Lecture III. Arthur's Modified Kernels II: The Spectral Terms

In order to describe the right hand side of Arthur's trace formula, we need first to recall the spectral expansion of the kernel

$$K_f(x,y) = \sum_{\chi \in \mathfrak{X}} K_\chi(x,y).$$

1. The Spectral Kernels K_χ. Henceforth, by a "Levi subgroup M of G" we understand the Levi component of a parabolic subgroup P of G; for any such M, we have

$$M^1(\mathbb{A}) = \{m \in M(\mathbb{A})\colon |\lambda(m)|_{\mathbb{A}} = 1 \ \ \forall \lambda \in X(M)_F\}.$$

In general, the decomposition of

$$L^2(Z(\mathbb{A})G(F)\backslash G(\mathbb{A}))$$

into right $G(\mathbb{A})$-invariant subspaces is determined by spectral data $\chi = \{(M,r)\} \in \mathfrak{X}$, where the pair (M,r) consists of a Levi subgroup M of G and a cuspidal representation r of $Z(\mathbb{A})\backslash M(\mathbb{A})$; the *class* $\{(M,r)\}$ derives from the equivalence relation

$$(M,r) \sim (M',r)$$

if and only if M is conjugate to M' by a Weyl group element w, and $r' = r^w$ on $Z(\mathbb{A})\backslash M^1(\mathbb{A})$. Then to each such cuspidal datum $\chi = \{(M,r)\}$ is associated a subspace L^2_χ, such that

$$L^2(Z(\mathbb{A})G(F)\backslash G(\mathbb{A})) = \bigoplus_\chi L^2_\chi,$$

and the corresponding decomposition of kernels

$$K_f(x,y) = \sum_\chi K_\chi(x,y)$$

is explicitly describable in terms of Eisenstein series.

For $G = \mathrm{GL}(2)$, cuspidal data comes in two possible forms:

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Case (i): $M = G, r =$ *a cuspidal representation* π *of* $Z(\mathbb{A})\backslash G(\mathbb{A})$. In this case, $L^2_\chi = L_\pi$, the irreducible subrepresentation of $L^2_0(Z(\mathbb{A})G(F)\backslash G(\mathbb{A}))$ realizing π, and

$$K_\chi(x,y) = K_\pi(x,y) = \sum_{\{\phi\}=\text{ o.n. basis of } L_\pi} R(f)\phi(x)\overline{\phi(y)} \tag{1.1}$$

Case (ii): $M = \left\{\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\right\}, r\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} = \mu\left(\frac{a_1}{a_2}\right)$ *with* μ *a character of* $F^x\backslash\mathbb{A}^1$ (*identified with the pair* $(M, \mu^{-1}$)*.* In this case,

$$K_\chi(x,y) = K_\mu(x,y) = \sum_{\{\phi_\mu\}} \int_{-\infty}^{\infty} E(x, \rho(\mu, it)(f)\phi_\mu, it, \mu)\, E(y, \phi_\mu, it, \mu)\, dt. \tag{1.2}$$

where $\{\phi_\mu\}$ runs through an ON basis for the induced representation space

$$\mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}\, \mu\left(\frac{a_1}{a_2}\right) = \rho(\mu, o),$$

consisting of functions $\phi : G(\mathbb{A}) \longrightarrow \mathbb{C}$ such that

$$\phi\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right) = \mu\left(\frac{a_1}{a_2}\right) |\frac{a_1}{a_2}|_{\mathbb{A}}^{1/2} \phi(g)$$

for all $\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix}$ in $B(\mathbb{A})$, and $\int_K |\phi(k)|^2\, dk < \infty$; $\rho(\mu, s)$ denotes the induced representation space $\mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}\, \mu\left(\frac{a_1}{a_2}\right) |\frac{a_1}{a_2}|_{\mathbb{A}}^s$, and $E(g, \phi_\mu, s, \mu)$ is the Eisenstein series

$$\sum_{\gamma \in B(F)\backslash G(F)} \phi_\mu^s(\gamma g)$$

attached to the function

$$\phi_\mu^s(g) = \phi_\mu(g) e^{s\alpha(H(g))}$$

in $\rho(\mu, s)$.

Remarks. (i) The series defining $E(g, \phi_\mu, s, \mu)$ converges only for $\mathrm{Re}(s) > 1/2$, but $E(g)$ itself has a meromorphic continuation to all of $\mathbb{C}$, with a possible pole at $s = 1/2$. These and other facts from the theory of Eisenstein series we shall simply assume, referring the reader to [GJ] for more details.

(ii) *If* $\mu^2 \equiv 1$, *then* (and only then) $K_\chi(x,y)$ *has an additional "discrete" term*, namely

$$(R(f)\mu)(x)\overline{\mu(y)} = \int_{Z(\mathbb{A})\backslash G(\mathbb{A})} f(g)\mu(\det g)\, dg\ \mu(\det x)\overline{\mu(\det y)}. \tag{1.3}$$

(this comes from the residue of the corresponding Eisenstein series at $s = 1/2\ldots$).

(iii) Above, $K_\chi(x,y)$ represents the kernel of $R(f)$ acting in the space L^2_χ, which (for any $\chi = \{(M,\mu)\}$) consists of all $\varphi(g)$ which are orthogonal to the space of cusp forms $L^2_0(Z(\mathbb{A})G(F)\backslash G(\mathbb{A}))$, and such that for almost every x in $G(\mathbb{A})$, the projection of the function

$$\varphi^x_N(m) = \int_{N(F)\backslash N(\mathbb{A})} \varphi(nmx)\,dn,$$

onto the space $L^2(Z(\mathbb{A})M(F)\backslash M^1(\mathbb{A}))$ transforms under $Z(\mathbb{A})\backslash M^1(\mathbb{A})$ as a sum or representations r, where (M,r) is a pair in χ.

The beauty of this description is that it generalizes to a general reductive group G, as we shall now describe.

For *any fixed* parabolic $P = M_P N_P$ in a reductive G, one may describe a decomposition

$$L^2(Z(\mathbb{A})N_P(\mathbb{A})M_P(F)\backslash G(\mathbb{A})) = L^2(P) = \bigoplus_{\chi\in\mathfrak{X}} L^2(P)_\chi, \tag{1.4}$$

where $L^2(P)_\chi$ consists of those φ in $L^2(Z(\mathbb{A})N_P(\mathbb{A})M_P(F)\backslash G(\mathbb{A}))$ with the following property: for *each* standard parabolic subgroup B of G, with $B \subset P$, and almost every x in $G(\mathbb{A})$, the projection of the function

$$m \longrightarrow \varphi_{B,x}(m) = \int_{N_B(F)\backslash N_B(\mathbb{A})} \varphi(nmx)\,dn$$

onto the space of cusp forms in $L^2(Z(\mathbb{A})M_B(F)\backslash M^1_B(\mathbb{A}))$ transforms under $M^1_B(\mathbb{A})$ as a sum of representations r_B, in which (M_B, r_B) is a pair in χ. If there is no such pair in χ, $\varphi_{B,x}$ will be orthogonal to the space of cusp forms on $Z(\mathbb{A})M_B(F)\backslash M^1_B(\mathbb{A})$ (as happens above for $G = \mathrm{GL}(2)$, $P = G = B$, and $\chi = \{(M,\mu)\}$).

Remark. For any fixed χ in $\mathfrak{X}$, let P_χ be the set of standard parabolics B, where (M_B, r_B) belongs to χ. Then from the theory of Eisenstein series it follows

$$L^2(Z(\mathbb{A})N_P(\mathbb{A})M_P(F)\backslash G(\mathbb{A}))_\chi$$

will be zero unless there is a group in P_χ contained in P.

Let us return now to the decomposition

$$L^2(Z(\mathbb{A})N_P(\mathbb{A})M_P(F)\backslash G(\mathbb{A})) = L^2(P) = \bigoplus_{\chi\in\mathfrak{X}} L^2(P)_\chi, \tag{1.4}$$

described above. If we let $K_{P,\chi}(x,y)$ denote the integral kernel of the restriction of $R_P(f)$ to $L^2(P)_\chi$, then one can still—in general—write down a formula for $K_{P,\chi}(x,y)$ in terms of Eisenstein series. Clearly

$$\sum_{\mathfrak{o}\in\mathfrak{O}} K_{P,\mathfrak{o}}(x,y) = \sum_{\chi\in\mathfrak{X}} K_{P,\chi}(x,y), \tag{1.5}$$

as each side equals the integral kernel of $R_P(f)$ (see §3 of Lecture II, where we computed the "geometric" expression for the kernel K_P, for $G = \mathrm{GL}(2)$ and $P =$ Borel).

Summing Up. We have

$$K_f(x,y)=\sum_{\chi\in x}K_\chi(x,y)$$

and (for any parabolic P of G),

$$K_P(\chi,y)=\sum_{\chi\in x}K_{P,\chi}(x,y).$$

Concluding Remarks. (a) When $P=G$,

$$K_P(x,y)=K_G(x,y)=K(x,y)$$

and

$$K_{P,\chi}(x,y)=K_\chi(x,y).$$

Thus the first spectral expansion is a special case of the second.
(b) For $G=\mathrm{GL}(2)$, and $P=B$ (the Borel), we have $L^2(P)_\chi=0$ if $\chi=(G,\pi)$. On the other hand, if $\chi=\{(M_P,\mu)\}$, then $L^2(P)_\chi$ just consists of φ in $L^2(Z(\mathbb{A})N_P(\mathbb{A})M_P(F)\backslash G(\mathbb{A})$, which under left action by $M^1(\mathbb{A})$ transforms according to a sum of μ and μ^{-1}. The corresponding kernel $K_{B,\chi}(x,y)$ is given by the formula

$$K_{B,\chi}(x,y)=\int_{Z(\mathbb{A})M(F)\backslash M^1(\mathbb{A})}K_B(x,my)(\mu(m)+\mu^w(m))\,dm. \tag{1.6}$$

This is easily checked by computing the composition of $R(f)$ with the projection operator P_χ defined on $L^2(Z(\mathbb{A})N(\mathbb{A})M(F)\backslash G(\mathbb{A}))$ by

$$(P_\chi\varphi)(y)=\int_{Z(\mathbb{A})M(F)\backslash M^1(\mathbb{A})}\varphi(my)(\mu(m)+\mu^w(m))\,dm.$$

Moreover, an analogue of (1.6) (without the subscript B) holds for the kernel $K_\chi(x,y)$.
(c) The series

$$\sum K_\chi(x,y)=\sum K_{G,\chi}(x,y)$$

and

$$\sum K_{P,\chi}(x,y)$$

converge absolutely (to $K(x,y)$ and $K_P(x,y)$ respectively).
(d) In general, for any G and $\chi=\{(M,r)\}$

$$\sum K_{G,\chi}(x,y)= \sum_P\frac{1}{n(A_P)}\sum_\sigma\int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*}\sum_\phi E(\chi,\rho(\sigma,\lambda)(f)\phi,\lambda)\overline{E(y,\phi,\lambda)}\,d\lambda, \tag{1.7}$$

where: the sum over P is over "associated" parabolics in G, $n(A_P)$ is the number of chambers in $\mathfrak{a}_P$, $\mathfrak{a}_P^* = X(M_P) \otimes \mathbb{R}$ is dual to $\mathfrak{a}_P = \mathrm{Hom}(X(M_P)_F, \mathbb{R})$, $\rho(\sigma, \lambda)$ is the right regular representation in $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \sigma \otimes e^{\lambda(H_P(p))}$, ϕ runs through a suitable K-finite) basis for $\rho(\sigma,\lambda)$, and—*critically*, σ runs through (classes of) irreducible unitary representations of $M(\mathbb{A})$ such that functions in $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \sigma$ belong to $L^2(P)_\chi$. Note that for $\chi = \{(G, \pi)\}$, (1.3) just reduces to

$$\sum_{\phi} R(f)\phi(x)\overline{\phi(y)},$$

since $L^2(P)_\chi \equiv 0$ unless $P = G$. Thus our original formula for $K_\pi(x,y)$ is indeed consistent with the statement that any $K_{P,\chi}(x,y)$ is expressible in terms of Eisenstein series (but now the Eisenstein series in question belong to ϕ in L_π, viewed as induced from themselves on $P = G$). At the same time, formula (1.7) is also general enough to include the extra term $R(f)(\mu)(x)\overline{\mu(y)}$ in case $G = \mathrm{GL}(2)$, and $\chi = \{(M, \mu)\}$ *with* $\mu = \mu^{-1}$; indeed, the parabolic $P = B$ in (1.7) contributes the leading term (1.2) to $K_\chi(x,y)$, while the parabolic $P = G$ contributes the crucial additional term (1.3).

2. Modified Kernels and Truncated Kernels.

2. Modified Kernels and Truncated Kernels. Recall the identity

$$\sum_{\mathfrak{o}\in\mathfrak{O}} K_{P,\mathfrak{o}}(x,y) = \sum_{\chi\in\mathfrak{X}} K_{P,\chi}(x,y), \tag{1.5}$$

introduced towards the end of the last section. By complete analogy with the geometric side (compare formulas (3.1) and (3.4) of Lecture II), Arthur defines the modified **spectral kernel functions**

$$k_\chi^T(x,f) = \sum_{P\subset G} (-1)^{\dim(A_P/Z)} \sum_{\delta\in P(F)\backslash G(F)} K_{P,\chi}(\delta x, \delta x)\hat{\tau}_P(H_P(\delta x) - T), \tag{2.1}$$

which in the case of GL(2) look like

$$k_\chi^T(x,f) = K_\chi(x,x) - \sum_{\delta\in B(F)\backslash G(F)} K_{B,\chi}(\delta x,\delta x)\hat{\tau}_B(H_B(\delta x) - T).$$

Because of the identity (1.5), one clearly has

$$\sum_{\mathfrak{o}\in\mathfrak{O}} k_{\mathfrak{o}}^T(x,f) = \sum_{\chi\in\mathfrak{X}} k_\chi^T(x,f), \tag{2.2}$$

i.e., the *modification* of the geometric expression for the kernel of R(f) is equal to the *modification* of the spectral expression for this kernel.

Despite its simplicity, this identity (2.2) is the starting point of Arthur's trace formula in the case of non-compact quotient. Of course, we emphasize (again!)

that neither side of (2.2) represents (in general) the kernel of $R(f)$. However, we have already seen in Lecture II that each of the functions $k_{\mathfrak{o}}^T(x,f)$ is (absolutely) integrable over $Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$, and that the sum of the resulting distributions

$$J_{\mathfrak{o}}^T(f) = \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} k_{\mathfrak{o}}^T(x,f)\,dx$$

also converges absolutely. Thus by (2.2), we have shown that

$$\sum_{\mathfrak{o}} J_{\mathfrak{o}}^T(f) = \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \sum_{\chi} k_{\chi}^T(x,f)\,dx, \tag{2.3}$$

i.e., to obtain the first form of Arthur's trace formula, we need "only" show that the integral (on the right side of (2.3) may be taken inside the sum over χ; in particular, each $k_{\chi}^T(x,f)$ is integrable, and the sum of the resulting distributions

$$J_{\chi}^T(f) = \int k_{\chi}^T(x,f)\,dx \tag{2.4}$$

converges absolutely to the left side of (2.3).

To prove the absolute integrability of each $k_{\chi}^T(x,f)$, and of the sum of such kernels, it turns out to be necessary to introduce a **truncation operator** on $G(F)\backslash G(\mathbb{A})$. Given T in $\mathfrak{a}^+$ as before, the *truncation* of a continuous function $\varphi(x)$ on $Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$ is the function

$$\Lambda^T\varphi(x) = \varphi(x) - \sum_{\delta\in B(F)\backslash G(F)} \varphi_N(\delta x)\hat{\tau}_B(H(\delta x)-T), \tag{2.5}$$

where

$$\varphi_N(\delta x) = \int_{N(F)\backslash N(\mathbb{A})} \varphi(nx)\,dn$$

denotes the **constant term** of φ. (Again, for $x\in\mathfrak{S}_c$, the sum over δ is finite, by Fact 1 of Lecture II ...) In general, for arbitrary reductive G,

$$\Lambda^T\varphi(x) = \sum_{P\subset G}(-1)^{\dim(A_P/Z)} \sum_{\delta\in P(F)\backslash G(F)} \varphi_{N_P}(\delta x)\hat{\tau}_P(H(\delta x)-T). \tag{2.6}$$

Note that if $\varphi(x)$ is **cuspidal**, i.e., $\varphi_{N_P}=0$ for every N_P, then

$$\Lambda^T\varphi = \varphi.$$

Of course, for a general automorphic φ, this is no longer true; however, the whole idea of **truncation** is that $\Lambda^T\varphi$ still equals φ on a large part of $\mathfrak{S}_c$ (how large depends on T), and suitably modifies φ "near infinity" so as to be integrable over all of $\mathfrak{S}_c$. In particular, for given "large" T, there is a compact (mod $Z(\mathbb{A})$) set Ω such that

$$\Lambda^T\varphi \equiv \varphi \text{ on } \Omega.$$

(Both this statement—and its proof—resembles its analogue for the modified kernels $k_{\mathfrak{o}}^T(x,f)$; see Remark 3.5 of Lecture II.)

Now for any χ in $\mathfrak{X}$, let $\Lambda_2^T K_\chi(x,x)$ denote the function obtained by truncating the function

$$K_{G,\chi}(x,y) = K_\chi(x,y)$$

with respect to the second variable and then setting $y = x$. The strategy Arthur follows in [A3] to prove that

$$\int \left(\sum_\chi k_\chi^T(x,f) \right) dx = \sum_\chi \int k_\chi^T(x,f)\, dx \tag{2.7}$$

is the following: Using properties of Λ^T, he first shows that

$$\int \sum_\chi \Lambda_2^T k_\chi(x,x)\, dx = \sum_\chi \int \Lambda_2^T k_\chi(x,x)\, dx \tag{2.8}$$

with

$$\sum_\chi \int \left| \Lambda_2^T k_\chi(x,x) \right| \, dx < \infty$$

and

$$\int \left| \sum_\chi \Lambda_2^T k_\chi(x,x) \right| dx < \infty;$$

then he shows that *for sufficiently large* T,

$$\int \sum_\chi \left| \Lambda_2^T k_\chi(x,x) - k_\chi^T(x,f) \right| \, dx = 0. \tag{2.9}$$

From these facts it follows that $\sum_{\mathfrak{o}} J_{\mathfrak{o}}^T(f)$ converges absolutely to $\sum_\chi J_\chi^T(f)$, where *for sufficiently large* T, $J_\chi^T(f)$ is given by *either* (absolutely convergent) integral

$$\int k_\chi^T(x,f)\, dx \quad \text{or} \quad \int \Lambda_2^T K_\chi(x,x)\, dx.$$

In particular, the first form of Arthur's trace formula is established.

Concluding Remarks. Set

$$K_{\text{cusp}}(x,y) = \sum_{\chi \in \mathfrak{X}(G)} K_\chi(x,y)$$

where $\mathfrak{X}(G)$ denotes the set of cuspidal data $\{(M,\sigma)\}$ with $M = G$, and R_0 be the regular representation of $G(\mathbb{A})$ restricted to

$$L_0^2 = \bigoplus_{\chi \in \mathfrak{X}(G)} L_\chi^2.$$

Then for any T,

$$\Lambda_2^T K_{\text{cusp}}(x,y) = K_{\text{cusp}}(x,y), \tag{2.10}$$

this kernel is integrable over the diagonal, and equals to $R_0(f)$. Thus Arthur's first form of the trace formula gives the more familiar **trace formula**

$$\operatorname{tr} R_0(f) = \sum_{\mathfrak{o}\in\mathfrak{O}} J_{\mathfrak{o}}^T(f) - \sum_{\mathfrak{X}\setminus\mathfrak{X}(G)} J_\chi^T(f). \tag{2.11}$$

Here only the proof of (2.10) is elementary; it results from the observation that $y \longrightarrow K_{\text{cusp}}(x,y)$ is cuspidal (see the beginning of §4 below). The remaining facts are discussed in [A1], at least for the case of rank 1. To prove that $\operatorname{tr} R_0(f)$ exists and equals $K_{\text{cusp}}(x,x)\,dx$ one uses arguments similar to those used in the (earlier) proof that $\int k_{\mathfrak{o}}^T(x,f)\,dx < \infty$.

3. Proof of (2.8) and the Absolute Convergence of $\displaystyle\sum_\chi \int \Lambda_2^T K_\chi(x,y)\,dx < \infty$. When Arthur described the spectral expansion

$$K(x,y) = \sum K_\chi(x,y). \tag{3.1}$$

in [A2], he stressed its absolute convergence by way of the following:

Lemma 3.2. (see Lemma 4.4 of [A2]) *There exists an N such that for any differential operators D_1 and D_2 in $\mathfrak{U}(\mathfrak{g})$,*

$$\sum_\chi \sum_P n(A)^{-1} \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \Big|\sum_{\phi_\chi} D_1 E(x,\rho(\sigma,\lambda)(f)(\phi_\chi),\lambda)\overline{D_2 E(y,\phi_\chi,\lambda)}\Big|\,d\lambda$$
$$\le c(D_1,D_2)\|x\|^N\|y\|^N.$$

Here $\|x\|$ denotes a "norm" or "height" function on $G(\mathbb{A})$, which we may (for GL(2)) take to be $\prod_v \|x_v\|$, with $\|x_v\| = \sup\{|(x_v)_{ij}|_v, |(x_v^{-1})_{ij}|_v\}$. This norm satisfies the properties $\|x\| = \|x^{-1}\|$, $\|x_1x_2\| \le \|x_1\|\|x_2\|$, and $\|x > \|c_1$, for some c_1; moreover, it is commensurable with $e^{\alpha(H(x))}$, *at least for x in a fixed Siegel domain.*

The above Lemma is essentially part of Langlands' theory of Eisenstein series; from it, we get

$$|D_{1,x}D_{2,y}K_\chi(x,y)| \le c(D_1,D_2)\|x\|^N\|y\|^N, \tag{3.3}$$

with a similar identity for $K(x,y)$ itself. (Here $D_{2,y}$ indicates that D_2 is operating only in the y variable, etc..) These identities say that $K(x,y)$ and $K_\chi(x,y)$ are "slowly increasing" in each variable. It is by way of these facts that the finiteness of

$$\sum_\chi \int |\Lambda_2^T K(x,x)|\,dx$$

is established.

Lemma 3.4. (see Lemma 1.4 of [A3]) *The truncation operator transforms "slowly increasing functions" φ into "rapidly decreasing" ones; more precisely, suppose φ in $C^\infty(Z(\mathbb{A})G(F)\backslash G(\mathbb{A}))$ is such that for some N (the "degree of slow increase"),*

$$|D\varphi(x)| \leq C_D \|x\|^N \qquad \text{for all } x \in \mathfrak{S}_c$$

and differential operators D. Then for any M

$$|\Lambda^T \varphi(x)| \leq C_M \|x\|^{-M}$$

for x in $\mathfrak{S}_c$ (and some C_M).

Remarks. (a) Since we are not proving Lemma (3.5), let us at least explain why it is plausible for $G = \mathrm{GL}(2)$. In this case, it is clear from Lemma 2 of Lecture II, that for x "near infinity", in particular, such that $\alpha(H(x) > \alpha(T) > C_0$,

$$\Lambda^T \varphi(x) = \varphi(x) - \varphi_N(x),$$

Thus Λ^T removes the constant term form ϕ, making it rapidly decreasing at ∞ (think roughly of a classical modular form for $\mathrm{SL}_2(\mathbb{Z})$ with vanishing constant term $\ldots$).

(b) What Lemma 1.4 of [A3] actually asserts is that for any pair of positive integers M and N, we can choose $D_1, \ldots D_n$ such that for any x in $\mathfrak{S}_c$,

$$|\Lambda^T \varphi(x)| \leq \sum_i \left(\sup_{x'} |D_i \varphi(x')| \cdot \|x'\|^{-N} \right) \cdot \|x'\|^{-M} \tag{3.5}$$

But if φ happens to be slowly increasing of degree N, *along with its derivatives*, then the expression in parentheses is bounded, and so Lemma 3.4 immediately follows. It is this latter form of the Lemma—that is, (3.5)—that we shall now apply.

Proposition 3.5. *The truncated kernels $\Lambda_2^T K(x,x)$ and $\Lambda_2^T K_\chi(x,x)$ are absolutely integrable, and*

$$\int \Lambda_2^T K(x,x)\, dx = \sum_\chi \int \Lambda_2^T K_\chi(x,x)\, dx.$$

Proof. Let us show that for any M, there exists constants c' and c_χ such that for x in $\mathfrak{S}_c$,

$$\Lambda_2^T K(x,x) \leq c' \|x\|^{N-M} \quad \text{and} \quad \Lambda_2^T K_\chi(x,x) \leq c'_\chi \|x\|^{N-M}.$$

Since $\|x\| \asymp e^{\alpha(H(x))}$ on any Siegel domain, this will suffice, by Iwasawa's decomposition, to prove that

$$\int |\Lambda_2^T K(x,x)|\, dx < \infty,$$

and hence by (Lebesgue's) dominated convergence theorem, that

$$\int \Lambda_2^T K(x,x)\,dx = \sum_\chi \int \Lambda_2^T K(x,x)\,dx,$$

an absolutely convergent sum.

Applying Lemma 3.4 (in the form of identity (3.5)) to the second variable of $K(x,y)$ yields

$$|\Lambda_2^T K(x,y)| \le C_0 \|y\|^{-M} \cdot \sup_{y'} \sup_{D_1,\dots D_n} (|D_{i,y} K(x,y')| \|y'\|^{-N}),$$

which by (3.3) is dominated by

$$C_0 \|y\|^{-M} \sup_{y',i} (C(1,D_i)\|x\|^N \|y'\|^N) \|y'\|^{-N} = C_0 \|y\|^{-N} \sup_i (C(1,D_i)) \|x\|^N.$$

So now setting $x = y$ gives the desired conclusions (for $K(x,x)$), and for $K_\chi(x,x)$ one argues similarly.

4. Proof of (2.9) Relating $\int \Lambda_2^T K_\chi(x,x)\,dx$ to $\int k_\chi^T(x,t)\,dx$.

Note first that when $\chi = (G,\pi)$, there is nothing to prove. Indeed, if K_π is the kernel of $R(f)$ restricted to any cuspidal subspace L_π of $L^2(Z(\mathbb{A})G(F)\backslash G(\mathbb{A}))$, then

$$y \longrightarrow K_\pi(x,y)$$

is itself **cuspidal**: for any φ^* in $(L_0^2)^\perp$,

$$\int K_\pi(x,y)\overline{\varphi^*(y)}\,dy = R(f) \circ (\text{Projection onto } L_\pi \subset L_0^2) = 0.$$

Thus it follows $\Lambda_2^T K_\pi(x,y) = K_\pi(x,y)$. On the other hand, as recalled in §1, $K_{P,\chi}(x,y) \equiv 0$ for all proper P in G (when $\chi = (G,\pi)$). Thus we also have (from the definition (2.1) of $k_\chi^T(x,f)$) that $k_\chi^T(x,f) = K_{G,\chi}(x,x) = K_\pi(x,x)$, i.e., $\Lambda_2^T K_\pi(x,x) \equiv k_\chi^T(x,f)$, and there is nothing to prove.

Henceforth, we shall fix $G = \mathrm{GL}(2)$, and assume $\chi = \{(M,\sigma)\}$ with $M = \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right\}$, and $\sigma\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \mu(a/b)$ (or rather its **class**, relative to the relation $\mu \sim \mu^{-1}$ in $(F^x\backslash\mathbb{A}^1)^\wedge$). Adapting the general arguments of §2 of [A3] to this case, we shall now (at least) prove that

$$\int |\Lambda_2^T K_\chi(x,x) - k_\chi^T(x,f)|\,dx = 0$$

for sufficiently large T.

By definition,

$$-\left(\Lambda_2^T K_\chi(x,x) - k_\chi^T(x,f)\right) = \sum_{\delta\in B(F)\backslash G(F)} \int_{N(F)\backslash N(\mathbb{A})} K_\chi(x,n\delta x)\hat{\tau}_B(H_B(\delta x)-T)\,dn - \sum_{\delta\in B(F)\backslash G(F)} K_{B,\chi}(\delta x,\delta x)\hat{\tau}_B(H_B(\delta x)-T).$$

Note that $K_\chi(x,n\delta x) = K_\chi(\delta x, n\delta x)$ (since $E(x,\phi)$ is left $G(F)$-invariant). Therefore the above difference equals

$$\sum_{\delta\in B(F)\backslash G(F)} \hat{\tau}_B(H_B(\delta x)-T)\left\{\int_{N(F)\backslash N(\mathbb{A})} K_\chi(\delta x,n\delta x)\,dn - K_{B,\chi}(\delta x,\delta x)\right\}. \tag{4.1}$$

Next recall our expression (1.6) for $K_{B,\chi}$ (and $K_{G,\chi}$) in the Concluding Remark (b) of §1. Applying it to (4.1), with $y=\delta x$, we get that the expression in parenthesis equals

$$\int_{N(F)\backslash N(\mathbb{A})} K_\chi(y,ny)\,dn - K_{B,\chi}(y,y) = \int_{Z(\mathbb{A})M(F)\backslash M^1(\mathbb{A})} \left\{\int_{N(F)\backslash N(\mathbb{A})} (K(y,nmy)-K_B(y,my))\,dn\right\} \times \mu(m)+\mu^{-1}(m)\,dm.$$

Applying the definition of K and K_B, we find, in turn, that the last expression in parentheses equals

$$\int_{N(F)\backslash N(\mathbb{A})} \sum_{\gamma\in Z(F)\backslash G(F)} f(y^{-1}\gamma\, n\, m\, y)\,dn - \int_{N(\mathbb{A})} \sum_{\mu\in Z(F)\backslash M(F)} f(y^{-1}\gamma\, n\, m\, y)\,dn$$

$$= \int_{N(\mathbb{A})} \left(\sum_{\gamma\in N(F)Z(F)\backslash G(F)} f(y^{-1}\gamma\, n\, m\, y) - \int_{N(\mathbb{A})} \sum_{\mu\in Z(F)\backslash M(F)} f(y^{-1}\gamma\, n\, m\, y)\right) dn$$

Now apply Bruhat's decomposition

$$Z(F)N(F)\backslash G(F) = Z(F)\backslash M(F) \cup (Z(F)\backslash M(F))wN(F)$$

to the first summation above. The result is that the entire last expression equals

$$\int_{N(\mathbb{A})} \sum_{\nu\in N(F)} \sum_{\mu\in Z(F)\backslash M(F)} f(y^{-1}\mu\, w\, \nu\, n\, m\, y)\,dn.$$

Using this, it remains to show that

$$\begin{aligned}
&\int |\Lambda_2^T K_\chi(x,x) - k_\chi^T(x,f)|\,dx \\
&\quad \le \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \sum_{\delta\in B(F)\backslash G(F)} \hat\tau_B(H(\delta x)-T)\ \times \\
&\qquad \int_{Z(\mathbb{A})M(F)\backslash M^1(\mathbb{A})} \int_{N(\mathbb{A})} \sum\sum f(x^{-1}\delta\mu\, w\,\nu\, n\, m\delta\chi)(\mu(m)+\mu^{-1}(m)\,dm\,d\chi.
\end{aligned}$$

$$\begin{aligned}
&\quad = \int_{Z(\mathbb{A})B(F)\backslash G(\mathbb{A})} \hat\tau_B(H(y)-T)\ \times \\
&\qquad \left(\iint \sum\sum f(y^{-1}\delta\mu\, w\,\nu\, n\, m\, y)(\mu(m)+\mu^{-1}(m)\right)\,dm\,dy \qquad (4.2)
\end{aligned}$$

$$= 0 \qquad \textit{for sufficiently large } T.$$

So suppose y is such that this expression (4.2) is *not* zero. Then our hypothesis on f implies

$$y^{-1}\mu\, w\,\nu\, n\, m\, y$$

belongs to the compact (mod $Z(\mathbb{A})$) subset Ω_f of $G(\mathbb{A})$. Writing out the Iwasawa decomposition n_1ak for y implies

$$g = a^{-1}n_1^{-1}\nu\, w\,\mu\, n\, m\, n_1 a \in \Omega_f'$$

with Ω_f' the compact (mod $Z(\mathbb{A})$) set $K\Omega_f K$, i.e., $\alpha(H(g))$ remains bounded (from below and above).

On the other hand, write

$$\begin{aligned}
g &= a^{-1}n_1^{-1}\nu\, w\,\mu\, n\, m\, n_1 a \\
&= (n_1^{-1}\nu)^{a^{-1}}a^{-1})w\,\mu\, m\, a\,(n^{m^{-1}}n_1)^{a^{-1}} \\
&= n^* a^* w n'
\end{aligned}$$

where $n^*, n' \in N(\mathbb{A})$ and $a^* = a^{-1}a^w(\mu m)^m$. Then

$$c < H(g) = H(a^*) + H(wn') = -2H(a) + H(wn')$$

implies

$$-2H(a) > c - H(wn'),$$

i.e.

$$H(a) < \frac{H(wn')}{2} - \frac{c}{2} < \frac{c_0 - c}{2} = c' \tag{4.3}$$

(using Lemma 4.1 of Lecture II).

To complete the proof, take T “sufficiently large” to mean that $\alpha(T) > c'$. Then the condition $\hat\tau_B(H(y)-T) = \hat\tau_B(H(a)-T) \neq 0$ will be incompatible with (4.3), i.e. (4.2) will be identically zero, as claimed.

5. An Alternate Strategy?. In any treatment of the trace formula prior to the 1978 paper [A2] (see [A1]), no *modified* kernels appear. For example, in [GJ] one simply and (formally) truncates both sides of the identity

$$K(x,y) = \sum_{\gamma} f(x^{-1}\gamma y) = \sum K_{\chi}(x,y)$$

to obtain a trace formula (for GL(2)) of the form

$$\int \Lambda_2^T K(x,x)\,dx = \sum_{\chi} \int \Lambda_2^T K_{\chi}(x,x)\,dx. \tag{5.1}$$

Our purpose now is to explain where the modified kernels are actually hiding in(5.1) and why they can *not* be avoided in the general theory.

Using the crucial Lemma 2.1 of Lecture II, one can easily prove (see p. 235 of [GJ]):

Lemma 5.2. *For T sufficiently large (i.e. $\alpha(T)$ larger than the constant d_{Ω_f} described in Lemma 2.1 of Lecture II),*

$$\Lambda_2^T K(x,x) = \sum_{\mathfrak{o} \text{ elliptic}} K_{\mathfrak{o}}^T(x,f) + \sum_{\substack{\mathfrak{o}\ unipotent \\ or \\ hyperbolic}} K_{\mathfrak{o}}^T(x,f) \equiv K^T(x,f)$$

Note that Lemma (5.2) implies that the *truncated* geometric kernel used in (5.1) is already a sum of the *modified* geometric kernels. Thus an alternate strategy for obtaining the trace formula

$$\sum_{\mathfrak{o}} J_{\mathfrak{o}}^T(f) = \sum_{\chi} J_{\chi}^T(f) \tag{5.3}$$

suggests itself: since we also have

$$\Lambda_2^T K(x,x) = \sum_{\chi} \Lambda_2^T K_{\chi}(x,x).$$

(the spectral analogue of the geometric formula for $\Lambda_2^T K(x,x)$ afforded by Lemma (5.2)) , why not just integrate each of these expressions term by term to obtain the trace formula identity (5.3) with

$$J_{\chi}^T(f) = \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \Lambda_2^T K_{\chi}(x,x)\,dx?$$

Indeed, by Lecture II we know the left hand side of (5.3) is nicely behaved, and by the first three sections of this lecture, we know the right hand side converges too.

This strategy was in fact carried out in [GJ] *for* GL(2), where it turns out that terms on either side of (5.3) can be computed *completely explicitly*, as (linear) polynomials in T (see §6 of [GJ] and §§1 and 2 of the next lecture). Then when these expressions are plugged into the formula

$$\operatorname{tr} R_0(f) = \sum_{\mathfrak{o}\in\mathfrak{O}} J^T_{\mathfrak{o}}(f) - \sum_{\chi\in\mathfrak{X}\setminus\mathfrak{X}(G)} J^T_\chi(f),$$

the terms depending on T *cancel out* (as they must, since the left hand side is independent of T). Thus we are left with our sought-after formula for $\operatorname{tr} R_0(f)$.

What goes wrong with this strategy in general? Well, for an arbitrary reductive group G, it would be hopeless to try and compute each term $J^T_{\mathfrak{o}}(f)$ and $J^T_\chi(f)$ completely explicitly; fortunately, however, it is also *unnecessary*. What Arthur's more indirect strategy (involving *modified spectral* as well as geometric kernels) allows one to prove is that $J^T_\chi(f)$ and $J^T_{\mathfrak{o}}(f)$ are *a priori polynomials* in T (for sufficiently large $\alpha(T)$). Thus we can concern ourselves only with the *constant terms* of these polynomials (since our goal is to make (5.3) explicit, and only terms not depending on T will survive on the right-hand side ...). These constant terms $((J_{\mathfrak{o}}(f) \equiv J^0_{\mathfrak{o}}(f)$ and $(J_\chi(f) \equiv J^0_\chi(f))$ are by no means trivial to compute, but they *do* turn out to be feasible to treat in Arthur's general theory, as we shall soon see.

Lecture IV. More Explicit Forms of the Trace Formula

In the first few lectures, we have described the first form of Arthur's trace formula, which in its symmetric formulation reads

$$\sum J_{\mathfrak{o}}^T(f) = \sum_{\chi} J_{\chi}^T(f); \tag{*}$$

more suggestively,

$$\operatorname{tr}(R_0(f)) = \sum_{\mathfrak{o}} J_{\mathfrak{o}}^T(f) - \sum_{\chi\in\mathfrak{X}\setminus\mathfrak{X}(G)} J_{\chi}^T(f). \tag{**}$$

After developing this formula in [A2] and [A3], Arthur devoted the next ten years (and several hundred pages of work) to rewriting this formula in a more explicit form, suitable for applications. In particular, in order to apply $(**)$ effectively to the theory of automorphic forms, it seems essential to know a lot about the distributions $J_{\mathfrak{o}}^T(f)$ and $J_{\chi}^T(f)$. How do they depend on T? How can they be expressed explicitly in terms of **weighted orbital integrals** or **weighted characters**? How can they be factored into local distributions on the groups G_v?

Our purpose in the remainder of these lectures is to explain some of Arthur's answers to these questions. But to make this task easier, we proceed in stages.

One of Arthur's first general results is that each distribution $J_{\mathfrak{o}}^T(f)$ or $(J_{\chi}^T(f))$ is a **polynomial** in T, of degree at most $\dim(A_{P_0}/Z)$. Another is that each "unramified" $J_{\mathfrak{o}}^T(f)$ (or $J_{\chi}^T(f)$) has a fairly explicit representation as a weighted orbital integral (or character). To motivate these and other general results, we shall first recall the familiar example of GL(2).

1. Explicit Results for GL(2) **(Geometric Side).** As we already observed in general in Lecture II, the elliptic terms $J_{\mathfrak{o}}^T(f)$ are independent of T, and simply equal ordinary (factorizable) orbital integrals (see Proposition 2.5 of Lecture II). On the other hand, the hyperbolic and unipotent terms are non-trivial (linear) polynomials in T, explicitly described in the propositions below.

Proposition 1.1. *If $\mathfrak{o}$ denotes a hyperbolic class with representative γ, then for $\alpha(T)$ sufficiently large,*

$$\begin{aligned}J_{\mathfrak{o}}^T(f) = (T_1 - T_2)m(F^*\backslash\mathbb{A}^1)&\int_{M(\mathbb{A})\backslash G(\mathbb{A})} f(x^{-1}\gamma x)\,dx\\ &- \frac{1}{2}m(F^*\backslash\mathbb{A}^1)\int_{M(\mathbb{A})\backslash G(\mathbb{A})} f(x^{-1}\gamma x)\alpha(H(wx)+H(x))\,dx\end{aligned}$$

Remarks. (1) The proof of this proposition is provided by the formal computations of [GJ, pp. 237–238] now justified by the already established *a priori* absolute convergence of $\int k_{\mathfrak{o}}^T(x,f)\,dx$. The assumption $\alpha(T) \gg 0$ is needed so as to be able to apply Lemma 4.1 of Lecture II (which in the case of GL(2) just requires $\alpha(T) > 0$).

(2) In §3 of Lecture I, we saw that the unmodified hyperbolic kernel $K_{\mathfrak{o}}$ was *not* integrable because of the appearance of the (divergent) volume of $F^x\backslash\mathbb{A}^x$; Proposition 1.1 shows that the integral of the *modified* kernel indeed "truncates" this volume to a finite interval of length $(T_1 - T_2)$, which appears as a linear term in T with coefficient $\int f(x^{-1}\gamma x)\,dx$; however, the constant term of this polynomial no longer involves an invariant orbital integral (because of the presence of the "weight function" $\alpha(H(wx)) + H(x)$).

Proposition 1.2. *For the unipotent class* $\mathfrak{o}$, *and* $\alpha(T)$ *sufficiently large,*

$$J_{\mathfrak{o}}(f) = m(Z(\mathbb{A})G(F)\backslash G(\mathbb{A}))f(1) + \underset{s=1}{\mathrm{f.\,p.}}\ (\zeta(F,s)) + (T_1 - T_2)m(F^*\backslash\mathbb{A}^1)\int_{\mathbb{A}} F(y)\,dy$$

where

$$F(y) = \int_K f(k^{-1}\left(\begin{smallmatrix}1 & a\\ 0 & 1\end{smallmatrix}\right)k)\,dk \qquad \textit{is in } \mathcal{S}(\mathbb{A}),$$

and $\mathrm{f.\,p.}_{s=1}\ (\zeta(F,s))$ *denotes the "finite part" at* $s=1$ *of the (meromorphic) zeta-function*

$$\zeta(F,s) = \int_{\mathbb{A}^x} F(a)|a|^s\,d^xa$$

of Tate.

Remarks. (1) Iwasawa's decomposition implies (formally) that

$$\begin{aligned}\zeta(F,s)\ (\text{at } s=1) &= \int_{\mathbb{A}^x}\int_K f\left(k^{-1}\left(\begin{smallmatrix}1 & a\\ 0 & 1\end{smallmatrix}\right)k\right)\,dk\ |a|\,d^xa\\ &= \int_{Z(\mathbb{A})N(\mathbb{A})\backslash G(\mathbb{A})} f\left(x^{-1}\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)x\right)\,dx,\end{aligned}$$

the non-trivial unipotent orbital integral of f. Thus the constant term of this $J_{\mathfrak{o}}^T(f)$ is indeed analogous to the constant term of the hyperbolic $J_{\mathfrak{o}}^T(f)$: one involves a **regularized** (unipotent) orbital integral, and the other a **weighted** (hyperbolic) orbital integral.

(2) Because the left hand side of (**) is independent of T, it must be that the linear terms in T appearing in both the hyperbolic and unipotent $J_{\mathfrak{o}}^T(f)$ cancel out with the linear terms appearing in $\sum J_\chi^T(f)$. Indeed one can check (using Iwasawa's decomposition) that the linear terms in Propositions 1.1 and 1.2 (for all hyperbolic $\mathfrak{o}$) combine into the single term

$$L(T) = (T_1 - T_2)m(F^*\backslash\mathbb{A}^1)\sum_{\gamma\in Z(F)\backslash M(F)}\int_K\int_{N(\mathbb{A})} f(k^{-1}\gamma nh)\,dn\,dh;$$

then this term is seen (later) to exactly cancel the linear term in T arising from $\sum_\chi J_\chi^T(f)$; see §2 below, also p. 239 of [GJ]).

(3) The proof of Proposition 1.2 again results from formal computations—see p. 236 of [GJ]—all of which are justified now by the *a priori* absolute convergence of $\int k_{\mathfrak{o}}^T(x,f)\,dx$. As in the hyperbolic case, the appearance of the linear term, and the "shearing off" (i.e., regularization) of the relevant orbital integral, are explained by the fact that the kernel $K_{\mathfrak{o}}(x,x)$ was modified before being integrated.

(4) The need to modify $K_{\mathfrak{o}}$ (in this unipotent case) is clarified by the exercise below.

Exercise. Show that for the unipotent class $\mathfrak{o}$,

$$\begin{aligned} J_\chi^T(f) &= \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} K_{\mathfrak{o}}(x,x)\,dx \\ &= \operatorname*{val}_{s=1}\,\{\zeta(F,s)\} + m(Z(\mathbb{A})G(F)\backslash G(\mathbb{A}))f(1), \end{aligned}$$

with $\zeta(F,s)$ as in the proposition above. In particular, $J_{\mathfrak{o}}(f) = \infty$ when the "principal part" of $\zeta(F,s)$, namely $m(F^*\backslash\mathbb{A}^1)\hat{F}(0)/(s-1)$, is non-zero.

Solution. Recall that

$$\mathfrak{o} = \{1\} \cup \mathfrak{o}',$$

where $\mathfrak{o}'$ consists of the elements

$$\left\{\delta^{-1}\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\delta \colon t\in F^*,\ \delta\in B(F)\backslash G(F)\right\}$$

Thus we compute

$$\begin{aligned} \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} k_{\mathfrak{o}'}(x,x)\,dx &= \int \sum_{t\neq 0}\sum_{\delta\in B(F)\backslash G(F)} \left(x^{-1}\delta^{-1}\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\delta x\right)dx \\ &= \sum_{t\neq 0}\int_{Z(\mathbb{A})B(F)\backslash G(\mathbb{A})} f\left(x^{-1}\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}x\right)dx, \end{aligned}$$

which, by Iwasawa's decomposition, equals

$$\begin{aligned} \int_K\int_{F^x\backslash\mathbb{A}^x}\sum_{t\in F^x} f\left(k^{-1}\begin{pmatrix}1 & a^{-1}t\\ 0 & 1\end{pmatrix}k\right)|a|^{-1}\,d^xa\,dk &= \int_{F^x\backslash\mathbb{A}^x}\left(\sum_{t\in F^x}F(at)\right)|a|\,d^xa \\ &= \int_{\mathbb{A}^*} F(a)|a|\,d^xa \end{aligned}$$

with $F(y)$ as defined in Proposition 1.2.

Concluding Remark. As we shall see in the next lecture, the explicit expressions just obtained for $J_{\mathfrak{o}}^T(f)$ (hyperbolic or unipotent) already suffice to express these distributions as (a sum of) products of *local* distributions $J_v(f_v)$; this fact will turn out to be crucial in yielding a *simple form* of the trace formula in Lecture V.

2. Explicit Results for GL(2) **(Spectral Side).** As already observed in general in Lecture III, the "elliptic" spectral terms $J_\chi^T(f)$, where $\chi = \{(G,\pi)\}$, are independent of T; they just reduce to the distributional characters

$$J_\chi^T(f) = \operatorname{tr}\pi(f).$$

As for the remaining "hyperbolic" and "unipotent" χ, we have the following:

Proposition 2.1. *Suppose $\chi = \{(M,\mu)\}$ is "hyperbolic" in the sense that $\mu \neq \mu^{-1}$ (i.e., there is no 1-dimensional residual spectrum contributing to $K_\chi(x,y)$). Then for T sufficiently large,*

$$(2.1)\quad J_\chi^T(f) = (T_1 - T_2)\int_{-\infty}^{\infty} \operatorname{tr}(\rho(\mu,it)(f))\,dt + \int_{-\infty}^{\infty} \operatorname{tr}(M(-it)M'(it)\rho(\mu,it)(f))\,dt$$

with the intertwining operators $M(s)$ (and their derivatives) to be explained below.

Because the proof of this result directly generalizes to any G, we include it.

By definition, for χ regular ($\mu \neq \mu^{-1}$),

$$(2.2)\quad J_\chi^T(f) = \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \Big(\sum_\phi \int_{-\infty}^{\infty} E(x,\rho(\mu,it)(f)\phi,it)\overline{\Lambda^T E(x,\phi,it)}\Big)\,dt\,dx.$$

However, because we have already established the absolute convergence of this integral, we can interchange the orders of integration and write

$$(2.3)\qquad J_\chi^T(f) = \sum_\phi \int_{-\infty}^{\infty} (\Lambda^T E(\rho(\mu,it)(f)\phi,it), \Lambda^T E(\phi,it))\,dt.$$

Here we have used also the well-known fact that Λ^T is an orthogonal projection operator on L^2, and hence $(\phi_1, \Lambda^T\phi_2) = (\Lambda^T\phi_1, \Lambda^T\phi_2)$, with $(\cdot,\cdot)$ denoting the usual inner product in L^2.

To continue with the proof, we obviously need a more explicit expression for the inner product of these truncated Eisenstein series. This is provided by the "Maass–Selberg relations" (generalized by Langlands to an arbitrary G in [La1]); in the present context they imply

(2.4)
$$\begin{aligned}(\Lambda^T E(\phi',it), \Lambda^T E(\phi,it)) &= (T_1 - T_2)(\phi',\phi) + (M(-it)M'(it)\phi',\phi)\\ &\quad + \frac{1}{2it}\{(\phi', M(it)\phi)e^{2it(T_1-T_2)} - (M(it)\phi',\phi)e^{-2it(T_1-T_2)}\}.\end{aligned}$$

Here $M(s) = M(s,\mu)$ is the operator

$$\phi \longrightarrow M(s)\phi(g) = \int_{N(\mathbb{A})} \phi(wng)\,dn$$

intertwining $\rho(\mu, s)$ with $\rho(\mu^w, -s)$. (Like $E(g, \phi, \mu, s)$, $M(s)$ is initially defined only for $\mathrm{Re}(s) > 1/2$, but is analytically continuable in $\mathbb{C}$, and holomorphic on $i\mathbb{R}$.) Note $M(-s) = M(-s, \mu^w)$ maps $\rho(\mu^w, -s)$ to $\rho(\mu, s)$, and the inner product (ϕ_1, ϕ_2) is defined by

$$(\phi_1, \phi_2) = \int_{Z(\mathbb{A})M(F)\backslash M^1(\mathbb{A})} \int_K \phi, (mk)\overline{\phi_2(mk)}\, dm\, dk.$$

Finally, $M'(s)$ is the derivative $\frac{d}{ds}M(s, \mu)\colon \rho(\mu, s) \longrightarrow \rho(\mu^w, -s)$ (defined by identifying $\rho(\mu, s)$ with $\rho(\mu, 0)\ldots$).

Now formula (2.4) is valid not just in our case (when μ is "regular"), but also when $\mu = \mu^{-1}$; in fact we shall require (2.4) below when we treat the "unipotent" contribution $J^T_\chi(f)$. However, when $\mu \neq \mu^{-1}$, it is a simple matter to observe that the third (and most complicated) term on the right side of (2.4) actually vanishes. Indeed, as $M(s) = M(s, \mu)$ maps ϕ into $\rho(-s, \mu^{-1})$, it follows that ϕ' and $M(it)\phi$ transform under $M^1(\mathbb{A})$ according to distinct characters (namely, μ and μ^{-1}). Thus the inner product

$$(\phi', M(it)\phi) = \int \phi'(k)\overline{M(it)\phi(h)}\, dh(\int_{Z(\mathbb{A})M(F)\backslash M^1(\mathbb{A})} \mu^2(m)\, dm)$$

vanishes (and so, similarly, does $(M(it)\phi', \phi)$). So plugging (2.4) into (2.3) simply yields

$$(2.5)\qquad (T_1 - T_2)\int_{-\infty}^{\infty} \mathrm{tr}(\rho(\mu, it)(f))\, dt + \int_{-\infty}^{\infty} \mathrm{tr}(M(-it)M'(it)\rho(\mu, it)(f))\, dt,$$

as required.

We now move on to the more problematic case when $\mu = \mu^{-1}$. In this case, plugging in (2.4) into (2.3) yields (in *addition* to the terms in (2.5)) the term

$$(2.6)\quad E(T) = \sum_\phi \int_{-\infty}^{\infty} \Big\{(\rho(\mu, it)(f)\phi, M(it)\phi)\frac{e^{2it(T_1-T_2)}}{2it} - (\rho(\mu, it)(f)\phi, M(-it)\phi)\frac{e^{-2it(T_1-T_2)}}{2it}\Big\}\, dt.$$

(Here we have used the fact that the adjoint of $M(s)$ is $M(-s)$.) But this still isn't enough to describe $J^T_\chi(f)$. In this singular situation, what's missing is the contribution from the one-dimensional residual spectrum, i.e., the contribution

$$J^{T,\mathrm{res}}_\mu(f) = \mu(f)\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \mu(x)\Lambda^T(\overline{\mu(x)})\, dx$$

from the residual kernel

$$K^{T,\mathrm{res}}_\mu(x, y) = \mu(f)\mu(x)\overline{\mu(y)}.$$

Analyzing both these functions of T (namely $E(T)$ and $J^{T,\mathrm{res}}_\mu(f)$) yields finally:

Proposition 2.7. *For $\alpha(T) \gg 0$, and $\chi = \{(M,\mu)\}$ with $\mu = \mu^{-1}$,*

$$J_\chi^T(f) = (T_1 - T_2)\int_{-\infty}^{\infty} \operatorname{tr}(\rho(\mu, it)(f))\,dt + \int_{-\infty}^{\infty} \operatorname{tr}(M(-it)M'(it)\rho(\mu,it)(f))\,dt, \\ + \tfrac{1}{4}\operatorname{tr}(M(0)\rho(\mu,0)(f)) + \mu(f)m(Z(\mathbb{A})G(F)\backslash G(\mathbb{A}))$$

plus $n_1^*(T)$, a negligible term (for T very large).

Proof. In [GJ, pp. 239–240], a Fourier analysis argument is given to show that

$$E(T) = \tfrac{1}{4}\operatorname{tr}(M(0)\rho(\mu,0)(f)) + r_1(T)$$

with $r_1(T)$ negligible for $\alpha(T)$ large. On the other hand, it is a pleasant exercise to check that

$$J_\mu^{T,\mathrm{res}}(f) \longrightarrow \mu(f)m(Z(\mathbb{A})G(F)\backslash G(\mathbb{A})) \qquad \text{as } T \to \infty.$$

Indeed,

$$\begin{aligned}
\mu(f)&\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \mu(x)\Lambda^T(\overline{\mu(x)})\,dx \\
&= \mu(f)\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \mu(x)\Big(\overline{\mu}(x) - \sum_{\delta\in B(F)\backslash G(F)} \overline{\mu}(x)\hat{\tau}_B(\delta x) - T)\Big)\,dx \\
&= \mu(f)\int dx - \mu(f)\int \sum_{\delta\in B(F)\backslash G(F)} \hat{\tau}_B(H(\delta x) - T)\,dx \\
&= \mu(f)m(Z(\mathbb{A})G(F)\backslash G(\mathbb{A})) - \mu(f)\int_{Z(\mathbb{A})B(F)\backslash G(\mathbb{A})} \hat{\tau}_B(H(x)-T)\,dx \\
&= \mu(f)m(Z(\mathbb{A})G(F)\backslash G(\mathbb{A})) - n_1(T)
\end{aligned}$$

with

$$n_1(T) = \mu(f)m(F^*\backslash\mathbb{A}^1)\frac{e^{-2(T_1-T_2)}}{2} \longrightarrow \infty \qquad \text{as } \alpha(T) \to \infty.$$

So setting $n_1^*(T) = r_1(T) + n_1(T)$ completes the proof. □

Concluding Remark. (a) Proposition 2.1 expresses $J_\chi^T(f)$ directly as a **polynomial** in T when χ is regular. But as already suggested, Arthur has proven in general that $J_\chi^T(f)$ is polynomial in T. Thus it must follow that the "negligible" term $n_1^*(T)$ (appearing in the expression for $J_\chi^T(f)$ for singular χ in Proposition 2.7) is actually zero identically. (Indeed, it is a polynomial in T, which goes to 0 as T goes to infinity ....) On the other hand, the expression $E(T)$ (or equivalently (6.28) in [GJ, p. 239] is definitely not a polynomial; rather $E(T)$ includes a negligible term $r_1(T)$ which must cancel the negligible (exponentially decreasing) term $n_1(T)$ appearing in the expression for $J_\mu^{T,\mathrm{res}}(f)$.

(b) Our computation of

$$J_\mu^{T,\mathrm{res}}(f) = \mu(f)\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \mu(x)\Lambda^T(\overline{\mu(x)})\,dx$$

really amounts to the computation of the inner product of two truncated "degenerate Eisenstein series" (induced from μ on $P = G \ldots$). In general, the explicit description of $J_\chi^{T,\mathrm{res}}(f)$ with χ "singular" will involve an inner product

$$(\Lambda^T E_P(\phi', \lambda'), \Lambda^T E_P(\phi, \lambda)) \tag{2.8}$$

where ϕ in $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \sigma$ belongs to $L^2(P)_\chi$, *but P does not belong to P_χ*. Thus the natural generalization of Proposition 2.7 must involve Arthur's generalization of Langlands' formula (valid for $P \notin P_\chi$, but like the formula for $J_\mu^{T,\mathrm{res}}(f)$ giving only an **asymptotic** expression for (2.8); see [A9]).

3. Results for General G. Arthur's generalization of the above results, and in fact the GL(2) results themselves, seem much easier to digest and understand when viewed not as a long chain of unrelated results, but rather as realizations of a few fundamental phenomena and principles. Of course Arthur knowingly developed his trace formula with these thoughts in mind. To begin to explain them, let us start with the symmetry of the formula itself.

Remark 3.1. Consider the formula

$$\sum_{\mathfrak{o} \in \mathfrak{O}} J_{\mathfrak{o}}^T(f) = \sum_{\chi \in \mathfrak{X}} J_\chi^T(f).$$

Although at first sight the indexing sets $\mathfrak{O}$ and $\mathfrak{X}$ seem asymmetric and unrelated, they can actually be viewed as mirror images of one another. Indeed, if $\mathfrak{o} \in \mathfrak{O}$, consider those (standard) parabolics B of G which are minimal with respect to the property that $\mathfrak{o}$ meets M_B. Then $\mathfrak{o} \cap M_B$ is a finite union of $M_B(F)$ conjugacy classes which are "elliptic" in the sense that they meet no proper parabolic subgroup of M_B. Moreover, if W_0 denotes the restricted Weyl group of (G, A_0), then $\mathfrak{O}$ is in **bijective** correspondence with the set of W_0-orbits of pairs (M_B, c_B), where B is a parabolic subgroup of G, and c_B is an elliptic conjugacy class in $M_B(F)$. Thus $\mathfrak{O}$ indeed corresponds naturally to $\mathfrak{X}$, itself defined to be the set of W_0-orbits of pairs (M_B, r_B), where r_B is an irreducible cuspidal automorphic representation of $M_B(\mathbb{A})$. This description of $\mathfrak{O}$ has the additional benefit that it highlights the analogy between "elliptic" classes and "cuspidal" representations. (For convenience here, we have ignored centers $\ldots$.)

In light of the symmetry between $\mathfrak{O}$ and $\mathfrak{X}$ (expressed by this Remark 3.1), it is not surprising that most of the general results Arthur proves for the distributions $J_{\mathfrak{o}}^T(f)$ have an immediate analogue for $J_\chi^T(f)$. For example, if $\mathfrak{o} = \{(M_B, c_B)\}$ is such that $J_{\mathfrak{o}}^T(f)$ is a polynomial of degree at most $\dim(A_B/Z)$, then so is $J_\chi^T(f)$, for $\chi = \{(M_B, r_B)\}$. Or, if $\mathfrak{o} = \{(M_B, c_B)\}$ is suitably **unramified** that $J_{\mathfrak{o}}^T(f)$ is expressible as a weighted orbital integral, then the corresponding $J_\chi^T(f)$ should be expressible as a weighted average of characters. Let us explain these matters in some more detail now, and along the way bring into play some other basic principles as well.

Proposition 3.1 (see Proposition 2.3 of [A4]). *For each fixed f, $J_{\mathfrak{o}}^T(f)$ is a polynomial in T of degree at most* $\dim(\mathfrak{a}_{P_0}/\mathfrak{a}_G)$ (*with P_0 the minimal parabolic*).

The proof has two main ingredients. First, for any parabolic $P \supset P_0$, and point X in $\mathfrak{a}_0$, a certain characteristic function

$$\Gamma'_P(H,X) = \sum_{\{R:\, R\supset P\}} (-1)^{\dim(A_R/Z)} \tau_P^R(H)\hat{\tau}_R(H-X)$$

is defined on $\mathfrak{a}_0$, with τ_P^R the characteristic function on $\mathfrak{a}_0$ of the set

$$\{H \in \mathfrak{a}_0 \colon \alpha(H) > 0,\ \ \alpha \in \Delta_P^R\}$$

and $\hat{\tau}_R$ of the set

$$\{H \in \mathfrak{a}_0 \colon \tilde{\omega}(H) > 0,\ \ \tilde{\omega} \in \hat{\Delta}_R\}.$$

(Here $\hat{\Delta}_R$ is the dual basis in $\mathfrak{a}_R^*$ corresponding to the basis of coroots $\{\alpha^\vee : \alpha \in \Delta_R\}$ in $\mathfrak{a}_R$.)

For example, for GL(2), and $X = (X_0, -X_0)$ in $\mathfrak{a}_0^+$, $\Gamma'_B(H,X) = \tau_B(H) - \hat{\tau}_B(H - X)$; as a function on the one-dimensional subspace $\mathfrak{a}_0^G = \{(T,-T)\} \approx \{T\}$, this is just the characteristic function of the interval $[0, X_0]$.

On the other hand, for $G = \mathrm{GL}(3)$ and $(X_1, X_2, -(X_1+X_2)) = X$ in $\mathfrak{a}_0^+$, the function $\Gamma'_{P_0}(\,\cdot\,, X)$ (viewed on the two-dimensional space $\{(r_1, r_2, -(r_1+r_2))\}$) is just the characteristic function of the shaded region below:

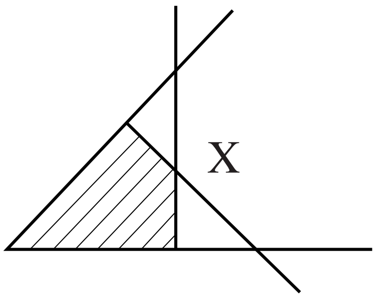

FIGURE 1

In general, the crucial property of $\Gamma'_P(H,X)$ is that it has compact support, and that its integral

$$\int_{\mathfrak{a}_P^G} \Gamma'_P(H,X)\, dH$$

is a polynomial in X of degree $q = \dim(A_P/Z)$. (Here $\mathfrak{a}_P^G$ is the subspace of $\mathfrak{a}_P$ annihilated by $\mathfrak{a}_G^*$, so in particular, $\mathfrak{a}_P = \mathfrak{a}_P^G \oplus \mathfrak{a}_G$.)

Second, certain geometric distributions $J_{\mathfrak{o}}^{M_Q,T}$ (analogous to $J_{\mathfrak{o}}^T$) are defined for each Levi **subgroup** M of G. Namely, for any fixed suitably regular point T_1 in $\mathfrak{a}_0^+$, class $\mathfrak{o}$ in $\mathfrak{O}$ and f_Q in $C_c^\infty(Z(\mathbb{A})\backslash M_Q(\mathbb{A}))$, define

$$J_{\mathfrak{o}}^{M_Q,T_1}(f_Q) = \sum_{i=1}^n J_{\mathfrak{o}_i}^{M_Q,T_1},$$

where $\mathfrak{o} \cap M_Q(F)$ is the union (possibly empty) of "classes" $\mathfrak{o}_1, \dots \mathfrak{o}_n$ in M_Q. (If $\mathfrak{o} \cap M_Q(F) = \emptyset$, then $J_{\mathfrak{o}}^{M_Q,T_1} \equiv 0$; on the other hand, if $M = G$, then $J_{\mathfrak{o}}^{M,T_1} = J_{\mathfrak{o}}^{T_1}$.)

To prove Proposition 3.1, Arthur derives the formula

$$J_{\mathfrak{o}}^T(f) = \sum_{Q \supset P_0} J_{\mathfrak{o}}^{M_Q,T_1}(f_Q) \int_{\mathfrak{a}_Q^G} \Gamma'_Q(H, T - T_1)\, dH, \tag{3.2}$$

where f_Q in $C_c^\infty(Z(\mathbb{A})\backslash M_Q(\mathbb{A}))$ is explicitly determined by f (and T varies freely through $T_1 + \mathfrak{a}_0^+$). Since each $\int \Gamma'_Q(H, T - T_1)\, dH$ is a polynomial in $T - T_1$ of degree *at most* $\dim(A_{P_0}/Z)$, it follows that the same is true of $J_{\mathfrak{o}}^T(f)$, as claimed. Note that $J_{\mathfrak{o}}^T(f)$ will be independent of T, i.e., be of degree zero, precisely when $\mathfrak{o}$ is elliptic, i.e., has empty intersection with every proper parabolic Q appearing in (3.2) above.

Remark 3.3. As expected, there is an analogue of Proposition 3.1 for $J_\chi^T(f)$, proved in just the same way (in keeping with the principle that the geometric and spectral terms are different facets of the same type of object). For χ in $\mathfrak{X}$, and M a Levi subgroup of G, one similarly defines distributions $J_\chi^{M,T}$ on $M(\mathbb{A})$ and proves (analogously) that

$$J_\chi^T(f) = \sum_{Q \supset P_0} J_\chi^{M_Q,T_1} \int_{\mathfrak{a}_Q^G} \Gamma'_Q(H, T - T_1)\, dH \tag{3.4}$$

is (again) a polynomial in T of degree $\leq \dim(A_0/Z)$. Interestingly, one also obtains the equality

$$\sum_{\mathfrak{o} \in \mathfrak{O}} J_{\mathfrak{o}}^{M,T}(f) = \sum_{\chi \in \mathfrak{X}} J_\chi^{M,T}(f), \tag{3.5}$$

which may be viewed as (a first form of) Arthur's trace formula "relativized" to an arbitrary Levi subgroup M_P of G! This relativization is reminiscent of Langlands' use of the spectral decomposition of $L^2(P)$ (P any parabolic of G) in order to describe $L^2(G)$ itself, and it constitutes another of the unifying themes in Arthur's work.

To turn next to explicit formulas for $J_{\mathfrak{o}}^T(f)$ (and $J_\chi^T(f)$), we need first to recall Arthur's notion of an **unramified** class of datum. For $\chi = \{(M_B, r_B)\}$ this just means that for any pair (M_B, r_B) in χ, the only element of $\Omega(\mathfrak{a}_P, \mathfrak{a}_P)$ fixing r_B is the trivial element. On the other hand, for $\mathfrak{o} = \{(M_B, c_B)\}$, the notion is a little less obvious.

Namely, fix a class $\mathfrak{o}$, and choose a parabolic subgroup P and a semisimple element γ in $\mathfrak{o}$ such that γ belongs to $M(F)$ but not to the Levi subgroup of parabolic properly contained in P. Then let $\sum(\gamma)$ denote the (possibly empty) set of roots α in $\sum = \sum(P, A_P)$ such that the centralizer of γ in $\mathfrak{y}_\alpha$ (the root subspace of the Lie algebra of N_P belonging to α) is not zero, and let A' be the intersection of the kernels of these roots α (regarded as characters of A). *Now assume that every element in* $\mathfrak{o}$ *is semi-simple*, and choose a parabolic subgroup P_1, and an element $w \in \Omega(\mathfrak{a}, \mathfrak{a}_1)$ such that $A_{P_1} = wA'w^{-1}$. Setting γ_1 equal to $w\gamma w^{-1}$, we call $\mathfrak{o}$ **unramified** if the centralizer $G(\gamma_1)$ is contained in M_1.

Note that

$$\gamma = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \gamma = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

determines a semi-simple class in GL(2) (or GL(3)) which is unramified, with P the Borel subgroup, $A' = A_P$, $P_1 = P$, and $\gamma_1 = \gamma$. On the other hand, any class in GL(3) containing a non-semisimple element, like say

$$\gamma = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix},$$

is automatically **ramified**, and its corresponding distribution $J_{\mathfrak{o}}^T(f)$ will not be of the nice type described in the proposition below.

Proposition 3.6. *Suppose $\mathfrak{o}$ is an unramified class in $\mathfrak{O}$, and γ_1 is a (semi-simple) element in $\mathfrak{o}$. Then for sufficiently large T,*

$$J_{\mathfrak{o}}^T(f) = m(G_{\gamma_1}(F)\backslash G_{\gamma_1}^1(\mathbb{A})) \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(x^{-1}\gamma_1 x) v(x,T)\, dx,$$

with $v(x,T)$ (the **weight function***) equal to the volume of the convex hull of the points*

$$\{w^{-1}T - w^{-1}H_{P_0}(wx) \colon w \in \bigcup_{P_2} \Omega(\mathfrak{a}_1, \mathfrak{a}_2)\}$$

projected onto $\mathfrak{a}_1/\mathfrak{a}_G$.

Let us describe $v(x,T)$ graphically for $G = \mathrm{GL}(2)$ or GL(3), with γ_1 a diagonal element (with distinct eigenvalues). We shall also sketch the proof of the proposition for GL(2), making clear how the volume $v(x,T)$ naturally arises.

N.B. We should really write $v_{\mathfrak{o}}(x,T)$ for $v(x,T)$, since this function indeed depends on the nature of $\mathfrak{o}$.

For GL(2) or GL(3), and γ_1 as just specified, $v(x,T)$ is precisely the volume of the convex hull of the **projection** of

$$\{w^{-1}T - w^{-1}H_{P_0}(wx) \colon w \in \Omega(\mathfrak{a}_{P_0}, \mathfrak{a}_{P_0})\}$$

onto $\mathfrak{a}_{P_0}/\mathfrak{a}_G$. For GL(2), this is just the length of the line segment determined by the points $(T_1,T_2) - H(x)$ and $(T_2,T_1) - wH(wx)$ in $\mathfrak{a}_{P_0}$ projected onto the one-dimensional space $\mathfrak{a}_{P_0}/\mathfrak{a}_G$, i.e.,

$$v(x,T) = 2(T_1 - T_2) - (\alpha(H(x)) + \alpha(H(wx))).$$

Thus Proposition 3.5 indeed reduces to Proposition 1.1.

On the other hand, for GL(3), $v(x,T)$ (at least for $T=0$) reduces to the area of the convex set

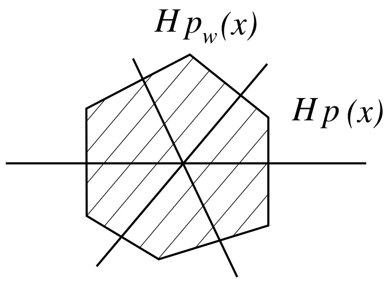

FIGURE 2

whose extreme points are indexed by the Weyl group elements $\{w_1 = e, w_2, \dots, w_6\}$. Note here that

$$w^{-1}H_{P_0}(wx) = H_{P_w}(x),$$

where P_w denotes the (not-necessarily standard) parabolic subgroup $w^{-1}P_0w$, and $H_{P_w}(x)$ is defined analogously to H_{P_0} using the decomposition $G = P_wK$. (Indeed $wx = nm_0k$ implies $w^{-1}H_{P_0}(wx) = w^{-1}H_{P_0}(m_0)$, and $x = w^{-1}nm_0k$ implies $H_{P_w}(x) = H_{P_0}(w^{-1}mw) = w^{-1}H_{P_0}(m_0)$, i.e., $H_{P_w}(x) = w^{-1}H_{P_0}(wx)$.) Thus we could have alternately described $v(x) = v(x,0)$ in terms of the points

$$\{H_P(x) \colon P \in \mathcal{P}(M_0)\},$$

where $\mathcal{P}(M_0)$ denotes the set of parabolics (not necessarily standard) in G whose Levi part *equals* M_0.

Remark 3.7. Weighted orbital integrals of the above type were first systematically studied by Arthur in the context of *real* groups. In [A8], Arthur studied general properties of these real weighted orbital integrals, motivated by a suggestion of Langlands that such integrals would arise in any projected general treatment of the trace formula. In particular, for matrix coefficients of discrete series representations, Arthur related $\int f(x^{-1}\gamma x)v(x)\,dx$ to the characters of the discrete series (whence the title of [A8]). On the other hand, Arthur's weight functions $v(x)$ were also interpreted in [A8] in terms of what would eventually be known as **$(\mathbf{G},\mathbf{M})$ families**. We shall return to this point in a later lecture, where (G,M) families are discussed in earnest.

On the Proof of Proposition 3.6 for GL(2). Precisely because $\mathfrak{o}$ is unramified (in this case, a hyperbolic class with semi-simple element γ_1), the centralizer $G(\gamma_1)$ is contained in M (in this case equals M) and we can express the modified kernel as

$$\begin{aligned}
&k_{\mathfrak{o}}^T(x,T)\\
&\quad=\sum_{\delta\in M(F)\backslash G(F)} f(x^{-1}\delta^{-1}\gamma_1\delta x)\\
&\qquad\qquad-\sum_{\delta\in B(F)\backslash G(F)}\int_{N(\mathbb{A})} f(x^{-1}\delta^{-1}\gamma_1 n\delta x)\,dn\;\hat{\tau}_B(H(\delta x)-T)\\
&\quad=\sum_{B(F)\backslash G(F)}\Bigg\{\sum_{\eta\in N(F)} f(x^{-1}\delta^{-1}\eta^{-1}\gamma_1\eta\delta x)\\
&\qquad\qquad\qquad-\int_{N(\mathbb{A})} f(x^{-1}\delta^{-1}\gamma_1 n\delta x)\,dn\;\hat{\tau}_B(H(\delta x)-T)\Bigg\}.
\end{aligned}$$

Thus

$$\begin{aligned}
&J_{\mathfrak{o}}^T(x,f)\\
&\quad=\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} k_{\mathfrak{o}}^T(x,T)\,dx\\
&\quad=\int_{Z(\mathbb{A})B(F)\backslash G(\mathbb{A})}\Bigg\{\sum_{\eta\in N(F)} f(x^{-1}\eta^{-1}\gamma_1 nx)\\
&\qquad\qquad\qquad-\int_{N(\mathbb{A})} f(x^{-1}n^{-1}\gamma_1 nx)\,dn\;\hat{\tau}_B(H(x)-T)\Bigg\}\,dx
\end{aligned}$$

(here we made the change of variables $n\longrightarrow(\gamma_1^{-1}n^{-1}\gamma_1)n$ in the second integral)

$$=\int_{Z(\mathbb{A})M(F)G(\mathbb{A})} f(x^{-1}\gamma_1 x)(1-\hat{\tau}_B(H(x)-T))\,dx,$$

(here we used the decomposition

$$Z(\mathbb{A})B(F)\backslash G(\mathbb{A})\approx N(\mathbb{A})Z(\mathbb{A})M(F)\backslash G(\mathbb{A})\cdot N(F)\backslash N(\mathbb{A}),$$

carrying out first the integration over $N(F)\backslash N(\mathbb{A})$). But now observe the following: because we know $k_{\mathfrak{o}}^T(x,T)$ is absolutely integrable, we also know that the last integral

$$A=\int_{Z(\mathbb{A})M(F)\backslash G(\mathbb{A})} f(x^{-1}\gamma_1 x)(1-\hat{\tau}_B(H(x)-T))\,dx$$

converges absolutely, and *equals* (from the change of variables $x\to wx$)

$$B=\int f(x^{-1}\gamma_1 x)(1-\hat{\tau}_B(H(wx)-T))\,dx,$$

i.e.,

$$\begin{aligned}
&J_{\mathfrak{o}}^T(x,f)\\
&\quad = A = \tfrac{1}{2}(A+B)\\
&\quad = \tfrac{1}{2}\int_{Z(\mathbb{A})M(F)\backslash G(\mathbb{A})} f(x^{-1}\gamma_1 x)(1-\hat{\tau}_B(H(x)-T)-\hat{\tau}_B(H(wx)-T))\,dx\\
&\quad = \int_{M(\mathbb{A})\backslash G(\mathbb{A})} f(x^{-1}\gamma_1 x)v^*(x,T)\,dx,
\end{aligned}$$

with

$$v^*(x,T) = \int_{Z(\mathbb{A})M(F)\backslash G(\mathbb{A})} (1-\hat{\tau}_B(H(mx)-T)) - \hat{\tau}_B(H(wmx)-T))\,dm,$$

and it remains only to prove:

Claim. *Set* $v_T^*(x) = 1-\hat{\tau}_B(H(x)-T)-\hat{\tau}_B(H(wx)-T)$. *Then*

$$v^*(x,T) = \int_{Z(\mathbb{A})M(F)\backslash M(\mathbb{A})} v_T^*(mx)\,dm = \textit{ Arthur's weight factor } v(x,T).$$

Proof. Note that $v_T^*(x)$ is identically zero unless

(a) Both $\hat{\tau}_B(H(x)-T)$ and $\hat{\tau}_B(H(wx)-T)$ are 1 (in which case $v_T^*(x)=-1$); or

(b) Both are zero (in which case $v_T^*(x) = 1$). But the first possibility for $v_T^*(mx)\neq 0$ implies that

$$\alpha(H(mx)) > \alpha(T) \quad \underline{\text{and}} \quad \alpha(H(wmx)) > \alpha(T),$$

which (for $\alpha(T)$ sufficiently large ...) implies (by Lemma 4.1 of Lecture II) that $w\in B(F)$, an obvious contradiction. On the other hand, it is straightforward to check that the second possibility for $v_T^*(mx)\neq 0$ implies

$$\alpha(H(mx)) - (T_1-T_2) < \alpha(H(m)) < (T_1-T_2) - \alpha(H(x)).$$

Thus the decomposition

$$Z(\mathbb{A})M(F)\backslash M(\mathbb{A}) = \left\{\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix} : a\in F^x\backslash\mathbb{A}^1\right\}\cdot\left\{\begin{pmatrix} e^t & 0\\ 0 & 1\end{pmatrix}\right\}$$

$$\begin{aligned}
\int v_T^*(mx)\,dx &= \int_{F^x\backslash\mathbb{A}^1} d^x a \int_{\alpha(H(wx))-(T_1-T_2)}^{(T_1-T_2)-\alpha(H(x))} dt\\
&= m(F^x\backslash\mathbb{A}^1)v(x,T).
\end{aligned}$$

∎

Remark 3.8. The analogue of Proposition 3.6 for a **spectral** unramified distribution $J_\chi^T(f)$ is also a "straightforward" generalization of the GL(2) situation (Proposition 2.1), namely, we have:

Proposition 3.9. *For $\chi = \{(M, r)\}$ unramified,*

$$J_\chi^T(f) = \sum_{P \in \mathcal{P}_\chi} \frac{1}{n(A_p)} \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \operatorname{tr}(M_P^T(\sigma)_\chi \rho(\sigma, \lambda)(f))\, d\lambda,$$

with $M_P^T(\sigma)_\chi$ the operator on $\operatorname{Ind}_P(\sigma, \lambda)$ defined by

$$c_P \lim_{\zeta \to 0} \sum_{P_2 \in \mathcal{P}_\chi} \sum_{w \in \Omega(\mathfrak{a}_{P_1}, \mathfrak{a}_{P_2})} \frac{e^{w\zeta(T)} M(w, 0, \sigma)^{-1} M(w, \zeta, \sigma)}{\prod_{\alpha \in \Delta_{P_2}} (w\zeta)(\alpha^\vee)}$$

and $M(w, \zeta, \sigma)$ the operator intertwining $\operatorname{Ind}_{P_1}^G \sigma| \quad |^{\lambda+\zeta}$ with $\operatorname{Ind}_{P_2}^G \sigma^w| \quad |^{w(\lambda+\zeta)}$.

As suggested in §2, the crucial point of the proof is Langlands' formula for the inner product of truncated Eisenstein series, and since this is valid for general G (for Eisenstein series E_P with $P \in \mathcal{P}_\chi$), the proof of Proposition 2.1 generalizes directly. The interesting point here is that the integrals appearing in the description of $J_\chi^T(f)$ above really should be viewed as **weighted** character sums. In fact, using the notion of (G, M) families, Arthur eventually views both $M_P^T(\sigma)$ and the weight functions $v(x)$ as special examples of functions constructed out of similar (G, M) families; we shall return to this point in Lecture VII.

(New) References

[A8] Arthur, J., *The characters of discrete series as orbital integrals*, Invent. Math. **32** (1976), 205–261.

[A9] ———, *On the inner product of truncated Eisenstein series*, Duke. Math. J. **49** (1982), no. 1, 35–70.

Lecture V. Simple Forms of the Trace Formula

In this lecture and the next, we explain (at least for GL(2)) how the trace formula can be molded into a simpler form, and then used to obtain dramatic results in representation theory, number theory, and the theory of automorphic forms. In addition to providing a welcome respite from the general theory, this detour will provide impetus and direction for the further discussion of Arthur's development of a general trace formula.

Roughly speaking, the trace formula takes on a simpler form when more restrictive hypotheses are put on the test functions f. The *more* restrictive the hypotheses, the easier it is to establish the corresponding "simple trace formula"; but the *less* restrictive the hypotheses, the more effective it is to apply the resulting trace formula to the theory of automorphic forms.

For example, the "very simple" trace formula of Deligne-Kazhdan (described in §3 below) has nice applications to local representation theory, but cannot give complete results on the functorial lifting of automorphic forms, and cannot give applications to the computation of Tamagawa numbers. For such applications one seems to need the "simple trace formula of Arthur", described in §2 below (in the context of GL(2)).

We start by showing (in §1) that each term on the right side of the formula

$$\operatorname{tr}(R_0(f)) = \sum_{\mathfrak{o}} J_{\mathfrak{o}}^T(f) - \sum_{\chi \notin \mathfrak{X}(G)} J_\chi^T(f)$$

may be expressed as a finite sum of products of *local* distributions on each G_v. Then we show (in §2) that for specially chosen $f = \prod_v f_v$, sufficiently many of these local distributions vanish to ensure that the resulting trace formula reduces to the simple formula

$$\operatorname{tr} R_d(f) = m(Z(\mathbb{A})G(F)\backslash G(\mathbb{A}))f(1) + \sum_{\substack{\mathfrak{o} \\ \text{elliptic}}} J_{\mathfrak{o}}(f)$$

(as in the case of compact quotient).

Henceforth, we assume

$$f = \prod_v f_v$$

is such that for all v, $f_v \in C_c^\infty(Z_v \backslash G_v)$, and for almost all v, f_v is the characteristic function of $Z_v K_v$.

1. Factorization into Local Distributions. We start with the geometric terms $J_{\mathfrak{o}}(f)$. Fix $f = \Pi f_v$, and let S_f denote the finite set of places of F outside of which f_v is the characteristic function of $Z_v K_v$.

Proposition 1.1. *Suppose γ is hyperbolic and belongs to $\mathfrak{o}$. Then $J_{\mathfrak{o}}(f)$ (the constant term of the polynomial $J^T_{\mathfrak{o}}(f)$) is expressible as the sum (over v in S_f) of the products*

$$c\Big(\prod_{w\neq v}\int_{M_w\backslash G_w} f_w(g^{-1}\gamma g)\,dg\Big)\int_{M_v\backslash G_v} f_v(g^{-1}\gamma g)v_v(g)\,dg.$$

Here $v_v(g) = \alpha(H_v(g) + H_v(wg))$ is the local weight function.

Proof. It is easy to check that for g in $G(\mathbb{A})$,

$$v(g) = \sum_v v_v(g_v),$$

with $v_v(g_v) \equiv 0$ on $Z_v K_v$. Thus, Proposition 1.1 of Lecture IV immediately implies

$$\begin{aligned}
J_{\mathfrak{o}}(f) &= c\sum_v \int_{M(\mathbb{A})\backslash G(\mathbb{A})} f(g^{-1}\gamma g)v_v(g)\,dg\\
&= \sum_v c\Big(\prod_{w\neq v}\int_{M_w\backslash G_w} f_w(g^{-1}\gamma g)\,dg\Big)\int_{M_v\backslash G_v} f_v(g^{-1}\gamma g)v_v(g)\,dg.
\end{aligned}$$

But if $v \notin S_f$, the second integrand is identically zero. Thus the proposition is clear. $\square$

The factorization of the unipotent term is a bit more complicated. Recall from Proposition 1.2 of Lecture IV that the constant term of the (non-trivial part of the) unipotent term $J^T_{\mathfrak{o}}(f)$ is computed by subtracting off from

$$\zeta(F,s) = \int_{Z^x}\int_K f(k^{-1}\left(\begin{smallmatrix}1 & a\\ 0 & 1\end{smallmatrix}\right)k)\,dk|a|^s\,d^x a$$

its principal part at $s = 1$, and then setting $s = 1$.

So first rewrite this last integral as $L(s, 1_F)\theta(s)$, where

$$\theta(s) = \frac{1}{L(s,1_F)}\iint f(k^{-1}\left(\begin{smallmatrix}1 & a\\ 0 & 1\end{smallmatrix}\right)k)\,dk|a|^s\,d^x a.$$

By Tate's theory, $\theta(s)$ is holomorphic at $s = 1$, whereas

$$L(s,1_F) = \frac{\lambda_{-1}}{s-1} + \lambda_0 + \cdots .$$

Therefore, the constant term $J_{\mathfrak{o}}(f)$ we need to compute is $\lambda_{-1}\theta'(1) + \lambda_0\theta(1)$, and arguing as in [GJ, pp. 242–243], we get:

Proposition 1.2. *With S_f as before, and $\mathfrak{o}$ unipotent,*

$$J_{\mathfrak{o}}(f) = \lambda_0 \prod_v \Big(\frac{1}{L(1,1_v)} \int_{Z_v N_v \backslash G_v} f_v\Big(g^{-1} \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) g\Big)\, dg \Big)$$
$$+ \lambda_{-1} \sum_{u \in S_f} \prod_{v \neq u} \Big(\frac{1}{L(1,1_v)} \int_{Z_v N_v \backslash G_v} \Big(g^{-1} \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) g\Big)\, dg_v \Big) \times \frac{d}{ds}\Big|_{s=1}$$
$$\frac{1}{L(s,1_u)} \iint f_u\Big(k^{-1} \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) k\Big)\, dk\ |a|^s\, d^x a$$
$$+ m(Z(\mathbb{A})G(F) \backslash G(\mathbb{A}) f(1).$$

Remarks. (1) Each of the local unipotent orbital integrals

$$\int_{Z_v N_v \backslash G_v} f_v(g^{-1} \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) g)\, dg = \zeta(F_v, 1)$$

converges, and for almost every v equals $(1 - N_v^{-1})^{-1}$. It is the appearance of the "convergence factors" $L(1, 1_v)^{-1}$ which makes possible the convergence of the **product** of these local integrals in Proposition 1.2. (The **divergence** of the global unipotent orbital integral

$$\int f\Big(g^{-1} \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) g\Big)\, dg = \prod_v \int f_v\Big(g^{-1} \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) g\Big)\, dg$$

was of course the reason for introducing the modified $J_{\mathfrak{o}}^T(f)$ in the first place ...).

(2) If $u \notin S_f$, then

$$\frac{1}{L(s,1_u)} \iint f_u\Big(k^{-1} \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) k\Big)\, dk |a|^s\, d^x a = \frac{1}{L(s,1_u)} \cdot \zeta(1_{O_v}, s) = 1.$$

and its derivative is zero; that's why we need only sum over $u \notin S_F$ in Proposition 1.2.

We turn now to the spectral contributions. Here we need to recall some crucial facts about *normalized* intertwining operators.

Recall the operator

$$(M(s)\phi)(g) = \int_{N(\mathbb{A})} \phi(wng)\, dn \qquad (\mathrm{Re}(s) > \tfrac{1}{2})$$

intertwining $\rho(\mu, s)$with $\rho(\mu^w, -s)$. This is the operator—*or rather its analytic continuation to* $i\mathbb{R}$—which appears in our description of the spectral "constant term" $J_\chi(f)(\chi = (M, \mu))$. The problem is that in analytically continuing $M(s)$ we lose its Euler product factorization (just as we lose the Euler product for $\zeta(s)$ in analytically continuing it to the left of $\mathrm{Re}(s) = 1$). It is to restore this factorization that one needs the *normalized* intertwining operators $R(s)$ recalled below.

For a fixed additive character $\psi = \prod_v \psi_v$ of $F\backslash\mathbb{A}$, and v any place of F, set

$$m_v(s,\mu_v,\psi_v) = \frac{L(s,\mu_v^2)}{L(s+1,\mu_v^2)\varepsilon(s,\mu_v^2,\psi_v)}.$$

Then define local **normalized** intertwining operators

$$R(s,\mu_v) : \rho(s,\mu_v) \longrightarrow \rho(-s,\mu_v^w)$$

by the equality

$$M(s,\mu_v) = m(s,\mu_v,\psi_v)R(s,\mu_v)$$

(where $M(s,\mu_v)\phi$ is just defined by the local integral $\int_{N_v}\phi(wng)\,dn$, convergent for $\mathrm{Re}(s) \gg 0$). The crucial property of this normalized operator $R(s,\mu_v)$ is that it takes the (normalized) K_v-fixed vector ϕ_v^0 in $\rho(s,\mu_v)$ (for μ_v unramified) to the unique normalized K_v-fixed vector in $\rho(-s,\mu_v^{-1})$; moreover, as an operator valued function of s, $R(s,\mu_v)$ is **holomorphic** in $\mathrm{Re}(s) > -\frac{1}{2}$. Thus we can define, for $\phi = \prod \phi_v$ in $\rho(s,\mu)$,

$$R(s,\mu)\phi = \prod_v R(s_v,\mu_v)\phi_v.$$

Since $R(s,\mu_v)(\phi_v) = \phi_v^0$ almost everywhere, this product makes perfect sense; moreover, $R(s,\mu)$ is holomorphic in $\mathrm{Re}(s) > -\frac{1}{2}$. But clearly (at least for $\mathrm{Re}(s) \gg 0$),

$$M(s,\mu) = m(s,\mu)R(s,\mu),$$

where

$$m(s,\mu) = \prod_v m_v(s,\mu_v,\psi)v).$$

So since $m(s,\mu)$ is meromorphic, we indeed obtain the analytic continuation of $M(s,\mu)$ to $i\mathbb{R}$ with its "Euler product" factorization intact.

N.B. By the (global) functional equation of Hecke L-functions,

$$m(s,\mu) = \frac{L(s,\mu^2)}{L(s+1,\mu^2)\varepsilon(s,\mu^2,\psi)} = \frac{L(1-s,\mu^{-2})}{L(s+1,\mu^2)}$$

Proposition 1.3. *For χ "unramified", the (constant term of the) spectral contribution $J_\chi(f)$ equals*

$$\int_{-\infty}^{\infty} \frac{m'(it,\mu)}{m(it,\mu)} \operatorname{tr}(\rho(it,\mu)(f))\,dt$$
$$+ \sum_{u\in S_f} \int_{-\infty}^{\infty} (\prod_{v\neq u} \operatorname{tr}(\rho(it,\mu_v)(f_v)) \operatorname{tr}(R_u(it,\mu_u)^{-1}R'_u(it,\mu_u)\rho(\mu,it)(f_u))\,dt.$$

In case χ is "ramified" ($\mu^2 \equiv 1$), there are the additional terms

$$\operatorname{tr}(M(0)\rho(\mu,0)(f)) \tag{1.4}$$

and

$$\mu(f)m(Z(\mathbb{A})G(F)\backslash G(\mathbb{A})).$$

Proof. Straightforward (see [GJ, p. 243]). □

2. Simplifications of the Formula. A first kind of simplification comes form making the relatively mild assumption on

$$f = \prod f_v$$

that its local hyperbolic orbital integrals

$$(*) \qquad \int_{M_v \backslash G_v} f\left(g^{-1} \left(\begin{smallmatrix} a_v & 0 \\ 0 & 1 \end{smallmatrix}\right) g\right) d^* g \equiv 0$$

for at least two place $v = v_1, v_2$.

Proposition 2.1. *With f satisfying the above assumption (*), we have*

$$\operatorname{tr} R_0(f) = m(Z(\mathbb{A})\backslash G(\mathbb{A}))f(1) + \sum_{\mathfrak{o} \text{ elliptic}} J_{\mathfrak{o}}(f) - \sum_{\mu^2=1} \mu(f) m(Z(\mathbb{A})G(F)\backslash G(\mathbb{A}))$$

Proof. First consider the spectral contributions $J_\chi(f)$, with $\chi = \{(M,\mu)\}$. If v_1 is a place where (*) holds, then $\operatorname{tr} \rho(it, \mu_{v_1})(f_v) = 0$ (by a well known computation of the trace of induced representations). Thus the first term appearing in Proposition 1.3 immediately disappears. But if (*) also holds at a *second* place v_2, then it is clear that the second term there must also always vanish. On the other hand, if $\mu^2 = 1$, then $M(0,\mu)$ intertwines the irreducible representation $\rho(0,\mu)$ with itself; hence $M(0,\mu)$ must be a scalar operator. In particular, $\operatorname{tr}(M(0)\rho(0,\mu)(f)) = \lambda \operatorname{tr}(\rho(0,\mu)(f)) = 0$ by our assumption on f, and we conclude from Proposition 1.3 that the full spectral contribution $\sum_\chi = (M,\mu) J_\chi(f)$ reduces to $(\sum_{\mu^2=1} \mu(f)) m((Z(\mathbb{A})G(F)\backslash G(\mathbb{A}))$.

Now we consider the possible geometric contributions $J_{\mathfrak{o}}(f)$, following Propositions 1.1 and 1.2. For the hyperbolic contributions, it is again obvious that if assumption (*) holds at *two* places, then each $J_{\mathfrak{o}}(f)$ is zero (just as for the unramified spectral forms). On the other hand, for the unipotent contribution, it suffices to make the following observation:

For any local f_v in $C_c^\infty(Z_v\backslash G_v)$,

$$\int_{Z_v N_v \backslash G_v} f_v\left(g^{-1} \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) g\right) dg = \lim_{a \to 1} |1 - a^{-1}| \int_{M_v \backslash G_v} f_v\left(g^{-1} \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) g\right) dg.$$

This is a simple yet important identity, which we encourage the reader to verify (using Iwasawa's decomposition and an appropriate change of variables in N). It implies that the local unipotent orbital integral of f_v vanishes as soon as its hyperbolic orbital integrals vanish. In particular, as soon as (*) holds for two places, it follows that all of the unipotent term $J_{\mathfrak{o}}(f)$ vanishes, except for the term involving $f(1)$. $\square$

In the next lecture, we shall explain how this simplification of the trace formula is used to prove the Jacquet–Langlands correspondence between automorphic representations between a quaternion algebra and GL(2), and to show that the corresponding Tamagawa numbers are equal.

3. The "Simple" Trace Formula. In the early 1980's, Deligne and Kazhdan introduced a remarkably simple trick into the trace formula repertoire, the so-called "simple trace formula" (see [BDKV]). As we shall see in the next lecture, it yields powerful *local* results on functorial lifting, with surprisingly little work.

Theorem 3.1 (The Simple Trace Formula of Deligne-Kazhdan). *Suppose* $f = \prod_v f_v$ *in* $C_c^\infty(Z(\mathbb{A})\backslash G(\mathbb{A}))$ *satisfies the following two properties:*

(i) *At one place* $v = v_1$, f_{v_1} *is the matrix coefficient of a supercuspidal representation of* G_{v_1}*; and*

(ii) *at a second place* $v = v_2$, f_{v_2} *is supported on the set of regular elliptic elements of* G_{v_2}.

Then

(a) $R(f)$ *has its image in* $L_0^2(Z(F)G(F)\backslash G(\mathbb{A}))$, *hence is of trace class, with*

$$\operatorname{tr} R(f) = \sum_{\pi \text{ cuspidal}} \operatorname{tr} \pi(f); \quad \textit{and}$$

(b) *we also have*

$$\operatorname{tr} R(f) = \sum_{\substack{\{\gamma\} \\ \text{regular} \\ \text{elliptic}}} m(Z(\mathbb{A})G_\gamma(F)\backslash G_\gamma(\mathbb{A}) \int_{G_\gamma\backslash G(\mathbb{A})} f(x^{-1}\gamma x)\, dx$$

Remarks. (1) The condition we really need in (i) is that f_{v_1} is a **supercusp form** in the sense of Harish-Chandra, i.e., (like the matrix coefficient of any supercuspidal representation),

$$\int_{N_{v_1}} f_{v_1}(gnh)\, dn = 0 \qquad \text{for all } g,\ h, \text{ in } G_v.$$

(2) Both the theorem, and its proof below, hold more generally for any G; see [Ro1], §1, for example.

(3) This simple trace formula is indeed less subtle than the simple form of Arthur's trace formula discussed in the last section. In particular, note that the distribution $f \to m(Z(F)G(F)\backslash G(\mathbb{A}))f(1)$ disappears entirely from this simpler form of the trace formula (and hence, for example, no application to Tamagawa numbers is possible ...). It is just this crudeness, however, that makes the formula much easier to prove!

Proof. For (a), it suffices to show that $R(f)$ has image in $L_0^2(Z(F)G(F)\backslash G(\mathbb{A})$ for then the theorem of Gelfand and Piatetski-Shapiro (see [GPS] and [Go2]) implies $R(f)$ is of trace class). To check that $R(f)\phi$ is a cusp form for any ϕ we simply compute:

$$\begin{aligned}\int_{N(F)\backslash N(\mathbb{A})} R(f)\phi(nx)\, dx &= \int_{N(F)\backslash N(\mathbb{A})} \Big(\int_{Z(\mathbb{A})\backslash G(\mathbb{A})} \sum_{v\in N(F)} f(x^{-1}nvg)\phi(g)\, dg\Big)\, dn \\ &= \int_{Z(\mathbb{A})N(F)\backslash G(\mathbb{A})} \Big(\int_{N(\mathbb{A})} f(x^{-1}ng)\, dn\Big)\phi(g)\, dg = 0\end{aligned}$$

since

$$\int_{N(\mathbb{A})} f(x^{-1}ng)\,dn = \prod_v \int_{N_v} f_v(x_v^{-1}ng_v)\,dn = 0$$

by our assumption (i) on f_{v_1}.

To prove (b), note that a γ in $G(F)$ is regular elliptic as soon as it is regular elliptic in $G_v = G(F_v)$ for some place v. But $f(x^{-1}\gamma x) \neq 0$ implies $\{\mathrm{supp}(f)\} \cap \{\text{Orbit of } \gamma\} \neq 0$, which in turn implies $\{\mathrm{supp}(f_{v_2})\} \cap \{\text{Orbit of } \gamma\} \neq 0$. So by hypothesis (ii), γ is regular elliptic in G_{v_2}, i.e.,

$$K_f(x,x) = \sum_{\substack{\{\gamma\} \\ \text{regular} \\ \text{elliptic}}} f(x^{-1}\gamma x), \tag{3.2}$$

and conclusion (b) follows. $\square$

Concluding Remark. Implicit in the proof above is the fact that $K_f(x,x)$ is an **absolutely integrable** smooth function on $Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$, and hence $\mathrm{tr}(R(f))$ $(= \mathrm{tr}(R_0(f)))$ is given by its integral. In fact (3.2) implies $K_f(x,x)$ is **compactly supported** on $Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$ (as we saw in Lecture II). Interestingly enough, if we drop the hypothesis (ii), so that (3.2) need no longer hold, it still follows (from (i) above) that $K_f(x,x)$ is absolutely integrable (in fact, rapidly decreasing), and that $\mathrm{tr}\,R(f) = \int K_f(x,x)\,dx$; for a proof, see [Ro1, §1].

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Lecture VI. Applications of the Trace Formula

Let $G = \mathrm{GL}(2)$, and let G' be the multiplicative group of a division quaternion algebra D over F.

In this lecture, we shall explain how the simple forms of the trace formula are used to prove the following three results:

(1) The Tamagawa number of G' equals the Tamagawa number of G;
(2) There is a bijection (the "Jacquet–Langlands correspondence") between the automorphic π' on G' (which are not one-dimensional) and the automorphic cuspidal π of G (such that π_v is square integrable for each place v of F ramified in D); and
(3) Given any finite set S of finite places v of F, and square-integrable π_v on G_v for $v \in S$, there exists a cuspidal π on G such that $(\pi)_v \cong \pi_v$ for all $v \in S$.

1. Tamagawa Numbers. At the end of the 1950's, Weil conjectured that the Tamagawa number of any simply connected semi-simple group G equals *one*. At the end of the 1960's Jacquet–Langlands proposed a two step program for proving this:

(1) Prove first that
$$\tau(G_0) = 1$$
for G_0 the **quasi-split** inner form of G, using the Eisenstein series method introduced by Langlands for split groups (see [La3] and [Lai]); then
(2) Use the trace formula to prove that
$$\tau(G) = \tau(G_0).$$
This second step was first carried out in §16 of [JL] (for GL_2 in place of SL_2), and then recently generalized to arbitrary G by Kottwitz to prove Weil's conjecture in general (see [Kot]).

We should stress that—in deriving a simple trace formula for an arbitrary semi-simple quasi-split G—Kottwitz had to appeal not only to all of Arthur's work on the trace formula through 1988, but also to his own earlier works on the **stable** form of the trace formula. By specializing Kottwitz's argument below to the case of $\mathrm{GL}(2)$ (in place of $\mathrm{SL}(2)$), we manage to avoid both these bodies of work.

We start by recalling the notion of Tamagawa measures for our G and G'.

Fix a non-trivial character $\psi = \prod \psi_v$ of $F\backslash\mathbb{A}$. For each place v of F, ψ_v determines a Fourier transform on $A_v = M_2(F_v)$ or D_v, hence also a self-dual Haar measure dy on A_v. The measure dy then determines a Haar measure $dy/\|y\|$ on A_v^x, where $\|y\|$ is the module homomorphism from A_v to $\mathbb{R}_+^x$. Hence the choice of ψ simultaneously determines Haar measures on each G_v and G'_v; similarly, Haar measures are determined on $F_v = Z_v \simeq Z'_v$. Such measures are called Tamagawa measures, locally and globally. The resulting measure of $Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$ is independent of ψ and called the Tamagawa number $\tau(G)$ of G, i.e.,

$$\tau(G) = m(Z(\mathbb{A})G(F)\backslash G(\mathbb{A})).$$

Similarly,

$$\tau(G') = m(Z'(\mathbb{A})G'(F)\backslash G'(\mathbb{A})).$$

The center Z' may of course be identified with the center $Z \simeq F^x$ of GL(2).

Theorem 1.1. *The Tamagawa numbers of G and G' are equal.*

We start the proof with a definition.

Suppose v in F is **ramified** in D, i.e., $D_v = D \otimes F_v$ is a division quaternion algebra over F_v. Then each quadratic extension L_v of F_v can be regarded simultaneously as a Cartan subgroup $T'_v = T'_{L_v}$ of G'_v *and* as an (elliptic) Cartan subgroup $T_v = T_{L_v}$ of $G_v = \mathrm{GL}_2(F_v)$.

Definition 1.2. We say a function f_v in $C_c^\infty(Z_v\backslash G_v)$ **matches** f'_v in $C_c^\infty(Z'_v\backslash G'_v)$ (and write $f_v \sim f'_v$) if:

(i) $f_v(1) = f'_v(1)$;

(ii) the **regular hyperbolic** orbital integrals

$$\int_{M_v\backslash G_v} f_v(g^{-1}\left(\begin{smallmatrix} a & 0 \\ 0 & 1\end{smallmatrix}\right) g)\, dg$$

vanish identically; and

(iii) for corresponding tori T_v and T'_v and $t \sim t'$,

$$\int_{T_v\backslash G_v} f_v(g^{-1}tg)\, dg = \int_{T'_v\backslash G'_v} f'_v(g^{-1}t'g)\, d'g.$$

N.B. The Haar measures on G_v and G'_v are (simultaneously) normalized by way of the Tamagawa measures recalled above; the measures on T_v and T'_v are fixed by a choice of Haar measure on L_v.

Fact 1 (Local Harmonic Analysis on G)**.** Given any f'_v in $C_c^\infty(Z'_v\backslash G'_v)$, there exist (infinitely many ...) f_v in $C_c^\infty(Z_v\backslash G_v)$ which match f'_v in the above sense.

This crucial fact can be proved using a characterization of the orbital integrals on GL_2 à la “Shalika germs”; see §6 of [La2]

Using Fact 1, we can relate Tamagawa numbers to traces, and then prove Theorem 1.1.

Proposition 1.3. *Let S_D denote the set of v in F ramified in D. Given $f' = \prod_v f'_v$ in $C_c^\infty(Z'(\mathbb{A})\backslash G'(\mathbb{A}))$, suppose $f = \prod f_v$ in $C_c^\infty(Z(\mathbb{A})\backslash G(\mathbb{A}))$ is such that for all v in S_D, $f_v \sim f'_v$ (in the sense of Definition 1.2), and for all $v \notin S_D$, $f_v = f'_v$ (via the natural isomorphism of G_v and G'_v). (In this case, we say f "globally matches" f', and write $f \sim f'$). Then*

$$\mathrm{tr}\, R_0(f) - \mathrm{tr}\, R'_0(f') = \{\tau(g) - \tau(G')\}\Big(f(1) + \sum_{\mu^2=1} \mu(f)\Big), \tag{1.4}$$

with R'_0 the representation of $G'(\mathbb{A})$ in the subspace of $L^2(Z(\mathbb{A})G'(F)\backslash G'(\mathbb{A}))$ orthogonal to all one-dimensional invariant subspaces.

Assuming this proposition (for the moment), we give the:

Proof of Theorem 1.1. If we rewrite (1.4) as

$$\mathrm{tr}\, R_0(f) - \mathrm{tr}\, R'_0(f') - \{\tau(G) - \tau(G')\} \sum_{\mu^2=1} \mu(f) = \{\tau(G) - \tau(G')\}(f(1), \tag{1.5}$$

then it clearly suffices to prove that both sides of this equation—valid for all $f \sim f'$ as above—equal zero. For this, we follow an argument first explicitly introduced by Langlands in [La2].

Fix a place v_0 outside S_D, and consider the Hecke algebra $\mathcal{H}_{v_0}(G_{v_0}, K_{v_0})$. If we fix all components of

$$f' = \prod_v f'_v$$

except the v_0-component, which we let vary through $\mathcal{H}_{v_0}$, then (1.5) reads

$$\sum_j c_j\, \mathrm{tr}(\pi_j)_{v_0}(f_{v_0}) - \sum_{\mu^2=1} c_\mu \mu_{v_0}(f_{v_0}) = c_0\{\tau(G) - \tau(G')\} f_{v_0}(1), \tag{1.6}$$

where the π_{jv_0} (and μ_{v_0}) are unramified unitary representations of G_{v_0}. But the distribution $f_{v_0} \to f_{v_0}(1)$ is given in terms of $\mathrm{tr}\, \pi_{v_0}(f_{v_0})$ (for all unramified tempered π_{v_0}) by integration against Plancherel's measure, which is **continuous** in the obvious sense. Similarly, the left-hand side of (1.6) defines a **discrete** measure (on this same unramified dual). Thus it follows from (the uniqueness part of) the Riesz representation theorem for measures that both sides of (1.6) must be zero. $\square$

Corollary (of Proposition 1.3 and the Proof of Theorem 1.1). *For matching f and f' on $G(\mathbb{A})$ and $G'(\mathbb{A})$,*

$$\mathrm{tr}\, R_0(f) = \mathrm{tr}\, R'_0(f'),$$

where R'_0 is the representation of $G'(\mathbb{A})$ in the subspace of $L^2(Z(\mathbb{A})G'(F)\backslash G'(\mathbb{A}))$ orthogonal to all one-dimensional invariant subspaces.

It remains now to complete the:

Proof of Proposition 1.3. Because $|S_D| \geq 2$, our assumption on f (that it matches f') implies that f_v has vanishing (regular) hyperbolic orbital integrals for at least *two* places v_1 and v_2. Thus by Proposition 2.1 of the last lecture (Lecture V)

$$\begin{aligned}(1.7)\quad \operatorname{tr} R_0(f) = {} & m(Z(\mathbb{A})G(F)\backslash G(\mathbb{A}))f(1) \\ & + \sum_{\substack{\gamma \text{ regular} \\ \text{elliptic}}} m(Z(\mathbb{A})G_\gamma(F)\backslash G_\gamma(\mathbb{A}) \int_{G_\gamma(\mathbb{A})\backslash G_\gamma(\mathbb{A})} f(x^{-1}\gamma x)\, dx \\ & - \sum_{\mu^2=1} \mu(f) m(Z(\mathbb{A})G(F)\backslash G(\mathbb{A})).\end{aligned}$$

On the other hand, for G', the trace formula for compact quotient in Lecture I implies

$$\begin{aligned}\operatorname{tr} R_0'(f') = {} & m(Z(\mathbb{A})G'(F)\backslash G(\mathbb{A}))f'(1) \\ & + \sum_{\{\gamma\} in Z'(F)\backslash G'(F)} m(Z(\mathbb{A})G'_\gamma(F)\backslash G'_\gamma(\mathbb{A}) \int_{G'_\gamma(\mathbb{A})\backslash G'(\mathbb{A})} f'(x^{-1}\gamma x)\, dx \\ & - \sum_{\mu^2=1} \mu(f') m(Z(\mathbb{A})G'(F)\backslash G'(\mathbb{A}))\end{aligned}$$

So to prove the proposition, it remains to check that: (i) the regular elliptic orbital integral terms match up on G and G'; and (ii) $\mu(f) = \mu(f')$ for each character μ.

As for (i), let $\{L\}$ run through a set of representatives for the classes of quadratic extensions L of F which *don't* split at any $v \in S_D$. Then each integral

$$m(Z(\mathbb{A})G_\gamma(F)\backslash G(\mathbb{A})) \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(x^{-1}\gamma x)\, dx$$

in (1.7) is of the form

$$(*)\qquad m(\mathbb{A}^x L^x\backslash L^x(\mathbb{A})) \int_{L^x(\mathbb{A})\backslash G(\mathbb{A})} f(g^{-1}\gamma g)\, dg$$

for some $\gamma \neq 1$ in $F^x\backslash L^x$. (Note that if γ belongs to a quadratic extension L which **splits** at some $v \in S_D$, then our hypothesis on f implies

$$\int_{T_{L_v}\backslash G_v} f(g_v^{-1}\gamma g_v)\, dg_v = 0,$$

i.e., the orbital integrals for such γ don't appear in the expression for $\operatorname{tr} R_0(f)$.) On the other hand, each regular orbital integral

$$m(Z(\mathbb{A})G'_\gamma(\mathbb{A})\backslash G'_\gamma(\mathbb{A})) \int_{G'_\gamma(\mathbb{A})\backslash G'(\mathbb{A})} f'(x^{-1}\gamma x)\, dx$$

appearing in the expression for $\operatorname{tr} R_0'(f')$ is also of the form (*), since quadratic L which don't split at $v \in S_D$ are precisely the quadratic extensions of F embeddable in D. Thus the regular elliptic orbital integrals indeed match up term by term.

As for (ii), let us verify that

$$\mu_v(f_v) = \mu_v(f_v')$$

for each place v of F. For $v \notin S_D$ this is a tautology. So let us assume $v \in S_D$. By Weyl's integration formula for $Z_v'\backslash G_v'$ we compute

$$\begin{aligned}\mu_v(f_v') &\cong \int_{Z_v\backslash G_v'} f_v'(g)\mu_v(\det g)\,dg \\ &= \sum_{\{L_v\}} \tfrac{1}{2}\int_{Z_v\backslash T_{L_v}'} \delta(t)\mu(\det t)\Big(\int_{T_{L_v}'\backslash G_v'} f_v'(g^{-1}tg)\,dg\Big)\,dt_v\end{aligned}$$

where $\{L_v\}$ indexes the Cartan subgroup T_{L_v}', and

$$\delta(t) = \left|\ \frac{(a_1-a_2)^2}{a_1a_2}\ \right|$$

if t has eigenvalues a_1 and a_2. But because f_v matches f_v', this last sum equals

$$\sum_{\{L_v\}} \tfrac{1}{2}\int_{Z_v\backslash T_{L_v}} \delta(t)\mu(\det t)\Big(\int_{T_{L_v}\backslash G_v} f_v(g^{-1}tg)\,dg\Big)\,dt_v = \int_{Z_v\backslash G_v} f_v(g)\mu_v(g)\,dg$$

(since the orbital integrals of f_v vanish off the elliptic Cartans). Thus $\mu_v(f_v') = \mu(f_v)$, as required, and the proof of Proposition 1.3 (and hence Theorem 1.1) is complete. $\square$

2. The Jacquet–Langlands Correspondence. Let $\mathcal{A}_0(G)$ denote the collection of irreducible invariant subspaces V_π of $R_0(g)$ in $L_0^2(Z(\mathbb{A})G(F)\backslash G(\mathbb{A}))$. According to the "multiplicity one" result of [JL], proved only using the theory of Whittaker models, each irreducible unitary representation

$$\pi = \otimes \pi_v$$

of $G(\mathbb{A})$ is realizable at most once in L_0^2. Thus the (distinct) spaces V_π in $\mathcal{A}_0(G)$ may be confused with the set of (classes of) irreducible automorphic cuspidal representations of GL(2). Moreover, "strong multiplicity one" (proved using "only" the theory of L-functions) implies that $V_\pi = V_{\pi'}$ as soon as π_v and π_v' are equivalent for almost all v.

Now consider the group $G' = D^x$, and the collection $\mathcal{A}_0(G')$ of irreducibly invariant subspaces of $R_0'(g)$ acting in the subspace of $L^2(Z(\mathbb{A})G'(F)\backslash G(\mathbb{A}))$ orthogonal to all the one-dimensional invariant subspaces. Then the trace formula implies that "multiplicity one" and "strong multiplicity one" hold in $\mathcal{A}_0(G')$ as well (even though Whittaker models disappear in this setting, and the theory of L-functions cannot be developed à la Hecke …). Indeed, this is a corollary of the following Jacquet–Langlands correspondence between G' and G:

Theorem 2.1. *There exists a 1–1 mapping*

$$V_{\pi'} \longrightarrow V_\pi$$

from $\mathcal{A}_0(G')$ to $\mathcal{A}_0(G)$, with the property that $\pi_v \equiv \pi'_v$ for all $v \notin S_D$; the image consists of all those π in $\mathcal{A}_0(G)$ with the property that π_v is square-integrable $(\mathrm{mod}\ Z)$ *for all $v \in S_D$.*

Remark. A weaker form of this correspondence asserts the existence of a map between the **representations** π' and π above; i.e., to each π' on G' realizable in some G' on $\mathcal{A}_0(G)$ there exists a π in $\mathcal{A}_0(G)$ such that $\pi'_v \equiv \pi_v$ for all $v \notin S_D$. As noted in Lecture I, this can be proved using L-functions but the argument does not yield a characterization of the image, nor the multiplicity one results for G'.

The trace formula proof of Theorem 2.1 starts from the basic identity

$$\operatorname{tr} R_0(f) = \operatorname{tr} R'_0(f')$$

valid for matching f and f'; see the Corollary to the proof of Theorem 1.1 in the preceding section. Equivalently, and more suggestively,

$$\sum_{\pi \in \mathcal{A}_0(G)} \operatorname{tr} \pi(f) = \sum_{\pi' \in \mathcal{A}'_0(G)} \operatorname{tr} \pi'(f'). \tag{2.2}$$

To extract the desired bijection between $V_{\pi'}$ and π from the identity (2.2) we need to refine this identity to one of the following type:

Fix any finite set S of places of F containing S_D and the archimedean places, and for each $v \notin S$, fix a given representation π_v^0 which is unramified; then

$$\sum_{V_\pi} \prod_{v \in S} \operatorname{tr} \pi_v(f_v) = \sum_{V_{\pi'}} \prod_{v \in S} \operatorname{tr} \pi'_v(f'_v), \tag{2.3}$$

with $f_v \sim f'_v$, and the sums taken over all V_π in $\mathcal{A}_0(G)$ (resp. $V_{\pi'}$ in $\mathcal{A}_0(G)$) such that π_v (resp. π'_v) is isomorphic to π_v^0 for all $v \notin S$. Note that by strong multiplicity one for $\mathrm{GL}(2)$, the left-hand side of (2.3) contains *at most one* term.

Assuming the truth of (2.3), let us explain the slick fashion in which Theorem 2.1 can be proved. Suppose first that *no* V_π in $\mathcal{A}_0(G)$ corresponds to a given $V_{\pi'}$ in $\mathcal{A}'_0(G')$. Then (2.3) (with S the set of places outside of which D and π'_v are p-adic and unramified, and $\pi_v^0 = \pi'_v$ for $v \notin S$) implies that the left-hand side of (2.3) is zero. Thus we also have

$$\sum \operatorname{tr} \pi'_S(f'_S) = 0, \tag{2.4}$$

where

$$\pi'_S = \underset{v \in S}{\otimes} \pi'_v, \quad f'_S = \prod_{v \in S} f'_v$$

is arbitrary in $C_c^\infty(Z_S(\mathbb{A})\backslash G'_S(\mathbb{A}))$, and the sum is over all elements of $\mathcal{A}'_0(G')$ (like, for example, $V_{\pi'}$) such that $(\pi)_v \cong \pi_v^0$ for $v \notin S$. But (2.4) contradicts a well-known result of local harmonic analysis known as "linear independence of characters" (see Lemma 16.1.1 of [JL] or Lemma 5.11 of [Ro1]).

Similarly, we can begin to characterize the image of this correspondence $V_{\pi'} \to \pi$. Indeed, suppose $\pi = \otimes \pi_v$ is in the image of this correspondence, but π_v is not square-integrable for some v_0 in S_D. Then it follows

$$\operatorname{tr} \pi_{v_0}(f_{v_0}) = 0,$$

since π_{v_0} would then be induced, and $f_{v_o} \sim f'_{v_0}$ implies that f_{v_0} has vanishing hyperbolic integrals. Thus the left-hand side of (2.3) would vanish, leading to the same contradiction as above.

To complete the characterization of the image, we need to recall some facts about the square-integrable representations of G_v and G'_v. In particular, let $\langle\ ,\ \rangle_e$ denote the inner product for the space of class functions on the set of regular elliptic elements of $Z_v \backslash G_v$ or $Z_v \backslash G'_v$ defined by

$$\langle f_1, f_2 \rangle_e = \sum_{T_v} \tfrac{1}{2} \int_{Z_v \backslash T_v^{\mathrm{reg}}} \delta(t) f_1(t) \overline{f_2(t)}\, dt,$$

the sum extending over the conjugacy classes of compact tori of G_v (conveniently confused with those of G'_v). For G'_v, the Peter–Weyl theorem and the Weyl integration formula imply that the characters of the irreducible representations of $Z_v \backslash G'_v$ comprise a **complete orthonormal set** with respect to $\langle\ ,\ \rangle_e$. On the other hand, for G_v the so-called "orthogonality relations for square integrable representations" imply that the characters of the square-integrable irreducible representations of $Z_v \backslash G_v$ comprise at least an **orthonormal** set with respect to $\langle\ ,\ \rangle_e$ Thus, given any square-integrable representation π_v on G_v, $v \in S_D$, we can determine an irreducible π''_v on G'_v by the condition

$$\langle \chi_{\pi_v}, \chi_{\pi''_v} \rangle_e = a_{\pi_v} \neq 0;$$

we also fix f'_v in $C_c^\infty(Z_v \backslash G'_v)$ such that

$$\operatorname{tr} \pi'_v(f'_v) = \begin{cases} 1 & \text{if } \pi'_v \cong \pi''_v; \\ 0 & \text{otherwise.} \end{cases}$$

Namely, we take $f'_v = \overline{\chi}_{\pi''_v}$.

So suppose now that π in $\mathcal{A}_0(G)$ is such that π_v is square-integrable for each $v \in S_D$. Then the trace formula identity (2.3) implies that

$$\prod_{v \in S} a_{\pi_v} = \sum_{V_{\pi'}} 1, \tag{2.5}$$

with the sum taken over $V_{\pi'}$ in $\mathcal{A}'_0(G)$ such that $\pi'_v \simeq \pi''_v$ for v in S, and $\pi'_v \simeq \pi_v$

for $v \notin S$. Indeed, for $v \in S$,

$$\begin{aligned}
\operatorname{tr} \pi_v(f_v) &= \int_{Z_v \backslash G_v} f(g)\chi_{\pi_v}(g)\, dg \\
&= \sum_{\{T_v\}} \tfrac{1}{2} \int_{Z_v \backslash T_v^{\mathrm{reg}}} \delta(t)\chi_{\pi_v}(t)\Phi_t(f)\, dt \\
&= \sum \tfrac{1}{2} \int \delta(t)\chi_{\pi_v}(t)\Phi_t(f'_v)\, dt \\
&= \langle \chi_{\pi_v}, \chi_{\pi''_v} \rangle_e = a_{\pi_v},
\end{aligned}$$

with

$$\Phi_t(f'_v) = \int_{T_v \backslash G_v} f_v(g^{-1}tg)\, dg = \int_{T_v \backslash G'_v} f'_v(g^{-1}tg)\, dg = \Phi_t(f'_v).$$

Claim. *The sum on the right side of (2.5) has exactly one term in it.*

Indeed, if there were none, the left-hand side of (2.5) would be zero (which is impossible by our assumption in a_{π_v}, whereas as if there were two or more terms, we would have

$$\prod |a_{\pi_v}| \geq 2$$

(again impossible, since each $|a_{\pi_v}| = |\langle \chi_{\pi_v}, \chi_{\pi''_v} \rangle_e| \leq \|\chi_{\pi_v}\|_e \|\chi_{\pi''_v}\|_e \leq 1$). Thus we conclude there must be *exactly* one $V_{\pi'}$ in $\mathcal{A}'_0(G')$ corresponding to π in $\mathcal{A}_0(G)$, and the theorem is finally proved.

2. Embedding of Local Discrete Series in Cusp Forms.

Theorem 2.1. *Let V be any finite number of finite places of F, and π_v a discrete series representation of $Z_v \backslash G_v$ for each v in V. Then there exists a cuspidal automorphic representation π of $G(\mathbb{A})$ such that $\pi_v = (\pi)_v$ for each v in V.*

Proof. We can assume that for at least one place $v_1 \in V$, π_{v_1} is actually supercuspidal (by adding such a pair (v_1, π_{v_1}) into the collection $\{v, \pi_v\}$ if necessary). Before applying the simple trace formula of Deligne-Kazhdan to this situation, we need to recall two important facts about **local harmonic analysis** on G_v.

For G_v non-archimedean, a function f_v in $C_c^\infty(Z_v \backslash G_v)$ is called a **pseudo-matrix coefficient** for the discrete series representation π_v of $Z_v \backslash G_v$ if $\operatorname{tr} \pi_v(f_v) = 1$, but $\operatorname{tr} \tau_v(f_v) = 0$ for any tempered irreducible admissible representation τ_v not equivalent to π_v. That *such pseudo-matrix coefficients exist* in the generality of a general reductive G is a theorem of Bernstein, Deligne and Kazhdan (see [BDK], [Clo] [BDKV]). Moreover, one knows that for such functions f_v and γ (regular) elliptic in G_v,

$$\Phi_\gamma(f_v) = \int_{T_v \backslash G_v} f_v(x^{-1}\gamma x)\, dx = c\overline{\chi_{\pi_v}}(\gamma)$$

for a non-zero constant c. In particular, the orbital integral $\Phi_\gamma(f_v)$ is not identically zero (since $\chi_{\pi_v} \not\equiv 0$ on the regular elliptic elements; see [Clo]).

Remark. Suppose π_v is supercuspidal, and we denote one of its matrix coefficients suitably normalized by $f_v(g)$. Then f_v belongs to $C_c^\infty(Z_v\backslash G_v)$,

$$\begin{aligned} \operatorname{tr}\pi_v(f_v) &= 1, \qquad \text{and} \\ \operatorname{tr}\tau_v(f_v) &= 0 \end{aligned}$$

for any irreducible admissible τ_v not equivalent to π_v. Thus the terminology pseudo-matrix coefficient is indeed apt.

Now given V and $\{\pi_v\}$ as in the hypothesis of Theorem 2.1, fix

$$f = \prod_v f_v \quad \text{in } C_c^\infty(Z(\mathbb{A})\backslash G(\mathbb{A}))$$

such that (a) for each $v \in V$, f_v is a pseudo-matrix coefficient; and (b) for a fixed finite place v_2 outside V, f_{v_2} is supported on the regular elliptic set of G_{v_2}. Then by the simple trace formula of Deligne-Kazhdan, for such f,

$$\sum_\pi \operatorname{tr}\pi(f) = \sum_\gamma m_\gamma \Phi_\gamma(f), \tag{2.2}$$

where the sum on the left is over the cuspidal automorphic π on $Z(\mathbb{A})\backslash G(\mathbb{A})$ *such that* $(\pi)_v \cong \pi_v$ *for each* $v \in V$, and the right-hand side is summed over the regular elliptic conjugacy classes in $Z(F)\backslash G(F)$. Thus to prove Theorem 2.1, it suffices to prove that this right-hand side (and hence the left-hand side) does not vanish for a *particular special choice of such* f.

Equivalently, write

$$\Phi_\gamma(f) = \prod_v \Phi_\gamma(f_v),$$

a product of local orbital integrals. It will suffice then to choose f such that $\Phi_\gamma(f) \neq 0$ for a particular elliptic regular γ_0, but is zero for all other γ.

To this end, first choose γ_0 and f_{v_2} (still assumed to have support in the regular elliptic set) such that

$$\Phi_{\gamma_0}(f_v) \neq 0 \qquad \text{for each } v \text{ in } V \cup \{v_2\}.$$

(Recall that for each pseudo-matrix coefficient, $\Phi_\gamma(f_v) = c\overline{\chi_{\pi_v}(\gamma)}$ is not identically zero on the elliptic regular set of G_v ....) This particular choice of γ_0 will intersect the maximal compact K_v of G_v for almost all finite places outside $V \cup \{v_2\}$. Let us denote this (infinite) set of places by V^*, and for each v in V^* set f_v equal to the unit element of $\mathcal{H}_v = \mathcal{H}(G_v, K_v)$. Finally, choose f_v at the remaining finite places (outside $V \cup \{v_2\} \cup V^*$) to be such that $\Phi_{\gamma_0}(f_v) \neq 0$.

With this refined choice of $f = \prod f_v$ (still arbitrary at the infinite places!) it is clear that

$$\Phi_\gamma(f) \neq 0 \tag{2.3}$$

for some elliptic regular γ only if the coefficients of the characteristic polynomial of γ are v-integral for each v in V^*, i.e. are rational with uniformly bounded denominator. But such rational numbers lie in a **lattice** of F. Thus we can choose f_v at the archimedean places to have support so small near γ_0 as to intersect the conjugacy class of no other γ satisfying (2.3). With this choice of $f = \prod f_v$, $\Phi_\gamma(f) \neq 0$ if and only if $\gamma = \gamma_0$, and the proof is complete. □

Concluding Remark. Crucial to our argument was the fact that the pseudo-matrix coefficients f_v, $v \in V$, eliminated from (2.2) all cuspidal $\tau = \otimes\tau_v$ with τ_v not equivalent to π_v for v in V. This worked in our case since the only non-tempered τ_v which might satisfy $\operatorname{tr}\tau_v(f_v) \neq 0$ is the trivial representation (when π_v is the Steinberg representation), and such a representation can never occur as a local component of a cusp form on GL(2). In general, an additional argument is necessary, involving some kind of limit-multiplicity argument (due to de George-Wallach); see [Ro1] or Appendix 3 of [BDKV].

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LECTURE VII. (G, M)-FAMILIES AND THE SPECTRAL $J_\chi(f)$

The transition from Arthur's first ("coarse") form of the trace formula to a more explicit ("fine") expansion results from formal operations on collections of functions called $(\mathbf{G}, \mathbf{M})$**-families**, operations which pervade the "fine $\mathfrak{o}$-expansion" as well as the "fine χ-expansion".

The purpose of this lecture is to describe some basic examples and properties of (G, M)-families, and to explain the pivotal role they play, especially in making explicit the spectral ramified terms $J_\chi(f)$.

1. Definitions and Basic Examples. Fix a Levi subgroup M of G, and let $\mathcal{P}(M)$ denote the set of not necessarily standard parabolic subgroups of G for which M is the Levi component.

Definition 1.1.

Suppose that for each P in $\mathcal{P}(M)$, $c_P(\Lambda)$ is a smooth function on $i\mathfrak{a}_M^*$. Then the collection

$$\{c_P(\Lambda)\colon P \in \mathcal{P}(M)\}$$

is called a $(\mathbf{G}, \mathbf{M})$**-family** if the following condition holds: if P and P' are adjacent groups in $\mathcal{P}(M)$, and Λ belongs to the hyperplane spanned by the common wall of the chambers of P and P', then

$$c_P(\Lambda) = c_{P'}(\Lambda). \tag{*}$$

Remark 1.2. The compatibility condition (*) here is equivalent to the property that whenever P and P' are elements of $\mathcal{P}(M)$ contained in a given parabolic subgroup Q, and Λ belongs to $i\mathfrak{a}_Q^*$, then

$$c_P(\Lambda) = c_{P'}(\Lambda).$$

In particular, let

$$\mathcal{F}(M)$$

denote the parabolic subgroups of G whose Levi components *contain* M; then for each Q in $\mathcal{F}(M)$ one may define a smooth function $c_Q(\lambda)$ on $i\mathfrak{a}_Q^* \subset i\mathfrak{a}_P^*$ through the formula

$$c_Q(\Lambda) = c_P(\Lambda).$$

for any parabolic P in $\mathcal{P}(M)$ contained in Q.

Example 1.3. Let $\{X_P \colon P \in \mathcal{P}(M)\}$ be a collection of points in $\mathfrak{a}_M$ with the property that for each pair (P, P') of adjacent groups in $\mathcal{P}(M)$, $X_P - X_{P'}$ is **perpendicular** to the hyperplane spanned by the common wall of the chambers of P and P'; pictorially:

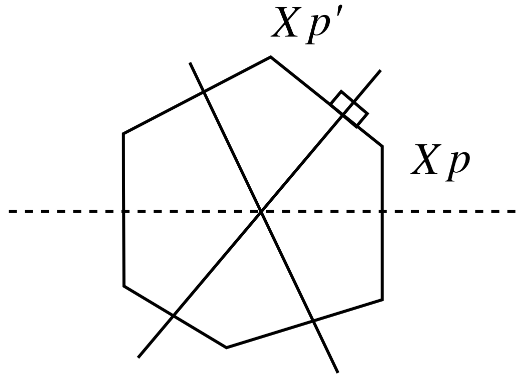

FIGURE 3

Then the resulting family of functions

$$c_P(\Lambda) = e^{\Lambda(X_P)}$$

on $i\mathfrak{a}_M^*$ is a (G, M)-family. Indeed for Λ on such a hyperplane,

$$\frac{c_P(\Lambda)}{c_{P'}(\Lambda)} = e^{\Lambda(X_P - X_{P'})} = 1.$$

Remark 1.4. The similarity here with Figure 2 of Lecture IV is not coincidental. Indeed, for any fixed x in $G(\mathbb{A})$, the points

$$\{-H_P(x) \colon P \in \mathcal{P}(M)\}$$

comprise an "orthogonal family" of points as depicted in the figure above; this is proved in Lemma 3.6 of [A8], where such "A_M-orthogonal" families of points were first introduced. (**N.B.** A collection of points $\{X_P\}$ in $\mathfrak{a}_M$ comprises an A_M-orthogonal family if an only if $X_P - X_{P'}$ is a multiple of the coroot associated to the unique root in $\Delta_P \cap -\Delta_{P'}$).)

Remark 1.5. If L is any Levi subgroup of G containing M, the more general notion of an (L, M)-family is defined exactly as above: for each P in $\mathcal{P}^L(M)$ (the parabolic subgroups of L with Levi component M), $c_P(\lambda)$ is smooth on $i\mathfrak{a}_M^*$, and if P' in $\mathcal{P}^L(M)$ is adjacent to P, then c_P and $c_{P'}$ agree on the wall between the corresponding adjacent chambers.

For the next Lemma, we need to recall a function $\theta_P(\Lambda)$ defined on $i\mathfrak{a}_M^*$ by

$$\theta_P(\Lambda) = (a_P)^{-1} \prod_{\alpha \in \Delta_P} \Lambda(\alpha^\vee),$$

where a_P is the covolume of a certain coroot lattice. (Again, there is the more general notion of θ_P^Q for any parabolic $Q \supset P \ldots$.)

Lemma 1.6. *For any (G, M)-family $\{c_P(\Lambda)\colon P \in \mathcal{P}(M)\}$, the function*

$$c_M(\Lambda) = \sum_{P \in \mathcal{P}(M)} c_P(\Lambda)\theta_P(\Lambda)^{-1},$$

initially defined away from the hyperplanes $\Lambda(\alpha^\vee) = 0$, extends to a smooth function on $i\mathfrak{a}_M^*$.

Proof. The only possible singularities along $\Lambda(\alpha^\vee) = 0$ occur in terms corresponding to those P for which α or $-\alpha$ is a simple root. But such groups occur in pairs of adjacent P, P', where c_P and $c_{P'}$ agree. I.e., the contribution of these P's to $c_M(\Lambda)$ is

$$\frac{c_P(\Lambda)}{\theta_P(\Lambda)} + \frac{c_{P'}(\Lambda)}{\theta_{P'}(\Lambda)},$$

where the numerators are equal, and the denominators appear as negatives of one another; thus the singularities cancel. □

Remark. The functions $c_M(\Lambda)$ arise naturally as the **weight functions** of the weighted orbital integrals described in Lecture IV. Indeed, consider the (G, M)-family

$$\{c_P(\Lambda) = e^{\Lambda(X_P)}; X_P = -H_P(x)\}$$

recalled above. In [A8] it was shown that the function

$$c_M(\Lambda) = \sum_{P \in \mathcal{P}(M)} e^{\Lambda(X_P)}\theta_P(\Lambda)^{-1}$$

equals the Fourier transform of the characteristic function of the convex hull of the points X_P (pictured in Figure 3 above). Thus it follows (without appealing to the more general Lemma 1.6 above) that $c_M(\Lambda)$ is a smooth function on $i\mathfrak{a}_M^*$ (being the Fourier transform of a compactly supported function). In particular, $c_M(0)$ is defined, and equal to the **volume of this convex hull**, namely the **weight function** $v_M(x)$.

It turns out that (G, M)-families pervade even more extensively the **spectral** terms $J_\chi(f)$. Here, as we shall explain below, (G, M) families arise which are **products** of families of the above (geometric) type, with (G, M)-families defined in terms of **intertwining operators**.

2. Motivation and Examples Arising from the Spectral Distributions $J_\chi^T(f)$.

Recall (from Lecture III) that

$$J_\chi^T(f) = \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \Lambda_2^T K_\chi(x, x)\, dx,$$

with

$$K_\chi(x, y) = \sum_P \frac{1}{n(A_P)} \sum_\sigma \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \sum_\phi E(x, \rho(\sigma, \lambda)(f)\phi, \lambda)\overline{E(y, \phi, \lambda)}\, d\lambda.$$

Here $P = M_P N_P$ runs over the "associated" parabolics of G, ϕ runs thru a K-finite basis for $\rho(\sigma, \lambda)$, and σ runs through (classes of) irreducible unitary representations of $M_P(\mathbb{A})$ such that functions in $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \sigma$ belong to $L^2(P)_\chi$.

Formally then,

$$J_\chi^T(f) = \sum_P \sum_\sigma \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \Psi_\sigma^T(\lambda, f, P)\, d\lambda,$$

where

$$\Psi_\sigma^T(\lambda, f, P) = \frac{1}{n(A_P)} \mathrm{tr}(\Omega_{\chi,\sigma}^T(P, \lambda)\rho(\sigma, \lambda)(f)),$$

and

$$(\Omega_{\chi,\sigma}^T(P, \lambda)\phi', \phi) = \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \Lambda^T E(x, \phi', \lambda)\overline{\Lambda^T E(x, \phi, \lambda)}\, dx.$$

But according to the inner product formula for truncated Eisenstein series, this operator $\Omega_{\chi,\sigma}^T(P, \lambda)$ equals the value at $\lambda' = \lambda$ of

$$\sum_{P_1} \sum_{\substack{t,t' \text{ in} \\ W(\mathfrak{a}_P, \mathfrak{a}_{P_1})}} M_{P_1|P}(t, \lambda)^{-1} M_{P_1|P}(t', \lambda') \frac{e^{(t'\lambda' - t\lambda)(T)}}{\theta_{P_1}(t'\lambda' - t\lambda)}. \tag{2.1}$$

Here $M_{P_1|P}(t, \lambda)$ is the intertwining operator defined on $\mathrm{Ind}(\sigma, \lambda)$ by

$$\int_{N_{P_1}(\mathbb{A}) \cap w_t N_P(\mathbb{A}) w_t^{-1} \backslash N_{P_1}(\mathbb{A})} \phi(w_t^{-1} n x)\, dx$$

(initially for λ with large real part, then for any λ by analytic continuation). Denote by $\omega_{\chi,\sigma}^T(P, \lambda)$ the value of the operator defined in (2.1) at $\lambda' = \lambda$. Actually, for $P \notin P_\chi$, the operators $\Omega_{\chi,\sigma}^T(P, \lambda)$ and $\omega_{\chi,\sigma}^T(P, \lambda)$ are only **asymptotically** equal (for large T); this is the thrust of [A9].

Claim.

$$\mathrm{tr}(\omega_{\chi,\sigma}^T(P, \lambda)\rho(\sigma, \lambda)(f)) \tag{2.2}$$

may be evaluated by setting $\lambda' = \lambda$ in the sum over $s \in W(\mathfrak{a}_P, \mathfrak{a}_P)$ of

$$\mathrm{tr}(\sum_{Q \in \mathcal{P}(M)} c_Q(\Lambda) d_Q(\Lambda) \theta_Q(\Lambda)^{-1} \rho(\sigma, \lambda)(f)),$$

where $\Lambda = \Lambda_s = s\lambda' - \lambda$, and c_Q and d_Q are two (G, M) families to be described below.

To see this, rewrite the expression (2.1) as

$$\sum_{\substack{s \in W_P = \\ W(\mathfrak{a}_P, \mathfrak{a}_P)}} \sum_{P_1} \sum_{t \in W(\mathfrak{a}_P, \mathfrak{a}_{P_1})} M_{P_1|P}(t, \lambda)^{-1} M_{P_1|P}(ts, \lambda') \frac{e^{(t(s\lambda' - \lambda))(T)}}{\theta_{P_1}(t(s\lambda' - \lambda))}.$$

For any P_1 and $t \in W(\mathfrak{a}_P, \mathfrak{a}_{P_1})$, set $Q = w_t^{-1} P_1 w_t$ for any representative w_t of t in $G(F)$. Then

$$(P_1, t) \leftrightarrow Q$$

is a bijection between pairs which occurs in the sum above and groups $Q \in \mathcal{P}(M)$. It can also be checked that $\theta_{P_1}(t(s\lambda' - \lambda)) = \theta_Q(s\lambda' - \lambda)$; moreover,

$$M_{P_1|P}(t,\lambda)^{-1} M_{P_1|P}(ts,\lambda') e^{(t(s\lambda'-\lambda))(T)} = M_{Q|P}(\lambda)^{-1} M_{Q|P}(s,\lambda') e^{(s\lambda'-\lambda)Y_Q(T)},$$

where $M_{Q|P}(\lambda) = M_{Q|P}(1,\lambda)$, and

$$Y_Q(T) = t^{-1}T$$

(or rather the projection of this point onto $\mathfrak{a}_M$). Thus the expression (2.1) indeed equals the value at $\lambda' = \lambda$ of the sum over s in $W(\mathfrak{a}_P, \mathfrak{a}_P)$ of

$$\sum_{Q \in \mathcal{P}(M)} c_Q(\Lambda) d_Q(\Lambda) \theta_Q(\Lambda)^{-1}$$

with

$$c_Q(\Lambda) = e^{\Lambda(Y_Q(T))}, \tag{2.3}$$

$$\Lambda = s\lambda' - \lambda,$$

and

$$d_Q(\Lambda) = (M_{Q|P}(\lambda)^{-1} M_{Q|P}(s,\lambda')). \tag{2.4}$$

Remark 2.5. Below we shall see that $\{c_Q(\Lambda)\}$ and $\{d_Q(\Lambda)\}$ both define (G, M)-families. Thus the **product**

$$\{e(\Lambda) = c(\Lambda) d(A)\}$$

is also a (G, M)-family, and the corresponding function

$$e_M(\Lambda) = \sum_{Q \in \mathcal{P}(M)} c_Q(\Lambda) d_Q(\Lambda) \theta_Q(\Lambda)^{-1}$$

is smooth on $i\mathfrak{a}_M^*$. In particular, the expression (2.2) may indeed be evaluated at $\lambda' = \lambda$ (which corresponds to evaluating $e_M(\Lambda)$ at $\Lambda = s\lambda - \lambda$).

Remark 2.6. In (2.3) and (2.4), set $\lambda' = \zeta + \lambda$ in $i\mathfrak{a}_M^*$, and $s =$ the identity. Then the value of $e_M(\Lambda)$ at $\lambda' = \lambda$ corresponds to the (operator) expression

$$\operatorname*{val}_{\lambda'=\lambda} \sum_{Q \in \mathcal{P}(M)} e^{\Lambda(Y_Q(T))} \frac{M_{Q|P}(\lambda)^{-1} M_{Q|P}(\lambda')}{\theta_Q(\zeta)}$$

$$= \lim_{\zeta \to 0} \sum_{P_1} \sum_{w \in W(\mathfrak{a}_P, \mathfrak{a}_{P_1})} e^{w\zeta(T)} \frac{M(w,\lambda)^{-1} M(w, \lambda+\zeta)}{\prod_{\alpha \in \Delta_{P_1}} (w\zeta)(\alpha^\vee)}$$

This is precisely the operator we denoted by $M_P^T(\sigma)_\chi$ in Proposition 3.9 of Lecture IV. Thus the "weight function" $M_P^T(\sigma)_\chi$ appearing in the weighted character expression for the **unramified** spectral term $J_\chi^T(f)$ in Proposition 3.9 is indeed realizable as a special value of $e_M(\Lambda)$ for an appropriate (G, M)-family e_P.

N.B. As we observed in the case of GL(2) (in §2 of Lecture IV), the expression for $J_\chi^T(f)$ simplifies considerably for **unramified** spectral data χ, since the sum over t, t' in $W(\mathfrak{a}_P, \mathfrak{a}_{P_1})$ in (2.1) need *only* be taken over $t = t'$, i.e., only the term corresponding to $s = 1$ really appears in the expression for (2.2), and we indeed have

$$J_\chi^T(f) = \sum_{P \in P_\chi} \sum_\sigma \frac{1}{n(A_P)} \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \operatorname{tr}(M_P^T(\sigma)_\chi \rho(\sigma, \lambda))\, d\lambda$$

as asserted in Proposition 3.9 of Lecture IV.

Proposition 2.7. *Both $c_Q(\Lambda)$ and $d_Q(\Lambda)$ define (G, M)-families.*

Proof. We have to show that if Q and Q' are adjacent in $\mathcal{P}(M)$, then

$$Y_Q(T) - Y_{Q'}(T)$$

is perpendicular to the hyperplane in $i\mathfrak{a}_M^*$ spanned by the wall common to the chambers of Q and Q'. Recall that for any Q in $\mathcal{P}(M)$, $Y_Q(T) = t^{-1}T$ if $Q = w_t^{-1} P_1 w_t$ for some fixed $P_1 \in \mathcal{P}(M)$. But if Q' is adjacent to Q, then $Q' = w_{t'}^{-1} P_1 w_{t'}$ with $t' = s_\alpha t$, and s_α the simple reflection corresponding to an $\alpha \in \Delta_{P_0}$. Therefore $Y_Q(T) - Y_{Q'}(T) = t^{-1}(T - s_\alpha^{-1}T)$, which is a multiple of $t^{-1}\alpha^\vee$ (see footnote[1] below). But $s^{-1}\alpha^\vee = \beta^\vee$, where $\beta^\vee$ is the unique root in $\Delta_Q \cap (-\Delta_{Q'})$. Thus $Y_Q(T) - Y_{Q'}(T)$ is perpendicular to the hyperplane between Q and Q'. □

As for $d_Q(\Lambda)$, we recall the standard functional equation

$$M_{Q'|P}(s, \lambda) = M_{Q'|Q}(1, s\lambda) M_{Q'|P}(s, \lambda),$$

valid for $s \in W(\mathfrak{a}_M, \mathfrak{a}_M)$ and $Q \in \mathcal{P}(M)$ (see equation (1.2) of [A8]). Using it gives

$$\begin{aligned} d_{Q'}(\Lambda) &= M_{Q'|P}(\lambda)^{-1} M_{Q'|P}(s, \lambda') \\ &= M_{Q|P}(\lambda)^{-1} M_{Q'|Q}(\lambda)^{-1} M_{Q'|Q}(s\lambda') M_{Q|P}(s, \lambda'). \end{aligned}$$

So it remains to show that *for Λ belonging to the hyperplane common to Q' and Q,*

$$M_{Q'|Q}(\lambda)^{-1} M_{Q'|Q}(s\lambda') = I.$$

For this, see §§1 and 2 of [A8].

[1] Formally, $\alpha^\vee = 2\alpha/(\alpha, \alpha)$, and $s_\alpha^{-1} = T - 2(T, \alpha)/(\alpha, \alpha)\ \alpha$. Therefore, for any β, $(\beta, T - s_\alpha^{-1}(T)) = (\beta, T) - (\beta, T) + 2(T, \alpha)/(\alpha, \alpha)\ (\beta, \alpha) = (T, \alpha)(\beta, \alpha^\vee)$, i.e., $T - s_\alpha^{-1}T$ is indeed a multiple of $\alpha^\vee$.

3. A Splitting Formula for Products of (G,M)-families. In the last section, we explained how $J_\chi^T(f)$ is expressed in terms of integrals of special values of the functions $e_M(\Lambda)$, where $e_P = c_P(\Lambda)d_P(\Lambda)$ is the product of two (G,M)-families. The result below is crucial in completing the evaluation of these integrals.

Proposition 3.1 (see Lemma 6.3 of [A4]). *In general,*

$$(cd)_M(\Lambda) = \sum_{Q\in\mathcal{F}(M)} c_M^Q(\Lambda)d'_Q(\Lambda),$$

where $\mathcal{F}(M)$ denotes the set of parabolics of G whose Levi component contains (but does not necessarily equal) M.

We will not prove this proposition, since the reader can find it in [A4]. But we shall at least define the terms c_M^Q and d'_Q appearing here, and explain their significance.

Given any (G,M)-family $\{c_P(\Lambda)\}$, and a parabolic Q containing some P in $\mathcal{P}(M)$, how first do we define the function c_Q?

Since $i\mathfrak{a}_Q^* \subset i\mathfrak{a}_P^*$, it is tempting to simply set

$$c_Q(\Lambda) = c_P(\Lambda_Q)$$

with Λ_Q the projection of Λ onto $i\mathfrak{a}_Q^*$. But is this well-defined? If P' is another element of $\mathcal{P}(M)$ contained in Q, we must check that

$$c_{P'}(\Lambda) = c_P(\Lambda)$$

as soon as $\lambda \in i\mathfrak{a}_Q^*$. But we have already remarked that the compatibility condition on the (G,M)-family $\{c_P\}$ implies this is the case (Exercise!), and thus c_Q *is* well-defined on $i\mathfrak{a}_Q^*$.

Now for any pair of parabolics $Q\subset R$, define

$$\hat\theta_Q^R(\Lambda) = (\hat a_Q^R)^{-1}\prod_{\tilde w\in\hat\Delta_Q^R}\Lambda(\tilde w^\vee),$$

with $\tilde w^\vee$ in $\mathfrak{a}_Q^R$ defined by

$$\alpha(\tilde w^\vee) = \tilde w(\alpha^\vee), \qquad \alpha\in\Delta_Q^R$$

(and the constant $\hat a_Q^R$ the covolume of a certain lattice ...). Then the functions

$$c'_Q(\Lambda), \qquad Q\supset P$$

are defined (initially on the complement of a finite set of hyperplanes in $i\mathfrak{a}_P^*$) by

$$c'_Q(\Lambda) = \sum_{\{R:\ R\supset Q\}} (-1)^{\dim(A_Q|A_R)}\hat\theta_Q^R(\Lambda)^{-1}c_R(\Lambda)\theta_R(\Lambda)^{-1}.$$

In Lemma 6.1 of [A4], it is shown that these functions extend to smooth functions on all of $i\mathfrak{a}_P^*$.

Now that the functions d'_Q in Proposition 3.1 are defined, there remains the simpler task of defining the $c_M^Q(\Lambda)$; these are just the c_M functions (1.5) derived from the following (L, M)-families of functions $\{c_R^Q(\Lambda)\}$.

Fix Q arbitrary in $\mathcal{F}(M)$, and let $L = L_Q$ denote its Levi component. By $\mathcal{P}^L(M)$ we denote the set of parabolic subgroups of L whose Levi component is M, and for each $R \in \mathcal{P}^L(M)$, we set $Q(R)$ equal to the unique group in $\mathcal{P}(M)$ which is contained in Q and such that $Q(R) \cap L = R$. Then the resulting family of functions

$$\{c_R^Q(\Lambda) = c_{Q(R)}(\Lambda) \colon R \in \mathcal{P}^L(M)\}$$

comprises an (L, M)-family, and the functions $c_M^Q(\Lambda)$ appearing in Proposition 3.1 are the corresponding c_M-functions

$$c_M^Q(\Lambda) = \sum_{R \in \mathcal{P}^L(M)} c_R^Q(\Lambda)\theta_R^Q(\Lambda)^{-1}.$$

4. GL(2) **Revisited.** Although we already discussed (in Lecture IV) the evaluation of $J_\chi^T(f)$ in this case, we wish to return to this calculation now from the more general standpoint of Arthur's theory.

According to the discussion in Section 2 above, $J_\chi^T(f)$ is given by the polynomial

$$\sum_P \int_{i\mathfrak{a}_P^* \backslash i\mathfrak{a}_G^*} \sum_\sigma \sum_{s \in W(\mathfrak{a}_P)} \Big\{\lambda \overset{\text{val}}{=} \lambda' \operatorname{tr}(e_M(\Lambda)\rho(\sigma, \lambda)(f))\Big\}\, d\lambda, \tag{4.1}$$

where $e(\Lambda)$ is the (G, M)-family given by the **product** of the family of functions (2.3) and (2.4). When $P = B$ (the Borel subgroup), we have

$$\lambda = \Big(\frac{it}{2}, \frac{-it}{2}\Big)$$

and

$$\lambda' = \Big(\frac{it'}{2}, \frac{-it'}{2}\Big),$$

with $t' = t + \zeta$ and $\Lambda = s\lambda' - \lambda$.

For $\chi = \{(M, \mu)\}$, the contribution from the term $s = 1$ reduces to

(4.2)

$$\int_{\mathbb{R}} \operatorname{tr}\left(\lim_{\zeta \to 0}\Big\{\frac{e^{i\zeta(T_1 - T_2)/2}}{i\zeta} + \frac{e^{-i\zeta(T_1-T_2)/2}M(w, it)^{-1}M(w, i(t+\zeta))}{-i\zeta}\Big\}\rho(\mu, it)(f)\right)\, dt$$

Indeed, the two elements of $\mathcal{P}(M)$ are just B and $\bar{B}$ (the subgroup wBw^{-1} opposite to B), and so

$$\begin{aligned} e_M(\Lambda) &= e_B(\Lambda)\theta_B(\Lambda)^{-1} + e_{\bar{B}}(\Lambda)\theta_B(\Lambda)^{-1} \\ &= \frac{c_B(\Lambda)d_B(\Lambda)}{i\zeta} + \frac{c_{\bar{B}}(\Lambda)d_{\bar{B}}(\Lambda)}{-i\zeta}. \end{aligned}$$

But (2.3) clearly implies that

$$c_B(\Lambda) = e^{\Lambda(T)} = e^{i\zeta(T_1-T_2)/2}$$

and

$$c_{\bar{B}}(\Lambda) = e^{-\Lambda(T)} = e^{-i\zeta(T_1-T_2)/2}.$$

Also $M_{B|B}(1,\lambda) = I$, so $d_B(\Lambda) = I$, and it remains only to compute $d_{\bar{B}}(\Lambda)$. For this, let T_w denote the linear transformation $\phi(x) \longrightarrow \phi(wx)$ taking $\operatorname{Ind}_{\bar{B}} \sigma^w \mid \;|^{w\lambda}$ to $\operatorname{Ind}_B \sigma \mid \;|^\lambda$. Then it is straightforward to check that

$$\begin{aligned} d_{\bar{B}}(\Lambda) &= M_{\bar{B}|B}(\lambda)^{-1} M_{\bar{B}|B}(1,\lambda') \\ &= (T_w M(w,\lambda))^{-1}(T_w M(w,\lambda')) \\ &= M(w,it)^{-1} M(w,i(t+t')), \end{aligned}$$

and thus (4.2) holds. To continue, there is no need to appeal to the fact that the expression in brackets in (4.2) comes from the (G,M)-family $\{e_P\}$. Instead, one can just compute the limit in question to be

$$(T_1 - T_2) + M(-it)M'(it),$$

and hence the contribution to $J_\chi^T(f)$ (from $s=1$) to be

$$(T_1 - T_2)\int_{=\infty}^{\infty} \operatorname{tr}(\rho(\mu,it)(f)\,dt + \int_{=\infty}^{\infty} \operatorname{tr}(M(-it)M'(it)\rho(\mu,it)(f)\,dt.$$

This of course agrees with formula (2.5) of Lecture IV, and for χ unramified, it accounts for all of $J_\chi^T(f)$.

Now let us concentrate on the more interesting contribution to (4.1), namely the one corresponding to $s = w$. In this case, $\Lambda = w\lambda' - \lambda$, and this equals $(-it, it)$ when $\lambda' = \lambda$. So as above, we compute

$$\begin{aligned} c_B(\Lambda) &= e^{\Lambda(T)} = e^{-2it(T_1-T_2)} \\ c_{\bar{B}}(\Lambda) &= e^{-\Lambda(T)} = e^{2it(T_1-T_2)} \\ d_B(\Lambda) &= M(w,\lambda') \end{aligned}$$

and

$$d_{\bar{B}}(\Lambda) = M(w,\lambda)^{-1}.$$

In other words, for $s = w$

$$\begin{aligned} &\left\{ \overset{\text{val}}{\lambda' = \lambda} \operatorname{tr}(e_M(\Lambda)\rho(\sigma,\lambda)(f) \right\} \\ &= \operatorname{tr}(M(w,-it)(\rho(\mu,it)(f))\frac{e^{2it(T_1-T_2)}}{2it} - \operatorname{tr}(M(w,it)(\rho(\mu,it)(f))\frac{e^{-2it(T_1-T_2)}}{2it}. \end{aligned}$$

This is exactly the expression (6.28) in [GJ]. But instead of evaluating its integral over $\mathbb{R}$ using the simple manipulations of [GJ], let us instead appeal to the splitting formula for $e_M(\Lambda)$.

Lemma. *For $s = w$*

$$e_M(\Lambda) = (cd)_M = d_B(0)\frac{e^{\Lambda(T)} - e^{-\Lambda(T)}}{-2it} + \frac{d_B(\Lambda) - d_B(0)}{-2it} \cdot e^{\Lambda(T)} + \frac{d_{\bar{B}}(\Lambda) - d_B(0)}{2it} \cdot e^{-\Lambda(T)}$$

with $e^{\pm\Lambda(T)} = e^{\mp 2it(T_1 - T_2)}$.

Proof. According to Proposition 3.1, we need to compute

$$c_M^Q(\Lambda)d_Q'(\Lambda)$$

for each $Q \in \mathcal{F}(M)$. So first fix $Q = G$. By definition, $d_G(\Lambda) = d_B(0) = d_{\bar{B}}(0)$, and then $d_G'(\Lambda) = d_G(\Lambda)$. On the other hand, we compute (from the definition of c_M^Q) that $c_M^G(\Lambda) = c_M(\Lambda)$. Thus the leading term of $(cd)M(\Lambda)$ is indeed

$$d_B(0)\frac{e^{\Lambda(T)} - e^{-\Lambda(T)}}{-2it},$$

as claimed.

Now fix $Q = B$. Then

$$d_B'(0) = \frac{d_B(\Lambda) - d_B(0)}{-2it},$$

while $c_M^B(\Lambda) = c_B(\Lambda)$. Indeed, in this case $L = M$ and $\mathcal{P}^L(M) = M$! Similarly

$$c_M^{\bar{B}}(\Lambda)d_{\bar{B}}'(\Lambda) = \frac{d_{\bar{B}}(\Lambda) - d_{\bar{B}}(0)}{-2it} \cdot c_{\bar{B}}(\Lambda),$$

and we're done. □

Proposition. *For T large, the contribution to $J_\chi^T(f)$ from (the Borel subgroup B and) $s = w$ is just*

$$\mathrm{tr}(d_B(0)\rho(0, \mu)(f)) = \mathrm{tr}(M(0, \mu)\rho(0, \mu)(f));$$

in particular, only the leading term in $(cd)_M(\Lambda)$ contributes to $J_\chi^T(f)$.

Proof.

Let us *suppose* for the moment that $J_\chi^T(f)$ can be calculated by taking a compactly supported Schwartz function $B(t)$ with $B(0) = 1$, and then computing

$$J_\chi^T(f) = \lim_{\varepsilon \to 0} \lim_{\alpha(T) \to \infty} \int_{-\infty}^{\infty} \mathrm{tr}(\omega_\chi^T(t)\rho(\mu, it)(f))B(\varepsilon t)\, dt.$$

By the last lemma, the contribution from $s = w$ (and $P = B$) to

$$\operatorname{tr}(\omega_\chi^T(t)\rho(\mu, it)(f))$$

is

$$\begin{aligned}\left\{\frac{e^{2it(T_1-T_2)} - e^{-2it(T_1-T_2)}}{2it}\right\} &\operatorname{tr}(d_B(0)\rho(\mu, it)(f)) \\ &+ e^{-2it(T_1-T_2)}\frac{\operatorname{tr}((d_B(\Lambda) - d_B(0))\rho(\mu, it)(f))}{-2it} \\ &\qquad + e^{2it(T_1-T_2)}\frac{\operatorname{tr}((d_{\bar{B}}(\Lambda) - d_B(0))\rho(\mu, it)(f))}{-2it}.\end{aligned}$$

But for any real T,

$$\begin{aligned}\frac{e^{iyT}}{iy} + \frac{e^{-iyT}}{-iy} &= \int_{-T}^{T} e^{iyx}\,dx \\ &= \int_{\infty}^{\infty} \chi_T(x)e^{-iyx}\,dx,\end{aligned}$$

with χ_T the characteristic function of $[-T, T]$. Thus we can write the contribution of the leading term as

$$\begin{aligned}&\lim_{\varepsilon\to 0}\lim_{\alpha(T)\to\infty}\left\{\int_{-\infty}^{\infty}\chi_{\alpha(T)}(x)\int_{-\infty}^{\infty} e^{2itx}\operatorname{tr}(d_B(0)\rho(\mu, it)(f))B(\varepsilon t)\,dt\,dx\right\} \\ &\quad = \lim_{\varepsilon\to 0}\lim_{\alpha(T)\to\infty}\left\{\int_{-\infty}^{\infty}\chi_{\alpha(T)}(x)\hat{F}_\varepsilon(x)\,dx\right\} \\ &\quad = \lim_{\varepsilon\to 0}\left\{\int_{-\infty}^{\infty}\hat{F}_\varepsilon(x)\,dx\right\} = \lim_{\varepsilon\to 0} F_\varepsilon(0),\end{aligned}$$

where

$$\begin{aligned}F_\varepsilon(t) &= \operatorname{tr}(d_B(0)\rho(\mu, it)(f))B(\varepsilon t) \\ &= \operatorname{tr}(M(0, \mu)\rho(\mu, 0)(f)) \qquad\qquad \text{at } t = 0,\end{aligned}$$

since $B(0) = 1$. On the other hand, for the second (or third) term, the contribution is

$$\lim_{\varepsilon\to 0}\lim_{\alpha(T)\to\infty}\int_{-\infty}^{\infty} e^{\mp 2it(T_1-T_2)}F_\varepsilon(t)\,dt$$

where for each ε,

$$F_\varepsilon(t) = \operatorname{tr}\left\{\frac{d_B(t) - d_B(0)\rho(\mu, it)(f)}{\pm 2it}\right\} B(\varepsilon t)$$

is a smooth Schwartz function on $\mathbb{R}$, whose Fourier transform vanishes at infinity. Thus the second and third terms contribute nothing to $J_\chi^T(f)$. $\square$

Concluding Remarks. (1) In treating the GL(2) calculation of $J_\chi^T(f)$ from the point of view of Arthur's general theory, we have made a big mountain out of a small mole hill. However, already for GL(3), there are compelling reasons for appealing to the general theory—not just for using the splitting formula for products of (G, M)-families, but also for exploiting the compactly supported function B in the calculation of $J_\chi(f)$. Indeed, without using this function B, there is the need to directly establish the integrability of a function like $\mathrm{tr}(M(w, \lambda)\rho(\sigma, \lambda)(f))$ (in order to justify applying a Riemann-Lebesgue Lemma as $T \to \infty$). In the case of GL(2), this can be checked directly, but already for GL(3) the required estimates on the growth of $M(w, \lambda)$ are non-trivial (see Section 3 of [Ja1]), and in general they are simply not available.

(2) It remains to prove that the above computation (with the insertion of the "smoothing function" $B(t)$) is justified. This is precisely the goal of Arthur's paper [A7] and—we hope—part of the subject matter of Lecture IX.

(New) Reference

[Ja1] Jacquet, H., *The continuous spectrum of the relative trace formula for* GL(3) *over a quadratic extension*, Israel Journal of Math. **89**, 1–59.

Lecture VIII. Jacquet's Relative Trace Formula

In deriving Arthur's trace formula

$$\sum_{\mathfrak{o}} J_{\mathfrak{o}}^T(f) = \sum_{\chi} J_{\chi}^T(f), \tag{*}$$

we started by restricting the kernel functions

$$K_f(x,y) = \sum_{\mathfrak{o}} K_{\mathfrak{o}}(x,y) = \sum_{\chi} K_{\chi}(x,y)$$

to the **diagonal subgroup**.

$$Z(\mathbb{A})G(F)\backslash G(\mathbb{A}) \times Z(\mathbb{A})G(F)\backslash G(\mathbb{A}).$$

Then after modifying these restricted kernels, we were able to integrate them and obtain the formula (*). The term "trace formula" seemed apt because—at least for cuspidal data $\chi = \{(M,r)\}$ with $M = G$, $J_{\chi}^T(f)$ actually represents the trace of $R(f)$ restricted to L_{χ}^2.

The idea of Jacquet's **relative** "trace formula" is to restrict the kernel functions $K_f(x,y)$ to interesting **subgroups** of the diagonal, and then integrate them against possibly **non-trivial automorphic forms** on these subgroups. Although the resulting integrals no longer represent the "trace" of anything, there results from this new approach a dramatic array of interesting possible applications to automorphic forms (See [JLR] and [Ja1] for a general discussion, and further references.)

In this lecture we shall concentrate on a particular example (cf. [Ye]) which already shows the power of this method, and indicates how the further development of the method is inextricably linked to the general theory developed by Arthur. Even in our exposition of this simplest example, we are influenced by Jacquet's more general ideas (see [Ja1]); moreover, the approach to the computations in §4 is adapted from joint work with him and Rogawski on similar computations for GL(3).

1. Description of a Base Change Theorem. Fix a quadratic extension E of the number field F, and let $\omega_{E/F}$ denote the corresponding quadratic character of $F^*\backslash\mathbb{A}_F$. Let us call a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ **stable** if it is not of the form $\pi(\mu')$ for any grossencharacter μ' of E.

Theorem (cf. [Ye]). *Every stable cuspidal π on $\mathrm{GL}_2(\mathbb{A}_F)$, with central character $\omega_{E/F}$, has a* **base change lift** *to a cuspidal representation π_E of $\mathrm{GL}_2(\mathbb{A}_E)$; moreover, the resulting map*

$$\pi \longrightarrow \pi_E$$

is a **bijection** *onto the cuspidal representations Π of $\mathrm{GL}_2(\mathbb{A}_E)$ which (have trivial central character and) are* **distinguished** *with respect to $\mathrm{GL}_2(F)$, i.e., the "period"*

$$\int_{Z(\mathbb{A})\,\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)} \phi'(h)\,dh \tag{1.2}$$

is non-zero for some ϕ' in the space of Π.

Remarks. (i) In [La2], Langlands established the existence of a base change lift for GL_2 for an **arbitrary cyclic** extension E of F, and in[AC] these results were dramatically generalized to GL_n. Both these works characterize the image of base change in terms of the Galois invariance of the representations "upstairs"; this is not surprising, since the proofs proceed from a comparison of the ("invariant") trace formula over F with the (Galois) "twisted" ("invariant") trace formula over E.

(ii) In [HLR] it is shown that an automorphic cuspidal representation Π of $\mathrm{GL}_2(E)$ (trivial on its center) is the base change lift of an automorphic cuspidal representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ (with central character $\omega_{E/F}$) if and only if Π is distinguished. The argument there combines the base change result of [La2] with properties of the so-called Asai L-function $L(s,\Pi,Asai)$ (which has a pole at $s=1$ if and only if π is distinguished ...). (By contrast, the relative trace formula proof of Theorem 1.1 (see below) makes no appeal to [La2].) The main thrust of [HLR] was to study algebraic cycles on certain Shimura varieties attached to GL_2 over E, whose Hasse–Weil zeta functions are computable in terms of

$$L(s,\Pi,\mathrm{Asai}).$$

It was precisely the desire to extend these results to the context of quaternion algebras that led to the idea of the relative trace formula introduced in [JLai].

(iii) In [JLai], Jacquet and Lai used a comparison of **relative** trace formulas on GL_2/E and the multiplicative group G' of a division quaternion algebra over E (the relative subgroups of the diagonal being the F-diagonal subgroups) to prove that Π' on G'/E is $G'(F)$ distinguished if and only if its Jacquet–Langlands correspondent Π on GL_2/E is $\mathrm{GL}_2(F)$ distinguished. This was applied in [Lai2] to extend the results of [HLR] to the case of certain compact Shimura varieties.

2. Outline of the Proof of Theorem 1.1. It is convenient to use the notation H for the group GL_2 over F, and G for the group $\mathrm{Res}_F^E\,\mathrm{GL}_2$ (so that $H(\mathbb{A})=\mathrm{GL}_2(\mathbb{A}_F)$ and $G(\mathbb{A})=\mathrm{GL}_2(\mathbb{A}_E)$). By

$$f=\prod f_v$$

we denote a test function in $C_c^\infty(H(\mathbb{A}), \omega_{E/F})$, where for almost every place v, f_v is the identity element of the local Hecke algebra $\mathcal{H}(H_v, K_v^H, \omega_v)$; similarly,

$$f' = \prod f'_v$$

belongs to $C_c^\infty(Z(\mathbb{A}_E\backslash G(\mathbb{A}_F))$, with f'_v almost everywhere the identity in

$$\mathcal{H}(Z_{G_v}\backslash G_v, K_v^G).$$

Corresponding to f there is the convolution operator

$$R(f) = \int_{Z(\mathbb{A})\backslash H(\mathbb{A})} f(h)R(h)\,dh$$

in the space $L^2(H(F)\backslash H(\mathbb{A}), \omega)$; it is an integral operator with kernel

$$K_f(x,y) = \sum_{\gamma \in Z(F)\backslash H(F)} f(x^{-1}\gamma y).$$

Similarly f' determines an integral operator $R'(f')$ in $L^2(Z_G(\mathbb{A})G(F)\backslash G(\mathbb{A}))$ with kernel

$$K_{f'}(g_1, g_2) = \sum_{\gamma \in Z_G(F)\backslash G(F)} f'(g_1^{-1}\gamma g_2).$$

The proof of Theorem 1.1 will come from comparing certain "relative traces" associated to these kernels, as we shall now explain.

Step I. Prove that for "matching" f and f',

$$\text{(2.1)}\quad \iint_{[N(F)\backslash N(\mathbb{A}_F)]^2} K_f(n_1,n_2)\psi_N(n_1^{-1}n_2)\,dn_1\,dn_2 = \int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} \int_{N'(F)\backslash N'(\mathbb{A}_F)} K_{f'}(n',h)\psi'_{N'}(n')\,dn'dh.$$

Here N (resp. N') is the standard maximal unipotent of H (resp. G), and ψ_N (resp. ψ'_N) the corresponding character of $N(F)\backslash N(\mathbb{A}_F)$ (resp. $N'(F)\backslash N'(\mathbb{A}_F)$) determined by a fixed additive character ψ (of $F\backslash\mathbb{A}$) (with $\psi' = \psi \circ \mathrm{tr}_{E/F}$).

Remarks. (i) A special feature of this particular relative trace formula approach is the **absolute convergence** of the integrals on either side of the identity (2.1). On the left hand side, this is clear from the compactness of the domains of integration; for the right side, this follows from the fact that *for sufficiently large T*,

$$\int_{N'(F)\backslash N'(\mathbb{A})} \Lambda_2^T K_{f'}(n',h)\psi'_{N'}(n')\,dn'$$

equals (as opposed to just approximates)

$$\int K_{f'}(n',h)\psi'_{N'}(n')\,dn'.$$

For a proof, see Proposition 2.1 of [Ja1].

(ii) At first sight, the equality in (2.1) seems surprising, since the double cosets

$$N\backslash H/N \quad \text{and} \quad N'\backslash G/H$$

which arise on each side do not match up. However, it seems to be a general feature of the relative trace formula that the "relevant" orbits—those which survive the relative integrations—do indeed match up. In the case at hand, it is the integration against ψ which allows for the matching up of relevant orbits. Once this is established, the assertion that (2.1) holds for matching f and f' reduces quickly to a collection of **purely local assertions**: two local function f_v and f'_v are "matching" if certain local relative orbital integrals (indexed by identical sets) are equal.

(iii) It is also crucial to prove a "fundamental lemma" asserting that this local matching is compatible with the base change map BC $*$ between the Hecke algebras of K_v and G_v. This map is dual to the natural base change morphism

$$\mathrm{BC} : (g, \sigma) \longrightarrow (g, g, \sigma)$$

from ${}^L H = \mathrm{GL}_2(\mathbb{C}) \times \mathrm{Gal}(E/F)$ to ${}^L G = (\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})) \rtimes \mathrm{Gal}(E/F)$. In particular, when f'_v belongs to the Hecke algebra of G_v, the function f_v in the Hecke algebra of H_v defined by

$$(f_v)^\vee(t) = (f'_v)^\vee(\mathrm{BC}(t))$$

will be a matching function to f'_v. For proofs of these matching results, see [Ye]. For more general results, and discussion of "relative fundamental lemmas", see [Ja2], [JaYe], [Mao], and [JLR].

Step II. Prove (again for these matching f and f') that the **relative** traces of the **continuous** parts of the kernels K_f and $K_{f'}$ cancel each other out.

To explain what this means, let us first write out the spectral expansion of K_f in crude form:

$$K_f(x, y) = K_{f,\,\mathrm{cont}} + K_{f,\mathrm{sp}} + K_{f,0},$$

where $K_{f,\mathrm{cont}}$ denotes the kernel of $R(f)$ restricted to the continuous spectrum of $L^2(H(F)\backslash H(\mathbb{A}), \omega_{E/F})$, and $K_{f,\mathrm{sp}}$ the kernel for the discrete non-cuspidal spectrum. Similarly,

$$K_{f'}(g_1, g_2) = K_{f',\mathrm{cont}} + K_{f',\mathrm{sp}} + K_{f',0}.$$

A simple but key observation is that the sp-part of the kernel contributes nothing to the relative trace. Indeed,

$$\int_{[N(F)\backslash N(\mathbb{A}_F)]^2} K_{f,\mathrm{sp}}(n_1, n_2)\psi(n_1^{-1}n_2)\, dn_1\, dn_2$$
$$= 0 = \int_{N'(F)\backslash N'(\mathbb{A})} K_{f',\mathrm{sp}}(n', h)\psi'_{N'}(n')\, dn',$$

since these parts of the spectrum are **non-generic**.

Thus, if we can show that the continuous kernels contribute roughly equal relative traces, we might then conclude

$$\iint K_{f,0}(n_1,n_2)\psi_N(n_1^{-1}n_2)\,dn_1\,dn_2 = \iint K_{f',0}(n',h)\psi_{N'}(n')\,dn'\,dh \tag{2.2}$$

Henceforth, we denote the maximal compact of H_v by K_v and the maximal compact of G_v by K'_v.

Step III. Use the equality of the relative **cuspidal** traces on H and G to establish the asserted base change bijection between the cuspidal (stable) π on H and the cuspidal (distinguished) Π on G.

To see why this is relatively (!) straightforward, let us write out the explicit spectral expansions for $K_{f,0}$ and $K_{f',0}$, namely,

$$K_{f,0}(x,y) = \sum_{\substack{\pi\\ \text{cuspidal}}} \sum_{\phi\in V_\pi} \pi(f)\phi(x)\overline{\phi(y)},$$

and

$$K_{f',0}(g_1,g_2) = \sum_{\substack{\Pi\\ \text{cuspidal}}} \sum_{\phi'\in V_\Pi} \Pi(f)\phi(g_1)\overline{\phi'(g_2)}.$$

If we assume f and f' are right $K^S = \prod_{v\notin S} K_v$ (resp. ${K'}^S$) invariant, where S is a finite set of places (including the infinite ones), then these identities read

$$K_{f,0}(x,y) = \sum_{\pi^{K^S}\neq\{0\}} (f^S)^\vee(t(\pi^S)) \sum_{\{\phi_\pi\}} \prod_{v\in S} \pi_v(f_v)\phi(x)\overline{\phi(y)} \tag{2.3}$$

and

$$K_{f',0}(g_1,g_2) = \sum_{\Pi^{{K'}^S}\neq\{0\}} ({f'}^S)^\vee(t(\Pi^S)) \sum_{\{\phi'_\Pi\}} \prod_{v\in S} \Pi_v(f'_v)\phi'(g_1)\overline{\phi'(g_2)}. \tag{2.3$'$}$$

If we further assume that for $v\notin S$ (where both f'_v and f_v are spherical), f_v **corresponds to f'_v via the natural base change map** between their Hecke algebras, then,

$$(f^S)^\vee(t(\pi^S)) = ({f'}^S)^\vee(\mathrm{BC}(t(\pi^S))).$$

So plugging these expressions into the relative trace formula identity (2.2) yields the more explicit identity

$$\sum_\pi A(\pi,f_S)({f'}^S)^\vee(\mathrm{BC}(t(\pi^S)) = \sum_\Pi B(\Pi,f'_S)({f'}^S)^\vee(t(\Pi^S)) \tag{2.4}$$

with

$$A(\pi, f_S) = \sum_{\{\phi_\pi\}} W^{\psi}_{\pi_S(f_S)\phi}(e)\overline{W^{\psi}_{\phi}(e)},$$

$$B(\Pi, f'_S) = \sum_{\{\phi'_\Pi\}} W^{\psi'}_{\Pi_S(f'_S)\phi'}(e)\overline{D(\phi')},$$

$$W^{\psi}_{\phi}(e) = \int_{N(F)\backslash N(\mathbb{A})} \phi(n)\overline{\psi_N(n)}\, dn, \qquad \text{(with a similar expression for } W^{\psi'}\text{)}$$

and

$$D(\phi') = \int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} \phi'(h)\, dh.$$

Note that (2.4) may be viewed as a (absolutely convergent) linear combination of irreducible characters of the group $(G)^S = \prod_{v \notin S} G_v$. Also, note that a particular cuspidal $\prod$ on $G(\mathbb{A})$ will appear on the right side of (2.4) only if it is distinguished, i.e., $D(\phi') \neq 0$ for some ϕ'. Thus it is not surprising that a "linear independence of characters argument" yields the desired bijection of Theorem 1.1. For more details, we refer the reader to [Ye], and §4 below.

Since the most subtle part of the proof outlined above is Step II, we focus on this step in some detail in the sections below.

3. Step II: Subtracting off the Continuous Spectrum. Once we know that the relative trace formulas on H and G agree for matching f and f', it follows (keeping in mind the previous discussion) that

$$\begin{aligned} (3.1) \quad & \iint K_{f,0}(n_1, n_2)\psi_N(n^{-1}n_2)\, dn_1\, dn_2 - \iint K_{f',0}(n', h)\psi_{N'}(n')\, dn'\, dh \\ &= \iint K_{f,\mathrm{cont}}(n_1, n_2)\psi_N(n^{-1}n_2)\, dn_1\, dn_2 - \iint K_{f',cont}(n', h)\psi_{N'}(n')\, dn'\, dh. \end{aligned}$$

"Subtracting off the continuous spectrum" from the initial relative trace formula identity thus amounts to being able to prove that both sides of this last identity are identically zero.

According to (2.3) and (2.4), the left hand side of (3.1) reduces to the discrete expression

$$\sum_{\pi} A(\pi, f_S)(f'^S)^\vee(\mathrm{BC}(t(\pi^S)) - \sum_{\Pi} B(\Pi, f'_S)(f'^S)^\vee(t(\Pi^S)).$$

So the natural strategy should be to show that the *right* hand side of (3.1) can be written as the difference of two absolutely convergent expressions of the form

$$(3.2) \qquad \int \Phi(\sigma)(f'^S)^\vee(\sigma)\, d\sigma$$

with $\Phi(\sigma)$ an integrable function on the unramified unitary dual of G^S. In fact it turns out that the right-hand side of (3.1) contributes a *discrete as well as continuous* distribution (i.e., the continuous spectrums cancel each other only modulo an "unstable" piece of the cuspidal spectrum). Nevertheless, it is possible (after rearranging terms) to apply the trick of Langlands from [La2, p. 211]: since (3.1) amounts to an equality between an "atomic" and a "continuous" measure on the unramified unitary dual of G^S, both sides must be identically zero.

Consider for example the expression

$$\iint_{[N(F)\backslash N(\mathbb{A})]^2} K_{f,\mathrm{cont}}(n_1, n_2)\psi_N(n_1^{-1}n_2)\, dn_1\, dn_2, \tag{3.3}$$

where

$$\begin{aligned} K_{f,\mathrm{cont}}(x,y) &= \sum_{\chi=\{(M,\mu)\}} K_\chi(x,y) \\ &= \sum_{\mu}\sum_{\{\phi_\mu\}} \int_{-\infty}^{\infty} E(x, \rho(\mu, it)(f)\phi, it, \mu)\overline{E(y,\phi,it,\mu)}\, dt. \end{aligned}$$

Because the integration is over a compact domain, it is easy to compute that (3.3) equals

$$\sum_{\substack{\mu \\ \text{unramified} \\ \text{outside } S}} \int_{-\infty}^{\infty} (f^S)^\vee(\mu, it) c(f_S, \mu, it)\, dt. \tag{3.4}$$

where (μ, it) denotes the conjugacy class in ${}^LG^S$ attached to $\operatorname{Ind}\mu^S\|\|^{it}$,

$$c(f_S, \mu, it) = \sum_{\{\phi_\mu\}} W(\rho(\mu_S, it)(f_S)\phi, \psi, \mu, it)\overline{W(\phi,\psi,\mu,it)},$$

and

$$W(\phi,\psi,\mu,it) = \int_{N(F)\backslash N(\mathbb{A})} E(n,\phi,it,\mu)\psi_N(n)\, dn.$$

Moreover, because f^S "matches" ${f'}^S$, we have

$$(f^S)^\vee(\mu, it) = {f'}^S)^\vee(\mathrm{BC}(\mu, it), \tag{3.5}$$

with $(\mathrm{BC})(\mu, it) = (\mu', it)$ and $\mu'(z) = \mu(z\bar{z})$. Thus the contribution from $K_{f,\mathrm{cont}}$ to the right hand side of (3.1) is indeed of the form (3.2).

Now consider the relative trace formula expression

$$\int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} \int_{N'(F)\backslash N'(\mathbb{A})} K_{f'_{\mathrm{cont}}}(n', h)\psi_{N'}(n')\, dn'\, dh$$

corresponding to the quadratic extension. As we already hinted (and shall explain in the next section), this turns out to produce **discrete** as well as continuous sums of traces $({f'}^S)^\vee(\sigma)$.

4. Step II Continued: Periods of Truncated Eisenstein Series, and the Continuous relative trace over G. Our purpose here is to describe some of the difficulties involved in proving the following result:

Theorem 4.1. *The contribution of the continuous spectrum to the relative trace formula for G is*

$$\sum_{\substack{\mu' \\ \mu'=\mu\circ N_{E/F}}} \int_{-\infty}^{\infty} ({f'}^S)^\vee(\mu', it)c_1(f'_S, \mu', it)\,dt + \sum_{\substack{\mu' \\ \mu'\neq\mu\circ N_{E/F}}} ({f'}^S)^\vee(\mu', 0)c_2(f'_S, \mu', 0).$$

Here both expressions are absolutely convergent,

$$\sum_{\mu'} |c_2(f'_S, \mu', 0)| < \infty, \quad \textit{and} \quad \sum_{\mu'} \int_{-\infty}^{\infty} |c_1(f'_S, \mu', it)|\,dt < \infty.$$

As we already explained in the last section, this theorem is exactly what is needed to complete the proof of Theorem 1.1. Indeed, if we subtract the **integral** expression here from

$$\iint K_{f,\mathrm{cont}}(n_1, n_2)\psi_N(n_1^{-1}n_2)\,dn_1\,dn_2$$

in (3.1) we obtain (using(3.4) and (3.5)) the expression

$$\sum_{\mu'} \int_{-\infty}^{\infty} ({f'}^S)(\mu', it)c_*(f'_S, \mu', it)\,dt. \tag{4.2}$$

On the other hand, bringing the **discrete** sum in (4.1) over to the left-hand side of (3.1) yields the expression

$$\sum_{\substack{\pi \\ \text{cuspidal}}} A(\pi, f_S)({f'}^S)(\mathrm{BC}(t(\pi^S))) \tag{4.3}$$

$$\sum_{\Pi \text{ cuspidal}} B(\Pi, f'_S)({f'}^S)^\vee(t(\Pi^S))$$

$$- \sum_{\mu\neq\mu\circ N_{E/F}} ({f'}^S)^\vee(\mu', 0)c_2(f'_S, \mu', 0).$$

Hence we indeed can conclude (by the arbitrariness of $({f'}^S)$) that both (4.2) and (4.3) are identically zero.

Remark. For any given grossencharacter μ' of E which does *not* factor through the norm map, let $\pi(\mu')$ denote the cuspidal representation of $H(\mathbb{A})$ whose Langlands parameter t_{π_v} almost everywhere base change lifts to the Langlands parameter of $\mathrm{Ind}_{B'}^G \mu'_v$. These $\pi(\mu')$ are the cusp forms constructed by Hecke and Maass (using either the theory of theta-functions or L-functions). The fact that (4.3) equals zero gives (by "linear independence of characters") an alternate proof of the existence of such forms, as well as establishing the bijection asserted in Theorem 1.1.

Now let us return to Theorem 4.1, giving a concrete expression for the relative trace of $R_{\text{cont}}(f')$. As we have already noted—for $\alpha(T)$ sufficiently large,

$$\int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} \int_{N'(F)\backslash N'(\mathbb{A})} K_{f'_{\text{cont}}}(n',h)\psi_{N'}(n')\,dn'\,dh$$
$$= \iint \Lambda_2^T K_{f',\text{cont}}(n',h)\psi_N'(n')\,dn'\,dh.$$

Moreover, it follows from Propositions 2.3 and 2.5 of [Ja1] that this last expression equals

$$\sum_{\chi=\{(M',\mu')\}} \iint \Lambda_2^T K_\chi(n',h)\,\psi_{N'}(n')\,dn'\,dh$$

and, ultimately, that

$$\text{(4.4)}\quad \int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} \int_{N'(F)\backslash N'(\mathbb{A})} K_{f'_{\text{cont}}}(n',h)\psi_{N'}(n')\,dn'\,dh$$
$$= \sum_{\mu'} \int_{-\infty}^{\infty} \sum_{\{\phi'_{\mu'}\}} \Big(\int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} \Lambda^T E(h,\rho(\mu',it)(f')\phi',it)\,dh\Big)$$
$$\times \overline{W(\phi',\psi,\mu',it)(e)}\,dt.$$

Thus our task is to obtain an explicit expression for the right-hand side of this last equation, as $T \to \infty$ (i.e., for T sufficiently large $\dots$); in this we follow [Ja1].

Remark. Introducing a truncation operation facilitates the explicit calculation of the relative continuous spectrum of G but, as with the ordinary trace formula, also produces "discrete traces" from within the continuous spectrum, as we shall now explain.

In (4.4), the sum is over the (classes of) unitary characters μ' of $E^*\backslash\mathbb{A}_E^x$ inducing the Eisenstein series

$$E(g,\phi',\mu',s) = \sum_{\gamma\in B'(F)\backslash G(F)} \phi_\mu(\gamma g),$$

with $\phi_\mu \in \operatorname{Ind}\mu'\|^s$, and $B' = \operatorname{Res}_F^E B = M'N'$. In computing the **periods**

$$\mathcal{P}(\phi',\mu',it) = \int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} \Lambda^T E(h,\rho(\mu,it)(f)\phi',\mu',it)\,dh,$$

one appeals to the Bruhat-type decomposition

$$(*)\qquad G(F) = B'(F)H(F)\bigcup B'(F)\eta H(F),$$

where η in $G(F)$ satisfies $\eta\bar\eta^{-1} = w$. (One can take $\eta = \left(\begin{smallmatrix}\sqrt{\tau} & 1\\ -\sqrt{\tau} & 1\end{smallmatrix}\right)$ if $\sqrt{\tau}$ generates E.) The proposition below (see [JLai] and [Ja1]) is to be viewed as a "relative analogue" of the Langlands formula for the inner product of two truncated Eisenstein series.

Proposition 4.5. *For each unitary character μ' of $E^*\backslash\mathbb{A}_E^*$, set*

$$\delta_1(\mu') = \begin{cases} m(F^x\backslash\mathbb{A}_F^1) & \text{if } \mu' \,|_{F^x\backslash\mathbb{A}_F^1} \equiv 1; \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\delta_2(\mu') = \begin{cases} m(\mathbb{A}_F^* E^x\backslash\mathbb{A}_E^*) & \text{if } \mu(a) \equiv \mu(\bar{a}); \\ 0 & \text{otherwise,} \end{cases}$$

Then for sufficiently large $\alpha(T)$,

$$\begin{aligned} &\mathcal{P}(\phi', \mu', it) \\ &\quad = \delta_1(\mu')\Big\{\frac{e^{it(T_1-T_2)}}{it}\int_K \phi'(k)\,dk - \frac{e^{-it(T_1-T_2)}}{it}\int_K M(w,it)\phi'(k)\,dk\Big\} \\ &\qquad\qquad + \delta_2(\mu')\int_{T\eta(\mathbb{A})\backslash H(\mathbb{A}_F)} \phi'(\eta h)\,dh \end{aligned}$$

with

$$T_\eta = \eta^{-1}B'(F)\eta\bigcap H(F).$$

Remark. The δ_1-term here corresponds to the contribution from the "small-cell" part of (*), whereas the δ_2-term corresponds to the "big cell" $(w \neq e)$. Note that the second term is independent of the truncation parameter T. Also, all terms must be understood in the sense of meromorphic continuation from s in some right half-plane to $s = it$.

Sketch of the Proof. From the definition of Λ^T, and the formula for the constant term of E, one computes that

$$\begin{aligned} (4.6)\quad \Lambda^T E(g,\phi',\mu',s) = \sum_{\gamma\in B'(F)\backslash G(F)} &\phi'(\gamma g)(1-\hat{\tau}_B(H(\gamma g)-T) \\ &- M(w,s)\phi'(\gamma g)\hat{\tau}_B(H)(\gamma g)-T). \end{aligned}$$

(This is a special case of Arthur's "second formula" for $\Lambda^T E$; see Lemma 4.1 of [A3].) The next step is to check that (for s in the domain of convergence of the Eisenstein series), the series

$$\sum_{\gamma\in B'(F)\backslash G(F)} |\phi'(\gamma g)|\,|1-\hat{\tau}_B(H(\gamma g)-T)| + |M(w,s)\phi'(\gamma g)|\hat{\tau}_B(H(\gamma g)-T)$$

is integrable over $Z(\mathbb{A})H(F)\backslash H(\mathbb{A})$. This justifies the term by term integration of (4.6) over $Z(\mathbb{A})H(F)\backslash H(\mathbb{A})$; see [Ja1] §3 for the case of GL(3). Then the decomposition (*), together with Iwasawa's decomposition for $H(\mathbb{A})$, ultimately yields the Proposition. Implicit here is the fact that for $\alpha(T)$ sufficiently large,

$$\hat{\tau}_B(H(\eta h)-T) \equiv 0 \qquad \text{for all } h;$$

this requires some proof (see Proposition 7.2 of [Ja1] for a more general setting), and explains why the contribution from η is indeed independent of T.

In applying Proposition 4.5 to the calculation of

$$\iint K_{f',\mathrm{cont}}(n',h)\psi_{N'}(n')\,dn'\,dh, \tag{4.1}$$

let us first consider the contributions from characters μ' with $\delta_2(\mu') \neq 0$. *These are precisely the grossencharacters of E which factor thru the norm map from F.* If in addition $\delta_1(\mu') = 0$, then Proposition 4.5 implies that the contribution of such μ' to (4.1) is

$$\sum_{\mu'} \int_{-\infty}^{\infty} \sum_{\{\phi'_{\mu'}\}} \theta_f(\mu', it, \phi') W(\phi', \psi, \mu', it)\,dt \tag{4.7}$$

where

$$\theta_f(\mu', it, \phi') = \delta_2(\mu') \int_{T_\eta(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} \rho(\mu', it)(f)\phi'(\eta h)\,dh.$$

Note we can rewrite (4.7) as

$$\sum_{\mu'} \int_{-\infty}^{\infty} ({f'}^S)^\vee(\mu', it) C_{f'_S}(\mu', it)\,dt$$

with

$$C_{f'_S}(\mu', it)\,dt = \sum_{\phi'} W(\phi', \psi, \mu', it)\theta_{f'_S}(\mu', it, \phi').$$

Now suppose $\delta_2(\mu') = 0$, i.e., μ' does *not* factor thru the norm map from F. Then Proposition 4.5 implies that the contribution of each such μ' to the relative trace (4.1) is

$$\int \sum_{\phi'} W(\phi', \psi, \mu', it) \cdot \tag{4.8}$$
$$\delta_1(\mu') \left\{ \frac{e^{it(T_1-T_2)}}{it} \int_K f\phi'(h)\,dh - \frac{e^{-it(T_1-T_2)}}{it} \int_K M(w, it) f\phi'(k)\,dk \right\} dt,$$

with

$$f\phi' = \rho(\mu', it)(f)\phi'.$$

Notice the similarity here with the formula encountered for the ordinary trace formula term $J_\chi^T(f)$ in §4 of Lecture VII. There we computed the limit as $T \to \infty$ by applying Arthur's splitting formula for products of (G, M)-families. Here we can do the same thing, taking (as before)

$$c_B(it) = e^{2it(T_1-T_2)} \quad \text{and} \quad c_{\bar{B}}(it) = e^{-2it(T_1-T_2)}.$$

But now we set

$$d_B(it) = W(\phi', \psi, \mu', it) \int_K f\phi'(k)\, dk$$

and

$$d_{\bar{B}}(it) = W(\phi', \psi, \mu', it) \int_K M(w, it)(f\phi')(k)\, dk.$$

It must of course be checked that this $d(\Lambda)$ is also a (G, M)-family (not obvious!). The main task, though, is to compute

$$\lim_{\varepsilon \to 0} \lim_{T \to \infty} \sum_{\phi'} (cd)_M(it) B(\varepsilon t)\, dt,$$

with $B(t)$ a compactly supported Schwartz function as in Lecture VII. Computing as before, we find that

$$d'_G(it) = d_B(0)(= d_{\bar{B}}(0)) = W(\phi', \psi, \mu', 0) \int_K \rho(\mu', 0)(f)\phi'(k)\, dk$$

with ϕ' in $\operatorname{Ind} \mu'$, and—ultimately—that the above limit equals

$$\sum_{\phi} W(\phi', \psi, 0, \mu') \int \rho(0, \mu')(f')\phi'(k)\, dk.$$

Thus we find that the total contribution (to the relative trace formula) from characters μ' of this form is

$$\sum_{\substack{\mu' \\ \mu' \neq N \circ \mu}} ({f'}^S)^\vee(0, \mu') C({f'}_S, \mu')$$

with

$$C(\psi, {f'}_S, \mu') = \sum_{\phi} W(\phi', \psi, 0, \mu') \int_K \rho(0, \mu'_S)(f'_S)\phi'(k)\, dk.$$

It remains to observe that when $\delta_2(\mu') \neq 0$ *and* $\delta_1(\mu') \neq 0$, the contribution to the relative trace formula is again just (4.5). There are apparent contributions

$$\frac{e^{2it(T_1 - T_2)}}{it} \int f\phi'(k)\, dk - \frac{e^{-2it(T_1 - T_2)}}{it} \int M(w, it)\phi'(k)\, dk$$

from $\mathcal{P}(\phi', \mu', t)$, but they integrate to zero over $\mathbb{R}$ by way of a Riemann–Lebesgue Lemma (just like the terms in $cd_M(t)$ not belonging to the leading term $Q = G$ in $\mathcal{F}(M)$ at the end of the last Lecture). We spare the reader the details.

Concluding Remarks. A host of natural problems present themselves in connection with any general development of the relative trace formula, especially as far as applications are concerned. In addition to the obvious problem of developing general local "matching" theorems (and related "fundamental lemmas"), there are these problems from the "spectral side":

(1) Develop a general formula for the period of a truncated Eisenstein series, analogous to Langlands' formula for the inner product of truncated Eisenstein series;
(2) Apply a general theory of (G, M)-families and smoothing functions $B(\lambda)$ to the calculation of the contribution of the "continuous spectrum" to the relative trace formula, and
(3) Prove in general that the resulting spectral expressions are sufficiently convergent to justify subtracting them off from the full relative trace formula. For the case of GL(2), this can done by direct estimates on the intertwining operators, but already for GL(3) such a direct approach is problematic.

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Lecture IX. Some Applications of Paley–Wiener, and Concluding Remarks

In Lecture VII, we explained how (G, M)-families arose in Arthur's explicit calculation of the general spectral terms $J_\chi(f)$. We also alluded to why a Paley–Wiener theorem was needed to complete this calculation, and promised to return to it. In fact, this Paley–Wiener theorem is also crucial in handling other convergence problems related to the trace formula. So our purpose now is to finally touch on these matters, albeit briefly. This being the last lecture, we shall also conclude with a few remarks orienting the reader to some papers of Arthur's that we so far haven't even mentioned.

1. The Paley–Wiener Theorem. The Fourier transform of a compactly supported smooth function f on $\mathbb{R}$ extends to an entire function $F(\Lambda)$ on $\mathbb{C}$, and satisfies a well-known growth condition, namely that there exists a constant N (depending on the support of f) such that

$$\sup\bigl\{|F(\Lambda)|e^{-N|\operatorname{Re}(\Lambda)|}(1+|\operatorname{Im}(\Lambda)|)^n\bigr\} < \infty$$

for any integer n. According to the classical Paley–Wiener Theorem, these properties characterize the image of the Fourier transform on the space of such test functions f.

For a real reductive group G, let $C_c^\infty(G, K)$ denote the "Hecke algebra" of K-finite compactly supported smooth functions. Then for any f in $C_c^\infty(G, K)$ and irreducible admissible representation π of G (on a Banach space U_π), the Fourier transform of f at π is defined to be the operator

$$\pi(f) = \int_G f(x)\pi(x)\,dx.$$

The resulting function

$$\pi \to \pi(f)$$

has as its domain the set of irreducible (admissible) representations of G, and for any (π, U_π) takes values in the space of operators on U_π. The purpose of the (operator) Paley–Wiener Theorem of [A10] is to characterize which functions

$$\pi \mapsto F(\pi)$$

are of the form $\pi \mapsto \pi(f)$ for some f.

More precisely, let us (following Arthur) fix a minimal parabolic subgroup B of G with Langlands decomposition $N_0A_0M_0^1$, and consider the nonunitary principal series of induced representations $\rho_B(\sigma,\Lambda)$, indexed by quasi-characters Λ of A_0 and irreducible representations σ of M_0^1. By a well-known theorem of Harish-Chandra, any π is equivalent to a subquotient of some $\rho_B(\sigma,\Lambda)$. This means that $\pi(f)$ will be completely determined by the **Fourier transform**

$$\hat{f}\colon (\sigma,\Lambda) \to \hat{f}_B(\sigma,\Lambda) = \rho_B(\sigma,\Lambda)(f).$$

What should the image of this Fourier transform be?

Arthur defines $PW(G,K)$ to be the space of functions

$$F\colon (\sigma,\Lambda) \mapsto F_B(\sigma,\Lambda),$$

which are entire, K-finite (in a sense I shall not explain here), and satisfy the growth condition

$$\sup_{\sigma,\Lambda}\{\|F(\sigma,\Lambda)\|e^{-N\|\operatorname{Re}(\Lambda)\|}(1+\|\operatorname{Im}(\Lambda)\|)^n\} < \infty, \tag{1.1}$$

for some N (and all n). Moreover, there is another, more complicated condition for $F(\sigma,\Lambda)$ which comes from the various intertwining maps between principal series. Namely, if there is a relation

$$\sum_{k=1}^{m} D_k(\rho_B(\sigma_k,\Lambda_k)(g)u_k,v_k) = 0$$

valid for all g in G, where D_k is a differential operator (acting through the variable Λ, and u_k, v_k are vectors in the space of $\rho(\sigma_k,\Lambda_k)$, then $F(\sigma,\Lambda)$ must also satisfy the relation

$$\sum_{k=1}^{m} D_k(F(\sigma_k,\Lambda_k)u_k,v_k) = 0. \tag{1.2}$$

(Such relations exist commonly, but are difficult to characterize explicitly; indeed, a listing of all such relations would be tantamount to a complete knowledge of the irreducible subquotients of the principal series.)

The main result of [A10] is that

$$f \to \hat{f}(\sigma,\Lambda)$$

provides an algebra isomorphism from $C_c^\infty(G,K)$ onto $\mathrm{PW}(G,K)$. The consequence of this result which is actually needed for the trace formula concerns the notion of **Paley–Wiener multipliers**.

By a "multiplier" of $C_c^\infty(G,K)$ is meant a linear operator C on $C_c^\infty(G,K)$ such that

$$C(f*g) = C(f)*g = f*C(g)$$

for all f, g in $C_c^\infty(G,K)$. (The algebra of all such "multipliers" coincides with $\mathrm{End}_{\mathfrak{U}(G)}(C_c^\infty(G,K))$. The **Paley–Wiener multipliers** are constructed as follows. Set

$$\mathfrak{h} = i\mathfrak{h}_K \oplus \mathfrak{h}_0, \tag{1.3}$$

where $\mathfrak{h}_K$ is a fixed Cartan subalgebra of the Lie algebra of $K \cap M_0$, and $\mathfrak{h}_0$ is the Lie algebra of the split component of M_0. (Here M_0 is the Levi component of a fixed minimal parabolic subgroup of G.) If $\mathfrak{G}$ is the Lie algebra of G, then $\mathfrak{h}_\mathbb{C}$ is a Cartan subalgebra of $\mathfrak{G}_\mathbb{C}$, and $\mathfrak{h} \subset \mathfrak{h}_\mathbb{C}$ is invariant under the Weyl group W of $(\mathfrak{G}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. Let $\mathcal{E}(\mathfrak{h})^W$ be the algebra of compactly supported distributions on $\mathfrak{h}$ which are invariant under W. Then for any γ in $\mathcal{E}(\mathfrak{h})^W$, the Fourier–Laplace transform

$$\hat{\gamma}(\nu), \qquad \nu \in \mathfrak{h}_\mathbb{C}^*$$

is an entire function on $\mathfrak{h}_\mathbb{C}^*$ which is W-invariant and satisfies a growth estimate of the form

$$\sup_{\nu \in \mathfrak{h}_\mathbb{C}^*} \left(|\hat{\gamma}(\nu)| e^{-N_\gamma \|\operatorname{Re}(\nu)\|}(1 + \|\operatorname{Im}\nu\|^{-n_\gamma}\right) < M \tag{1.4}$$

for some integers N_γ and n_γ. According to the result below, these $\hat{\gamma}$ provide us with multipliers C_γ (called **Paley–Wiener multipliers**).

Theorem 1.5 (see Thm. 4.2 of [A10]). *For each distribution γ in $\mathcal{E}(\mathfrak{h})^W$, and f in $C_c^\infty(G,K)$, there is a unique f_γ in $C_c^\infty(G,K)$ such that*

$$\pi(f_\gamma) = \hat{\gamma}(\nu_\pi)\pi(f) \tag{1.5}$$

for any irreducible admissible representation π. (Here $\{\nu_\pi\}$ is the W-orbit in $h_\mathbb{C}^*$ associated to the infinitesimal character of π.) In particular, if we define

$$C_\gamma(f) = f_\gamma,$$

then the map

$$\gamma \to C_\gamma, \qquad \gamma \in \mathcal{E}(\mathfrak{h})^W,$$

is a homomorphism from the algebra $\mathcal{E}(\mathfrak{h})^W$ to the algebra of multipliers

$$\mathrm{End}_{\mathfrak{U}(G)}(C_c^\infty(G,K)).$$

Remarks. (i) This Theorem is almost an immediate consequence of the Paley–Wiener Theorem described earlier. Indeed, the uniqueness is clear (since (1.5) completely determines the Fourier transform of f_γ). On the other hand, the existence of f_γ results from checking that the function

$$F_B(\sigma,\Lambda) = \hat{\gamma}(\nu_\sigma + \Lambda)\hat{f}_B(\sigma,\Lambda)$$

belongs to $\mathrm{PW}(G,K)$, and hence is the Fourier transform of some f_γ; then since $\pi(f_\gamma)$ is given by the action of $\rho(\sigma,\Lambda)(f_\gamma)$ on the appropriate subquotient, it follows that

$$\pi(f_\gamma) = \hat{\gamma}(\nu_\sigma + \Lambda)\pi(f) = \hat{\gamma}(\nu_\pi)\pi(f),$$

as required.

(ii) That $C_\gamma(f) = f_\gamma$ defines a "multiplier" of $C_c^\infty(G,K)$ follows formally from (1.5): $C_\gamma(f * g) = C_\gamma(f) * g$, since

$$\begin{aligned}\pi((f*g)_\gamma) &= \hat{\gamma}(\nu_\pi)\pi(f*g)\\ &= \hat{\gamma}(\nu_\pi)\pi(f)\pi(g)\\ &= \pi(f_\gamma)\pi(g).\end{aligned}$$

(iii) This Paley–Wiener theorem may be viewed as an analogue for real groups of the p-adic results of [BDK] and [Ro3].

2. Applications of Paley–Wiener to the Calculation of $J_\chi(f)$.. Recall how $J_\chi^T(f)$ was defined (for sufficiently large T) to be the integral over the diagonal of the truncated kernel function $\Lambda_2^T(K_\chi(x,y)$. Using the explicit formula for $\Lambda_2^T(K_\chi(x,y)$ in terms of Eisenstein series, we then wrote

$$J_\chi^T(f) = \sum_P \sum_\sigma \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \Psi_\sigma^T(\lambda, f)\, d\lambda, \tag{2.1}$$

where

$$\Psi_\sigma^T(\lambda,f) = \frac{1}{n(A_P)}\,\mathrm{tr}(\Omega_{\chi,\sigma}^T(P,\lambda)\rho(\sigma,\lambda)(f)),$$

and $\Omega_{\chi,\sigma}^T(P,\lambda)$ is the operator on the space of $\rho(\sigma,\lambda)$ defined by

$$(\Omega_{\chi,\sigma}^T(P,\lambda)\phi',\phi) = \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \Lambda^T E(x,\phi',\lambda)\overline{\Lambda^T E(x,\phi,\lambda)}\, dx.$$

We stress that the integral formula (2.1) for $J_\chi^T(f)$ is **absolutely convergent**, and that $J_\chi^T(f)$ is known to be **polynomial in** T. The problem is to *explicitly* compute $J_\chi^T(f)$ for a conveniently chosen value T_0 of T.

At the end of Lecture IV, we used Arthur's general theory to derive an explicit formula for $J_\chi^T(f)$ *in the case of* $G = \mathrm{GL}(2)$. The strategy consisted of three steps:

(1) plug into (2.1) Langlands' explicit formula for $\Omega_{\chi,\sigma}^T(P,\lambda)$ in terms of intertwining operators;
(2) use the theory of (G,M)-families to rewrite $\Psi_\sigma^T(\lambda,f)$ in a form still more amenable to integration over $d\lambda$; and
(3) compute the resulting integrals over λ (and show that some of the contributions are negligible as $T\to\infty$).

It was in this last step that we resorted to a Riemann–Lebesgue Lemma whose application is best justified by way of Paley–Wiener multipliers; indeed these multipliers allow for the insertion of a compactly supported function $B(\lambda)$ into the formula for $J_\chi T$, as explained in Theorem 2.3 below.

For general G, there is an additional, more fundamental reason for requiring the Paley–Wiener Theorem in the calculation of $J_\chi(f)$. It derives from the fact that Langlands' explicit formula for $\Omega^T_{\chi,\sigma}(P,\lambda)$ is *not* valid for Eisenstein series induced from non-cuspidal σ. (Equivalently, it is not valid when P lies outside P_χ, the set of parabolics attached to the collection of pairs (M_B, r_B) describing the cuspidal datum χ.) In these cases, Arthur proved "only" that the "nice" formula for $\Omega^T_{\chi,\sigma}(P,\lambda)$ is an **asymptotic** one for large T, valid uniformly only on compact subsets of λ.

More precisely, let $\omega^T_{\chi,\sigma}(P,\lambda)$ denote the operator on $\rho(\sigma,\lambda)$ defined explicitly in terms of intertwining operators as the value at $\lambda' = \lambda$ of

$$\sum_{P_1}\sum_{t,t'} M_{P_1|P}(t,\lambda)^{-1} M_{P_1|P}(t',\lambda') \frac{e^{(t'\lambda'-t\lambda)(T)}}{\theta_{P_1}(t'\lambda'-t\lambda)}$$

(cf. formula (2.1) of Lecture VII for complete details). Then (cf. Corollary 9.2 of [A9]) the difference between $(\Omega^T_{\chi,\sigma}(P,\lambda)\rho(\lambda,\sigma)(f)\phi,\phi)$ and (the "nicer" expression) $(\omega^T_{\chi,\sigma}(P,\lambda)\rho(\lambda,\sigma)(f)\phi,\phi)$ is bounded in absolute value by

$$r(\lambda)\|\phi\|^2 e^{-\epsilon\|T\|}, \tag{2.2}$$

where $\epsilon > 0$, and $r(\lambda)$ is a locally bounded function on $i\mathfrak{a}^*_P$. Thus the integrals of these expressions over *all* of $i\mathfrak{a}^*_P/i\mathfrak{a}^*_G$ need not approach one another as $T \to \infty$. In short, the substitution of $\omega^T_{\chi,\sigma}$ for $\Omega^T_{\chi,\sigma}$ in (2.1) is simply *not* justified.

Example. For $G = \mathrm{GL}(2)$, we explicitly derived Arthur's estimate(2.2) in case $\chi = \{(M,\mu)\}$, with $\mu = \mu^{-1}$, and $P = G$. Namely, in the proof of Prop. 2.7 in Lecture IV, we computed the difference of these expressions to be $Me^{-2(T_1-T_2)}$. Of course, in this case the problem of **uniformity** alluded to above is *a priori* absent (the variable λ being constrained to lie in the space $i\mathfrak{a}^*_P/i\mathfrak{a}^*_G = \{0\}\ldots$).

In general, the way out of this problem, like the problem of showing certain terms to be "negligible" in step (3) above, is by way of Arthur's Paley–Wiener Theorem, as we shall now (finally) explain.

For any B in $\mathcal{S}(i\mathfrak{h}^*/i\mathfrak{a}^*_G)^W$, and irreducible unitary representation σ on $M(\mathbb{A})$, define

$$B_\sigma(\lambda) = B(iY_\sigma + \lambda),$$

Here Y_σ is the imaginary part of the orbit $\{\nu_\sigma\}$ in $\mathfrak{h}^*_{\mathbb{C}}$ associated to the infinitesimal character of σ_∞ (and we have fixed an embedding of any $\mathfrak{a}^*_P$ in $\mathfrak{h}^*$). Recall that $\{\nu_\sigma + \lambda\}$ is also the orbit in $\mathfrak{h}^*_{\mathbb{C}}$ associated to the infinitesimal character of the induced representation $\rho(\sigma_\infty,\lambda)$ of G_∞.

Theorem 2.3 (see Thm. 6.3 of [A5]). *Suppose B in $\mathcal{S}(i\mathfrak{h}^*/i\mathfrak{a}_G^*)^W$ is such that $B(0) = 1$. For any $\epsilon > 0$, write B^ϵ for the function $B^\epsilon(\nu) = B(\epsilon\nu)$. Then*

$$J_\chi^T(f) = \lim_{\epsilon\to 0}\lim_{T\to\infty}\sum_{P,\sigma}\int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*}\Psi_\sigma^T(\lambda, f)B_\sigma^\epsilon(\lambda)\,d\lambda.$$

Remarks. (i) The importance of this result is that it allows us to invoke the **asymptotic** formula for $\Omega_{\chi,\sigma}^T$. Indeed, if we take B in the Theorem to be **compactly** supported, then we can indeed substitute $\omega_{\chi,\sigma}^T$ for $\Omega_{\chi,\sigma}^T$ and write

$$J_\chi^T(f) = \lim_{\epsilon\to 0}\lim_{T\to\infty}\sum_{P,\sigma}\int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*}\mathrm{tr}(\omega_{\chi,\sigma}^T(P,\lambda)\rho(\lambda,\sigma)(f))B_\sigma^\epsilon(\lambda)\,d\lambda \tag{2.4}$$

(since the error term (2.2), multiplied by such a compactly supported B, will indeed approach 0 uniformly in λ).

(ii) For large T, the expression

$$\sum_{P,\sigma}\int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*}\Psi_\sigma^T(\lambda, f)B_\sigma(\lambda)\,d\lambda$$

is asymptotic to a polynomial $P^T(B)$ in T. So Theorem 2.3 really says

$$J_\chi^T(f) = \lim_{\epsilon\to 0}P^T(B^\epsilon),$$

or

$$J_\chi^T(f) = \lim_{\epsilon\to 0}\lim_{T\to\infty}\sum_{P,\sigma}\int\Psi_\sigma^T(\lambda, f)B_\sigma^\epsilon(\lambda)\,d\lambda,$$

where the limit in T (as in the statement of the Theorem) is interpreted as the polynomial $P^T(B^\epsilon)$ which is asymptotic to the given function as T approaches ∞. On the other hand, since

$$\lim_{\epsilon\to 0}B_\sigma^\epsilon(\lambda) = \lim_{\epsilon\to 0}B(\epsilon(iY_\sigma+\lambda)) = B(0) = 1,$$

the dominated convergence theorem implies

$$\begin{aligned}
J_\chi^T(f) &= \sum_{P,\sigma}\int\Psi_\sigma^T(\lambda)\,d\lambda \qquad \text{(for } T \text{ large)}\\
&= \lim_{\epsilon\to 0}\sum_{P,\sigma}\int\Psi_\sigma^T(\lambda)B_\sigma^\epsilon(\lambda)\,d\lambda \qquad (T \text{ large})\\
&= \lim_{T\to\infty}\lim_{\epsilon\to 0}\sum_{P,\sigma}\int\Psi_\sigma^T(\lambda)B_\sigma^\epsilon(\lambda)\,d\lambda
\end{aligned}$$

(with $\lim_{T\to\infty}$ interpreted as above). Thus all of the work of [A5] essentially boils down to a justification of the interchange of order of two limits (over ϵ and T)!

(iii) (**Concerning the Proof of Theorem 2.3**). The basic idea of the proof is explained clearly on pp. 1258–60 of [A5]. The starting point is an application of the Paley–Wiener multiplier Theorem 1.5 to the **archimedean** component of any K-finite function f in $C_c^\infty(G(\mathbb{A}))$. For any γ in $\mathcal{E}(\mathfrak{h})^W$, this yields a function f_γ in $C_c^\infty(G(\mathbb{A}))$ such that

$$\Psi_\sigma^T(\lambda, f_\lambda) = \hat{\gamma}(\nu_\sigma + \lambda)\Psi_\sigma^T(\lambda, f).$$

Using this relation, we write

$$\begin{aligned} J_\chi^T(f_\lambda) &= \sum_{P,\sigma} \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \hat{\gamma}(\nu_\sigma + \lambda)\Psi_\sigma^T(\lambda, f)\, d\lambda \qquad \text{(for } T \text{ large)} \\ &= \sum_{P,\sigma} \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \Psi_\sigma^T(\lambda, f)\Big(\int_{\mathfrak{h}} \gamma(H)e^{(\nu_\sigma+\lambda)(H)}\, dH\Big)\, d\lambda \\ &= \int_{\mathfrak{h}} \Big(\sum_{P,\sigma} \Psi_\sigma^T(H)e^{\nu_\sigma(H)}\Big)\gamma(H)\, dH \end{aligned} \tag{2.5}$$

where

$$\Psi_\sigma^T(H) = \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \Psi_\sigma^T(\lambda, f)e^{\lambda(H)}\, d\lambda.$$

In particular, if we fix H in $\mathfrak{h}$, and set

$$\gamma^H = |W|^{-1} \sum_{s\in W} \gamma_{s^{-1}H}$$

with $\gamma_{s^{-1}H}$ the Dirac measure at $s^{-1}H$, then (2.5) implies

$$J_\chi^T(f_{\gamma^H}) = |W|^{-1} \sum_{s\in W} \sum_{P,\sigma} \Psi_\sigma^T(s^{-1}H)e^{\nu_\sigma(s^{-1}H)} \tag{2.6}$$

Note here that, for each value of H, the right hand side of (2.6) is a polynomial in T (since the left hand side is). Thus we have constructed a family of polynomials in T, call them $p^T(H)$, *whose value at* $H = 0$ *is exactly* $J_\chi^T(f)$ (since then $f_{\gamma^H} = f$).

Now to compute $p^T(H)$ at 0, the natural thing to do is to integrate this function against an arbitrary Schwartz function $\beta(H)$. By the Plancherel Theorem for $\mathfrak{h}$, the resulting inner product $\int p^T(H)\beta(H)\, dH$ can be replaced by one over $i\mathfrak{h}^*/i\mathfrak{a}_G^*$ involving the Fourier transform

$$B(\nu) = \int \beta(H)e^{\nu(H)}\, dH.$$

In particular, $J_\chi^T(f)$ should be obtained by having β approximate the Dirac measure at 0, i.e., by using

$$\beta_\epsilon(H) = \epsilon^{-\dim(\mathfrak{h})}\beta(\epsilon^{-1}H)$$

with ϵ small, and $\int \beta(H)\,dH = 1$; Equivalently, this brings into play

$$B^\epsilon(\nu) = B(\epsilon\nu)$$

with $B(0) = 1$, and this is *roughly* how one ends up with the sought-after formula

$$J_\chi^T(f) = \lim_{\epsilon\to 0} \sum_{P,\sigma} \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \Psi_\sigma^T(\lambda, f) B_\sigma^\epsilon(\lambda)\,d\lambda.$$

The only problem is that $p^T(H)$ is not actually tempered, and hence cannot be integrated against any β! Thus a lengthy detour is required in Sections 4–6 of [A5], involving additional polynomials and analysis (but nothing approaching either the Paley–Wiener theorem, or the theorem on the polynomial nature of $J_\chi^T(f)$).

3. A Final Formula for $\sum_\chi J_\chi^T(f)$. At the end of Lecture VII, we explained how, for GL(2), the formula

$$J_\chi^T(f) = \lim_{\epsilon\to 0} \lim_{T\to\infty} \int_{-\infty}^{\infty} \operatorname{tr}(\omega_\chi^T(t)\rho(\mu, it)(f) B(\epsilon t)\,dt$$

could be computed explicitly using (G, M) families. Now, thanks to formula 2.4, we have a similar strategy for computing $J_\chi^T(f)$ in general, and the end result is a formula directly generalizing the explicit formula

$$\begin{aligned} J_\chi^T(f) = (T_1 - T_2) \int_{-\infty}^{\infty} \operatorname{tr}(\rho(\mu, it)(f))\,dt + \int_{-\infty}^{\infty} \operatorname{tr}(M(-it)M'(it)\rho(\mu, it)(f)\,dt \\ + \tfrac{1}{4}\operatorname{tr}(M(0)\rho(\mu, 0)(f)) + \mu(f)\tau(G). \end{aligned}$$

(Recall that the "limit in T" is to be interpreted as the polynomial asymptotic to the given integral as T approaches ∞.)

N.B. In Lecture IV, we were interested only in the constant term of the polynomial $J_\chi^T(f)$, i.e., $J_\chi^T(f) = J_\chi^0(f)$. In Arthur's general theory, one fixes on a convenient point T_0 (which in many "classical" situations is just zero; see Lemma 1.1 of [A4]), and sets

$$J_\chi(f) = J_\chi^{T_0}(f).$$

Theorem 3.2 (see Theorem 8.2 of [A6]). *Suppose $f \in C_c^\infty(Z(\mathbb{A})\backslash G(\mathbb{A}))$. Then $J_\chi^T(f)$ equals the sum over $M \in \mathcal{L}(M_0)$, $L \in \mathcal{L}(M)$, σ on $M(\mathbb{A})$ (compatible with χ), and w in $W^L(\mathfrak{a}_M)_{\mathrm{reg}}$ of the product of*

$$|W_0^M|\,|W_0|^{-1}|\det(w-1)_{\mathfrak{a}_M^L})|^{-1}$$

with

$$\int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} |\mathcal{P}(M)|^{-1} \sum_{P\in\mathcal{P}(M)} \operatorname{tr}(\mathcal{M}_L(P, \lambda) M(P, w)\rho(\sigma, \lambda)(f))\,d\lambda.$$

Remarks. (i) The exact definition of each of the objects appearing here is to be found in Sections 4 through 8 of [A6]. But the point is that they are all more or less familiar objects related to the intertwining operators $M_{Q|P}$ discussed in Lecture VII. For GL_2, there will be only three terms in the sum above: one for $M = L = M_0$, with $w = e$ and $\sigma = \mu$ a character of M_0 (this gives the term in (3.1) with the logarithmic derivative); one for $M = M_0, L = G$, w the non-trivial Weyl element, and σ a character of M_0 (this gives the term involving $\mathrm{tr}(M(0)\rho(\mu, 0)(f))$); and one with $M = L = G$, $w = e$, and μ the character $\mu(\det g)$ of $G(\mathbb{A})$ (this gives the contribution $\mu(f)\tau(G)$).

(ii) The proof of Theorem 3.2 consists of two non-trivial steps. First one uses (G, M) families to carry out the explicit computation of

$$\operatorname*{val}_{T=T_0}\left(\lim_{\epsilon\to 0} \text{“}\lim_{T\to\infty}\text{”} \sum_P \sum_\sigma \int \omega^T_{\sigma,\chi}(\lambda, f) B_\sigma(\epsilon\lambda)\, d\lambda\right),$$

where $\omega^T_{\sigma,\chi}(\lambda, f)$ is a sum of special values of functions $e_M(\Lambda)$ for particular (G, M)-families $e(\Lambda) = c(\Lambda)d(\Lambda)$ (involving the intertwining operators $M_{Q|P}$); this is carried out in Sections 3 through 5 of [A6], and the end result is a sum of terms of the form

$$\lim_{\epsilon\to 0} \int_{i\mathfrak{a}^*_L/i\mathfrak{a}^*_G} \sum_{P\in\mathcal{P}(M)} \mathrm{tr}(\mathcal{M}_L(P, \lambda)M(P, w)\rho(\lambda, \sigma)B^\epsilon_\sigma(\lambda)d$$

with B in $C^\infty_c(i\mathfrak{h}^*/i\mathfrak{a}^*_G)^W$ equal to 1 at 0. Clearly the second step must be to show that this (limit and) test function can be made to disappear!

Equivalently, by the dominated convergence theorem, one must show that the sum

$$\sum_\sigma \int_{i\mathfrak{a}^*_L/i\mathfrak{a}^*_G} \|\mathcal{M}_L(P, \lambda)\rho(\lambda, \sigma)(f)\|\, d\lambda$$

is finite. This is carried out in Sections 6 through 8 of [A6], using normalized intertwining operators, and estimates on the scalar functions defining these normalizations. (Some assumptions are made on these normalizations at p-adic places, but at least for $\mathrm{GL}(n)$ these assumptions are now proved; see [Sha1] and [Sha2]).

Finally, having reached the exalted plateau which Theorem 3.2 represents, we should mention that it is really just the *beginning*! In particular it is not yet known if the explicit spectral expansion

$$\sum_\chi J_\chi(f) = \sum_\chi \sum_{\substack{M,L\\ w}} \sum_\sigma \int_{\lambda\in i\mathfrak{a}^*_L/i\mathfrak{a}^*_G} \sum_P \mathrm{tr}(\mathcal{M}^*_L(P, \lambda, \sigma, w, f)\, d\lambda \tag{3.3}$$

converges *absolutely* as a multiple integral. (The problem is that the estimates on the normalizing factors alluded to above are not uniform in χ!).

In other words, we know that

$$\sum_\chi |J_\chi(f)| < \infty,$$

and that

$$\sum_{\sigma\in\prod(M(\mathbb{A}))_\chi} \int \|\mathcal{M}_L(P,\lambda)\rho(\lambda,\sigma)(f)\|\, d\lambda < \infty,$$

but not that this last sum can be summed over all χ. This lack of knowledge causes problems in *applying* the trace formula, say to base change, where on needs absolutely convergent expressions (in order to use tricks like Langlands' to subtract off the non-discrete spectrum). In order to get around this problem, Arthur proves certain estimates on the convergence of

$$\sum_\chi |J_\chi(f)|$$

which may be viewed as a weak form of the *conjectured* absolute convergence of the multiple integrals (3.3). This involves the infinitesimal characters of the infinite components σ_∞ of the representations σ appearing in the spectral expansion, and an application of the Paley–Wiener multiplier theorem. The end result is a technique called "separation of infinitesimal characters via multipliers" which allows one to obtain the required equality of discrete traces from the matching of the full traces; details appear in [A12], [AC], and [La3].

4. Related Works. Here are *some* of the topics from Arthur's work which we have ignored till now.

In comparing trace formulas for different groups (for example, in proving base change for GL_n), one wants to rewrite the basic formula

$$\sum_{\mathfrak{o}} J_{\mathfrak{o}}(f) = \sum_\chi J_\chi(f) \tag{4.1}$$

in **invariant** form, i.e., as an identity

$$\sum_{\mathfrak{o}} I_{\mathfrak{o}}(f) = \sum_\chi I_\chi(f), \tag{4.2}$$

where each of the terms is an **invariant** distribution. The reason for this is that one needs to assert the equality of two trace formulas for "associated" (or "matching") functions, given only by their orbital integrals (which are invariant distributions ...). Such an invariant trace formula is obtained already in [A4], where a subtle rearrangement and modification of the terms $J_{\mathfrak{o}}(f)$ and $J_\chi(f)$ yields the new collection of invariant distribution $I_{\mathfrak{o}}(f)$ and $I_\chi(f)$ appearing in (4.2). However, this is only the "coarse" form of the invariant trace formula, analogous to the "coarse" formula $\sum J_{\mathfrak{o}}(f) = \sum J_\chi(f)$. To get the "fine" expansion of the invariant trace formula requires the papers [A11] and [A12]

Explicit formulas for $I_{\mathfrak{o}}(f)$ are obtained from the analogous fine expansions of $J_{\mathfrak{o}}(f)$ discussed in [A13] and [A14]. (In Lecture IV we described the exact form of $J_{\mathfrak{o}}(f)$—for **unramified** $\mathfrak{o}$—in terms of weighted orbital integrals; the extreme opposite case—of $\mathfrak{o}$ corresponding to $\{1\}$—is the subject matter of [A13], and the general case—a mixture of these extremes—is handled in [A14].) On the other hand, explicit formulas for $I_\chi(f)$ are obtained by another delicate mixture of the techniques of Paley–Wiener multipliers and (G, M)-families, somewhat analogously to the case of $\sum J_\chi(f)$.

Remarks. (i) In a very recent work, Labesse has explained how one can apply the trace formula to the base change problem for $\mathrm{GL}(n)$ *without* first putting the trace formula in invariant form. This gives an alternate, simpler approach to [AC]; see [La3].

(ii) Recall from our discussion of Jacquet's *relative* trace formula that no truncation operator was needed to define the "relative" distributions $J_{\mathfrak{o}}(f)$ and $J_\chi(f)$; hence these terms are already invariant!

(iii) Outside the scope of these "introductory" Lectures are the works of Kottwitz and others on the stable trace formula (see [Kot2] and [La5]), and the more recent papers of Arthur on unipotent representations, A-packets, etc. (see [A15], [A16]).

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