq \emptyset$ then there exists $\mathcal{P} \in \mathbf{F}_{\mathfrak{B}}$ such that $\mathcal{P}$ is the range packing of a discrete complex polynomial and $\mathbf{F}_{\mathfrak{B}} = \{z\mathcal{P}\}_{z\in\mathbf{C}}$, i.e. all packings in $\mathbf{F}_{\mathfrak{B}}$ are copies of $\mathcal{P}$ under similarities of $\mathbf{C}$.*

Proof. From Theorem 2.5 we have that there exists a univalent circle packing $\mathcal{O}$ for $\mathbb{K}$ such that $\operatorname{carr}(\mathcal{O}) = \mathbf{C}$. Since $\mathbf{F}_{\mathfrak{B}} \neq \emptyset$ there is $\mathcal{P} \in \mathbf{F}_{\mathfrak{B}}$ of finite valence with branch set $\mathfrak{B}$. From Theorem 2.3 it follows that the cp-map from $\mathcal{O}$ to $\mathcal{P}$ is a discrete complex polynomial. As $\mathcal{P}$ was arbitrary we get that all packings in $\mathbf{F}_{\mathfrak{B}}$ are range packings of discrete complex polynomials. We also see that $\{z\mathcal{P}\}_{z\in\mathbf{C}} \subset \mathbf{F}_{\mathfrak{B}}$. To finish the proof it is sufficient to show that for each $\mathcal{Q} \in \mathbf{F}_{\mathfrak{B}}$ there exists a constant $c_{\mathcal{Q}}$ such that $r_{\mathcal{Q}}(v) = c_{\mathcal{Q}} r_{\mathcal{P}}(v)$ for all $v \in \mathbb{K}^{(0)}$. However, by Corollary 3.2 it is enough to show that for each $\mathcal{Q} \in \mathbf{F}_{\mathfrak{B}}$ there is $M_{\mathcal{Q}}$ such that

$$(*) \qquad \frac{1}{M_{\mathcal{Q}}} \leq \frac{r_{\mathcal{Q}}(v)}{r_{\mathcal{P}}(v)} \leq M_{\mathcal{Q}} \qquad \text{for all } v \in \mathbb{K}^{(0)}.$$

The remainder of our proof is devoted entirely to verification of (*).

Fix $\mathcal{Q} \in \mathbf{F}_{\mathfrak{B}}$, $\mathcal{Q} \neq \mathcal{P}$. By applying similarities to $\mathcal{O}$, $\mathcal{P}$, and $\mathcal{Q}$ if necessary, we can assume that $S_{\mathcal{O}}(v_0) = S_{\mathcal{P}}(v_0) = S_{\mathcal{Q}}(v_0) = 0$ and $r_{\mathcal{O}}(v_0) = r_{\mathcal{P}}(v_0) = r_{\mathcal{Q}}(v_0) = 1$ for a designated vertex $v_0 \in \mathbb{K}^{(0)}$, where the notation is as in Section 2. Denote by f_1 and f_2 the cp-maps from $\mathcal{O}$ to $\mathcal{P}$ and from $\mathcal{O}$ to $\mathcal{Q}$, respectively. Notice that $f_1(0) = 0 = f_2(0)$ and, by Theorem 2.3, $f_1 = \psi_1 \circ h_1$ and $f_2 = \psi_2 \circ h_2$, where h_1, h_2 are entire K-quasiconformal mappings, ψ_1, ψ_2 are complex polynomials of degree, say, l, and $h_1(0) = 0 = h_2(0)$, $h_1(\infty) = \infty = h_2(\infty)$. Moreover, for each j, $j = 1, 2$,

$$(4.1) \qquad \frac{1}{K} \leq \frac{r_j(w)}{r_j(v)} \leq K \quad \text{for all pairs of adjacent vertices } v, w \in \mathbb{K}^{(0)},$$

where r_1 and r_2 are the radius functions of $\mathcal{P}$ and $\mathcal{Q}$, respectively. It follows from quasiconformal arguments (Theorem 2.4 of [L]) that there exists a constant $c = c(K)$ depending only on K such that

$$\frac{\max_\alpha |h_j(\delta e^{i\alpha})|}{\min_\alpha |h_j(\delta e^{i\alpha})|} \leq c \qquad \text{for every } \delta > 0,\ j = 1, 2.$$

Since ψ_j's are polynomials of degree l, there is $R > 0$ such that for each j, $j = 1, 2$,

$$(4.2) \qquad \frac{\max_\alpha |f_j(\delta e^{i\alpha})|}{\min_\alpha |f_j(\delta e^{i\alpha})|} = \frac{\max_\alpha |h_j(\delta e^{i\alpha})|}{\min_\alpha |h_j(\delta e^{i\alpha})|} \frac{|1 + O(\frac{1}{\delta})|}{|1 - O(\frac{1}{\delta})|} \leq 2c \quad \text{for all } \delta > R.$$

For each $n > R$ let $\mathcal{O}_n$ denote the largest, simply connected portion of $\mathcal{O}$ contained in $D(n)$, $D(\delta) := \{|z| < \delta\}$. Write $\mathbb{K}_n \subset \mathbb{K}$ for the complex of $\mathcal{O}_n$. Let $\mathcal{P}_n$ and $\mathcal{Q}_n$ be portions of $\mathcal{P}$ and $\mathcal{Q}$, respectively, associated with the complex $\mathbb{K}_n$. By Theorem 2.2(ii) there exists a Bl-type packing $\mathcal{B}_n$ for $\mathbb{K}_n$ with branch set $\mathfrak{B}$, and normalized so that the circle $C_{\mathcal{B}_n}(v_0)$ is centered at 0. Define $\rho_j(n) := \max_\alpha |f_j(ne^{i\alpha})|$ for $j = 1, 2$. Write $\tilde{\mathcal{P}}_n := \frac{1}{\rho_j(n)}\mathcal{P}_n$ and $\tilde{\mathcal{Q}}_n := \frac{1}{\rho_j(n)}\mathcal{Q}_n$.

Since $\tilde{\mathcal{P}}_n, \tilde{\mathcal{Q}}_n \subseteq D(1)$ and packings $\mathcal{B}_n$, $\mathcal{P}_n$, and $\mathcal{Q}_n$ have identical branch sets, by discrete Schwarz Lemma of [BSt2] one has

$$\frac{1}{\rho_j(n)} = \frac{r_j(v_0)}{\rho_j(n)} \leq \eta_n(v_0) \qquad \text{for all } n, \text{ and } j = 1, 2, \tag{4.3}$$

where η_n is the radius function of $\mathcal{B}_n$.

From (4.2) we obtain that $\frac{1}{\rho_j(n)} f_j\big(\partial D(n)\big) \subseteq \Omega := \{|z| \geq \frac{1}{2c}\}$. However, it is not generally true that all boundary circles of $\tilde{\mathcal{P}}_n$ and $\tilde{\mathcal{Q}}_n$ (i.e., circles in $\tilde{\mathcal{P}}_n$ and $\tilde{\mathcal{Q}}_n$ corresponding to boundary vertices of $\mathbb{K}_n$) intersect Ω. Nevertheless, we will now show that they are not far away from the set Ω. To be more precise, for $v \in \operatorname{bd}\mathbb{K}_n^{(0)}$ we define

$$\delta_j^n(v) = \begin{cases} 0 & \text{if } \frac{1}{\rho_j(n)}\left(|f_j(v)| + r_j(v)\right) \geq \frac{1}{2c} \\ \frac{1}{2c} - \frac{1}{\rho_j(n)}\left(|f_j(v)| + r_j(v)\right) & \text{otherwise,} \end{cases}$$

which is the distance between the boundary circle $C_{\tilde{\mathcal{P}}_n}(v)$ and Ω when $j = 1$ or between the boundary circle $C_{\tilde{\mathcal{Q}}_n}(v)$ and Ω when $j = 2$ (Figure 4). We are only interested in vertices $v \in \operatorname{bd}\mathbb{K}_n^{(0)}$ with $\delta_j^n(v) > 0$; for if $\delta_j^n(v) = 0$ then $C_{\tilde{\mathcal{P}}_n}(v) \cap \Omega \neq \emptyset$ when $j = 1$ and $C_{\tilde{\mathcal{Q}}_n}(v) \cap \Omega \neq \emptyset$ when $j = 2$.

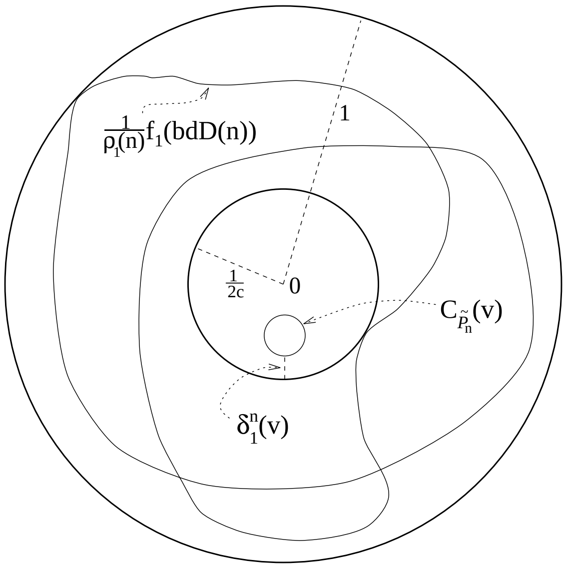

Figure 4. $\delta_j^n(v)$ for $j = 1$.

It is immediate that

$$\frac{1}{\rho_j(n)} r_j(v) \leq \frac{1}{2c} - \delta_j^n(v) \qquad \text{for } \delta_j^n(v) > 0. \tag{4.4}$$

From the definitions of $\mathbb{K}_n$ and the cp-maps f_j it is easy to see that if $v \in \operatorname{bd}\mathbb{K}_n^{(0)}$ and $C_{\tilde{\mathcal{P}}_n}(v) \cap \Omega = \emptyset$ (respectively, $C_{\tilde{\mathcal{Q}}_n}(v) \cap \Omega = \emptyset$) then there is at least one neighbor

$w \in \mathbb{K}^{(0)}$ of v such that the circle in $\frac{1}{\rho_1(n)}\mathcal{P}$ (respectively, $\frac{1}{\rho_2(n)}\mathcal{Q}$) corresponding to w intersects Ω. Therefore, it follows that

$$\delta_j^n(v) \leq \frac{2}{\rho_j(n)} \max_{\substack{w \in \mathbb{K}^{(0)} \\ w \text{ adjucent to } v}} r_j(w) \qquad \text{whenever } \delta_j^n(v) > 0,\ j = 1, 2.$$

Thus, by (4.1)

$$\delta_j^n(v) \leq \frac{2}{\rho_j(n)} K r_j(v) \qquad \text{when } \delta_j^n(v) > 0. \tag{4.5}$$

Hence, (4.4) and (4.5) imply

$$\delta_j^n(v) \leq \frac{K}{c}\,(2K+1)^{-1}\,, \tag{4.6}$$

i.e. the boundary circles of $\tilde{\mathcal{P}}_n$ and $\tilde{\mathcal{Q}}_n$ that do not intersect Ω are "not far away" from Ω.

It follows from (4.6) that $2c(2K+1) \geq (\frac{1}{2c} - \delta_j^n(v))^{-1}$. The last inequality implies that packings $\hat{\mathcal{P}}_n := 2c(2K+1)\tilde{\mathcal{P}}_n$ and $\hat{\mathcal{Q}}_n := 2c(2K+1)\tilde{\mathcal{Q}}_n$ have their boundary circles intersecting $\{|z| \geq 1\}$, and therefore, in the language of [DSt], $\hat{\mathcal{P}}_n$ and $\hat{\mathcal{Q}}_n$ properly cover $D(1)$. Hence, by Discrete Distortion Lemma of [DSt]

$$\eta_n(v_0) \leq 2c(2K+1)\frac{r_j(v_0)}{\rho_j(n)} \qquad \text{for all } n > R, \text{ and } j = 1, 2. \tag{4.7}$$

Thus, (4.3) and (4.7) give

$$\frac{1}{2c(2K+1)}\frac{r_1(v_0)}{\rho_1(n)} \leq \frac{r_2(v_0)}{\rho_2(n)} \leq 2c(2K+1)\frac{r_1(v_0)}{\rho_1(n)} \qquad \text{for all } n > R. \tag{4.8}$$

We will now show that (4.8) can be extended to every vertex of $\mathbb{K}$ with, perhaps, slightly larger but uniform constant instead of $2c(2K+1)$. Denote by $\tilde{\mathcal{O}}_n$ a univalent Bl-type packing (i.e., with empty branch set) for $\mathbb{K}_n$ with the circle $C_{\tilde{\mathcal{O}}_n}(v_0)$ centered at 0. Since $\mathbb{K}$ is recurrent, it follows from the proof of Discrete Uniformization Theorem of [BSt1,§4] and Theorem 2.5 that for each $v \in \mathbb{K}^{(0)}$, $C_{\tilde{\mathcal{O}}_n}(v)$ converges to a point which is the origin as $n \to \infty$. The monotonicity results of [BSt2, Theorem 4] imply that for each $v \in \mathbb{K}^{(0)}$, $C_{\mathcal{B}_n}(v)$ converges to a point which is the origin as $n \to \infty$.

Let $\epsilon > 0$ be small. Fix $w \in \mathbb{K}^{(0)}$, $w \neq v_0$. From the above convergence results there exists $R_w \geq R$ such that

$$C_{\mathcal{B}_n}(w) \subset D(\epsilon), \quad \tfrac{1}{\rho_1(n)}|f_j(w)| < \epsilon, \quad \text{and } \tfrac{1}{\rho_1(n)}|r_j(w)| < \epsilon \quad \text{for all } n \geq R_w.$$

Let $\mathcal{B}_n(w)$ be a Bl-type packing for $\mathbb{K}_n$ with branch set $\mathfrak{B}$ and normalized so that its circle associated with w is centered at 0. We also define $\tilde{\mathcal{P}}_n(w)$ and $\tilde{\mathcal{Q}}_n(w)$ as packings that are obtained by translating packings $(1-\epsilon)\tilde{\mathcal{P}}_n$ and $(1-\epsilon)\tilde{\mathcal{Q}}_n$, respectively, so that their circles corresponding to the vertex w are centered at 0. Thus, by discrete Schwarz Lemma of [BSt2],

$$(1-\epsilon)\frac{r_j(w)}{\rho_j(n)} \le \eta_n^w(w) \qquad \text{for all } n \ge R_w, \text{ and } j=1,2, \tag{4.9}$$

where η_n^w is the radius function of $\mathcal{B}_n(w)$. It is clear from the definition of $\tilde{\mathcal{P}}_n(w)$ that the boundary circle of $\tilde{\mathcal{P}}_n(w)$ associated with a vertex $v \in \operatorname{bd}\mathbb{K}_n^{(0)}$ is within the distance $\delta_1^n(v) + \epsilon(\frac{1}{2c} - \delta_1^n(v)) + \epsilon$ from Ω for all $n \ge R_w$. Similarly for boundary circles of $\tilde{\mathcal{Q}}_n(w)$. Thus it follows from (4.6) that $\hat{\mathcal{P}}_n(w) := c(K,\epsilon)\tilde{\mathcal{P}}_n(w)$ and $\hat{\mathcal{Q}}_n(w) := c(K,\epsilon)\tilde{\mathcal{Q}}_n(w)$ properly cover $D(1)$, where $c(K,\epsilon) := \frac{2c(2K+1)}{1-\epsilon-\epsilon 2c(2K+1)}$. Hence, by Discrete Distortion Lemma of [DSt],

$$\eta_n^w(w) \le c(K,\epsilon)(1-\epsilon)\frac{r_j(w)}{\rho_j(n)} \qquad \text{for all } n \ge R_w, \text{ and } j=1,2. \tag{4.10}$$

Therefore, (4.9) and (4.10) give

$$\frac{1}{c(K,\epsilon)}\frac{\rho_2(n)}{\rho_1(n)} \le \frac{r_2(w)}{r_1(w)} \le c(K,\epsilon)\frac{\rho_2(n)}{\rho_1(n)} \qquad \text{for all } n \ge R_w. \tag{4.11}$$

Now (4.8) and (4.11) imply

$$\frac{1}{2c(2K+1)c(K,\epsilon)}\frac{r_2(v_0)}{r_1(v_0)} \le \frac{r_2(w)}{r_1(w)} \le 2c(2K+1)c(K,\epsilon)\frac{r_2(v_0)}{r_1(v_0)}.$$

Since w is arbitrary vertex of $\mathbb{K}$, the last double inequality proves the assertion (*). $\square$

From Theorem 2.5 and the above result we obtain

Corollary 4.3. *Let $\mathcal{U}$ be a univalent, bounded degree circle packing with* $\operatorname{carr}(\mathcal{U}) = \mathbf{C}$. *Denote by $\mathbb{K}$ the complex of $\mathcal{U}$. Then the set of all finite valence circle packings for $\mathbb{K}$ with finite branch sets is equal to the set of all range packings of discrete complex polynomials for $\mathbb{K}$. Moreover, each such a packing is determined uniquely, up to similarities of* $\mathbf{C}$, *by its branch set.*

Concluding Remarks. (1) We conjecture that Theorem 4.2 should also be true for triangulations with recurrent 1-skeletons but not necessarily of bounded degree.

(2) For complete analogy with the univalent setting, one should be able to show the uniqueness result, similar to Corollary 4.3, for branched circle packings whose corresponding univalent packings cover $\mathbf{C}$ but underlying complexes are not of bounded degree. As it was shown in [HSc], such complexes might have 1-skeletons which are not recurrent even though univalent packings associated with them have carriers equal to $\mathbf{C}$.

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