# LECTURES ON THE ARTHUR–SELBERG TRACE FORMULA

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These are Notes prepared for nine lectures given at the Mathematical Sciences Research Institute, MSRI, Berkeley during the period January–March 1995. It is a pleasant duty to record here my gratitude to MSRI, and its staff, for making possible this 1994–95 Special Year in Automorphic Forms, and for providing such a setting for work.

The purpose of these Notes is to describe the contents of Arthur's earlier, foundational papers on the trace formula. In keeping with the introductory nature of the lectures, we have sometimes illustrated the ideas of Arthur's general theory by applying them in detail to the case of GL(2); we have also included a few lectures on the "simple trace formula" (and its applications), and on Jacquet's relative trace formula.

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## Lecture I. Introduction to the Trace Formula

The ultimate purpose of the trace formula is to better understand the structure of automorphic forms on a general reductive group $G$. In particular, one wants to understand the decomposition of the right regular representation $R_0(g)$ of $G(\mathbb{A})$ in the space of cusp forms $L_0^2(G(F) \setminus G(\mathbb{A}))$. The original idea of the (Selberg) trace formula—roughly speaking—was to describe the trace of the integral operator $R_0(f) = \int_{G(\mathbb{A})} f(g)R_0(g)dg$ for $f$ in $C_c^\infty(G(\mathbb{A}))$; this Selberg succeeded to do for certain special $G$. On the other hand, *Arthur's trace formula* is valid for arbitrary $G$, and asserts the equality of two different sums of distributions on $G$, namely:

$$\sum_{\mathfrak{o}\in\mathfrak{O}} J_{\mathfrak{o}}^T(f) = \sum_{\chi\in\mathfrak{X}} J_\chi^T(f). \tag{0.1}$$

Here the left side of (0.1) is summed over certain equivalence classes in the group of rational points $G(F)$; it is called the *geometric side* of the trace formula. On the right side of (0.1), the sum is over certain "cuspidal data" $\chi = \{(M,\sigma)\}$ of $G$; this is the *spectral side* of the trace formula (one of whose terms is $\mathrm{tr}R_0(f)$).

Our purpose in these lectures is to describe and develop this formula in complete detail, but mostly in the context of GL(2). We also wish to describe some refinements of (0.1) necessary for applications to automorphic forms. The basic references are Arthur's original papers [A1]–[A10]. Earlier expositions of the GL(2) trace formula, such as [Ge] and [GJ], will be referred to occasionally; but as they mostly predate Arthur's general development of the trace formula, their *point of view* will largely be ignored, and their arguments not reproduced here.

**1. Some history.** "The" trace formula was introduced by Selberg, in the context of a semi-simple Lie group $G$ and discrete subgroup $\Gamma$, in his famous 1956 paper [Se]. In that paper Selberg first of all described a general formula in the case of *compact quotient* $\Gamma \setminus G$ (see the next section below for a brief description). Secondly, he treated in detail certain special *non-compact* quotient cases such as $SL_2(\mathbb{Z}) \setminus SL_2(\mathbb{R})$; here Selberg analytically continued Eisenstein Series $E(z,s)$ in order to handle the continuous spectrum of $L^2(\Gamma \setminus G)$, and derived a formula for the trace of $R(f)$ in the space of cusp forms. In both the case of compact and non-compact quotient, Selberg was motivated by applications to geometry and number theory (lengths of geodesics in the Riemann surface in the case of compact quotient, traces of Hecke operators in case of non-compact quotient, etc.). But despite the obvious desirability of generalizing these results in the case of non-compact quotient, the obstacles—such as analytically continuing the Eisenstein series in higher rank—were formidable. A big breakthrough came in the 1960's when Langlands developed a theory of Eisenstein series valid for any reductive group $G$, and used it to describe the continuous spectrum of $L^2(G(F) \setminus G(\mathbb{A}))$ (cf. [La 1]). As an outgrowth of this theory, Langlands was led to his amazing conjectures about automorphic forms and their $L$-functions, i.e., the "Langlands program", with its cornerstone the *principle of functoriality of automorphic forms.* A typical result in this program is the following:

**Theorem 1 ([JL]).** *Let $D$ be a division quaternion algebra over a number field $F$ and $G' = D^*$. Then to each automorphic cuspidal representation $\sigma$ of $G'$ (i.e., an irreducible $G'(\mathbb{A})$-submodule of $L^2(Z(\mathbb{A})G'(F)\setminus G'(\mathbb{A}))$ of dimension greater than one) there exists a corresponding automorphic cuspidal representation $\pi = \pi(\sigma)$ of $G(\mathbb{A}) = \mathrm{GL}_2(\mathbb{A})$ (an irreducible $G(\mathbb{A})$-submodule of the space of cusp forms $L_0^2(Z(\mathbb{A})G(F)\setminus G(\mathbb{A}))$ with the property that for each place $v$ in $F$ unramified for $D$ (i.e. where $G'(F_v) \simeq \mathrm{GL}_2(F_v)$), $\sigma_v \simeq \pi_v$.*

Interestingly enough, this correspondence (the well-known *Jacquet–Langlands correspondence*) can be proved using any one of the three basic tools of the theory of automorphic forms: the theory of automorphic *L-functions*, the theory of *theta-series* liftings, *or* the trace formula (for both $G'$ and $G = \mathrm{GL}_2$, essentially as developed by Selberg). However, in this case at least, the trace formula approach gives the strongest payoff, namely a *characterization* of the image of the correspondence $\sigma \to \pi(\sigma)$: a cuspidal representation $\pi$ of $G(\mathbb{A})$ is of the form $\pi(\sigma)$ for some $\sigma$ on a given $G'_D$ if and only if for each *ramified* prime $v$, $\pi_v$ is a square integrable (discrete series) representation of $G_v = \mathrm{GL}_2(F_v)$.

Subsequently, a ("twisted") trace formula for $\mathrm{GL}_2$ was developed by Saito-Shintani and Langlands to prove another important instance of functoriality, namely *base change* (with spectacular applications to Artin's conjecture for Galois representations, and hence ultimately to Wiles' proof of Fermat ... ). Thus the need to generalize the trace formula to a general reductive group $G$ was clear, and this is precisely what Arthur did in his papers between the early 1970's and late 1980's.

**2. The Case of Compact Quotient.** In order to motivate Arthur's work, let us explain how the form (2) takes in the case of compact quotient. Thus $G$ is a semi-simple group over (the number field) $F$, with $G(F)\backslash G(\mathbb{A})$ compact. In particular, $G$ is anisotropic, without unipotent elements. With $R(g)$ the right regular representation in $L^2(G(F)\setminus G(\mathbb{A}))$ and $f$ in $C_c^\infty(G(\mathbb{A}))$, a straightforward computation gives

$$(R(f)\phi)(x) = \int_{G(\mathbb{A})} f(y)\phi(xy)\,dy =$$

$$\int_{G(\mathbb{A})} f(x^{-1}y)\phi(y)\,dy = \int_{G(F)\setminus G(\mathbb{A})} \Big(\sum_{\gamma\in G(F)} f(x^{-1}\gamma y)\Big)\phi(y)\,dy,$$

i.e., $R(f)$ is an integral operator on $\phi$ in $L^2(G(F)\setminus G(\mathbb{A}))$, with kernel

$$K_f(x,y) = \sum_{\gamma\in G(F)} f(x^{-1}\gamma y).$$

Since $G(F)\setminus G(\mathbb{A})$ is compact, it is then a well known theorem (cf. [GGPS], Chapter I), that $R(f)$ is of trace class, and that

$$\mathrm{tr}R(f) = \int_{G(F)\setminus G(\mathbb{A})} K_f(x,x)\,dx,$$

the integral of the kernel over the "diagonal subgroup" $G(F) \setminus G(\mathbb{A})^{\Delta}$. In this case, *the (Selberg) trace formula amounts to a calculation of trace$R(f)$ using two different expressions for this kernel.* On the one hand, we can write

$$K_f(x,y) = \sum_{\mathfrak{o} \in \mathfrak{O}} K_{\mathfrak{o}}(x,y)$$

where $\mathfrak{O}$ denotes the set of conjugacy classes in $G(F)$, and

$$K_{\mathfrak{o}}(x,y) = \sum_{\gamma \in \mathfrak{o} = \{\gamma_0\}} f(x^{-1}\gamma y) = \sum_{\delta \in G_{\gamma_0}(F) \setminus G(F)} f(x^{-1}\delta^{-1}\gamma_0 \delta x)$$

where $G_{\gamma_0}(F)$ is the centralizer of $\gamma_0$ in $G(F)$. Then

$$J_{\mathfrak{o}}(f) = \int_{G(F)\setminus G(\mathbb{A})} K_f(x,x)\,dx = \int_{G(F)\setminus G(\mathbb{A})} \sum_{\delta \in G_{\gamma_0}(F)\setminus G(F)} f(x^{-1}\delta^{-1}\gamma_0\delta x)\,dx =$$

$$\int_{G_{\gamma_0}(F)\setminus G(\mathbb{A})} f(x^{-1}\gamma_0 x)\,dx = meas(G_{\gamma_0}(F) \setminus G_{\gamma_0}(\mathbb{A})) \int_{G_{\gamma_0}(\mathbb{A})\setminus G(\mathbb{A})} f((x^{-1}\gamma_0 x)\,dx,$$

and thus

$$\mathrm{tr}R(f) = \sum_{\mathfrak{o} \in \mathfrak{O}} J_{\mathfrak{o}}(f) = \sum_{\{\gamma\}} meas(G_{\gamma}(F) \setminus G_{\gamma}(\mathbb{A})) \int_{G_{\gamma}(\mathbb{A})\setminus G(\mathbb{A})} f((x^{-1}\gamma x)\,dx.$$

On the other hand, the compactness of $G(F) \setminus G(\mathbb{A})$ also implies a discrete decomposition $R(g) = \sum_{\pi} m_{\pi}\pi(g)$ where $\pi$ runs through the irreducible "automorphic" representations of $G(\mathbb{A})$, and $m_{\pi} < \infty$ is the multiplicity of $\pi$ in $R$. Equivalently, $L^2(G(F) \setminus G(\mathbb{A})) = \sum L_{\pi}$ where $L_{\pi}$ is the invariant subspace realizing the $m_{\pi}$ copies of $\pi$, and $R(f) = \sum R_{\pi}(f)$ where each $R_{\pi}(f)$ is an integral operator in $L_{\pi}$ with kernel

$$K_{\pi}(x,y) = \sum_{\phi \text{ an ON basis for} L_{\pi}} (R(f)\phi)(x)\overline{\phi(y)}.$$

Thus we also have

$$\mathrm{tr}R(f) = \sum_{\chi \in \{(G,\pi)\}} J_{\chi}(f),$$

where

$$J_{\chi}(f) = \int_{G(F)\setminus G(\mathbb{A})} K_{\pi}(x,x)\,dx.$$

The power of the trace formula derives from the fact that this equality

$$\sum J_{\mathfrak{o}}(f) = \sum J_{\chi}(f), \tag{2.1}$$

namely

$$\sum_{\{\gamma\}} m_{\gamma} \int_{G_{\gamma}(\mathbb{A})\setminus G(\mathbb{A})} f(x^{-1}\gamma x)\,dx = \sum_{\pi} \int_{G(F)\setminus G(\mathbb{A})} (\sum_{\phi} (R(f)\phi)(x)\overline{\phi(x)})\,dx$$

relates two completely different kinds of data, geometric and spectral (the latter of which we rarely know much about, but very much want to!).

**3. Difficulties in the Non-Compact Case.** Arthur's great contribution was to figure out how to overcome all the obstacles that present themselves in generalizing (2.1) from the compact to the non-compact quotient setting. Some of the principal problems in the non-compact setting are these:

(a) Although $R(f)$ is still an integral operator, with kernel

$$K_f(x,y) = \sum_{\gamma} f(x^{-1}\gamma y),$$

this kernel is *no longer* integrable over the diagonal; and

(b) The regular representation $R(g)$ no longer decomposes discretely, Eisenstein series are required to describe the continuous spectrum, and $R(f)$ is of course no longer of trace class. Nevertheless, Arthur derives a formula (0.1) which still relates "geometric" and "spectral" distributions attached to $G$—and has powerful applications to number theory (even if neither side of (0.1) represents a formula for $\mathrm{tr}R(f)$!). For example, in this formula,

$$J_{\mathfrak{o}}^T(f) = \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} K_{\mathfrak{o}}^T(x,x)\,dx,$$

where $K_{\mathfrak{o}}^T$ is a *modification* of the kernel

$$K_{\mathfrak{o}}(x,x) = \sum_{\gamma\in\mathfrak{o}} f(x^{-1}\gamma x)$$

which *can* be integrated over the diagonal. To see why this modification is really needed, and what it accomplishes, let us look at the example of $G = \mathrm{GL}(2)$, and $\mathfrak{o}$ the hyperbolic conjugacy class

$$\mathfrak{o} = \{\delta^{-1}\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)\delta : \delta\in M(F)\setminus G(F)\},$$

where $M = \{\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix}\}$. In this case,

$$J_{\mathfrak{o}}(f) = \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} K_{\mathfrak{o}}(x,x)\,dx$$

(where $Z$ is the center of scalar matrices). A formal computation then shows

$$\begin{aligned} J_{\mathfrak{o}}(f) &= \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \sum_{\delta\in M(F)\backslash G(F)} f(x^{-1}\delta^{-1}\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)\delta x)\,dx \\ &= \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} f(x^{-1}\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)x)\,dx \\ &= \int_{M(\mathbb{A})\backslash G(\mathbb{A})} (\int_{Z(\mathbb{A})M(F)\backslash M(\mathbb{A})} dy) f(x^{-1}\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)x)\,dx, \end{aligned}$$

i.e.,

$$J_{\mathfrak{o}}(f) = m(Z(\mathbb{A})M(F)\setminus M(\mathbb{A}))F_f(\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)),$$

with $F_f(\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right))$ the nice "orbital integral" $\int_{M(\mathbb{A})\backslash G(\mathbb{A})} f(x^{-1}\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)x)\,dx$ of a compactly supported function $f$ in $C_c^\infty(Z(\mathbb{A})\backslash G(\mathbb{A}))$. Thus $J_{\mathfrak{o}}(f)$ is *infinite* for the simple reason that

$$m(Z(\mathbb{A})M(F)\setminus M(\mathbb{A})) = m(F^*\setminus\mathbb{A}^*) = \infty.$$

To compensate for this, Arthur "modifies" $K_{\mathfrak{o}}$ to $K_{\mathfrak{o}}^T$, using a *truncation parameter* $T = (T_1, T_2)$ in the Lie algebra of $M$. As we shall see, this modification operator (in this case) "truncates" $F^* \setminus \mathbb{A}^*$ to a finite interval of length $T_2 - T_1$, but turns $F_f$ into a "weighted" orbital integral! In general, for an arbitrary $G$, it will turn out that $K_{\mathfrak{o}}$ will be integrable "over the diagonal" only when $\mathfrak{o}$ intersects *no* proper parabolic subgroup $P$ of $G$. Of course for anisotropic $G$ this always happens, but in general one really needs to modify $K_{\mathfrak{o}}$ (as in the above example), prove that the modified kernels $K_{\mathfrak{o}}^T$ are indeed integrable, and then evaluate them (as generalized weighted orbital integrals).

What about the spectral side? Here it turns out that $L^2(Z(\mathbb{A})G(F) \setminus G(\mathbb{A})) = \sum_{\chi \in \mathfrak{X}} L^2_\chi$, where each $L^2_\chi$ is a $G(\mathbb{A})$-invariant (not necessary discretely decomposable) submodule indexed by certain cuspidal data $\mathfrak{X} = \{(M, \sigma)\}$, where $M$ is a Levi subgroup of a parabolic $P \subset G$, and $\sigma$ is a cuspidal automorphic representation of $M(\mathbb{A})$. Thus we have $K(x,y) = \sum K_\chi(x,y)$, the spectral decomposition of $K_f(x,y)$ corresponding to (0.1). Here it also turns out that $K_\chi(x,y)$ will *not* be integrable over the diagonal, unless the cuspidal data $(M,\sigma) = (G,\pi)$ (i.e. $P = G$). Thus we again need to deal with *modified* $K_\chi^T$ (which are integrable ... ), eventually arriving at the trace formula (0.1) by integrating the equality

$$\sum_{\mathfrak{o}} K_{\mathfrak{o}}^T(x,y) = \sum_{\chi} K_\chi^T(x,y) \tag{3.1}$$

over the diagonal.

**Remarks.** (1) The expressions on either side of (3.1) no longer (generally) represent the kernel $K_f(x,y) = \sum f(x^{-1}\gamma y)$, and hence (0.1) is no longer a real trace formula at all. Nevertheless, what is true is the following. Let $\mathfrak{X}(G)$ be the subset of cuspidal data $\chi = (M,\sigma) \in \mathfrak{X}$ with $M = G$. Then $\sum_{\chi \in \mathfrak{X}(G)} K_\chi(x,y)$ is the kernel of $R(f)$ restricted to $L^2_0(Z(\mathbb{A})G(F) \setminus G(\mathbb{A}))$, it is integrable over the diagonal, and (0.1) asserts that

$$\mathrm{tr} R_0(f) = \sum_{\chi \in \mathfrak{X}(G)} J_\chi(f) = \sum_{\mathfrak{o}} J_{\mathfrak{o}}^T(f) - \sum_{\mathfrak{X} \setminus \mathfrak{X}(G)} J_\chi^T(f), \tag{3.2}$$

i.e., Arthur's trace formula may be viewed as a trace formula after all.

(2) In most applications of the trace formula to automorphic forms we want to *compare* the formula for one group with the trace formula for another. In that case the form of (0.1) is no less useful than (3.2).

**4. Possible Applications.** We have already mentioned the applications to quaternion algebras and base change, both of which we plan to discuss (at the level of GL(2), not GL($n$) - which is the subject matter of [AC], using the more difficult GL($n$) trace formulas). We also plan to discuss (i) Jacquet's *relative trace formula*, where the idea is to integrate the (different expressions for the) kernel over interesting *non*-diagonal subgroups of $G \times G$; and (ii) applications of the *simple trace formula* such as to the embedding of a (local) discrete series representation of $\mathrm{GL}_2(F_v)$ into a (global) cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$. In all these applications, one needs to further refine Arthur's first form of the trace formula. This is because (as already hinted) the "truncation" i.e., modification process used to make $K_{\mathfrak{o}}$ or $K_\chi$ integrable introduces various defects into the distributions $J_{\mathfrak{o}}^T(f)$ and $J_\chi^T(f)$. For example, they may no longer be *invariant* distributions (invariant under conjugation by $G(\mathbb{A})$), or *factorizable* as the product of local distributions over $G(F_v)$. Thus one needs to replace and rearrange the sums of distributions appearing in (0.1) by still more complicated but ultimately more practical expressions.

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## Lecture II. Arthur's Modified Kernels I: The Geometric Terms

Our purpose here is to develop the geometric side of Arthur's first form of the trace formula

$$\sum_{\mathfrak{o}\in\mathfrak{O}} J_{\mathfrak{o}}^T(f) = \sum_{\chi\in\mathfrak{X}} J_{\chi}^T(f),$$

for the group $G = GL(2)$. In particular, we must introduce the modified kernels $k_{\mathfrak{o}}^T(x, f)$, and prove they are integrable.

N.B. The treatment of the trace formula in [Ge] and [GJ] largely ignored the integrability of these modified kernels.

**1. The Definition of $\mathbf{K}_{\mathfrak{o}}$.** In the case of compact quotient, the geometric side of the trace formula is summed over ordinary $G(F)$-conjugacy classes $\mathfrak{o}$ in $G(F)$ (or in $Z(F)\backslash G(F)$, where $Z$ is the center of $G$). For arbitrary $G$, it turns out to be best to make the following definition: $\gamma_1$ and $\gamma_2$ in $Z(F)\backslash G(F)$ are called *equivalent*, or "*conjugate*", if their semi-simple components are $G(F)$ conjugate in the usual sense. That is, $\gamma_1 \sim \gamma_2$ means, with the Jordan decompositions $\gamma_i = \gamma_i^s\gamma_i^u$ $(i = 1, 2)$, that $\gamma_1^s$ is $G(F)$-conjugate to $\gamma_2^s$. For each such equivalence class $\mathfrak{o}$, and for $f \in C_c^\infty(Z(\mathbb{A})\backslash G(\mathbb{A}))$, set

$$K_{\mathfrak{o}}(x, y) = \sum_{\gamma\in\mathfrak{o}} f(x^{-1}\gamma y),$$

and let $\mathfrak{O}$ denote the collection of all such classes $\mathfrak{o}$ in $Z(\mathbb{A})\backslash G(\mathbb{A})$.

Note that if $\gamma$ is elliptic, i.e. if $\gamma$ is not conjugate in $G(F)$ to an element of any proper parabolic subgroup $P(F)$, then it is automatically semi-simple. (Its eigenvalues do not lie in $F$, and hence are distinct.) Thus our new notion of conjugacy reduces to ordinary conjugacy.

On the other hand, suppose $\gamma$ is $G(F)$-conjugate to an element $p$ in a proper parabolic subgroup $P(F)$. For $GL(2)$, this means $P$ is the Borel subgroup $B = \left\{\left(\begin{smallmatrix} a & x \\ 0 & b\end{smallmatrix}\right)\right\}$, so modulo $Z(F)$,

$$p = p^s p^u \quad \text{with} \quad p^s = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1\end{smallmatrix}\right), \quad \text{and} \quad p^u = \left(\begin{smallmatrix} 1 & y \\ 0 & 1\end{smallmatrix}\right).$$

**Case (a):** $\alpha = 1$. In this case, the class $\mathfrak{o}$ containing $\gamma$ consists of two ordinary unipotent conjugacy classes in $Z(F)\backslash G(F)$: the trivial class $\{1\}$, and the ordinary conjugacy class of any non-trivial unipotent element $\left(\begin{smallmatrix} 1 & x_0 \\ 0 & 1\end{smallmatrix}\right)$ (since $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1\end{smallmatrix}\right)$ already conjugates $\left(\begin{smallmatrix} 1 & x_0 \\ 0 & 1\end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} 1 & \alpha x_0 \\ 0 & 1\end{smallmatrix}\right)$ ).

**Case (b):** $\alpha \neq 1$. In this case, $\mathfrak{o}$ is an ordinary hyperbolic conjugacy class of the form

$$\mathfrak{o}_\alpha = \left\{\delta^{-1}\left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1\end{smallmatrix}\right)\delta \mid \delta \in M(F)\backslash G(F)\right\}.$$

Note that $M(F)$ is just the centralizer $G_{\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)}(F)$, and that $\left(\begin{smallmatrix}\alpha & x\\ 0 & 1\end{smallmatrix}\right)$ also belongs to $\mathfrak{o}_\alpha$ (since $\left(\begin{smallmatrix}\alpha & x\\ 0 & 1\end{smallmatrix}\right) = \left(\begin{smallmatrix}1 & y\\ 0 & 1\end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & y\\ 0 & 1\end{smallmatrix}\right)^{-1}$ where $y = x/(\alpha-1)$ ).

**2. The Integrability of Elliptic $K_{\mathfrak{o}}$.** As suggested in Lecture I, it is a general phenomenon that for an *elliptic* class $\mathfrak{o}$, the kernel $K_{\mathfrak{o}}(x,y)$ is integrable over the diagonal subset $Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$. We now prove this for $G = GL(2)$, following the general arguments of [A1]. The crucial step is a lemma which first requires some preliminary notions to be recalled.

According to the Iwasawa decomposition for $G$, we have

$$G(\mathbb{A}) = N(\mathbb{A})\,M(\mathbb{A})\,K, \tag{2.1}$$

where $N$ is the standard unipotent subgroup $\{\left(\begin{smallmatrix}1 & y\\ 0 & 1\end{smallmatrix}\right)\}$, and $K$ is the standard maximal compact subgroup $\prod K_\nu$ of $G(\mathbb{A})$. If we denote by $\mathfrak{a}_M$ the Lie algebra of $M = \{\left(\begin{smallmatrix}a & 0\\ 0 & b\end{smallmatrix}\right)\}$, i.e. ,

$$\mathfrak{a}_M = \mathrm{Hom}(X(M), \mathbb{R}) \approx \mathbb{R}^2,$$

then we can define the familiar function

$$H_M : M(\mathbb{A}) \to \mathfrak{a}_M \quad \text{by} \quad H_M\left(\begin{smallmatrix}a & 0\\ 0 & b\end{smallmatrix}\right) = (\log|a|_{\mathbb{A}}, \log|b|_{\mathbb{A}}),$$

and extend it to $G(\mathbb{A})$ by way of the decomposition 2.1, i.e. if $g = nak$ in $G(\mathbb{A})$, then

$$H_M(g) = H_M(a).$$

**Lemma 2.1** ( see [A1, p 362] ). *Suppose $\Omega$ is a subset of $G(\mathbb{A})$ which is compact modulo $Z(\mathbb{A})$. Then there exists a number $d_\Omega > 0$ with the following property: if $\gamma \in G(F)$ is such that*

$$g^{-1}\gamma n g \in \Omega$$

*for some $n \in N(\mathbb{A})$ and $g \in G(\mathbb{A})$ with $\alpha H(g) > d_\Omega$, then $\gamma \in B(F)$.*

Assuming this Lemma (for the moment), let us quickly dispose of the integrability of $K_{\mathfrak{o}}$. For this, and for other purposes later on, we also need to recall the notion of a *Siegel domain*, together with an explicit integration formula for its volume.

First we refine the Iwasawa decomposition 2.1 by writing, for each $g$ in $G(\mathbb{A})$,

$$g = znh_tmk \tag{2.2}$$

where $z \in Z(\mathbb{A})$, $n \in N(\mathbb{A})$, $k \in K$,

$$h_t = \left(\begin{smallmatrix}\mathrm{e}^t & 0\\ 0 & \mathrm{e}^{-t}\end{smallmatrix}\right),$$

with $\mathrm{e}^t$ an idele equal to 1 at every finite place and the real number $\mathrm{e}^t$ at each infinite place, and

$$m = \left(\begin{smallmatrix}a & 0\\ 0 & 1\end{smallmatrix}\right),$$

with $a \in \mathbb{A}^1 = \{a \in \mathbb{A}^\times \mid |a|_{\mathbb{A}} = 1\}$. The corresponding integration formula, for $f$ on $Z(\mathbb{A})\backslash G(\mathbb{A})$, is

$$\int_{Z(\mathbb{A})\backslash G(\mathbb{A})} f(g)\,dg = \int\!\!\int\!\!\int\!\!\int_{-\infty}^{\infty} f(natk)\mathrm{e}^{-2t}\,dt\,d^\times a\,dn\,dk. \tag{2.3}$$

Now for each compact subset $C_1$ of $N(\mathbb{A})$, compact subset $C_2$ of $\mathbb{A}^1$, and real number $c$ (not necessarily positive), let $\mathfrak{S}_c$ denote the set of points $g$ in $G(\mathbb{A})$ of the form (2.2) with $n \in C_1$, $a \in C_2$ and $t > c/2$. Such a set is called a *Siegel set*. Note that

$$\mathfrak{S}_c \subset \{\, g \in G(\mathbb{A}) \mid H(g) > c \,\}.$$

**Fact 1** (see [Go, Thm 9]). For any $c_1$, $c_2$,

$$\mathfrak{S}_{c_1} \cap \gamma \mathfrak{S}_{c_2} \neq \emptyset$$

for only finitely many $\gamma$ in $G(F)$ *modulo* $B(F)$.

**Fact 2** (see [Go, p 16]). If $\mathfrak{S}_c$ is sufficiently large (i.e. $C_1$ and $C_2$ are sufficiently large, and $c$ is sufficiently *small*), then

$$G(\mathbb{A}) = G(F) \cdot \mathfrak{S}_c. \tag{2.4}$$

In other words, $\mathfrak{S}_c$ is essentially a fundamental domain for $G(F)$ acting on $G(\mathbb{A})$; more precisely, Fact 2 says only that certain $\mathfrak{S}_c$ *contain* a fundamental domain. Henceforth, we always assume that $\mathfrak{S}_c$ is so chosen that 2.4 holds, and we refer to such a $\mathfrak{S}_c$ as a *Siegel domain*.

**Remark.** From the above, it is clear that

$$Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$$

has *finite* volume. In fact, by 2.3 and 2.4,

$$\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} dg \leq \int_{Z(\mathbb{A})\backslash \mathfrak{S}_c} dg = \int\int\int \left( \int_c^\infty \mathrm{e}^{-2t}\, dt \right) d^\times a\, dn\, dk,$$

with the outer three integrals over compact domains.

**Proposition 2.2.** *If* $\mathfrak{o}$ *is* elliptic, *then*

$$x \mapsto K_{\mathfrak{o}}(x,x) = \sum_{\gamma \in \mathfrak{o}} f(x^{-1}\gamma x)$$

*is absolutely integrable over* $Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$, *and its integral is*

$$J_{\mathfrak{o}}(f) = m(Z(\mathbb{A})G(F)\backslash G(\mathbb{A})) \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(x^{-1}\gamma x)\, dx.$$

*Proof.* It suffices to prove the integrability assertion, since the ensuing formula for $J_{\mathfrak{o}}(f)$ follows just as in the case of compact quotient.

First note that for $x$ lying in any fixed set $\Omega_1$, which is compact modulo $Z(\mathbb{A})$, the sum defining $K_{\mathfrak{o}}(x,x)$ is actually finite and hence $K_{\mathfrak{o}}$ defines a nice, smooth function on $Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$. Indeed, $f(x^{-1}\gamma x) \neq 0$ implies $x^{-1}\gamma x \in \Omega_f =$ support of $f$. Thus $\gamma$ itself must lie in a compact set modulo $Z(\mathbb{A})$, namely $\Omega_1 \Omega_f \Omega_1^{-1}$, and such a set can contain only finitely many $\gamma$ in $Z(F)\backslash G(F)$.

On the other hand, if $\gamma$ is *elliptic*, then $f(x^{-1}\gamma x) \neq 0$ implies (by Lemma 2.1) that $\alpha H(g) < d_{\Omega_f}$. I.e., the support of

$$K_{\mathrm{ell}}(x,x) = \sum_{\gamma \text{ elliptic}} f(x^{-1}\gamma x)$$

lies in the "doubly truncated" Siegel set

$$\mathfrak{S}_c{}^d = Z(\mathbb{A})C_1\{h_t\}C_2K \quad \text{with} \quad c < 2t < d_{\Omega_f}.$$

Equivalently, the support of $K_{\text{ell}}$ is compact modulo $Z(\mathbb{A})$, and the integrability is clear. $\square$

**Remark.** The argument above actually proves that the *full* elliptic contribution is absolutely integrable, i.e. , that

$$\int |K_{\text{ell}}(x,x)|\,dx < \infty.$$

Thus, by the Lebesgue dominated convergence theorem, it follows that

$$\sum_{\mathfrak{o}\ \text{elliptic}} \int |K_{\mathfrak{o}}(x,x)|\,dx < \infty,$$

and hence that the sum

$$\sum_{\mathfrak{o}\ \text{elliptic}} J_{\mathfrak{o}}(f)$$

converges absolutely.

It remains now to prove Lemma 2.1. It might be tempting to do so for $GL(2)$ by simply trying to multiply out some $2 \times 2$ matrices (as we did in [Ge, p 201]). However, this more quickly than not leads to an *incomplete* proof (as happened in fact in [Ge]), whereas the approach of [A1, p 362] works simply and generally, as we shall show below.

*Proof of Lemma* 2.1. Let

$$\rho : G \to G\ell(V)$$

denote the adjoint representation of $G$ on the space of *trace zero* $2 \times 2$ matrices over $F$; its highest weight vector is $\Lambda = \alpha$, the unique simple root of $G$ with respect to the torus $M$. Fix a basis $\{\mathrm{e}_0, \mathrm{e}_1, \mathrm{e}_2\}$ of $V$ such that

$$\begin{aligned} &\rho(a)\mathrm{e}_j = a^{\Lambda - j\alpha}\mathrm{e}_j \quad \text{for} \quad a \in M \quad (j = 0,1,2) \\ &\rho(n)\mathrm{e}_0 = \mathrm{e}_0 \qquad\quad\ \text{for} \quad n \in N \quad \text{and} \\ &\rho(w)\mathrm{e}_0 = \mathrm{e}_2 \end{aligned}$$

For $\xi$ in $V(\mathbb{A})$, define

$$\|\,\xi\,\| = \prod_{\nu} \|\,\xi_\nu\,\|_\nu,$$

where, for finite $\nu$, $\|\,\xi_\nu\,\|_\nu = \max_j\{|\xi_\nu^j|_\nu\}$ (where $(\xi_\nu^j)$ denote the coordinates of $\xi_\nu$ with respect to the basis $\{\mathrm{e}_j\}$), and for infinite $\nu$, $\|\,\xi_\nu\,\|_\nu$ is defined by the Hilbert space structure that makes $\{\mathrm{e}_j\}$ into an orthonormal basis. (To make sense out of the infinite product, we restrict $\xi$ to be "primitive", i.e. , $\|\,\xi_\nu\,\|_\nu = 1$ for all but finitely many $\nu$.)

To prove the Lemma, suppose that

$$(*) \qquad g^{-1}\gamma n g \in \Omega$$

with $n \in N(\mathbb{A})$ and $\Omega$ compact modulo $Z(\mathbb{A})$. Write $g = n_1 a k$, so that $(*)$ implies

$$a^{-1} n_1^{-1} \gamma n n_1 a \in K\Omega K.$$

Because the map $g \mapsto \rho(g)\mathrm{e}_0$ is continuous, and $\rho$ is trivial on $Z(\mathbb{A})$, it follows that

$$\left\| \rho(a^{-1}n_1^{-1}\gamma n n_1 a) \cdot \mathrm{e}_0 \right\| \leq \mathrm{e}^{2d_0}$$

for some $d_0 > 0$. So suppose that $\gamma$ does not belong to $B(F)$. Then by Bruhat's decomposition,

$$\gamma = b_0 w n_0$$

for some $b_0$ in $B(F)$ and $n_0$ in $N(F)$. But for any $b \in B(\mathbb{A})$ and $g \in G(\mathbb{A})$, our assumptions on $\rho$ and $\mathrm{e}_0$ imply

$$\| \rho(gb) \cdot \mathrm{e}_0 \| = \mathrm{e}^{\alpha H(b)} \| \rho(g) \cdot \mathrm{e}_0 \|.$$

Thus (with $\gamma = b_0 w n_0$) we compute that

$$\begin{aligned} \left\| \rho((a^{-1}n_1^{-1}b_0 w)(n_0 n n_1 a)) \cdot \mathrm{e}_0 \right\| &= \mathrm{e}^{\alpha H(a)} \left\| \rho(a^{-1}n_1^{-1}b_0) \cdot \mathrm{e}_2 \right\| \\ &= \mathrm{e}^{2\alpha H(a)} \| \rho(n^*) \cdot \mathrm{e}_2 \| \\ &\geq \mathrm{e}^{2\alpha H(a)} \| \mathrm{e}_2 \| = \mathrm{e}^{2\alpha H(g)} \end{aligned}$$

(since $H(g) = H(a)$, $\rho(a)\,\mathrm{e}_2 = a^{-\alpha}\,\mathrm{e}_2$, and $\|\rho(n)\,\mathrm{e}_2\| \geq \|\mathrm{e}_2\|$ for any $n$ in $N(\mathbb{A})$). Thus the Lemma is proved for $d_\Omega > d_0$. $\square$

**3. The Definition of $\mathbf{K}_{\mathfrak{o}}^{\mathbf{T}}(\mathbf{x}, \mathbf{y})$.** For $\mathfrak{o}$ the (unique non-trivial) unipotent class in $Z(F)\backslash G(F)$, we saw in Lecture I that $K_{\mathfrak{o}}(x, y)$ is *not* integrable over the diagonal. A somewhat less trivial computation (see Lecture III) shows that the hyperbolic $K_{\mathfrak{o}}(x, y)$ are also not integrable. So let us finally introduce the required modifications of these kernels.

For $P = B$, let $\widehat{\tau}_B$ denote the characteristic function of the *positive Weyl chamber*

$${\mathfrak{a}_B}^+ = \{ (r_1, r_2) \in \mathfrak{a}_B = \mathfrak{a}_M \mid r_1 - r_2 > 0 \},$$

i.e. , the cone in $\mathbb{R}^2$ where $\alpha(r_1, r_2) = r_1 - r_2$ is positive. Then define, for any $\mathfrak{o} \in \mathfrak{O}$ and $T = (T_1, T_2)$ in ${\mathfrak{a}_B}^+$,

$$k_{\mathfrak{o}}^T(x, f) = K_{\mathfrak{o}}(x, x) - \sum_{\delta \in B(F)\backslash G(F)} K_{B,\mathfrak{o}}(\delta x, \delta x)\ \widehat{\tau}_B(H(\delta x) - T) \tag{3.1}$$

where

$$K_{B,\mathfrak{o}}(x, y) = \sum_{\gamma \in \mathfrak{o} \cap Z(F)\backslash M(F)} \int_{N(\mathbb{A})} f(x^{-1}\gamma n y)\, dn.$$

This is Arthur's *modified kernel* $k_{\mathfrak{o}}^T(x, f)$.

Note that if $\mathfrak{o}$ is elliptic, then $\mathfrak{o} \cap M(F) = \emptyset$. In this case $K_{B,\mathfrak{o}} \equiv 0$, and

$$k_{\mathfrak{o}}^T(x, f) = K_{\mathfrak{o}}(x, x).$$

In general, the term subtracted in 3.1 from $K_{\mathfrak{o}}(x,x)$ is the "correction term at $\infty$" which ends up making $k_{\mathfrak{o}}^T(x,f)$ integrable. Before proving this is so, let us note the inductive nature of the definition of $k_{\mathfrak{o}}^T$ (thereby making more plausible the appearance of $K_{B,\mathfrak{o}}$). Set

$$K_B(x,y) = \sum_{\mathfrak{o}} K_{B,\mathfrak{o}}(x,y) \equiv \sum_{\gamma\in Z(F)\backslash M(F)} \int_{N(\mathbb{A})} f(x^{-1}\gamma n y)\,dn. \tag{3.2}$$

Then $K_B(x,y)$ is just the kernel of $R(f)$ acting in $L^2(Z(\mathbb{A})N(\mathbb{A})B(F)\backslash G(\mathbb{A}))$ in place of $L^2(Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$. Indeed for $\varphi$ in $L^2(Z(\mathbb{A})N(\mathbb{A})B(F)\backslash G(\mathbb{A}))$,

$$\begin{aligned} R(f)\varphi(x) &= \int_{Z(\mathbb{A})\backslash G(\mathbb{A})} f(x^{-1}y)\varphi(y)\,dy \\ &= \int_{Z(\mathbb{A})N(\mathbb{A})B(F)\backslash G(\mathbb{A})} \left(\sum_{Z(F)\backslash M(F)} \int_{N(\mathbb{A})} f(x^{-1}\gamma n y)\,dn\right)\varphi(y)\,dy, \end{aligned} \tag{3.3}$$

as claimed. In general, for an arbitrary $G$, the definition of $k_{\mathfrak{o}}^T(x,f)$ takes into account *all* the proper parabolics $P$ of $G$, subtracting off from $K_{\mathfrak{o}}(x,x)$ not just one term (for the minimal parabolic $B$), but the sum

$$\sum_{P\subset G} (-1)^{\dim A_P/Z} \sum_{\delta\in P(F)\backslash G(F)} K_{P,\mathfrak{o}}(\delta x,\delta x)\ \widehat{\tau}_P(H(\delta x)-T), \tag{3.4}$$

where

$$K_P(x,y) = \sum_{\mathfrak{o}\in\mathfrak{O}} K_{P,\mathfrak{o}}(x,y)$$

is the kernel of $R_P(f)$ in $L^2(Z(\mathbb{A})N_P(\mathbb{A})M_P(F)\backslash G(\mathbb{A}))$. (Here $A_P$ is the maximal split torus in $M_P$, the Levi component of $P=M_PN_P$, and $R_P$ is the right regular representation of $G(\mathbb{A})$ in $L^2(Z(\mathbb{A})N_P(\mathbb{A})M_P(F)\backslash G(\mathbb{A}))$.)

**Remark.** Observe that

$$k_{\mathfrak{o}}^T(x,f) = K_{\mathfrak{o}}(x,x).$$

for $x$ in a compact (modulo $Z(\mathbb{A})$) set $\Omega$ (how large depends on T). Indeed, for $x$ in an appropriate such set $\Omega$, $\widehat{\tau}_B(H(\delta x)-T)$ is identically zero*. Thus $x$ is indeed a modification of $K_{\mathfrak{o}}(x,x)$ only "near infinity" (and in higher rank, more than one parabolic is needed to effect these modifications near infinity … ).

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**Proof:* Fix $c$ and $T$ such that $\alpha(T)>c$. Fact 1 of Section 2 implies that in $\mathfrak{S}_c$, $\widehat{\tau}_B(H(\delta x)-T)\neq 0$ only for finitely many classes $\delta_i$ in $B(F)\backslash G(F)$. Then for each such class, $\widehat{\tau}_B(H(\delta_i x)-T)=0$ on the compact set $\Omega_i^T$ in $\mathfrak{S}_c$ where $c<H(\delta_i x)<\alpha(T)$. Thus $\widehat{\tau}_B(H(\delta x)-T)$ is identically zero on $\Omega_T=\cap_{i=1}^n\Omega_i^T$.

**4. The Integrability of $k_{\mathfrak{o}}^T(x,f)$.** Henceforth, let us assume that $\mathfrak{o}$ is unipotent or hyperbolic. To prove that $k_{\mathfrak{o}}^T(x,f)$ is integrable over $Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$, it suffices to show that it is integrable over a Siegel domain $\mathfrak{S}_c$. So pick any $T \in \mathfrak{a}_B{}^+$, and write

$$\mathfrak{S}_c = \mathcal{B}_T \cup \widetilde{\mathcal{B}}_T,$$

where

$$\mathcal{B}_T = \{\, g \in \mathfrak{S}_c \mid \alpha H(g) > \alpha(T) > 0 \,\},$$

and $\widetilde{\mathcal{B}}_T$ is its complement in $\mathfrak{S}_c$. Clearly $k_{\mathfrak{o}}^T(x,f)$ is integrable over $\widetilde{\mathcal{B}}_T$, since

$$c < \alpha H(g) \le \alpha(T).$$

Thus it suffices to prove that $k_{\mathfrak{o}}^T(x,f)$ is integrable over $\mathcal{B}_T$. (N.B. In general, the partition of a Siegel domain requires more non-trivial "reduction theory" than is evident in the case of $GL(2)$; this is what leads to the "geometric" combinatorial lemmas in §6 of [A2].)

**Lemma 4.1.** *There exists a constant $c_0$ such that $\alpha H(wn) \le c_0$ for all $n \in N(\mathbb{A})$; moreover, if $\gamma$ in $G(F)$ satisfies the inequality*

$$\alpha H(\gamma x) > c_0$$

*for some $x$ in $G(\mathbb{A})$ with $\alpha H(x) > c_0$, it must follow that $\gamma \in B(F)$.*

As the proof is similar to that of the crucial Lemma 2.1 in §2 of this Lecture, we leave the details to the reader (see [A1, pp 334–335]).

Now *fix* $T$ in $\mathfrak{a}_B{}^+$ such that $\alpha(T) > c_0$. Then for $x$ in $\mathcal{B}_T$, Lemma 4.1 implies that

$$\hat{\tau}_B(H(\delta x) - T) \ne 0$$

(if and) only if $\delta \in B(F)$. Thus (for such $x$),

$$\sum_{\gamma\in\mathfrak{o}} f(x^{-1}\gamma x) = \sum_{\delta\in B(F)\backslash G(F)} \sum_{\gamma\in\mathfrak{o}} f(x^{-1}\delta^{-1}\gamma\delta x)\hat{\tau}_B(H(\delta x) - T),$$

and hence

$$\begin{aligned} k_{\mathfrak{o}}^T(x,f) = \sum_{\delta\in B(F)\backslash G(F)} \Bigg\{ &\sum_{\gamma\in\mathfrak{o}} f(x^{-1}\delta^{-1}\gamma\delta x) \\ &- \int_{N(\mathbb{A})} \sum_{\mu\in M(F)\cap\mathfrak{o}} f(x^{-1}\delta^{-1}\mu n\delta x)\,dn \Bigg\} \hat{\tau}_B(H(\delta x) - T) \end{aligned}$$

Next fix $T$ such that $\alpha(T) > \max\{c_0, d_{\Omega_f}\}$, with $d_{\Omega_f}$ as in Lemma 2.1. Then by Lemma 2.1, we may replace the sum here over $\gamma \in \mathfrak{o}$ by a sum over

$$B(F) \cap \mathfrak{o} = N(F) \cdot M(F) \cap \mathfrak{o}$$

(This last equality holds since the class of $\gamma$ in $B(F)$ is determined by its semi-simple part, i.e. its $M(F)$-part.) Thus we get

$$\begin{aligned} k_{\mathfrak{o}}^T(x,f) &= \sum_{\delta\in B(F)\backslash G(F)} \sum_{\mu\in M(F)\cap\mathfrak{o}} \Big\{ \sum_{\nu\in N(F)} f(x^{-1}\delta^{-1}\mu\nu\delta x) \\ &\qquad - \int_{N(\mathbb{A})} f(x^{-1}\delta^{-1}\mu n\delta x)\,dn \Big\} \widehat{\tau}_B(H(\delta x)-T) \\ &= \sum_{\delta}\sum_{\mu}\sum_{\nu\in N(F)-\{1\}} \widehat{F}_{\delta x,\mu}(\nu)\,\widehat{\tau}_B(H(\delta x)-T) \end{aligned} \tag{4.1}$$

where

$$F_{x,\mu}(n) = f(x^{-1}\mu n x)$$

defines a Schwartz-Bruhat function on $N(\mathbb{A}) \approx \mathbb{A}$ with Fourier transform

$$\widehat{F}_{x,\mu}(\nu) = \int_{N(\mathbb{A})} f(x^{-1}\mu n x)\psi_N(n\nu)\,dn$$

(Here $\psi_N(\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)) = \psi(x)$ for some fixed non-trivial character on $F\backslash\mathbb{A}$, and we have applied the Poisson summation formula to $F_{\delta x,\mu}$.) Finally we are ready to prove

**Proposition 4.2.** *For any $\mathfrak{o}\in\mathfrak{O}$, $k_{\mathfrak{o}}^T(x,f)$ is absolutely integrable over*

$$Z(\mathbb{A})G(F)\backslash G(\mathbb{A}).$$

*Proof.* We may assume $\mathfrak{o}$ is not elliptic. By the above arguments, it suffices to check the integrability over $\mathcal{B}_T$, with $\alpha(T) > \max\{c_0, d_{\Omega_f}\}$ as above. As in that domain formula (4.1) holds, we conclude

$$\begin{aligned} &\int \left|k_{\mathfrak{o}}^T(x,f)\right| dx \le \\ &\int_{Z(\mathbb{A})\,B(F)\backslash G(\mathbb{A})} \sum_{\mu\in M(F)\cap\mathfrak{o}} \sum_{\nu\in N(F)-\{1\}} \left|\widehat{F}_{x,\mu}(\nu)\right| \widehat{\tau}_B(H(x)-T)\,dx. \end{aligned} \tag{4.2}$$

Note that for any $\mu = \left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)$ in $M(F)\cap\mathfrak{o}$, $\nu = \left(\begin{smallmatrix}1 & w\\ 0 & 1\end{smallmatrix}\right)$ in $N(F)-\{1\}$, and $x = znh_tmk$,

$$\begin{aligned} \widehat{F}_{x,\mu}(\nu) &= \int_{\mathbb{A}} f\left(x^{-1}\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & y\\ 0 & 1\end{smallmatrix}\right)x\right)\psi(yw)\,dy \\ &= \mathrm{e}^{2t}\widehat{F}_{k^*,\mu}(\mathrm{e}^{2t}\nu) \end{aligned}$$

with $k^* = (h_t^{-1}nh_t)mk$. (Here $\mathrm{e}^{2t}\nu = \left(\begin{smallmatrix}1 & \mathrm{e}^{2t}w\\ 0 & 1\end{smallmatrix}\right)$, and we confuse $N(F)$ with $F$.) Note also that the variable $x$ in (4.2) runs through a Siegel set. Thus $k^*$ runs through a compact set, and we have, for any $N>0$

$$\sum_{\nu\neq 1} \widehat{F}_{k^*,\mu}(\mathrm{e}^{2t}\nu) \le c_N(\mathrm{e}^{-2t})^N$$

Applying Iwasawa's decomposition (2.3) to (4.2), we then conclude that

$$\int \left|k_{\mathfrak{o}}^T(x,f)\right| dx \le C\int_{N(F)\backslash N(\mathbb{A})}\int_{F^\times\backslash\mathbb{A}^1}\int_K \left(\int_{\alpha(T)}^{\infty} \mathrm{e}^{-2tN}\,dt\right) dk\,d^\times a\,dn < \infty$$

as required. $\square$

Summing up, we have

**Theorem 4.3.** *For $\alpha(T)$ sufficiently large,*

$$\sum_{\mathfrak{o}\in\mathfrak{O}} \int_{Z(\mathbb{A})\,G(F)\backslash G(\mathbb{A})} \left|k_{\mathfrak{o}}^T(x,f)\right| dx < \infty. \quad \square$$

Hence the left side $\sum_{\mathfrak{o}\in\mathfrak{O}} J_{\mathfrak{o}}^T(f)$ of (1) of Lecture I is defined and absolutely convergent.

Indeed, it suffices to remark that for a given $f \in C_c^\infty(Z(\mathbb{A})\backslash G(\mathbb{A}))$, $k_{\mathfrak{o}}^T(x,f)$ is identically zero except for finitely many hyperbolic $\mathfrak{o}$. This is clear from the defining equation (3.1), and the fact that $x^{-1}\delta^{-1}\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)\delta x$ and $x^{-1}\delta^{-1}\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)n\delta x$ can belong to the fixed compact mod $Z(\mathbb{A})$ set $\Omega_f$ for only finitely many $\alpha$; to see this, observe that conjugating by $\delta x$ preserves eigenvalues. Thus the sum over $\mathfrak{o}$ in $\mathfrak{O}$ in Theorem 4.3 is infinite only over the elliptic classes, where we already know the relevant sum $\sum\int|K_{\mathfrak{o}}(x,f)|\,dx$ is finite (see the Remark after the proof of Proposition 2.2).

**Concluding Remarks.** (a) The terms $J_{\mathfrak{o}}^T(f)$ are indeed generalizations of the orbital integrals appearing on the geometric side of Selberg's compact quotient trace formula. Indeed, for $\mathfrak{o}$ elliptic (which is automatic for $G$ anisotropic), $J_{\mathfrak{o}}^T(f)$ reduces to a multiple of the usual orbital integral $\int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(x^{-1}\gamma x)\,dx$.

We have seen—for $GL(2)$—that the kernel $K_{\mathfrak{o}}(x,x)$ is already integrable on $\mathfrak{S}_c$ if $\mathfrak{o}$ is an *elliptic class* (as opposed to hyperbolic or unipotent). For general $G$, the proper distinction is between those $\mathfrak{o}$ which never intersect any proper parabolic subgroup subgroup $P$ of $G$ and those that do. Indeed Arthur does not talk about "elliptic" $\mathfrak{o}$ at all, but rather defines a modified kernel

$$k_{\mathfrak{o}}^T(x,f)$$

for any $\mathfrak{o}$, and proves that $k_{\mathfrak{o}}^T(x,f)$ is integrable. But of course—as we have already observed—$k_{\mathfrak{o}}^T(x,f) = K_{\mathfrak{o}}(x,x)$ precisely when $\mathfrak{o}$ is elliptic in this sense (that it intersects no proper $P$ ... ).

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