The second bounded cohomology of a group with infinitely many ends *

Koji Fujiwara †
Mathematics Department, Keio University
Yokohama, 223, Japan
fujiwara@math.keio.ac.jp

Abstract

We study the second bounded cohomology of an amalgamated free product of groups, and an HNN extension of a group. As an application, we have a group with infinitely many ends has infinite dimensional second bounded cohomology.

Contents

---

*1991 *Mathematics Subject Classification*, Primary: 20F32, Secondary: 55N20.
†This work was done when the author visited MSRI supported in part by NSF grant DMS-9022140 and a JSPS grant.

1 Introduction

Bounded cohomology was defined by F. Trauber for groups and by M. Gromov for spaces. We review the definition of bounded cohomology of a group G. Let

$$C_b^k(G;A) = \{f : G^k \to A \mid f \text{ has bounded range}\},$$

where $A = Z$ or R. The boundary $\delta : C_b^k(G;A) \to C_b^{k+1}(G;A)$ is given by

$$\begin{aligned}\delta f(g_0,\ldots,g_k) &= f(g_1,\ldots,g_k) + \sum_{i=1}^{k}(-1)^i f(g_0,\ldots,g_{i-1}g_i,\ldots g_k) \\ &\quad +(-1)^{k+1} f(g_0,\ldots,g_{k-1}).\end{aligned}$$

The cohomology of the complex $\{C_b^k(G;A),\delta\}$ is the *bounded cohomology* of G, denoted by $H_b^*(G;A)$. See [G], [I] as general references for the theory of bounded cohomology.

For any group G, the first bounded cohomology, $H_b^1(G;A)$ is trivial. For an amenable group G, $H_b^n(G;R)$ is trivial for any n. The first example of a group with non-trivial second bounded cohomology was given by R. Brooks, [B]. He showed a free group of rank greater than 1 has infinite dimensional second bounded cohomology. R.I. Grigorchuk investigated the structure of the second bounded cohomology of free groups, torus knot groups and surface groups, [Gr]. T. Yoshida [Y] and T. Soma [So1], [So2] studied the third bounded cohomology of surfaces and hyperbolic three manifolds. D.B.A. Epstein and the author showed a non-trivial word-hyperbolic group has infinite dimensional second bounded cohomology, [EF].

We state results on the second bounded cohomology of an amalgamated free product of groups:

Theorem 1 *Let $G = A *_C B$. If $|C\backslash A/C| \geq 3$ and $|B/C| \geq 2$, then the cardinality of the dimension of $H_b^2(G;R)$ as a vector space over $\boldsymbol{R}$ is continuum.*

Corollary 1 *Let $G = A * B$ with $A \neq \{1\}, B \neq \{1\}$. If $G \neq Z_2 * Z_2$, then the cardinality of the dimension of $H_b^2(G;R)$ as a vector space over $\boldsymbol{R}$ is continuum.*

Remark. (1) Cor 1 is a generalization of R. Brooks' result on free groups.
(2) Since $Z_2 * Z_2$ is amenable, $H_b^2(Z_2 * Z_2;R)$ is trivial.

Corollary 2 *Let $G = A *_C B$. If $|A| = \infty$, $|C| < \infty$, and $|B/C| \geq 2$, then the cardinality of the dimension of $H_b^2(G;R)$ as a vector space over $\boldsymbol{R}$ is continuum.*

Corollary 3 *Let $G = A *_C B$. If A is abelian, $|A/C| \geq 3$, and $|B/C| \geq 2$, then the cardinality of the dimension of $H_b^2(G;R)$ as a vector space over $\boldsymbol{R}$ is continuum.*

Example. $PSL_2(Z) = Z_2 * Z_3$ and $SL_2(Z) = Z_4 *_{Z_2} Z_6$ satisfy the assumption of Th 1. They are non-elementary word-hyperbolic groups too.

For an HNN extension of a group, we show:

Theorem 2 *Let $G = A *_C \varphi$. If $|A/C| \geq 2$, $|A/\varphi(C)| \geq 2$, then then the cardinality of the dimension of $H_b^2(G;R)$ as a vector space over $\boldsymbol{R}$ is continuum.*

Remark. In Th 1 and 2, and Cor 1, 2 and 3, we don't have to assume G is finitely generated.

We apply Th 1 and 2 to a group with infinitely many ends, namely, due to Stallings' structure theorem [S], we have:

Theorem 3 *If G is a finitely generated group with infinitely many ends, then the cardinality of the dimension of $H_b^2(G;R)$ as a vector space over $\boldsymbol{R}$ is continuum.*

To conclude, we state a conjecture.

Conjecture. Let G be a group. If $H_b^2(G;R) \neq \{0\}$, then $H_b^2(G;R)$ is infinite dimensional as a vector space over **R**.

2 Quasi homomorphism and the counting function

We will review the counting function of a group w.r.t. a word. This was defined in [EF] as a generalization of R. Brooks' counting function for free groups in [B]. Let G be a group with a (finite or infinite) set of generators and $\Gamma(G)$ the Cayley graph. For a word $w = x_1 x_2 \ldots x_n$ in these generators, define $|w| = n$. Let $\overline{w}$ be the element of G which is represented by the word

w. Define $w^{-1} = x_n^{-1} \dots x_1^{-1}$. We sometimes identify a word w and the path starting at 1 and labeled by w in $\Gamma(G)$. For a path α labeled by w, define $|\alpha| = |w|$ and $\overline{\alpha} = \overline{w}$. For an element g in G, define $|g| = \inf_\alpha |\alpha|$, where α ranges over all the paths with $\overline{\alpha} = g$.

Let α be a finite path in $\Gamma(G)$. Define $|\alpha|_w$ to be the maximal number of times that w can be seen as a subword of α without overlapping. We define

$$c_w(\alpha) = \sup_{\alpha'}\{|\alpha'|_w - (|\alpha'| - |\overline{\alpha}|)\} = |\overline{\alpha}| - \inf_{\alpha'}(|\alpha'| - |\alpha'|_w),$$

where α' ranges over all the paths with the same starting point as α and the same finishing point. If the infimum in the definition of $c_w(\alpha)$ is attained by α', we say that α' *realizes* c_w *at* α. We have $|\overline{\alpha}|/|w| \geq c_w(\alpha) \geq 0$. If α is a geodesic, then $c_w(\alpha) \geq |\alpha|_w$.

We define $h_w = c_w - c_{w^{-1}}$. For each g in G, we choose γ_g to be a path from 1 to g and set $h_w(g) = h_w(\gamma_g)$. Then $h_w(g)$ does not depend on the choice of γ_g. Thus $h_w \in C^1(G; Z)$.

Let $f \in C^1(G; R)$. If there exists a constant $D < \infty$ s.t.

$$|f(gh) - f(g) - f(h)| \leq D,$$

for any $g, h \in G$, then we say f is a *quasi homomorphism* with *defect* D. Let f be a quasi homomorphism with defect D. Then $|\delta f| \leq D$ and $\delta(\delta f) = 0$, thus $\delta f \in Z_b^2(G; R)$, which defines $[\delta f] \in H_b^2(G; R)$. Remark that we always have $[\delta f] = 0$ in $H^2(G; R)$, however, we may have $[\delta f] \neq 0$ in $H_b^2(G; R)$ since f is not necessarily in $C_b^1(G; R)$.

3 Quasi homomorphisms of $A *_C B$

Let $G = A *_C B$ with $|A/C| \geq 2$, $|B/C| \geq 2$. Take the set $\{A \cup B\}\backslash\{1\}$ as a set of generators of G and denote its Cayley graph by $\Gamma(G)$. Note that if a generating set is infinite, then Γ is not locally finite.

If a word $w = x_1 \dots x_n$ satisfies $n = 1$ or $x_1, x_3, \dots \in A\backslash C$ (or $B\backslash C$) and $x_2, x_4, \dots \in B\backslash C$ (or $A\backslash C$, resp.), then we say w is *reduced.*

Lemma 3.1 *A word $w = x_1 \dots x_n$ is reduced iff it is a geodesic in Γ.*

Proof. Assume w is not reduced, then there exists a subword $x_i x_{i+1}$ with $x_i, x_{i+1} \in A$(or $x_i, x_{i+1} \in B$). Then $\overline{x_i x_{i+1}} \in A$(or B resp.), thus $|\overline{x_i x_{i+1}}| \leq 1$. Therefore $x_i x_{i+1}$ is not a geodesic, hence w is not a geodesic.

On the other hand, assume w is not a geodesic. To show w is not reduced by contradiction, suppose w is reduced. Take a geodesic γ, hence reduced, s.t. $\overline{w} = \overline{\gamma}$. Then $|w| = |\gamma|$ since reduced words representing a same element have same length. Thus w is a geodesic. This is a contradiction. We showed w is not reduced. q.e.d.

Lemma 3.2 *Let w be a word and α a path. If w^2 is reduced, then there is a geodesic which realizes c_w at α.*

Proof. Since w^2 is reduced, w is reduced. Let γ be a path which realizes c_w at α s.t. $|\gamma|_w$ is minimal among all the realizing paths at α. We will show that γ is a geodesic. If $|\gamma|_w = 0$, then γ is a geodesic. Assume $|\gamma|_w = n > 0$. Write γ as

$$\gamma_1 w_1 \gamma_2 \ldots w_n \gamma_{n+1},$$

where w_i is a copy of w and γ_i may be an empty word. First, to show every γ_i is reduced by contradiction, suppose some γ_i is not reduced. Replace γ_i by a reduced word γ_i' with $\overline{\gamma_i} = \overline{\gamma_i'}$, then we have a new path

$$\gamma' = \gamma_1 w_1 \gamma_2 \ldots \gamma_i' \ldots w_n \gamma_{n+1},$$

which satisfies $|\gamma'| < |\gamma|$, $|\gamma'|_w \geq |\gamma|_w$ and $\overline{\gamma} = \overline{\gamma'}$. This contradicts that γ is a realizing path. Thus every γ_i is reduced. Next, in order to show that γ is reduced by contradiction, suppose not. Since w^2 is reduced, there is a subword $w_i \gamma_{i+1} w_{i+1}$ of γ which is not reduced and γ_{i+1} is not empty. Since $w_i, w_{i+1}, \gamma_{i+1}$ are reduced, one of the following four cases is occurring.
(i) The last letter of w_i and the initial letter of γ_{i+1} are in A.
(ii) The last letter of w_i and the initial letter of γ_{i+1} are in B.
(iii) The last letter of γ_{i+1} and the initial letter of w_{i+1} are in A.
(iv) The last letter of γ_{i+1} and the initial letter of w_{i+1} are in B.

We treat the case (i). Write

$$w_i = \ldots b_1 a_1,\ \gamma_{i+1} = a_2 b_2 \ldots,\ a_i \in A, b_i \in B.$$

Rewrite the subword $w_i \gamma_{i+1}$ in γ as

$$w_i \gamma_{i+1} = \ldots b_1 a_1 a_2 b_2 \ldots = \ldots b_1 a' b_2 \ldots,$$

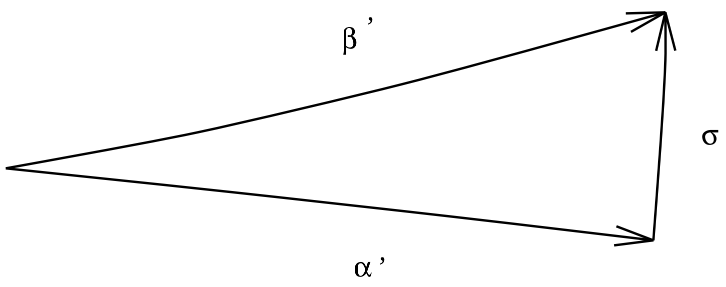

Figure 1: Geodesic triangle. This illustrates Lemma 3.3.

where $a' = a_1 a_2 \in A$. This gives a new word γ' with $|\gamma'| = |\gamma| - 1$, $|\gamma'|_w = |\gamma|_w - 1$. Since $\overline{\gamma'} = \overline{\gamma}$ and $|\gamma| - |\gamma|_w = |\gamma'| - |\gamma'|_w$, γ' is another realizing path with $|\gamma'|_w < |\gamma|_w$. This contradicts the choice of γ, thus γ is reduced. By Lemma 3.1, γ is a geodesic. A similar argument applies to the other three cases, and we omit it. q.e.d.

Lemma 3.3 *Let α, β be paths starting at 1. We have*

$$|c_w(\alpha) - c_w(\beta)| \leq 2|\overline{\alpha^{-1}\beta}|.$$

Proof. Take realizing geodesics of c_w at α, α', and at β, β'. Then $c_w(\alpha) = |\alpha'|_w$, $c_w(\beta) = |\beta'|_w$. Take a geodesic σ with $\overline{\alpha^{-1}\beta} = \overline{\sigma}$.

Since the path $\alpha'\sigma$ satisfies $\overline{\alpha'\sigma} = \overline{\beta'}$,

$$|\beta'| - |\beta'|_w \leq |\alpha'\sigma| - |\alpha'\sigma|_w.$$

Since $|\alpha'\sigma| = |\alpha'| + |\sigma|$ and $|\alpha'\sigma|_w \geq |\alpha'|_w + |\sigma|_w$,

$$|\beta'| - |\beta'|_w \leq |\alpha'| + |\sigma| - |\alpha'|_w - |\sigma|_w.$$

Thus

$$c_w(\beta) = |\beta'|_w \geq |\alpha'|_w + |\beta'| - |\alpha'| - |\sigma| + |\sigma|_w \geq c_w(\alpha) - 2|\sigma|,$$

since $|\alpha'|_w = c_w(\alpha)$, $|\beta'| - |\alpha'| \geq -|\sigma|$, $|\sigma|_w \geq 0$. Similarly, $c_w(\alpha) \geq c_w(\beta) - 2|\sigma|$. Thus $|c_w(\alpha) - c_w(\beta)| \leq 2|\sigma| = 2|\overline{\alpha^{-1}\beta}|$. q.e.d.

Lemma 3.4 *Let α, β be paths starting at 1. We have*

$$|h_w(\alpha) - h_w(\beta)| \leq 4|\overline{\alpha^{-1}\beta}|.$$

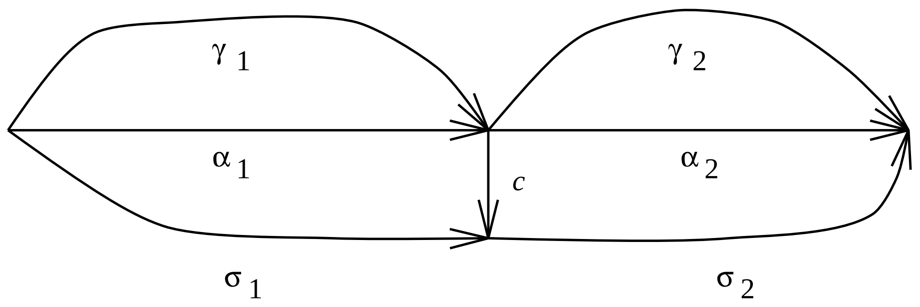

Figure 2: Dividing geodesics. This illustrates Lemma 3.7. $\alpha = \alpha_1\alpha_2$, $\gamma = \gamma_1\gamma_2$, $\sigma = \sigma_1\sigma_2$.

Proof. $h_w = c_w - c_{w^{-1}}$ and Lemma 3.3. q.e.d.

Lemma 3.5

$$c_w(\alpha) = c_{w^{-1}}(\alpha^{-1}).$$

Proof. Clear from the definition of c_w. q.e.d.

Lemma 3.6

$$h_w(\alpha) = -h_w(\alpha^{-1}).$$

Proof. $h_w(\alpha^{-1}) = c_w(\alpha^{-1}) - c_{w^{-1}}(\alpha^{-1}) = c_{w^{-1}}(\alpha) - c_w(\alpha) = -h_w(\alpha)$. q.e.d.

Lemma 3.7 *Let α be a reduced path. If $\alpha = \alpha_1\alpha_2$, then*

$$|h_w(\alpha) - h_w(\alpha_1) - h_w(\alpha_2)| \leq 10.$$

Proof. We will show

$$|c_w(\alpha) - c_w(\alpha_1) - c_w(\alpha_2)| \leq 5.$$

Take realizing geodesics of c_w at α_1, γ_1, and at α_2, γ_2. We have $|\gamma_1|_w = c_w(\alpha_1)$, $|\gamma_2|_w = c_w(\alpha_2)$. Put $\gamma = \gamma_1\gamma_2$, then $\overline{\gamma} = \overline{\alpha}$. Since α is reduced, it is a geodesic. Since $|\alpha| = |\alpha_1| + |\alpha_2| = |\gamma_1| + |\gamma_2| = |\gamma|$, γ is a geodesic. Since γ is a geodesic with $\overline{\gamma} = \overline{\alpha}$, $c_w(\alpha) \geq |\gamma|_w$. Thus

$$c_w(\alpha) \geq |\gamma|_w \geq |\gamma_1|_w + |\gamma_2|_w = c_w(\alpha_1) + c_w(\alpha_2).$$

On the other hand, take a realizing geodesic σ at α, then $c_w(\alpha) = |\sigma|_w$. Since α and σ are reduced, there exists subdivision of σ, $\sigma = \sigma_1\sigma_2$ s.t. $\overline{\alpha_1^{-1}\sigma_1} = \overline{\alpha_2\sigma_2^{-1}} = c$, for some $c \in C$. Since $|c| \le 1$, from Lemma 3.3 and 3.5.

$$|c_w(\alpha_i) - c_w(\sigma_i)| \le 2,\ i = 1, 2.$$

Since σ_1, σ_2 are geodesics, $|\sigma_i|_w \le c_w(\sigma_i)$, $i = 1, 2$. Thus

$$c_w(\alpha) = |\sigma|_w \le |\sigma_1|_w + |\sigma_2|_w + 1 \le c_w(\sigma_1) + c_w(\sigma_2) + 1 \le c_w(\alpha_1) + c_w(\alpha_2) + 5.$$

We showed $|c_w(\alpha) - c_w(\alpha_1) - c_w(\alpha_2)| \le 5$. $|c_{w^{-1}}(\alpha) - c_{w^{-1}}(\alpha_1) - c_{w^{-1}}(\alpha_2)| \le 5$ is similar. Since $h_w = c_w - c_{w^{-1}}$, we get $|h_w(\alpha) - h_w(\alpha_1) - h_w(\alpha_2)| \le 10$. q.e.d.

Proposition 1 *Let w be a word. We have*

$$|\delta h_w| \le 78.$$

Proof. Let x, y be elements in G. $\delta h_w(x, y) = h_w(x) + h_w(y) - h_w(xy)$. We will show $|h_w(xy) - h_w(x) - h_w(y)| \le 78$. Take geodesics, hence reduced paths, α, β and γ with $\overline{\alpha} = x, \overline{\beta} = y, \overline{\gamma} = xy$. Since $\overline{\alpha\beta} = \overline{\gamma}$ and α, β, γ are reduced, there exist subdivisions of α, β, γ,

$$\alpha = \alpha_1\alpha_2\alpha_3,\ \beta = \beta_1\beta_2\beta_3,\ \gamma = \gamma_1\gamma_2\gamma_3,$$

s.t.

$$\overline{\gamma_1^{-1}\alpha_1} = c_1,\ \overline{\alpha_3\beta_1} = c_2,\ \overline{\beta_3\gamma_3^{-1}} = c_3,$$

for some $c_1, c_2, c_3 \in C$ and $\overline{\alpha_2}, \overline{\beta_2}, \overline{\gamma_2}$ are (simultaneously) in A or B.

Since $\overline{\alpha_2}$ is in A or B, $c_w(\alpha_2), c_{w^{-1}}(\alpha_2) \le |\overline{\alpha_2}| \le 1$. Thus $|h_w(\alpha_2)| \le 2$. Similarly, $|h_w(\beta_2)|, |h_w(\gamma_2)| \le 2$.

Using Lemma 3.7 twice for $\alpha = ((\alpha_1\alpha_2)\alpha_3)$,

$$|h_w(\alpha) - h_w(\alpha_1) - h_w(\alpha_2) - h_w(\alpha_3)| \le 20,$$

thus $|h_w(\alpha) - h_w(\alpha_1) - h_w(\alpha_3)| \le 22$. Similarly,

$$|h_w(\beta) - h_w(\beta_1) - h_w(\beta_3)| \le 22,$$

$$|h_w(\gamma) - h_w(\gamma_1) - h_w(\gamma_3)| \le 22.$$

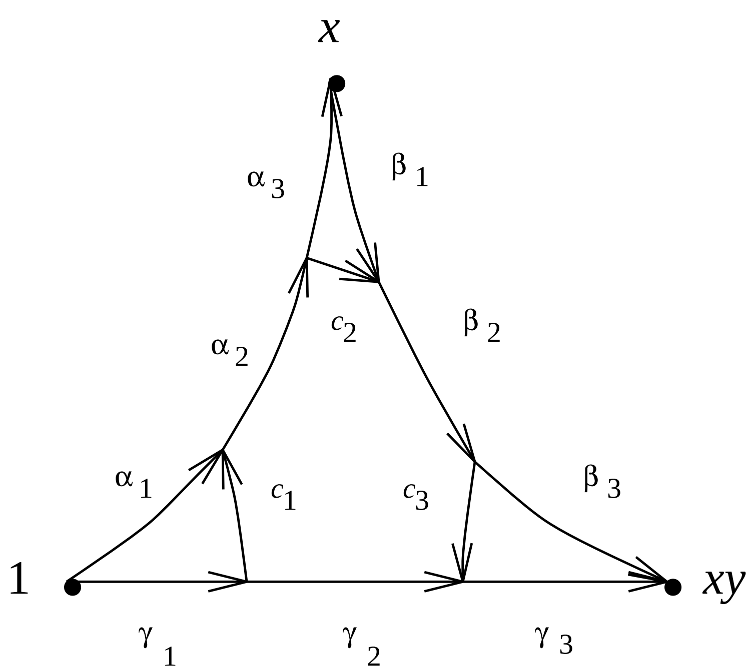

Figure 3: Dividing geodesic triangle. This illustrates Prop 1. $c_1, c_2, c_3 \in C$.

Thus

$$\begin{aligned}
& |h_w(\alpha)+h_w(\beta)-h_w(\gamma)| \\
\leq\ & |h_w(\alpha_1)+h_w(\alpha_3)+h_w(\beta_1)+h_w(\beta_3)-h_w(\gamma_1)-h_w(\gamma_3)|+66 \\
\leq\ & |h_w(\alpha_1)-h_w(\gamma_1)|+|h_w(\beta_1)-h_w(\alpha_3^{-1})|+|h_w(\gamma_3^{-1})-h_w(\beta_3^{-1})|+66 \\
\leq\ & 4(|c_1|+|c_2|+|c_3|)+66 \leq 78,
\end{aligned}$$

since $|c_i| \leq 1$ and Lemma 3.4 and 3.6.

By definition, $h_w(\alpha)=h_w(x), h_w(\beta)=h_w(y), h_w(\gamma)=h_w(xy)$, thus

$$|h_w(x)+h_w(y)-h_w(xy)| \leq 78.$$

q.e.d.

4 Choice of words for $A *_C B$

The goal of this section is to prove the following proposition.

Proposition 2 *Let $G = A *_C B$ with $|C\backslash A/C| \geq 3$ and $|B/C| \geq 2$. Then there exist words w_i, $0 \leq i < \infty$ which satisfy the following conditions.*
(1) For any $i \geq 0$ and any $n \geq 1$, $h_{w_i}(w_i^n) = n$.
(2) For any $j > i \geq 0$ and any $n \geq 1$, $h_{w_j}(w_i^n) = 0$.
(3) For any $i \geq 0$, $\overline{w_i} \in [G, G]$.

Let $G = A *_C B$ with $|C\backslash A/C| \geq 3$ and $|B/C| \geq 2$. Let w and w' be reduced paths starting at a same point and finishing at a same point,

$$w = a_1 b_1 \dots a_n b_n,$$

$$w' = a'_1 b'_1 \dots a'_m b'_m,$$

where a_1, b_n, a'_1, b'_m may be empty.

Since $\overline{w} = \overline{w'}$ and w and w' are reduced, $n = m$ and there exist $c_1, \dots c_n$ and $c'_0, c'_1, \dots c'_n$ in C s.t.

$$c'_{i-1} a_i c_i^{-1} = a'_i,\ c_i b_i c_i'^{-1} = b'_i,$$

for $1 \leq^{\forall} i \leq n$, where $c'_0 = c'_n = 1$. See Figure 4.

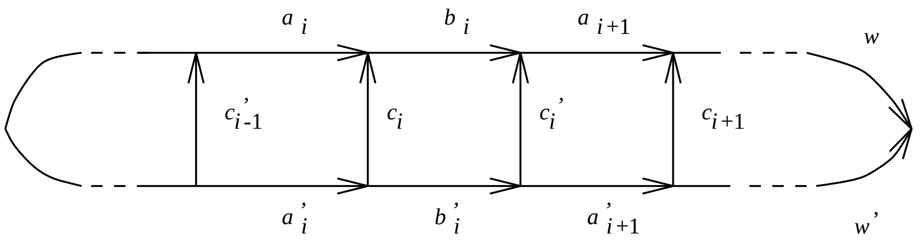

Figure 4: Reduced paths w, w' with common end points. $\overline{w} = \overline{w'}$.

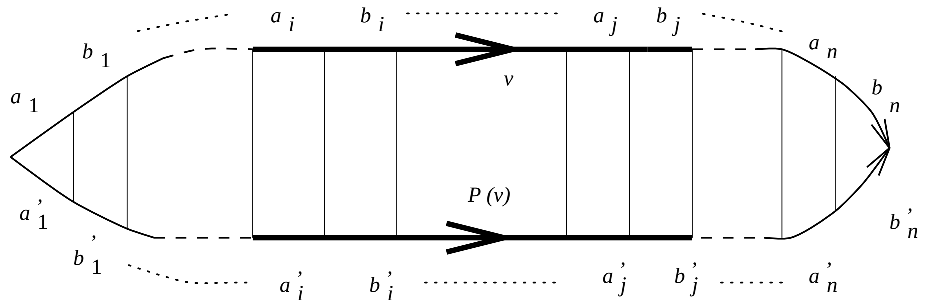

Figure 5: Definition of $P(v)$.

Let v be a subword of w,

$$v = a_i b_i \dots a_j b_j,$$

where a_i, b_j may be empty. Define a subword of w', denoted by $P(v)$, by

$$P(v) = a'_i b'_i \dots a'_j b'_j.$$

Let v' be a subword of w'. If v' is a subword of $P(v)$, we say v *covers* v'. If $v' = P(v)$, then we say v *faces* v'.

Since $|C \backslash A / C| \geq 3$, we can choose $a_1, a_2 \in A$ s.t. $a_1, a_2 \in A \backslash C$, and $a_2 \notin C a_1 C$. Take $b \in B \backslash C$. Define words $w_i, 0 \leq i < \infty$ by

$$\begin{aligned} w_i &= (a_1 b)^{10^i} (a_1^{-1} b^{-1})^{10^i} (a_2 b)^{10^i} (a_2^{-1} b^{-1})^{10^i} \\ &\quad (a_1 b)^{4 \cdot 10^i} (a_1^{-1} b^{-1})^{4 \cdot 10^i} (a_2 b)^{4 \cdot 10^i} (a_2^{-1} b^{-1})^{4 \cdot 10^i}. \end{aligned}$$

We write the subword $(a_1 b)^{4 \cdot 10^i}$ by $w_i(1,+)$ and the subword $(a_1^{-1} b^{-1})^{4 \cdot 10^i}$ by $w_i(1,-)$.

Lemma 4.1 *The words $w_i, 0 \leq i < \infty$ satisfy the following conditions.*
(1) For any $i \geq 0$ and any $n \geq 1$, w_i^n is reduced.
(2) For any $i \geq 0$, $|w_i| = 40 \cdot 10^i$ and $|w_i(1, \pm)| = 8 \cdot 10^i$.
(3) For any $i \geq 0$, $\overline{w_i} \in [G, G]$.

Proof. (1) is clear from the choice of a_1, a_2, and b. (2) is obvious. For (3), use $\overline{(a_i b)^p (a_i^{-1} b^{-1})^p} \in [G, G]$ for $i = 1, 2$ and any $p \geq 1$. q.e.d.

Lemma 4.2 *In any pair of paths having a same starting point and a same finishing point, the following conditions hold.*
(1) a_1 cannot face a_2.
(2) For any $i \geq 0$, w_i^2 cannot cover w_i^{-1}.
(3) For any $i < j$, $w_j(1, +)$ cannot cover w_i and $w_j(1, -)^{-1}$ cannot cover w_i.
(4) For any $k > 0$ and any $i < j$, w_i^k cannot cover w_j nor w_j^{-1}.

Proof. (1) If a_1 faces a_2 in some pair of paths, then there exist $c_1, c_2 \in C$ s.t. $c_1 a_1 c_2 = a_2$, thus $a_2 \in C a_1 C$, which contradicts our choice of a_1 and a_2.
(2) Let W_0 symbolize the order of the occurrence of elements labeled by $a_1^{\pm 1}$ or $a_2^{\pm 1}$ in w_0 in the following way:

$$W_0 = 1\overline{1}2\overline{2}1111\overline{1111}2222\overline{2222},$$

where 1 stands for a_1, 2 for a_2, $\overline{1}$ for a_1^{-1}, and $\overline{2}$ for a_2^{-1}. Here, remark that in W_0, we ignore $b^{\pm 1}$'s of w_0. Let W_0^{-1} represent for w_0^{-1}:

$$W_0^{-1} = 2222\overline{2222}1111\overline{1111}2\overline{2}1\overline{1}.$$

In order to show the conclusion by contradiction, assume w_0^2 covers w_0^{-1}. Then W_0^2 covers W_0^{-1}. For each possible position for W_0^2 covering W_0^{-1}, one can find some 1(or 2, $\overline{1}$, $\overline{2}$) in W_0^{-1} facing some 2 (or 1, $\overline{2}$, $\overline{1}$ resp.) in W_0^2, see Figure 6. This means a_1 is facing some a_2, which contradicts (1). Thus we showed (2) for $i = 0$. A similar argument works for w_i for any $i \geq 1$, and we omit it.
(3) Observe that $w_j(1, +)$ consists of a_1 and b, while w_i contains a_2. If $w_j(1, +)$ covers w_i, then each a_2 in w_i faces some a_1 in $w_j(1, +)$, which contradicts (1). The same argument goes for $w_j(1, -)^{-1}$, and we omit it.
(4) Assume w_i^k covers w_j for some $i < j$ and some $k > 0$. Then $w_j(1, +)$ covers some w_i of w_i^k since $2|w_i| \leq |w_j(1, +)|$, which contradicts (3). Assume w_i^k covers w_j^{-1} for some $i < j$ and some $k > 0$. Then $w_j(1, -)^{-1}$ covers some w_i of w_i^k, which contradicts (3). q.e.d.

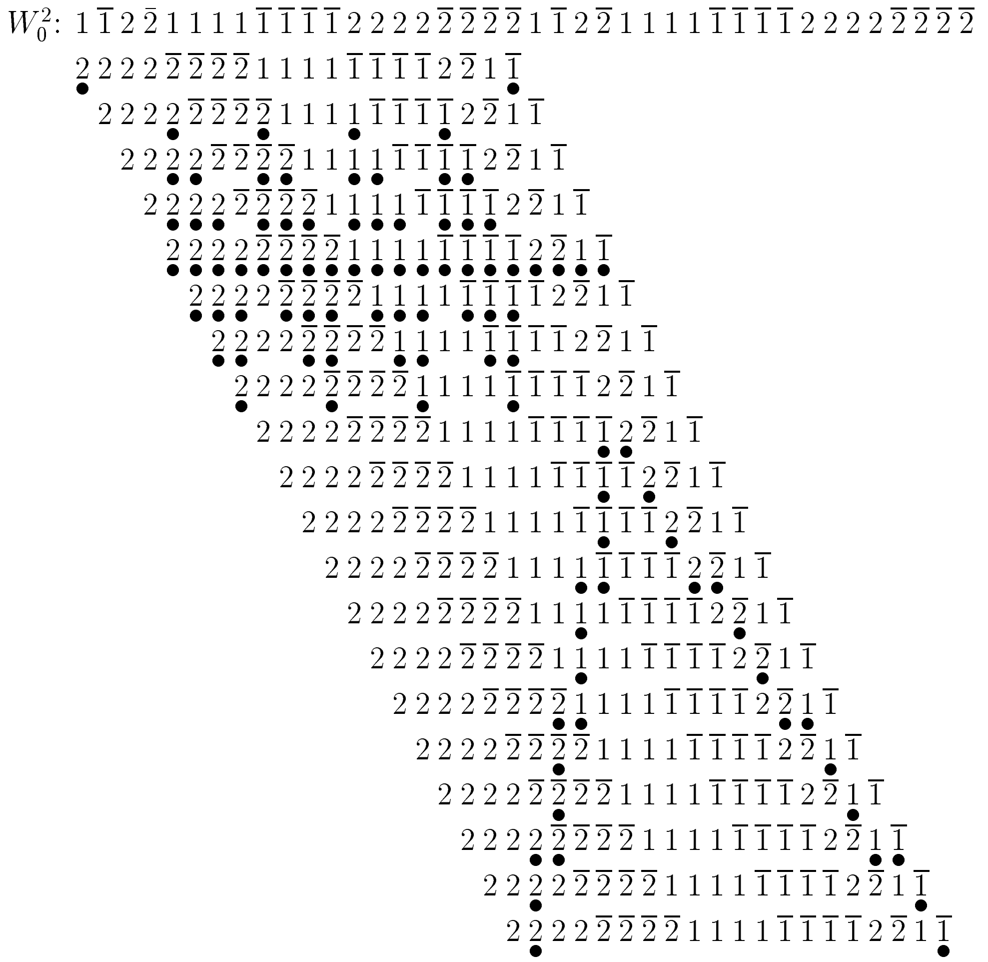

Figure 6: W_0^2 cannot cover W_0^{-1}. The first row is W_0^2. The second to the last rows describe all the possible positions for W_0^{-1}. We put $\bullet$ below illegal pairs of $(1,2)$ or $(\overline{1},\overline{2})$. In each position for W_0^{-1}, we find at least one such pair, which is a contradiction.

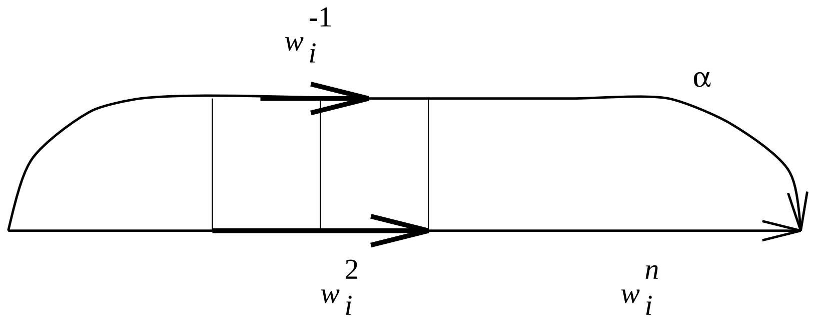

Figure 7: w_i^2 cannot cover w_i^{-1}.

Lemma 4.3 *c_{w_i} and $c_{w_i^{-1}}$ satisfy the following conditions.*
(1) For any $n \geq 1$ and any $i \geq 0$, $c_{w_i}(w_i^n) = n$.
(2) For any $n \geq 1$ and any $i \geq 0$, $c_{w_i^{-1}}(w_i^n) = 0$.
(3) For any $j > i \geq 0$ and any $n \geq 1$, $c_{w_j}(w_i^n) = 0$.
(4) For any $j > i \geq 0$ and any $n \geq 1$, $c_{w_j^{-1}}(w_i^n) = 0$.

Proof. (1) Since w_i^n is reduced, w_i^n is a geodesic. Thus $c_{w_i}(w_i^n) \geq |w_i^n|_{w_i} = n$. On the other hand, $c_{w_i}(w_i^n) \leq |\overline{w_i^n}|/|w_i| = n$. Thus $c_{w_i}(w_i^n) = n$.
(2) To show $c_{w_i^{-1}}(w_i^n) = 0$ by contradiction, assume $c_{w_i^{-1}}(w_i^n) > 0$. Take a realizing geodesic of $c_{w_i^{-1}}$ at w_i^n, α. Then $|\alpha|_{w_i^{-1}} > 0$. Fix a subword labeled by w_i^{-1} in α. Since w_i^n and α are reduced, there is a subword of w_i^n labeled by w_i^2 which covers the w_i^{-1} in α. But w_i^2 cannot cover w_i^{-1} by Lemma 4.2(2), a contradiction. See Figure 7.
(3) To show the claim by contradiction, assume $c_{w_j}(w_i^n) > 0$ for some $i < j$ and n. Take a realizing geodesic α of c_{w_j} at w_i^n. Then $|\alpha|_{w_j} > 0$. Fix some subword labeled by w_j in α.

Since α and w_i^n are reduced, there exists a subword labeled by w_i^k in w_i^n which covers the w_j in α. But w_i^k cannot cover w_j by Lemma 4.2(4), a contradiction.
(4) Similar to (3). q.e.d.

Now we are in a position to prove Proposition 2.
Proof of Proposition 2. Let $w_i, 0 \leq i < \infty$ be the set of words defined before in this section. We will show they are the words we want. Since $h_{w_i} = c_{w_i} - c_{w_i^{-1}}$, $h_{w_i}(w_i^n) = n$ by Lemma 4.3(1) and (2). By Lemma 4.3(3)

and (4), $h_{w_j}(w_i^n) = 0$ for $i < j$. For a homomorphism $\phi : G \to R$, $\phi(\overline{w_i}) = 0$ since $\overline{w_i} \in [G, G]$ by Lemma 4.1(3). q.e.d.

5 Proofs of Th 1 and Cor 1,2,3

Proof of Th 1. Let w_i, $0 \leq i < \infty$ be the words in Prop 2. By Prop 1, all the cocycles δh_{w_i} have the same bound. It follows that if $(a_i)_i \in l^1$, then $\sum_i a_i \delta h_{w_i}$ is also a well-defined cocycle. We get a real linear map

$$\omega : l^1 \to H_b^2(G; R),$$

which sends $(a_i)_i$ to the cohomology class of $\sum_i a_i \delta h_{w_i}$. In order to show ω is injective, suppose $\omega((a_i)) = 0$. Then

$$\delta(\sum_{i=0}^{\infty} a_i h_{w_i}) = \delta b$$

for some $b \in C_b^1(G; R)$. This means

$$\sum_i a_i h_{w_i} - b = \phi,$$

for some homomorphism $\phi : G \to R$. Applying this to $\overline{w_0^n} \in G$, we find

$$a_0 n - b(\overline{w_0^n}) = \phi(\overline{w_0^n}) = 0,$$

for all $n \geq 1$ by Prop 2. Since b is bounded, $a_0 = 0$. Similarly, $a_i = 0$ for all $i \geq 1$. Thus ω is injective. It is well-known that the cardinality of the dimension of l^1 as a vector space is continuum. q.e.d.

Proof of Cor 1. W.l.o.g., $|A| \geq 3$. Since $C = \{1\}$, $|C \backslash A/C| = |A| \geq 3$. Apply Th 1. q.e.d.

Proof of Cor 2. Since $|C| < \infty$ and $|A| = \infty$, $|C \backslash A/C| = \infty$. Apply Th 1. q.e.d.

Proof of Cor 3. Since A is abelian, $|C \backslash A/C| = |A/C| \geq 3$. Apply Th 1. q.e.d.

6 Quasi homomorphisms of $A *_C \varphi$

Let $G = A *_C \varphi =< A, t; c = t^{-1}\varphi(c)t, {}^{\forall}c \in C >$ with $|A/C| \geq 2$, $|A/\varphi(C)| \geq 2$. Let g be

$$g = a_1 t^{n_1} a_2 t^{n_2} \dots a_I t^{n_I} a_{I+1},$$

with $a_i \in A$ and $n_i \neq 0$, where a_1, a_{I+1} may be empty. Suppose g satisfies the following two conditions (i) and (ii) for $1 \leq^{\forall} i \leq I-1$.
(i) If $n_i > 0, n_{i+1} < 0$, then $a_{i+1} \notin C$.
(ii) If $n_i < 0, n_{i+1} > 0$, then $a_{i+1} \notin \varphi(C)$.
Then we say g is *reduced.*

The following fact is known as Britton's lemma.

Lemma 6.1 (Britton, [LS]) *Suppose* $1 \leq I$. *If* g *is reduced, then* $g \neq 1$ *in* G.

As an application of Britton's lemma, we have:

Lemma 6.2 *Let*

$$g = a_1 t^{n_1} a_2 t^{n_2} \dots a_I t^{n_I} a_{I+1},\ a_i \in A$$

$$h = b_1 t^{m_1} b_2 t^{m_2} \dots b_J t^{m_J} b_{J+1},\ b_j \in A$$

be reduced with $n_i = \pm 1$, $m_j = \pm 1$. *If* $g = h$ *in* G, *then*

$$I = J,\ n_1 = m_1, \dots, n_I = m_I,$$

and

$$a_{I+1}b_{I+1}^{-1},\ t^{n_I}a_{I+1}b_{I+1}^{-1}t^{-m_I},\ a_I t^{n_I} a_{I+1} b_{I+1}^{-1} t^{-m_I} b_I^{-1}, \dots$$
$$\dots, t^{n_1} a_2 t^{n_2} a_3 \dots a_{I+1} b_{I+1}^{-1} \dots b_3^{-1} t^{-m_2} b_2^{-1} t^{-m_1}$$

are in C *or* $\varphi(C)$, *especially, in* A.

Proof. W.l.o.g., we assume $J \leq I$. $h^{-1} = b_{J+1}^{-1} t^{-m_J} b_J^{-1} t^{-m_{J-1}} \dots b_1^{-1}$ is reduced since h is reduced. Put

$$d_{I+1} = a_{I+1} b_{J+1}^{-1},$$

then $d_{I+1} \in A$ and

$$gh^{-1} = (a_1 t^{n_1} a_2 t^{n_2} \dots a_I t^{n_I}) d_{I+1} (t^{-m_J} b_J^{-1} t^{-m_{J-1}} \dots b_1^{-1})$$

in G. Since $gh^{-1} = 1$ in G, the expression on the right hand side is not reduced by Britton's lemma. But, since $a_1 t^{n_1} a_2 t^{n_2} \dots a_I t^{n_I}$ and $t^{-m_J} b_J^{-1} t^{-m_{J-1}} \dots b_1^{-1}$ are reduced, we have

$$n_I = m_J = 1(\text{or} -1), \ d_{I+1} \in C(\varphi(C), \text{resp.}),$$

and $t^{n_I} d_{I+1} t^{-m_J} \in \varphi(C)(C, \text{ resp.})$.

Put

$$c_I = t^{n_I} d_{I+1} t^{-m_J}, \ d_I = a_I c_I b_J^{-1}.$$

Clearly $d_I \in A$ and

$$gh^{-1} = (a_1 t^{n_1} a_2 t^{n_2} \dots a_{I-1} t^{n_{I-1}}) d_I (t^{-m_{J-1}} b_{J-1}^{-1} t^{-m_{J-2}} \dots b_1^{-1})$$

in G. By Britton's lemma, the expression on the right hand side is not reduced. Thus

$$n_{I-1} = m_{J-1} = 1(\text{or} -1),$$

$$d_I \in C(\varphi(C), \text{resp.}), \ \text{and } c_{I-1} = t^{n_{I-1}} d_I t^{-m_{J-1}} \in \varphi(C)(C, \text{resp.}).$$

We define $d_{I-1}, d_{I-2}, \dots, d_{I-J+2}$ and $c_{I-1}, c_{I-2}, \dots, c_{I-J+1}$ inductively by

$$c_i = t^{n_i} d_{i+1} t^{-m_{J-I+i}}, \ d_i = a_i c_i b_{J-I+i}^{-1}.$$

See Figure 8.

Using Britton's lemma repeatedly, we have

$$n_{I-J+j} = m_j, \ 1 \leq^{\forall} j \leq J$$

and

$$d_{I+1}, d_I, \dots, d_{I-J+2}, c_I, c_{I-1}, \dots, c_{I-J+1} \in C \text{ or } \varphi(C).$$

Put

$$d_{I-J+1} = a_{I-J+1} c_{I-J+1} b_1^{-1}.$$

Clearly $d_{I-J+1} \in A$. To complete the proof, it suffices to show $J = I$. In order to show it by contradiction, suppose $J < I$. Then

$$gh^{-1} = a_1 t^{n_1} a_2 \dots t^{n_{I-J-1}} a_{I-J} t^{n_{I-J}} d_{I-J+1}$$

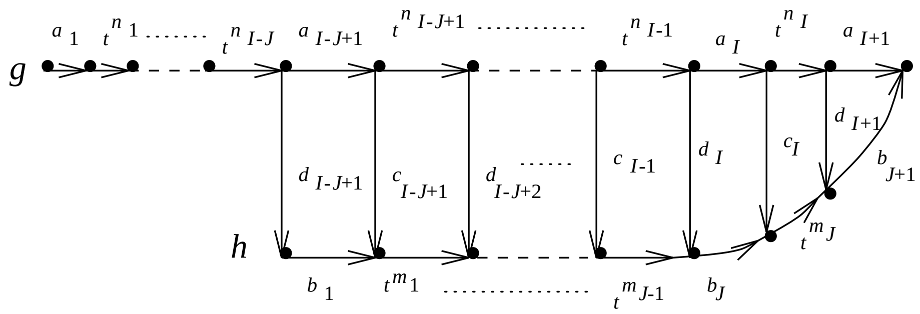

Figure 8: This illustrates Lemma 6.2. $d_i, c_i \in A$ for $I - J + 1 \leq^{\forall} i \leq I$. If $J < I$, then it contradicts Britton's lemma.

in G and the expression on the right hand side is reduced, which contradicts Britton's lemma, since $gh^{-1} = 1$ in G. We got $I = J$. q.e.d.

We take $\{t\} \cup A \backslash \{1\}$ as a set of generators of G and write the Cayley graph of G w.r.t. this set by Γ.

Lemma 6.3 *If a path α is a geodesic in Γ, then it is reduced.*

Proof. If a path α is not reduced, then we can make it shorter using $tct^{-1} = \varphi(c)$, hance α is not a geodesic. q.e.d.

Remark. Compare Lemma 6.3 with Lemma 3.1. A reduced path is not always a geodesic in this case. For example, the left hand side of $\varphi(c)^{-1}tc = t$ for $c \in C \backslash \{1\}$ is reduced but not a geodesic.

Lemma 6.4 *Let α be a path and w a word. If w^2 is reduced, then there is a reduced path β which realizes c_w at α.*

Proof. Take a path β which realizes c_w at α s.t. $|\beta|_w$ is minimal among all the realizing paths. Then a similar argument to the proof of Lemma 3.2 shows β is reduced. We omit the detail. q.e.d.

Lemma 6.5 *Let α, β be paths starting at* 1. *We have*

$$|c_w(\alpha) - c_w(\beta)| \leq 2|\overline{\alpha^{-1}\beta}|.$$

$$|h_w(\alpha) - h_w(\beta)| \leq 4|\overline{\alpha^{-1}\beta}|.$$

Proof. Similar to Lemma 3.3 and 3.4. q.e.d.

Lemma 6.6 *Let α be a reduced path. If $\alpha = \alpha_1\alpha_2$, then*

$$|h_w(\alpha) - h_w(\alpha_1) - h_w(\alpha_2)| \leq 10.$$

Proof. Similar to Lemma 3.7. q.e.d.

Proposition 3

$$|\delta h_w| \leq 78.$$

Proof. The outline is similar to the proof of Proposition 1. Let x, y be elements in G. We show $|h_w(xy) - h_w(x) - h_w(y)| \leq 78$. Take reduced paths α, β, γ with $\overline{\alpha} = x, \overline{\beta} = y, \overline{\gamma} = xy$. Since $\overline{\alpha\beta} = \overline{\gamma}$ and α, β, γ are reduced, by Lemma 6.1 and 6.2, there exist subdivisions of α, β, γ,

$$\alpha = \alpha_1\alpha_2\alpha_3,\ \beta = \beta_1\beta_2\beta_3,\ \gamma = \gamma_1\gamma_2\gamma_3,$$

s.t. $\overline{\gamma_1^{-1}\alpha_1} = c_1, \overline{\alpha_3\beta_1} = c_2, \overline{\beta_3\gamma_3^{-1}} = c_3$, for some $c_1, c_2, c_3 \in C \cup \varphi(C)$, and $\overline{\alpha_2}, \overline{\beta_2}, \overline{\gamma_2} \in A$. By Lemma 6.6,

$$|h_w(\alpha) - h_w(\alpha_1) - h_w(\alpha_2) - h_w(\alpha_3)| \leq 20.$$

Since $|h_w(\alpha_2)| \leq 2$, $|h_w(\alpha) - h_w(\alpha_1) - h_w(\alpha_3)| \leq 22$. Similarly,

$$|h_w(\beta) - h_w(\beta_1) - h_w(\beta_3)| \leq 22,\ |h_w(\gamma) - h_w(\gamma_1) - h_w(\gamma_3)| \leq 22.$$

By a similar argument to the proof of Prop 1,

$$|h_w(x) + h_w(y) - h_w(xy)| = |h_w(\alpha) + h_w(\beta) - h_w(\gamma)| \leq 78.$$

q.e.d.

7 Choice of words for $A *_C \varphi$

Let $G = A *_C \varphi$ with $|A/C| \geq 2$, $|A/\varphi(C)| \geq 2$.

Lemma 7.1 *Let w be*

$$w = t^{n_1}a_1t^{n_2}a_2\dots t^{n_I}a_I,$$

s.t. $a_i \in A\backslash\{1\}, n_i \in Z\backslash\{0\}$ for $1 \leq^{\forall} i \leq I$. We denote the set of the following conditions (1.1), ..., (1.4) by Condition I and (2.1), ..., (2.4) by Condition II.

(1.1) $0 < n_1, n_3, n_5, \dots$.

(1.2) $0 > n_2, n_4, n_6, \dots$.

(1.3) $a_1, a_3, a_5, \dots \notin C$.

(1.4) $a_2, a_4, a_6, \dots \notin \varphi(C)$.

(2.1) $0 > n_1, n_3, n_5, \dots$.

(2.2) $0 < n_2, n_4, n_6, \dots$.

(2.3) $a_1, a_3, a_5, \dots \notin \varphi(C)$.

(2.4) $a_2, a_4, a_6, \dots \notin C$.

If either the Condition I or II holds, then w is a geodesic in Γ.

Proof. We treat the case that the Condition I holds. The other case is similar. Under this condition, clearly w is reduced. Let γ be a geodesic with $\overline{w} = \overline{\gamma}$. Then by Lemma 6.3, γ is reduced. By Lemma 6.2, we have

$$\gamma = b_0\tau_1b_1\tau_2b_2\dots\tau_Ib_I,$$

s.t. $b_i \in A\backslash\{1\}$ for $1 \leq i \leq I-1$, and $b_0, b_I \in A\backslash\{1\}$ or is empty, and

$$\begin{aligned}
\tau_i &= tb_{i,1}tb_{i,2}\dots tb_{i,n_i-2}tb_{i,n_i-1}t, \text{ if } i \text{ is odd,}\\
\tau_i &= t^{-1}b_{i,1}t^{-1}b_{i,2}\dots t^{-1}b_{i,n_i-2}t^{-1}b_{i,n_i-1}t^{-1}, \text{ if } i \text{ is even,}
\end{aligned}$$

where $b_{i,j} \in A\backslash\{1\}$ or is empty, for $1 \leq^{\forall} i \leq I$ and $1 \leq^{\forall} j \leq n_i - 1$.

We will show b_I is not empty. To show this by contradiction, assume b_I is empty. Then

$$\begin{aligned}
w\gamma^{-1} &= \dots ta_It^{-1}\dots, \text{ if } I \text{ is odd,}\\
w\gamma^{-1} &= \dots t^{-1}a_It\dots, \text{ if } I \text{ is even.}
\end{aligned}$$

Since w and γ^{-1} are reduced, $w\gamma^{-1}$ is reduced by (1.3) if I is odd or by (1.4) if I is even. Then it follows from Britton's lemma that $\overline{w\gamma^{-1}} \neq 1$, a contradiction. We got b_I is not empty. Thus

$$|\gamma| \geq \sum_{i=1}^{I} |\tau_i| + I \geq \sum_{i=1}^{I} |n_i| + I = |w|.$$

We have the first inequality because $b_1, \ldots, b_I$ are not empty, and the second one since the number of t's in τ_i is n_i for $1 \leq^\forall i \leq I$. Since γ is a geodesic, we have $|\gamma| = |w|$, hence w is a geodesic. q.e.d.

Lemma 7.2 *Take $g \in A\backslash C$ and $h \in A\backslash\varphi(C)$ and fix them.*
Let $w_i, 0 \leq i < \infty$, be words s.t.

$$w_i = t^{10^i} g t^{-10^i} h t^{10^i} g^{-1} t^{-10^i} h^{-1} t^{2\cdot 10^i} g t^{-2\cdot 10^i} h t^{3\cdot 10^i} g^{-1} t^{-3\cdot 10^i} h^{-1}.$$

Then the words satisfy the following conditions.
(1) For any $i \geq 0$ and any $n \geq 1$, $c_{w_i}(w_i^n) = n$.
(2) For any $i \geq 0$ and any $n \geq 1$, $c_{w_i^{-1}}(w_i^n) = 0$.
(3) For any $j > i \geq 0$ and any $n \geq 1$, $c_{w_j^{\pm 1}}(w_i^n) = 0$.
(4) For any $i \geq 0$, $\overline{w_i} \in [G, G]$.

Proof. (1) By Lemma 7.1, w_i^n is a geodesic. Thus $c_{w_i}(w_i^n) = n$.
(2) In order to show $c_{w_0^{-1}}(w_0^n) = 0$ for any $n \geq 1$ by contradiction, suppose $c_{w_0^{-1}}(w_0^n) > 0$ for some n. By Lemma 6.4, take a reduced path α s.t. $\overline{\alpha} = \overline{w_0^n}$ and $|\alpha|_{w_0^{-1}} > 0$. We describe the order of t's and t^{-1}'s appearing in w_0 by W_0,

$$W_0 = + - + - + + - - + + + - - -,$$

where $+$ stands for t, $-$ for t^{-1} and we ignore $g^{\pm 1}, h^{\pm 1}$ in w_0. Then W_0^n represents w_0^n. Let Λ be the order of t's and t^{-1}'s in α. Applying Lemma 6.2 for α and w_0^n, we get $W_0^n = \Lambda$. Λ contains W_0^{-1} as a subpiece since $|\alpha|_{w_0^{-1}} > 0$, where

$$W_0^{-1} = + + + - - - + + - - + - + - \ .$$

But it is easy to check that W_0^n cannot contain W_0^{-1} as a subset, thus we get a contradiction. Hence $c_{w_0^{-1}}(w_0^n) = 0$. Similarly, we can show that $c_{w_i^{-1}}(w_i^n) = 0$ for any $i \geq 1$ and any $n \geq 1$.

(3) Let W_i, $1 \le i < \infty$ be the order of t's and t^{-1}'s in w_i. To show $c_{w_j}(w_i^n) = 0$ for all $n \ge 1$ and all $j > i > 0$ by contradiction, assume $c_{w_j}(w_i^n) > 0$ for some $n \ge 1$ and some $j > i > 0$. Then, by our assumption, W_j must be contained in W_i^n as a subword. This is a contradiction since W_j has $3 \cdot 10^j$ consecutive t's, but W_i has at most $3 \cdot 10^i$ consecutive t's. Thus we have $c_{w_j}(w_i^n) = 0$. $c_{w_j^{-1}}(w_i^n) = 0$ is similar.
(4) Set $T = t^{10^i}$, then

$$w_i = [T, g][gh, [T, g^{-1}]][T, g^{-1}][g, h][T^2, g][gh, [T^3, g^{-1}]][T^3, g^{-1}][g, h],$$

thus $\overline{w_i} \in [G, G]$. q.e.d.

Proposition 4 *Let $G = A *_C \varphi$ with $|A/C| \ge 2$, $|A/\varphi(C)| \ge 2$. There exist words w_i, $0 \le i < \infty$ which satisfy the following conditions.*
(1) For any $i \ge 0$ and any $n \ge 1$, $h_{w_i}(w_i^n) = n$.
(2) For any $j > i \ge 0$ and any $n \ge 1$, $h_{w_j}(w_i^n) = 0$.
(3) For any $i \ge 0$, $\overline{w_i} \in [G, G]$.

Proof. The words w_i, $0 \le i < \infty$ in Lemma 7.2 clearly satisfy (1), (2) and (3). q.e.d.

8 Proofs of Th 2 and Th 3

Proof of Th 2. We can show Th 2 using Prop 3 and Prop 4 as we showed Th 1 using Prop 1 and Prop 2. We omit the detail. q.e.d.
Proof of Th 3. By J. Stallings' structure theorem [S], G is either
(1) $A *_C B$ with $|C| < \infty$, $|A/C| \ge 3$ and $|B/C| \ge 2$,
or
(2) $A *_C \varphi$ with $|C| < \infty$ and $|A/C| \ge 2$.

In the case (1), if $|A| = \infty$ or $|B| = \infty$, then we have the conclusion by Cor 2. If $|A| < \infty$ and $|B| < \infty$, then G is word-hyperbolic. Since G has infinitely many ends, it is non-elementary. It is known that the conclusion of Th 3 holds for a non-elementary word-hyperbolic group, [EF]. In the case (2), since $|A/C| \ge 2$ and $|C| < \infty$, we have $|A/\varphi(C)| \ge 2$. Apply Th 2. q.e.d.

References

[BaGh1] J. Barge, E. Ghys, Surfaces et cohomologie bornée, Invent. Math., 92, 1988, 509-526.

[BaGh2] J. Barge, E. Ghys, Cocycles bornés et actions de groupes sur les arbres réels, in "Group Theory from a Geometric Viewpoint", World Sci. Pub., New Jersey, 1991, 617-622.

[BaGh3] J. Barge, E. Ghys, Cocycles d'Euler et de Maslov, Math. Ann., 294, no 2, 1992, 235-265.

[Bav] C. Bavard, Longueur stable des commutateurs, L'Enseignement Math., 37, 1991, 109-150.

[B] R. Brooks, Some remarks on bounded cohomology, Ann. Math. Studies, 97, 1881, 53-63.

[BS] R. Brooks, C. Series, Bounded cohomology for surface groups, Topology, 23, no 1, 1884, 29-36.

[Br] K.S. Brown, "Cohomology of groups", Springer, New York, 1982.

[EF] D.B.A. Epstein, K. Fujiwara, The second bounded cohomology of word-hyperbolic groups, preprint.

[Gh] E. Ghys, Groupes d'homéomorphismes du cercle et cohomologie bornée, Contemp. Math. 58, AMS, 1987, 81-106.

[Gr] R.I. Grigorchuk, Some remarks on bounded cohomology, LMS Lecture Note 204, Cambridge Univ Press, Cambridge, 1995, 111-163.

[G] M. Gromov, Volume and bounded cohomology, Publ. Math. IHES, 56, 1982, 5-99.

[I] N.V. Ivanov, Foundations of the theory of bounded cohomology, Zap. Nauchn. Sem. Leningr. Otd. Mat. Inst., 143, 1985, 69-109.

[LS] R.C. Lyndon, P.E. Schupp, "Combinatorial Group Theory", Berlin, Springer, 1977.

[MaMo] S. Matsumoto, S. Morita, Bounded cohomology of certain groups of homomorphisms, Proc. A.M.S., 94, 1985, 539-544.

[Mi] Y. Mitsumatsu, Bounded cohomology and l^1-homology of surfaces, Topology, 23, 1984, 465-471.

[So1] T. Soma, Bounded cohomology of closed surfaces, preprint.

[So2] T. Soma, Bounded cohomology and topologically tame Kleinian groups, preprint.

[S] J. Stallings, "Group theory and three-dimensional manifolds", Yale Univ. Press, New Haven, 1971.

[Y] T. Yoshida, On 3-dimensional bounded cohomology of surfaces, in "Homotopy Theory and Related Topics", Advanced Studies in Pure Math 9, Kinokuniya, Tokyo, 1986, 173-176.