EXCEPTIONAL Θ-CORRESPONDENCES I

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Abstract. Let G be a split simply laced group defined over a p-adic field F. In this paper we study the restriction of the minimal representation of G to various dual pairs in G. For example, the restriction of the minimal representation of E_7 to the dual pair $G_2 \times Sp(6)$ gives the non-endoscopic Langlands lift of irreducible representations of G_2 to $Sp(6)$.

Introduction

Let F be a p-adic field and G be a split simple group of type D_n or E_n, $(n = 6, 7, 8)$ over F. If the type is D_n then G is $SO(2n)$. In [KS] the minimal representation V of G was introduced and in [S] some further properties of it were studied. It is an analogue of the Weil representation of $Sp(2n)$ and in some ways it is better behaved then the Weil representation. Therefore it makes sense to ask if one can use it to obtain dual pair correspondences.

We have two types of dual pairs in G. In $SO(2n + 2)$ we have a dual pair

$$SO(2n - 1) \times SO(3).$$

If G is exceptional then we have a dual pair

$$G_2(F) \times H_D$$

with H_D of type A_2, C_3 or F_4, respectively.

Let $A \times B$ be a dual pair in G and X an irreducible representation of A. We say that an irreducible representation Y of B is a Θ-lift of X if $X \otimes Y$ is a quotient of V (see [H1]). Let $\Theta(X)$ be the set of all such Y.

In this paper we compute Jacquet functors V_U of V where U is a unipotent radical of a maximal parabolic subgroup of $SO(3)$ or H_D, with U commutative, or a unipotent radical of a maximal parabolic subgroup of G_2 such that U is a Heisenberg group. The results are reminiscent of those of Kudla [Ku] and Rallis [Ra] in the classical case. We then use the knowledge of Jacquet functors to obtain dual pair correspondences. For example, we compute Θ-lifts of tempered spherical representations of $SO(3)$, A_2, C_3 and G_2. The lifts from A_2 to G_2 and from G_2 to C_3 obtained by restricting the minimal representation of E_6 and E_7 respectively, are Langlands correspondences.

1991 *Mathematics Subject Classification.* Primary 22E35 and 50, Secondary 11F70.

Research at MSRI is supported in part by NSF grant DMS-9022140

In a sequel to this paper we plan to compute Jacquet functors V_U for unipotent radicals of remaining maximal parabolic subgroups. Once this is done, it will be possible to compute lifts of most Langlands quotients and check the conjectural answer for the exceptional correspondences as given by B. Gross [G2]. Local computations are, in a way, a preparation for global correspondences (i.e. correspondences of automorphic forms). So it is worth mentioning that in a forthcoming work, D. Ginzburg, S. Rallis and D. Soudry are studying a global variant of exceptional correspondences. Another possible global application has recently been initiated by B. Gross in connection to a realization of a G_2-motive [G1].

The paper is organized as follows. In Section 1 we recall the definition and few properties of the minimal representation from [S]. In Section 2 we study the dual pair $SO(3)\times SO(2n-1)$. It is the simpliest case and as such it is a good introduction to exceptional dual pairs which form a more interesting part of this work. In Sections 3 and 4 we study the dual pair $G_2 \times A_2$. We compute Θ-lifts of spherical tempered representations of A_2. In Section 5 we study the dual pair $G_2 \times C_3$. We compute Θ-lifts of spherical tempered representations of C_3. In Section 6 we recall some facts about simple Jordan algebras or rank 3. Exceptional groups can be described nicely in terms of Jordan algebras. We use this description in Section 7 to compute V_U where U is the unipotent radical of the "Heisenberg" maximal parabolic subgroup of G_2 in all three cases. We finish the paper by computing Θ-lifts of spherical tempered representations of G_2 in all three cases.

1. Minimal representation

Let G be a simple, split, group of type A_{2n-1}, D_n or E_n. Let ${}^LG$ be the complex L-group of G (see [B]). It is well known that conjugacy classes of unipotent elements in ${}^LG$ correspond to conjugacy classes of homomorphisms

$$\varphi : SL_2(\mathbf{C}) \to {}^LG.$$

Assume now that φ corresponds to the subregular unipotent orbit. Let q be the order of the residual field of F. Let

$$s = \varphi \begin{pmatrix} q^{\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} \end{pmatrix}.$$

Then V is the spherical representation of G with the Satake parameter s (see [Ca]).

Let $\mathfrak{g}$ be the Lie algebra of G. Let $< \cdot, \cdot >$ be the Killing form on $\mathfrak{g}$. Fix $\psi : F \to \mathbb{C}^\times$, a non-trivial character. Let $\mathcal{C}_c^\infty(\mathfrak{g})$ denote the space of locally constant, compactly supported functions on $\mathfrak{g}$. Define the Fourier transform on $\mathcal{C}_c^\infty(\mathfrak{g})$ by

$$\hat{f}(y) = \int_{\mathfrak{g}} f(x)\psi(< x, y >)dx.$$

Let $\mathcal{O}_{\min}$ be the unique minimal nilpotent G-orbit in $\mathfrak{g}$ and $\mu_{\mathcal{O}_{\min}}$ a G-invariant measure on $\mathcal{O}_{\min}$ normalized as in [MW]. Let $\mathfrak{g}_{\mathbb{Z}}$ be the $\mathbb{Z}$-span of a Chevaley basis of $\mathfrak{g}$. Let $\mathcal{R}$ be the ring of integers of F and ϖ the uniformizing element. Let

$$\mathfrak{g}_n = \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \varpi^n \mathcal{R}.$$

In [S] we have shown that there exists $m > 0$ such that

$$tr \int_{\mathfrak{g}} f(x)\pi(\exp x)dx = \int \hat{f}\mu_{\mathcal{O}_{\min}}$$

for any $f \in \mathcal{C}_c^\infty(\mathfrak{g})$ supported on $\mathfrak{g}_m$ and such that $\hat{f}(0) = 0$.

This implies "smallness" of the N-spectrum of V, where N is a unipotent radical of a maximal parabolic subgroup of G. More precisely, let Δ be the Dynkin diagram of G. Mark the diagram Δ as follows. Attach 0 to the unique branch vertex (or to the middle vertex of Δ if the type of G is A_{2n-1}) and 2 to all other. This marking corresponds to the subregular nilpotent orbit [D]. Let $P = MN$ be a maximal parabolic subgroup of G. Let Δ_M be the Dynkin diagram of M. Assume that we are in the following favorable situation:

(1) N is a commutative group.
(2) The marking of Δ corresponding to the subregular nilpotent orbit of G restricts to the marking of Δ_M corresponding to the subregular nilpotent orbit of M.

The possible cases are given by the following table:

G	M	N
D_{n+1}	D_n	F^{2n}
E_6	D_5	F^{16}
E_7	E_6	F^{27}

Here F^{2n} is the standard representation of $SO(2n)$, F^{16} is a spin-representation of $Spin_{10}$ and F^{27} is isomorphic to the exceptional Jordan algebra. We say that a point in N is singular if it is a highest weight vector for a Borel subgroup of M. Let ω and $\overline{\omega}$ be the sets of singular vectors in N and $\overline{N}$. Note that ω is the smallest non-trivial M-orbit in N. If $G = SO(2n)$ then ω is the null-cone of the invariant quadratic form (with 0 excluded).

Theorem 1.1. *The minimal representation V of G has a P-invariant filtration*

$$0 \to \mathcal{C}_c^\infty(\overline{\omega}) \to V \to V_N \to 0.$$

Here $\mathcal{C}_c^\infty(\overline{\omega})$ denotes the space of locally constant, compactly supported functions on $\overline{\omega}$ and V_N is the space of N-coinvariants of V (Jacquet functor).

(1) *Let $f \in \mathcal{C}_c^\infty(\overline{\omega})$. The action of P is given by*

$$\pi(n)f(x) = \psi(< x, n >)f(x), \qquad n \in N$$

and

$$\pi(m)f(x) = |\det(m)|^{\frac{s}{d}} f(m^{-1}x), \qquad m \in M.$$

(2)

$$V_N \cong V_M \otimes |\det|^{\frac{t}{d}} + |\det|^{\frac{s}{d}}$$

where V_M is the minimal representation of M (center acting trivially).

In the above formulas $< \cdot, \cdot >$ *is the* F*-valued pairing between* N *and* $\overline{N}$ *induced by the Killing form on* $\mathfrak{g}$, det *is determinant of the representation of* M *on* N, d *is the dimension of* N *and the values of* s *and* t *are given in the following table.*

G	s	t
D_{n+1}	$n-1$	1
E_6	4	2
E_7	6	3

Proof. This is just Theorem 6.5 in [S] if G is E_7. The other two cases also satisfy conditions of Proposition 4.1 in [S]. Hence the proof carries over with no changes. The proof given in [S], however, is valid only when the residual characterisitic is odd. It remains to discuss the case when the residual characteristic is 2.

Let $x \in \overline{N}$, and define $\psi_x(n) = \psi(< x, n >)$. Let V_{N,ψ_x} be the quotient of V by the space spanned by the elements $\{\pi(n)v - \psi_x(n)v | n \in N, v \in V\}$. The key point in the proof of Theorem 6.5 in [S] is to show that

$$V_{N,\psi_x} = 0$$

for $x \neq 0$ and not in $\overline{\omega}$, i.e. the N-spectrum of V is concentrated on the closure of the smallest M-orbit in $\overline{N}$. This follows from the work of Moeglin and Waldspurger [MW], but only if the residual characteristic of F is odd.

Let k be a number field and $\mathbb{A}$ its ring of adeles. Ginzburg, Rallis and Soudry [GR] have constructed a square integrable automorphic form $\Pi = \otimes_v \Pi_v$ on $G_{\mathbb{A}}$ isomorphic to V at every finite place. But arguing exactly as Howe ([H2] Lemma 2.4), one shows that if the N-spectrum is concentrated on the closure of $\overline{\omega}$ at one place v, it has to be concentrated on the closure of $\overline{\omega}$ at all places. This completes the proof of the theorem.

2. Dual pair $SO(2n-1) \times SO(3)$

Let $G = SO(2n+2)$. Let e, h, f be the standard basis for $sl(2)$, the Lie algebra of $SO(3) = PGL_2(F)$. Let $P = MN$ be the maximal parabolic of G as in Section 1. Let $\mathfrak{g}$ be the Lie algebra of G and let

$$\mathfrak{g} = \overline{\mathfrak{n}} + \mathfrak{m} + \mathfrak{n}$$

be the corresponding decomposition of $\mathfrak{g}$. The embedding $SO(3) \subseteq SO(2n+2)$ corresponds to $sl(2) \subseteq \mathfrak{g}$ such that

(1) $\overline{\mathfrak{n}} = \{x \in \mathfrak{g} \mid [h, x] = -2x\}$.
(2) $\mathfrak{m} = \{x \in \mathfrak{g} \mid [h, x] = 0\}$.
(3) $\mathfrak{n} = \{x \in \mathfrak{g} \mid [h, x] = 2x\}$.

Note that $e \in \mathfrak{n}$, $f \in \overline{\mathfrak{n}}$ and $SO(2n-1)$ is the centralizer of e in M.

Let $Q = LU = P \cap SO(3)$ be a Borel subgroup of $SO(3)$.

Proposition 2.1. *Let* $\begin{pmatrix} a & \\ & b \end{pmatrix} \in L \subset PGL_2(F) = SO(3)$.

(1)
$$\pi(\begin{pmatrix} a & \\ & b \end{pmatrix})f(x) = |\frac{a}{b}|^{n-1} f(\frac{a}{b}x), \qquad f \in \mathcal{C}_c^\infty(\overline{\omega}).$$

(2) *The eigenvalues of* $\pi(\begin{pmatrix} a & \\ & b \end{pmatrix})$ *on* V_N *are* $|\frac{a}{b}|$ *and* $|\frac{a}{b}|^{n-1}$.

Proof. This is just Theorem 1.1.

Let χ be a multiplicative character of F. Let $\rho\chi$ denote the character of L defined by

$$\rho\chi(\begin{pmatrix} a & \\ & b \end{pmatrix}) = |\frac{a}{b}|^{\frac{1}{2}}\chi(\frac{a}{b}).$$

Let $\tau_\chi = \mathrm{Ind}_Q^{SO(3)}\,\rho\chi$. If χ is unitary than τ_χ is an irreducible tempered spherical representation of $SO(3)$. Let $\sigma \in \Theta(\tau_\chi)$. By Frobenius reciprocity

$$Hom_{SO(2n-1)\times SO(3)}(V, \sigma\otimes\tau_\chi) = Hom_{SO(2n-1)\times L}(V_U, \sigma\otimes\rho\chi).$$

Therefore, $\sigma \in \Theta(\tau_\chi)$ is a quotient of V if and only if $\sigma\otimes\rho\chi$ is a quotient of V_U. Since

$$0 \to \mathcal{C}_c^\infty(\overline{\omega})_U \to V_U \to V_N \to 0$$

we need to understand $\mathcal{C}_c^\infty(\overline{\omega})_U$. Let NN and $\overline{NN}$ be the complements of $\overline{U}$ and U in N and $\overline{N}$. Let $\overline{\omega}$, as before, be the set of singular vectors in $\overline{N}$. Put

$$\overline{\omega\omega} = \overline{\omega}\cap\overline{NN}.$$

Lemma 2.2.

$$\mathcal{C}_c^\infty(\overline{\omega})_U = \mathcal{C}_c^\infty(\overline{\omega\omega}).$$

Proof. Let us recall few known facts about Jacquet functors. Let E be a U-module. Then $E_U = E/E(U)$ where $E(U)$ can be defined either as the space spanned by the elements $\{\pi(u)v - v | u \in U, v \in E\}$ or the space of all v such that

$$\int_{U_c} \pi(u)v\,du = 0$$

for some open compact subgroup $U_c \subset U$ depending on v (2.33 [BZ]).

Obviously, $\mathcal{C}_c^\infty(\overline{\omega\omega})$ is a quotient of $\mathcal{C}_c^\infty(\overline{\omega})$ and by Theorem 1.1 (1), U acts trivially on $\mathcal{C}_c^\infty(\overline{\omega\omega})$. Let $f \in \mathcal{C}_c^\infty(\overline{\omega})$ such that $f|_{\overline{\omega\omega}} = 0$. To prove the lemma, we need to find an open compact subgroup U_c such that

$$\int_{U_c} \psi(< x, u >) f(x)\,du = 0$$

for all $x \in \overline{\omega}$. Let x be such that $f(x) \neq 0$. Since x is not in $\overline{NN}$, there exists an open compact neighborhood $\mathcal{O}$ of x and an open compact subgroup $U_{\mathcal{O}}$ such that $\psi(< y, u >)$ is a non-trivial character of $U_{\mathcal{O}}$ for any $y \in \mathcal{O}$. Since the support of f is compact, a finite collection of $\mathcal{O}$ covers the support of f. The union of the corresponding $U_{\mathcal{O}}$ is the desired U_c. The lemma follows.

We can, therefore, summarize the situation with the following proposition.

Proposition 2.3. *V_U has a filtration with two quotients:*

$$\mathcal{C}_c^\infty(\overline{\omega\omega}), \quad \text{and } V_N.$$

As $SO(2n-1)\times L$-modules:

(1) *the action on $\mathcal{C}_c^\infty(\overline{\omega\omega})$ is geometric, twisted with $|\frac{a}{b}|^{n-1}$ for $\begin{pmatrix} a & \\ & b \end{pmatrix} \in L$.*

(2)
$$V_N \cong V_M \otimes |\frac{a}{b}| + 1 \otimes |\frac{a}{b}|^{n-1}$$
where V_M is the minimal representation of M (center acting trivially).

Note that $\overline{NN} = F^{2n-1}$ and $\overline{\omega\omega}$ is the null-cone of the $SO(2n-1)$-invariant quadratic form (with 0 excluded). Let $\mathcal{C}^\infty(\overline{\omega\omega})$ be the space of locally constant functions on $\overline{\omega\omega}$. We can define degenerate principal series representations σ_χ by

$$\sigma_\chi = \{f \in \mathcal{C}^\infty(\overline{\omega\omega}) \mid f(cx) = \chi(c)|c|^{\frac{3}{2}-n} f(x)\}.$$

Analogously, σ_χ can be defined as a quotient of $\mathcal{C}_c^\infty(\overline{\omega\omega})$ consisting of $\overline{f}$ such that

$$\overline{f}(cx) = \chi(c)|c|^{\frac{3}{2}-n}\overline{f}(x).$$

If χ is unramified and unitary than σ_χ is an irreducible unitarizable spherical representation by a result of Tadić [T2], Theorem 9.2. We are now ready to state and prove the main result of this section.

Proposition 2.4. *Let χ be an unramified, unitary multiplicative character. Then*

$$\Theta(\tau_\chi) = \{\sigma_\chi\}.$$

Proof. Let $\sigma \in \Theta(\tau_\chi)$. Then $\sigma \times \rho\chi$ is a quotient of V_U. Since $\rho\chi$ is different from $|\cdot|^{n-1}$ and $|\cdot|$ if χ is unitary, it follows that $\sigma \times \rho\chi$ is quotient of $\mathcal{C}_c^\infty(\overline{\omega\omega})$. Therefore σ must be isomorphic to σ_χ. To finish the theorem, we have to show that $\sigma_\chi \in \Theta(\tau_\chi)$. By Frobenius reciprocity, it suffices to show that $\sigma_\chi \times \rho\chi$ is a quotient of V_U. We know that it is a quotient of $\mathcal{C}_c^\infty(\overline{\omega\omega})$. We need the following lemma.

Lemma 2.5. *Let H be a group and*

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

a sequence of H-modules. Let W be a H-invariant quotient of V_1. Assume that there exists a H-invariant linear map T on V_2 reducing to scalar multiplications by λ and μ on W and V_3. If $\lambda \neq \mu$ then W is a quotient of V_3.

Proof. Let V_0 be such that
$$0 \to V_0 \to V_1 \to W \to 0.$$

Then

$$0 \to W \to V_2/V_0 \to V_3 \to 0$$

Since the eigenvalues of T are different, the last sequence splits and the lemma follows.

Let ϖ be the uniformizing element of F. Put

$$T = \pi \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix}.$$

If χ is unitary, T satisfies the conditions of Lemma, hence $\sigma_\chi \times \rho\chi$ is a quotient of V_U and the theorem is proved.

3. Group E_6

Let J be the 27-dimensional exceptional Jordan algebra and (x, y, z) be the Dickson trilinear form on J (see [J2]). Let G_S be the isometry group of the Dickson form. It is a split, simply-connected groups of type E_6 [A1].

Let $\mathfrak{g}_S$ be the Lie algebra of G_S. Then

$$\mathfrak{g}_S = \bar{\mathfrak{n}} + \mathfrak{m}_S + \mathfrak{n}$$

where $[\mathfrak{m}_S, \mathfrak{m}_S] = so(10)$ and $\bar{\mathfrak{n}}$, $\mathfrak{n}$ are two spin-representations.

Let $P_S = M_S N$ be the corresponding maximal parabolic subgroup in G_S. Then $[M_S, M_S] = Spin_{10}$ and $Spin_{10}$ can be defined as the subgroup of G_S stabilizing the 10-dimensional subspace J_{10} of J consisting of matrices in J such that the coefficients in the third row and the third column are zero and fixing the vector

$$d = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In particular, $Q_d(x, y) = (x, y, d)$ is the invariant quadratic form on J_{10} and N (one of the spin-modules) can be identified with the space in J consisting of matrices

$$\begin{pmatrix} 0 & 0 & y \\ 0 & 0 & z \\ \overline{y} & \overline{z} & 0 \end{pmatrix}.$$

The group G_S has a unique smallest, non-trivial orbit Ω on J. It is the orbit of d and its dimension is 17. On the other hand, M_S has two non-trivial orbits on N and the smaller (call it ω) can be identified with $\Omega \cap N$ [A1].

4. Dual pair $G_2(F) \times PGL_3(F)$

Let D be a nondegenerate subalgebra of the Octonions and let J_D denote the subalgebra of J obtained by restricting the off-diagonal entries of the matrices of J to D. When $\dim(D) = 1$, i.e. $D =< 1_{\mathbb{O}} >= F$ then J_F is the algebra of symmetric 3×3 matrices over

F and $C_{G_S}(J_F) = G_2(F)$, the split group of type G_2. The centralizer of $G_2(F)$ in G_S is $SL_3(F)$ [A1].

Let G, $G_S \subset G$, be a reductive group of type E_6 appearing as a Levi factor of a maximal parabolic subgroup of a split, simply-connected group of type E_7. Then $C_G(G_2(F)) = GL_3(F)$. The action of G_S on J can be extended to G. Indeed, the inclusion $GL_3 \subset G$ induces an isomorphism $GL_3/[GL_3, GL_3] \cong G/[G,G]$. Since $[G,G] = G_S$, it suffices to extend the action to $GL_3 \subset SL_3$. This is done by the formula

$$gxg^t$$

where $x \in J$, $g \in GL_3(F)$ and g^t is the transpose of g.

Let $\mathfrak{g}$ be the Lie algebra of G. Then, as in Section 3,

$$\mathfrak{g} = \bar{\mathfrak{n}} + \mathfrak{m} + \mathfrak{n}$$

with $\mathfrak{m}_S \subset \mathfrak{m}$. Let $P = MN$ be the corresponding maximal parabolic subgroup of G. Then $Q = LU = P \cap GL_3(F)$ is a maximal parabolic subgroup of GL_3 with $L = GL_2(F) \times GL_1(F)$. One can write

$$gl(3) = \bar{\mathfrak{u}} + \mathfrak{l} + \mathfrak{u}$$

where $\mathfrak{l} \subset \mathfrak{m}$, $\overline{\mathfrak{u}} \subset \overline{\mathfrak{n}}$ and $\mathfrak{u} \subset \mathfrak{n}$.

Proposition 4.1. *Let V be the minimal representation of G, the center acting trivially. Identify $\overline{N}$ with the set of pairs of Octonions (y,z). Let $f \in \mathcal{C}_c^\infty(\overline{\omega})$. Then:*

(1)
$$\pi(g)f((y,z)) = f((g^{-1}y, g^{-1}z)), \qquad g \in G_2(F).$$

(2)
$$\pi(l_2 \times l_1)f((y,z)) = \frac{|\det l_2|^2}{|l_1|^4} f(l_1^{-1}(y,z)l_2), \qquad l_2 \times l_1 \in GL_2(F) \times GL_1(F)$$

where det *denotes the usual determinant of* 2×2 *matrices.*

Proof. This is just Theorem 1.1.

Let Y be an irreducible representations of $GL_3(F)$ and $\tilde{Y}$ an irreducible representation of $G_2(F)$ such that $\tilde{Y} \in \Theta(Y)$. Assume that Y is a submodule of $\mathrm{Ind}_Q^{GL_3} X$ for some irreducible representation X of L. By Frobenius reciprocity

$$Hom_{G_2 \times GL_3}(V, \tilde{Y} \otimes \mathrm{Ind}\, X) = Hom_{G_2 \times L}(V_U, \tilde{Y} \otimes X).$$

Therefore, $\tilde{Y} \otimes Y$ is a quotient of V only if $\tilde{Y} \otimes X$ is a quotient of V_U. Since

$$0 \to \mathcal{C}_c^\infty(\overline{\omega})_U \to V_U \to V_N \to 0$$

we need to understand $\mathcal{C}_c^\infty(\overline{\omega})_U$. Let NN and $\overline{NN}$ be the orthogonal complements of $\overline{U}$ and U in N and $\overline{N}$. Since U can be identified with the space in J consisting of matrices

$$\begin{pmatrix} 0 & 0 & y \\ 0 & 0 & z \\ y & z & 0 \end{pmatrix}$$

where $y, z \in F$, it follows that

$$NN = \{ \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & z \\ \overline{y} & \overline{z} & 0 \end{pmatrix} \mid \overline{y} = -y \text{ and } \overline{z} = -z \}.$$

Let $\overline{\omega\omega} = \overline{\omega} \cap \overline{NN}$. As in Lemma 2.2 $\mathcal{C}_c^\infty(\overline{\omega})_U = \mathcal{C}_c^\infty(\overline{\omega\omega})$, and we have to understand the structure of $G_2(F) \times L$ orbits on $\overline{\omega\omega}$ which is, of course, the same as the structure of orbits on $\omega\omega = \omega \cap NN$.

Proposition 4.2.

(1) $\overline{\omega\omega} = \{(y, z) \neq (0, 0) | \overline{y} = -y, \overline{z} = -z$ *and* $y^2 = z^2 = yz = 0\}$.

(2) *Let AA and BB be the subsets of $\overline{\omega\omega}$ consisting all pairs (y, z) such that the space $Fy + Fz$ has dimension 2 and 1 respectively. Then AA and BB are $G_2(F) \times GL_2(F)$-orbits.*

Proof. Let us prove the corresponding statement for $\omega\omega$. Let

$$n = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & z \\ -y & -z & 0 \end{pmatrix} \in \omega\omega.$$

Since n is a traceless matrix in J it will be singular iff $n^2 = 0$. But this is equivalent to $y^2 = z^2 = yz = 0$. The first part of the proposition is proved.

We go on to observe that G_2 has three orbits on the set of spaces of traceless Octonions with the property that the Octonion multiplication is trivial. These are characterized by their dimension; the possible choices being $0, 1, 2$. The stabilizers of the nontrivial spaces are the maximal parabolics of G_2.

Let x be a traceless Octonion such that $x^2 = 0$. Let P_1 be the maximal parabolic subgroup of $G_2(F)$ stabilizing the line Fx. The Levi factor of P_1 is "spanned" by a long root. Consider

$$B = \{(ax, 0) | a \in F \text{ and } a \neq 0\}.$$

Let QQ be the maximal parabolic subgroup of $GL_2(F)$ stabilizing B. Then $P_1 \times QQ$ acts transitively on B and

$$BB = (G_2(F) \times GL_2(F)) \times_{(P_1 \times QQ)} B.$$

Let y and z be two traceless and linearly independent Octonions such that $y^2 = z^2 = yz = 0$. Let P_2 be the maximal parabolic subgroup of $G_2(F)$ stabilizing the space $Fy + Fz$. The Levi factor of P_2 is "spanned" by a short root. Consider

$$A = \{(ay + bz, cy + zd) | a, b, c, d \in F \text{ and } ad - bc \neq 0\}.$$

Then $P_2 \times GL(2)$ acts transitively on A and since $G_2(F)$ acts transitively on the set of all two-dimensional spaces of traceless Octonions with trivial multiplication,

$$AA = G_2(F) \times_{P_2} A.$$

The proposition is proved.

We can now summarize the structure of V_U as a $G_2(F) \times GL_2(F)$-module in the following theorem (compare [Ku]).

Theorem 4.3. *V_U has a filtration with three quotients:*

$$\mathcal{C}_c^\infty(AA),\ \mathcal{C}_c^\infty(BB), \ \text{ and } V_N.$$

Moreover

(1)
$$\mathcal{C}_c^\infty(AA) = \mathrm{ind}_{P_2}^{G_2}(\mathcal{C}_c^\infty(A)) \otimes |\det|^2$$

(2)
$$\mathcal{C}_c^\infty(BB) = \mathrm{ind}_{P_1 \times QQ}^{G_2 \times GL_2}(\mathcal{C}_c^\infty(B)) \otimes |\det|^2$$

(3)
$$V_N \cong V_M \otimes |\det| + 1 \otimes |\det|^2$$

as $G_2(F) \times GL_2(F)$-modules.

We are now ready to state and prove the main theorem.

Theorem 4.4. *Let $\Phi : SL_3(\mathbb{C}) \to G_2(\mathbb{C})$ be the standard inclusion of the L-groups of $PGL_3(F)$ and $G_2(F)$. Let Y be a tempered spherical representation of $PGL_3(F)$. Let $s \in SL_3(\mathbb{C})$ be its Satake parameter. Let $\tilde{Y}$ be the tempered spherical representation of $G_2(F)$ whose Satake parameter is $\Phi(s)$. Then*

$$\Theta(Y) = \{\tilde{Y}\}.$$

Proof. Write $Y = \mathrm{Ind}_Q^{GL_3} X$ (note that there are up to three different choices for X). Then the restriction of X to $GL_2(F)$ (a factor of L) is the irreducible spherical representation whose parameter is $(\chi|\cdot|^{1/2}, \mu|\cdot|^{1/2})$ where χ and μ are unitary characters. Let $\tilde{X}$ be the irreducible spherical representation of $GL_2(F)$ whose parameter is $(\chi^{-1}|\cdot|^{3/2}, \mu^{-1}|\cdot|^{3/2})$. Note that $\mathcal{C}_c^\infty(A)$ is the regular representation of $GL(2)$. After taking into account the twist with $|\det|^2$ it follows that there is a unique quotient of $\mathcal{C}_c^\infty(AA)$ isomorphic to $\tilde{X} \otimes X$ as a $GL_2(F) \times GL_2(F)$-module. Here the first copy of $GL_2(F)$ is the Levi factor of P_2 and the second is the factor of L. Therefore,

$$\mathrm{Ind}_{P_2}^{G_2}(\tilde{X}) \otimes X$$

is a quotient of $\mathcal{C}_c^\infty(AA)$. Write $\tilde{Y} = \mathrm{Ind}_{P_2}^{G_2} \tilde{X}$. Let U_2 be the unipotent radical of P_2. It is Heisenberg group. Let Z be the center of U_2. The action of $GL_2(F)$ (Levi factor of P_2)

on Z is det and on U_2/Z it is isomorphic to $S^3(F^2)\otimes \det^{-1}$. It follows that $\rho_{U_2} = |\det|^3$. Hence $\tilde{Y}$ is tempered. It is irreducible by a result of Keys [Ke]. We have to show that $\tilde{Y}\otimes X$ is a quotient of V_U and that X does not appear as a quotient composition factors of V_U different from $\mathcal{C}_c^\infty(AA)$.

We again use Lemma 2.5, but in this case T will be in the Bernstein center (see [BD]) of $GL_2(F)\subseteq L$. Recall that the component of the Bernstein center corresponding to representations generated by their Iwahori-fixed vectors is isomorphic to

$$\mathbb{C}[x,x^{-1},y,y^{-1}]^W$$

where $W=\{1,w\}$, $w(x)=y$ and $w(y)=x$ is the Weyl group of GL_2. Let I be the Iwahori subgroup of $GL_2(F)$. Let ϖ be the uniformizing element in F. Then any unramified character χ of F^* is determined by its value on ϖ. If E is a subquotient of an induced representation with the parameter (χ_1,χ_2) then

$$(x+y)=\chi_1(\varpi)+\chi_2(\varpi) \text{ and } xy=\chi_1(\varpi)\chi_2(\varpi)$$

on E. Let

$$T_1 = q^{3/2}(x^{-1}+y^{-1})-(xy)^{-1},$$

where $q=|\varpi|^{-1}$. On X T_1 acts as the scalar

$$q^2(\chi(\varpi)^{-1}+\mu(\varpi)^{-1})-q\chi(\varpi)^{-1}\mu(\varpi)^{-1},$$

which if real, is strictly less then $2q^2$. On the other hand, $\mathcal{C}_c^\infty(B)$ is the regular representation of GL_1, so $\mathcal{C}_c^\infty(B)$ as a $GL_2(F)$-module consists of induced representations whose inducing parameters are $(|\cdot|^{3/2},\chi)$. It has an Iwahori-fixed vector only when χ is an unramified character. On such induced representations T_1 acts as

$$q^3 = q^{3/2}(q^{3/2}+\chi(\varpi)^{-1})-q^{3/2}\chi(\varpi)^{-1}.$$

Hence T_1 acts on $\mathcal{C}_c^\infty(B)^I$ as the scalar q^3, which is always bigger then $2q^2$. Therefore X can not be a quotient of $\mathcal{C}_c^\infty(B)$.

Let

$$T_2 = xy.$$

Then T_2 acts on X as $|\varpi|\chi(\varpi)\mu(\varpi)$ which is different from $|\varpi|^2$ and $|\varpi|^4$, the eigenvalues of T_2 on V_N^I. Therefore X can not be a quotient of V_N. Finally, Lemma 2.5 applied to T_1 and T_2 shows that $\tilde{Y}\otimes X$ is a quotient of V_U.

To finish the proof we have to check that the correspondence $Y\to\tilde{Y}$ is Langlands. Since $\tilde{Y}=\mathrm{Ind}_{P_2}^{G_2}\tilde{X}$ and P_2 is spanned by a short root, the Satake parameter of $\tilde{Y}$ sits in a Levi factor of a parabolic subgroup of $G_2(\mathbb{C})$ spanned by a long root. Since $SL_3(\mathbb{C})$ is spanned by long roots of G_2, $\tilde{Y}$ must be a lift of a representation of $PGL_3(F)$ induced from Q: Y or Y^*. Note that replacing the pair (χ,μ) by (χ^{-1},μ^{-1}) does not change $\tilde{Y}$ but replaces Y by Y^*. The theorem is proved.

5. Dual pair $G_2(F) \times PGSp_6(F)$

Let G be the simply connected group of type E_7. Let $P = MN$ be the maximal parabolic subgroup of G such that M is the reductive group of type E_6 discussed in the last section. The nilpotent radical N is commutative and isomorphic to J as an M-module. Let $G_2(F) \times GL_3(F)$ be the dual pair in M, described in Section 4. The centralizer of $G_2(F)$ in G is $Sp(6)$. This can be easily seen on the level of Lie algebras. Let $\mathfrak{g}$ be the Lie algebra of G. Then

$$\mathfrak{g} = \bar{\mathfrak{n}} + \mathfrak{m} + \mathfrak{n}$$

and write

$$C_{\mathfrak{g}}(G_2(F)) = \bar{\mathfrak{u}} + \mathfrak{l} + \mathfrak{u}$$

where $\mathfrak{l} \subset \mathfrak{m}$, $\overline{\mathfrak{u}} \subset \overline{\mathfrak{n}}$ and $\mathfrak{u} \subset \mathfrak{n}$. Then $\mathfrak{l} = gl(3)$ and $\mathfrak{u} \subset \mathfrak{n}$ corresponds to the inclusion $J_F \subset J$ since

$$J^{G_2(F)} = J_F.$$

Therefore $C_{\mathfrak{g}}(G_2(F)) = sp(6)$ whose Siegel parabolic subalgebra is $\mathfrak{l} + \mathfrak{u}$. Note also that we have a distinguished SL_2,

$$SL_2(F) \subset Sp_6(F) \subset G$$

such that if (e, h, f) is the standard basis of $sl(2)$ (the Lie algebra of SL_2) then $h \in \mathfrak{m}$, $f \in \overline{\mathfrak{n}}$ and $e \in \mathfrak{n}$. Moreover e corresponds to the unit element of J under the identification of $\mathfrak{n}$ and J. Let G_A be the split adjoint group of type E_7, and let $P_A = M_A N$ be the corresponding maximal parabolic subgroup. Then M_A can be defined as the group of isogenies of the Dickson form. We have a dual pair $G_2(F) \times PGSp_6(F) \subset G_A$. Note that $PGSp_6(F)$ is generated by the distinguished $PGL_2(F) \subset PGSp_6(F)$ and the image of $Sp_6(F)$.

Let $Q = GL_3(F)U$ be the Siegel parabolic subgroup of $Sp_6(F)$ and $Q_A = LU$ be the corresponding maximal parabolic of $PGSp_6(F)$. Let V be the minimal representation of G_A. We want to compute V_U. Since

$$0 \to \mathcal{C}_c^\infty(\overline{\omega})_U \to V_U \to V_N \to 0$$

we need to understand $\mathcal{C}_c^\infty(\overline{\omega})_U$. Let NN and $\overline{NN}$ be the orthogonal complements of $\overline{U}$ and U in N and $\overline{N}$. Since U can be identified with J_F, it follows that

$$NN = \{\begin{pmatrix} 0 & x & y \\ \overline{x} & 0 & z \\ \overline{y} & \overline{z} & 0 \end{pmatrix} \mid \overline{x} = -x, \overline{y} = -y \text{ and } \overline{z} = -z\}.$$

Let $\overline{\omega\omega} = \overline{\omega} \cap \overline{NN}$. As in Lemma 2.2 $\mathcal{C}_c^\infty(\overline{\omega})_U = \mathcal{C}_c^\infty(\overline{\omega\omega})$ and we have:

Proposition 5.1. *Identify $\overline{NN}$ with the set of triples of traceless Octonions (x, y, z). Let $f \in \mathcal{C}_c^\infty(\overline{\omega\omega})$. The action of $G_2(F) \times L$ is given by*

(1)
$$\pi(g)f((x, y, z)) = f((g^{-1}x, g^{-1}y, g^{-1}z)), \qquad g \in G_2(F).$$

(2)

$$\pi\begin{pmatrix} a & \\ & b \end{pmatrix} f((x,y,z)) = |\frac{a}{b}|^6 f(\frac{a}{b}(x,y,z)), \qquad \begin{pmatrix} a & \\ & b \end{pmatrix} \in PGL_2(F) \subset PGSp_6(F).$$

(3)

$$\pi(g)f((x,y,z)) = |\det g|^4 f(\det g(x,y,z)\,{}^t g^{-1}), \qquad g \in GL_3(F).$$

Proof. It follows from Theorem 1.1. Note, however, that in this case the formulas are describing the action on $\mathcal{C}_c^\infty(\overline{\omega\omega})$.

To keep our notation simple we work in G. We have to understand the structure of $G_2(F) \times GL_3(F)$ orbits on $\overline{\omega\omega}$ which is, of course, the same as the structure of orbits on $\omega\omega = \omega \cap NN$.

Proposition 5.2.

(1) $\overline{\omega\omega} = \{(x,y,z) | \overline{x} = -x, \overline{y} = -y, \overline{z} = -z$ *and* $x^2 = y^2 = z^2 = xy = xz = yz = 0\}$.

(2) *Let* AA *and* BB *be the subsets of* $\overline{\omega\omega}$ *consisting of all triples* (x,y,z) *such that the space* $Fx + Fy + Fz$ *has dimension* 2 *and* 1 *respectively. Then* AA *and* BB *are* $G_2(F) \times GL_3(F)$*-orbits and*

$$\overline{\omega\omega} = AA \cup BB.$$

Proof. Let us prove the corresponding statement for $\omega\omega$. Let

$$n = \begin{pmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{pmatrix} \in \omega\omega.$$

Since n is a traceless matrix in J it will be singular iff $n^2 = 0$. But this is equivalent to $x^2 = y^2 = z^2 = xy = xz = yz = 0$. The first part of the proposition is proved.

Again, recall that G_2 has three orbits on the set of spaces of traceless Octonions with the property that the Octonion multiplication is trivial. These are characterized by their dimension; the possible choices being $0, 1, 2$. The stabilizers of the nontrivial spaces are the maximal parabolics of G_2. It follows that x, y and z are linearly dependent. hence $\overline{\omega\omega} = AA \cup BB$. It remains to show that AA and BB are single orbits. The proof is analogous to the proof of Proposition 4.2.

Let x be a traceless Octonion such that $x^2 = 0$. Let P_1 be the maximal parabolic subgroup of $G_2(F)$ stabilizing the line Fx. Consider

$$B = \{(ax, 0, 0) | a \in F \text{ and } a \neq 0\}.$$

Let Q_1 be the maximal parabolic of $GL_3(F)$ stabilizing B. Then $P_1 \times Q_1$ acts transitively on B and

$$BB = (G_2(F) \times GL_3(F)) \times_{(P_1 \times Q_1)} B.$$

Let y and z be two traceless and linearly independent Octonions such that $y^2 = z^2 = yz = 0$. Let P_2 be the maximal parabolic subgroup of $G_2(F)$ stabilizing the space $Fy + Fz$. Consider

$$A = \{(ay + bz, cy + dz, 0) | a, b, c, d \in F \text{ and } ad - bc \neq 0\}.$$

Let Q_2 be the maximal parabolic subgroup of $GL_3(F)$ stabilizing A. Then $P_2 \times Q_2$ acts transitively on A and

$$AA = (G_2(F) \times GL_3(F)) \times_{(P_2 \times Q_2)} A.$$

The proposition is proved.

We can now summarize the structure of V_U as a $G_2(F) \times GL_3(F)$-module.

Theorem 5.3. *V_U has a filtration with three subquotients:*

$$\mathcal{C}_c^\infty(AA),\ \mathcal{C}_c^\infty(BB),\ \text{ and } V_N.$$

Moreover

(1)
$$\mathcal{C}_c^\infty(AA) = \mathrm{ind}_{P_2 \times Q_2}^{G_2 \times GL_3}(\mathcal{C}_c^\infty(A)) \otimes |\det|^4$$

(2)
$$\mathcal{C}_c^\infty(BB) = \mathrm{ind}_{P_1 \times Q_1}^{G_2 \times GL_3}(\mathcal{C}_c^\infty(B)) \otimes |\det|^4$$

(3)
$$V_N \cong V_M \otimes |\det|^2 + 1 \otimes |\det|^4$$

as $G_2(F) \times GL_3(F)$-modules. Here det *denotes the usual determinant of* 3×3 *matrices.*

We are now ready to state and prove the main theorem.

Theorem 5.4. *Let $\Phi : G_2(\mathbb{C}) \to Spin_7(\mathbb{C})$ be the standard inclusion of the L-groups of $G_2(F)$ and $PGSp_6(F)$; $G_2(\mathbb{C})$ fixes a non-zero vector in the* 8*-dimensional spin representation of $Spin_7(\mathbb{C})$. Let Y' be a tempered spherical representation of $PGSp_6(F)$. Then $\Theta(Y')$ is not empty only if the Satake parameter of Y' is $\Phi(s)$ for some s, a Satake parameter of a tempered spherical representation $\tilde{Y}$ of $G_2(F)$. In that case*

$$\Theta(Y') = \{\tilde{Y}\}.$$

Proof. Let X' be a spherical tempered representation of $PGSp_6(F)$. Recall that every tempered spherical representation of $PGSp_6(F)$ is fully induced (see [T1]). To simplify notation we work with $Sp_6(F)$. The restriction of X' to $Sp_6(F)$ can be written as $\mathrm{Ind}_Q^{Sp_6} X \otimes |\det|^2$ where X is a tempered spherical representation of $GL_3(F)$ (note that $\rho_U = |\det|^2$).

Let Y' be a spherical tempered representation of $PGSp_6(F)$ whose Satake parameter is $\Phi(s)$. This means that the restriction of Y' to $Sp_6(F)$ can be written as $\mathrm{Ind}_Q^{Sp_6}(Y \otimes |\det|^2)$

where Y is a tempered spherical representation of $PGL_3(F)$. Also $\tilde{Y}$, the representation of $G_2(F)$ whose parameter is s, is the lift of Y from $PGL_3(F)$. By Theorem 5.3, $V_M \otimes |\det|^2$ is a quotient of V_U (M is a group of type E_6). It follows from the Frobenius reciprocity that $\tilde{Y} \in \Theta(Y')$.

The rest of the theorem follows form the knowledge of V_U. Indeed, let X' be a tempered spherical representation of $PGSp_6(F)$. Again, it is a fully induced representation so its restriction to $Sp_6(F)$ can be written as $\mathrm{Ind}_Q^{Sp_6} X \otimes |\det|^2$ where X is a tempered spherical representation of $GL_3(F)$. If $\tilde{Y} \otimes X'$ is a quotient of V for some irreducible representation $\tilde{Y}$ of $G_2(F)$, then $\tilde{Y} \otimes (X \otimes |\det|^2)$ is a quotient of V_U, i.e. it is a quotient of one of the three pieces in Theorem 5.3. For example, if it is a quotient of $\mathcal{C}_c^\infty(AA)$ then

$$\tilde{Y} \otimes (X \otimes |\det|^2) = \mathrm{Ind}_{P_2}^{G_2}(\tilde{X}) \otimes (X \otimes |\det|^2)$$

where the parameter of $\tilde{X}$ is $(|\cdot|^{3/2}\chi^{-1}, |\cdot|^{3/2}\mu^{-1})$ and the parameter of X is $(\chi, \mu, \chi\mu)$ for some unitary characters χ and μ. Since $\mathrm{Ind}\, X \otimes |\det|^2 = \mathrm{Ind}\, Y \otimes |\det|^2$, where Y is a representation of $PGL_3(F)$ with parameter $(\chi, \mu, \chi^{-1}\mu^{-1})$, we are back in the situation as in the beginning of the proof. The theorem is proved.

With only minor modifications of the preceeding proof one can easily show the following proposition.

Proposition 5.5. *Let Y be a spherical representation of $PGSp_6(F)$ which appears as a quotient of V, the minimal representation of E_7. Then the Satake parameter of Y is $\Phi(s)$ for some $s \in G_2(\mathbb{C})$.*

6. Some facts about Jordan algebras

In this section we shall describe structure of exceptional groups using simple Jordan algebras of rank 3.

Let D be a non-degenerate subalgebra of Octonions. Let $J_D \subseteq J$ be the corresponding Jordan algebra of rank 3. The Dickson form on J_D is given by

$$6(x,y,z) = 2tr(xyz) - tr(x)tr(yz) - tr(y)tr(zx) - tr(z)tr(xy) - tr(x)tr(y)tr(z).$$

Define

$$\mathfrak{l}_D = \{x \in gl(J_D) \,|\, (xv, v, v) = 0 \text{ for all } v \in J_D\}.$$

Now assume that $D = F$, $F + F$, $M_2(F)$ (2×2-matrices) or $\mathbb{O}$. Let G_D be a simply connected group of type F_4 or E_n, $n = 6, 7$ or 8 respectively. Let $\mathfrak{g}_D$ be its Lie algebra. Let Δ be the Dynkin diagram of $\mathfrak{g}_D$. We shall identify it with a set of simple roots. Let $\tilde{\alpha}$ be the highest positive root. Let α be the unique simple root not perpendicular to $\tilde{\alpha}$. Let $\mathfrak{m}_D \subset \mathfrak{g}_D$ be the simple algebra whose Dynkin diagram is $\Delta \setminus \{\alpha\}$. Extend Δ by adding $-\tilde{\alpha}$. Let β be the unique simple root not perpendicular to α. Remove the vertex corresponding to the simple root β. The extended diagram breaks into several pieces, one of which is an A_2 diagram corresponding to $\{\alpha, -\tilde{\alpha}\}$. Then $\mathfrak{l}_D \subset \mathfrak{g}_D$ corresponds to the rest of the diagram. Consider the adjoint action of $sl(3) + \mathfrak{l}_D$ on $\mathfrak{g}_D$. One has a decomposition

$$\mathfrak{g}_D = sl(3) + \mathfrak{l}_D + V \otimes J_D + V^* \otimes J_D^*$$

where V is the standard 3-dimensional representation of $sl(3)$ (see [HP]). Let M_D and L_D be the corresponding simply connected groups and let H_D be a subgroup of L_D centralizing the identity of J_D. The possible choices are given by the Freudenthal's magic square:

G_D	M_D	L_D	H_D	J_D
F_4	Sp_6	SL_3	SO_3	S^2V
E_6	SL_6	$SL_3 \times SL_3$	SL_3	$V \otimes V^*$
E_7	E_6	SL_6	Sp_6	$\wedge^2 F^6$
E_8	E_7	E_6	F_4	F^{27}

In the above table we have identified J_D with a "usual" representation of L_D whenever possible. Since the centralizer of H_D in J_D is one-dimensional, it follows that the centralizer of H_D in $\mathfrak{g}_D$ is (see [FH])

$$\mathfrak{g}_2 = sl(3) + V + V^*.$$

Assume that $D \neq F$. We shall need few facts about the action of H_D on J_D^0, the set of traceless elements in J_D. Recall that a point $x \in J_D^0$ is singular if $x^2 = 0$. The group H_D acts transitively on the set of singular vectors in J_D^0. Denote by Q_1 a parabolic subgroup of H_D stabilizing a singular line in J_D^0.

Let $Fy + Fz \subset J_D^0$ be a singular space, i.e. each point is singular. This is equivalent to

$$y^2 = z^2 = yz = 0.$$

We need to know H_D-orbits of singular two-dimensional subspaces of J_D^0. We have two different cases.

If $\dim D = 2$ then J_D is isomorphic to the algebra of all 3×3-matrices with coefficients in F and $H_D = SL_3$ acts by conjugation. The restriction of the Dickson form to J_D is given by

$$(x, x, x) = \det(x)$$

where det is the determinant of 3×3 matrices. In this case, a singular point in J_D^0 is a nilpotent rank-one matrix. There are two SL_3-orbits of singular two-dimensional spaces. Let $Fy+Fz$ be a singular space. Then either the images or the kernels of y and z coincide. Denote by Q^+ or Q^- the stabilizer of the singular space depending on the orbit. It is a maximal parabolic subgroup of SL_3.

In other cases, a stabilizer of a singular two-dimensional space is a parabolic subgroup only if the space is "amber" ([A]).

Definition 6.1. *Let $S \subset J_D^0$ be a singular two-dimensional space. We say that S is amber if $S \subset xJ_D$ for every $x \in S$.*

Proposition 6.2. *If $\dim D \geq 4$ then H_D acts transitively on the space of amber two-dimensional subspaces of J_D^0.*

Proof. If $\dim D = 8$ this is a result of Aschbacher, 9.3-5 [A1]. Let $Fy + Fz = S$ be an amber space. Since H_D acts transitively on the set of singular vectors, we can assume that $z = x$, where x is a fixed singular vector in J_D^0. Let Q_1 be a parabolic subgroup of H_D

stabilizing the line Fx. Let L_1 be a Levi factor of Q_1. Let $I_x \subset J_D$ be an L_1-invariant space such that

$$xJ_D \cap J_D^0 = Fx + I_x.$$

To finish the proof, we have to show that L_1 acts transitively on the set of singular points in I_x.

If $\dim D = 4$ then $L_1 = GL_2 \times SL_2$ and I_x can be identified with the space of 2×2-matrices. If $\dim D = 8$ then $[L_1, L_1] = Spin_7$ and I_x is the corresponding 8-dimensional spin-representation. In both cases L_1 has two non-trivial orbits in I_x, the smaller consisting of singular points. The proposition is proved.

Let Q_2 be the stabilizer of an amber plane. We finish this section with a description of Q_2 in both cases. Identify the amber space with F^2. Let L_2 be a Levi factor of Q_2.

If $\dim D = 4$ then Q_2 is contained in a Siegel parabolic subgroup of $Sp(6)$ and $L_2 = GL_2 \times GL_1$. The action of L_2 on the amber space is the standard action of $GL_2 \times GL_1$ on the space of 2×1 matrices.

If $\dim D = 8$. Then L_2 is a quotient of $GL_3 \times GL_2$ obtained by identifying centers of both factors (consider $GL_3 \times GL_2$ acting on 3×2 matrices, for example). Let $(l_3, l_2) \in GL_3 \times GL_2$. The action of L_2 on the amber space is given by

$$\det(l_3)\det(l_2)l_2x \;\; x \in F^2.$$

In particular, the action of L_2 on the amber space induces a non-split exact sequence

$$1 \to SL_3 \to L_2 \to GL_2 \to 1.$$

We have $[L_2, L_2] = SL_3 \times SL_2$ where SL_3 is spanned by long roots and SL_2 by short roots.

7. Jacquet functor for G_2 - Heisenberg parabolic

We continue to use the notation and definitions introduced in the previous section. Let

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in sl(3) \subset \mathfrak{g}_D.$$

Let $\mathfrak{g}(k) = \{y \in \mathfrak{g}_D | [x, y] = ky\}$. Then $\mathfrak{g}(k) \neq 0$ for $k = -2, -1, 0, 1, 2$. Let $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ be the maximal parabolic subalgebra with $\mathfrak{m} = \mathfrak{g}(0)$ and $\mathfrak{n} = \mathfrak{g}(1) + \mathfrak{g}(2)$. Note that $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{m}_D$ and $\mathfrak{n}$ is a Heisenberg Lie algebra with center $\mathfrak{g}(2)$. Let $P = MN$ be the corresponding maximal parabolic subgroup. Let Z be the center of N and let N_D be the quotient of N by Z. Then N_D is an irreducible M_D-module. Since $N_D \cong \mathfrak{g}(1)$, the restriction to L_D is isomorphic to

$$F + J_D + J_D^* + F.$$

The centralizer of H_D in G_D is $G_2(F)$, and

$$G_2(F) \cap P = P_2 = GL_2(F)U_2$$

is the Heisenberg parabolic of $G_2(F)$. Then $Z \subset U_2$ and let U be the quotient of U_2 by Z.

Let Ω be the set of singular vectors in N_D. Let $\overline{N}$ denote the opposite nilpotent subgroup. Let $\overline{N}_D$, $\overline{Z}$... be the corresponding objects attached to $\overline{N}$. Note that the Killing form on $\mathfrak{g}_D$, the Lie algebra of G_D, defines a non-degenerate pairing $< \cdot, \cdot >$ between N_D and $\overline{N}_D$.

Theorem 7.1. *Assume that the residual characteristic of F is odd. Let V be the minimal representation of G_D. Let Z be the center of N as above. Then V_Z has a P-invariant filtration*

$$0 \to \mathcal{C}_c^\infty(\overline{\Omega}) \to V_Z \to V_N \to 0$$

where $\mathcal{C}_c^\infty(\overline{\Omega})$ denotes the space of locally constant, compactly supported functions on $\overline{\Omega}$.

(1) *Let $f \in \mathcal{C}_c^\infty(\overline{\Omega})$. Then*

$$\pi(n)f(x) = \psi(< x, n >)f(x), \qquad n \in N$$

and

$$\pi(m)f(x) = |\det(m)|^{\frac{s}{d}} f(m^{-1}x), \qquad m \in M.$$

(2)

$$V_N \cong V_M \otimes |\det|^{\frac{t}{d}} + |\det|^{\frac{s}{d}}$$

where V_M is the minimal representation of M (center acting trivially).

In the above formulas det *is the determinant of the representation of M on N_D, d is the dimension of N_D and the values of s and t are given in the following table.*

G	s	t
E_6	4	3
E_7	6	4
E_8	10	6

Proof. Part (2) is Prop. 4.1 [S]. Let W be the kernel of the projection of V_Z onto V_N. Let $x \in \overline{N}_D$ and let

$$\psi_x(y) = \psi(< x, y >)$$

be a character of N. The theorem of Moeglin and Waldspurger [MW] implies that $\dim W_{N,\psi_x} = 0$ or 1 and it is one if and only if $x \in \overline{\Omega}$. Let $x \in \overline{\Omega}$. Let M_x be the stabilizer of x in M and χ the character of M_x describing the action of M_x on W_{N,ψ_x}. By Frobenius reciprocity there exists a non-trivial P-homomorphism

$$T : W \to \mathrm{Ind}_{M_x N}^P(\chi \otimes \psi_x).$$

Let $\mathcal{C}^\infty(\overline{\Omega})$ denote the space of locally constant functions on $\overline{\Omega}$. Note that we have an inclusion

$$\mathrm{Ind}_{M_x N}^P(\chi \otimes \psi_x) \subseteq \mathcal{C}^\infty(\overline{\Omega}).$$

Let $w \in W$ and $f = T(w)$. We need to show that f is a compactly supported function on $\overline{\Omega}$. Let $N_{D,k}$, $k \in \mathbb{Z}$ be a family of lattices of N_D such that $\cup_k N_{D,k} = N_D$ and $\cap_k N_{D,k} = 0$. Let $\overline{N}_{D,k}$ be their dual lattices in $\overline{N}_D$. Since W is a smooth module, there exists a small integer k depending on w such that $\pi(n)w = w$ for all $n \in N_{D,k}$. This implies that f is supported inside $\overline{N}_{D,k}$. Since $W_N = 0$ there exists an integer l depending on w such that

$$\int_{N_{D,l}} \pi(n)w dn = 0$$

(see 2.33 [BZ]). This implies that f is supported outside $\overline{N}_{D,k}$. Since $\overline{\Omega}$ is locally closed and the boundary is $\{0\}$, it follows that $f \in \mathcal{C}_c^\infty(\overline{\Omega})$. Let ind denote smooth induction with compact support. By the Bernstein-Zelevinsky analogue of Mackey Theory (see [BZ], pages 46-47)

$$\mathrm{ind}_{M_x N}^P(\chi \otimes \psi_x) = \mathcal{C}_c^\infty(\overline{\Omega})$$

is irreducible P-module. Hence $T(W) = \mathcal{C}_c^\infty(\overline{\Omega})$. Let W' be the kernel of T. Since

$$\dim W_{N,\psi_x} = \dim \mathcal{C}_c^\infty(\overline{\Omega})_{N,\psi_x}$$

for any $x \in \overline{N}_D$, it follows that $W'_{N,\psi_x} = 0$ for any $x \in \overline{N}_D$ (2.35 [BZ]). Therefore $W' = 0$ by 5.14 [BZ].

Note that the inclusion $M_x \to M$ induces an isomorphism $M_x/[M_x, M_x] \cong M/[M, M]$ This can be easily checked by choosing x to be in $\mathfrak{g}_{D,\alpha} \subset \mathfrak{g}(1)$, α is defined in the previous section. Hence χ is a character of M and to finish the proof we have to show that

$$\chi(m) = |\det(m)|^{\frac{s}{d}} \qquad m \in M.$$

Let $GL_2(F)$ be the Levi factor of P_2 defined above. The inclusion of $GL_2(F)$ into M induces in isomorphism $GL_2(F)/SL_2(F) \cong M/[M, M]$. This can be seen by looking at the action of $GL_2(F)$ and M on Z. Therefore it suffices to find the restriction of χ to $GL_2(F)$. In Section 8 we shall use the information on correspondences obtained in previous sections to find the character.

Let NN be the orthogonal complement of $\overline{U}$ in N_D. Let J_D^0 be the set of traceless elements of J_D. Then NN as a $GL_2(F) \times H_D$-module is isomorphic to

$$F^2 \otimes J_D^0.$$

Write

$$\Omega\Omega = \Omega \cap NN.$$

Proposition 7.2.

(1) $\overline{\Omega\Omega} = \{(y, z) \neq (0, 0) |\ y, z \in J_D^0$ *and the space* $Fy + Fz$ *is amber* $\}$.

(2) *Let* AA *and* BB *be the subsets of* $\overline{\Omega\Omega}$ *consisting of all pairs* (y, z) *such that the space* $Fy + Fz$ *has dimension* 2 *and* 1 *respectively. Then* BB *is a* $GL_2(F) \times H_D$*-orbit. If* $\dim D \geq 4$ *then* AA *is a* $GL_2(F) \times H_D$*-orbit. If* $\dim D = 2$ *then* AA *is a union of two orbits.*

Proof. Recall that a two-dimensional space $S \subset J_D^0$ is amber if every element is singular and

$$S \subset xJ_D$$

for all $x \in S$. As before, write

$$N_D = F + J_D + J_D^* + F.$$

Let $Q_D = LU_D \subseteq M_D$ be the stabilizer of the last summand in the above decomposition. Then $[L, L] = L_D$, $U_D \cong J_D$ as an L_D-module and Q_D stabilizes the partial flag

$$N_D \supset J_D + J_D^* + F \supset J_D^* + F \supset F.$$

More precisely, let $u \in U_D$, $(a, y, z, b) \in N_D$ and $(a', y', z', b') = u(a, y, z, b)$. Then

$$a' = a,\ y' = y + au,\ z' = z + 2u \times y + au \times u,$$

$$b' = b + tr(uy) + 3(u, u, z) + a \det u,$$

where $\det u = (u, u, u)$ and $u \times y$, the cross product, is an element of J_D such that

$$tr((u \times y)x) = 3(x, y, u)$$

for all $x \in J_D$ (see [Ki] page 143).

Lemma 7.3. *The group Q_D has 4 orbits on Ω. Their representatives are*

$$v_1 = (1, 0, 0, 0),\ v_2 = (0, x, 0, 0)$$

$$v_3 = (0, 0, x, 0),\ v_4 = (0, 0, 0, 1)$$

where x is any singular element in J_D.

Proof. Note that $\mathbb{P}(\Omega) = M_D/Q_D$. We have to compute $Q_D \backslash M_D/Q_D$ which is the same as $W_L \backslash W_M/W_L$, here W_M and W_L denote the Weyl groups of M_D and L_D. Since N_D is a miniscule representation i.e. weight vectors are all contained in one W_M-orbit, it follows that they are parametrized by W_M/W_L. On the other hand J_D is a miniscule representation of L_D so $W_L \backslash W_M/W_L$ has four orbits and it follows easily that v_i, $(1 \leq i \leq 4)$ form a complete set of representatives of Q_D-orbits. The lemma is proved.

Let $(0, y, z, 0) \in \overline{\Omega\Omega}$. If $y \neq 0$ then Lemma 7.3 implies that it is Q_D conjugated to v_2. Therefore y is in the L-orbit of x, hence y is singular. Since the action of $GL_2(F)$ is

$$(y, z) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ay + cz, by + dz)$$

the same argument implies that any element of $Fy + Fz$ is singular.

Note that $(0, y, z, 0)$ is Q_D, hence U_D-conjugated to $(0, y, 0, 0)$. But this means that $z = 2u \times y$ for some $u \in U_D \cong J_D$ such that $tr(uy) = 0$. Since $tr((u \times y)v) = 3(u, y, v)$ for all elements $v \in J_D$ and

$$6(u, y, v) = 2tr(uyv) - tr(u)tr(yv)$$

(use $tr(y) = tr(uy) = 0$), it follows that

$$tr((2u \times y)v) = tr(u^* yv)$$

for all $v \in J_D$, here $u^* = 2u - tr(u)$. Therefore $z = u^* y \in yJ_D$. Since the same argument can be repeated for any linear combination of y and z, the first part of the proposition follows.

Let $x \in J_D^0$ such that $x^2 = 0$. Let Q_1 be the parabolic subgroup of H_D stabilizing the line Fx. Consider

$$B = \{(ax, 0) | a \in F \text{ and } a \neq 0\}.$$

Let $QQ \subset GL_2(F)$ be the Borel subgroup stabilizing the line B. Then $QQ \times Q_1$ acts transitively on B and

$$BB = (GL_2(F) \times H_D) \times_{(QQ \times Q_1)} B.$$

Assume now that $\dim D \geq 4$. Let $y, z \in J_D^0$ such that the space $Fy + Fz$ is amber. Let Q_2 be the parabolic subgroup of H_D stabilizing the space $Fy + Fz$. Consider

$$A = \{(ay + bz, cy + zd) | a, b, c, d \in F \text{ and } ad - bc \neq 0\}.$$

Then $GL_2(F) \times Q_2$ acts transitively on A and

$$AA = H_D \times_{Q_2} A.$$

If $\dim D = 2$ then we have two orbits of singular two-dimensional spaces. Let $Fy^+ + Fz^+$ and $Fy^- + Fz^-$ be their representatives and Q^+ and Q^- their stabilizers in $SL_3(F)$. One can define A^+ and A^- as above, hence

$$AA = SL_3(F) \times_{Q^+} A^+ \cup SL_3(F) \times_{Q^-} A^-.$$

The proposition is proved.

Since $\mathcal{C}_c^\infty(\overline{\Omega})_{U_2} = \mathcal{C}_c^\infty(\overline{\Omega\Omega})$, the structure of V_{U_2} as $GL_2(F) \times H_D$ is given as follows.

Theorem 7.4. *Assume that the residual characteristic of F is odd. V_{U_2} has a filtration with three quotients:*

$$\mathcal{C}_c^\infty(AA),\ \mathcal{C}_c^\infty(BB),\ \text{ and } V_N.$$

Moreover, as $GL_2(F) \times H_D$-modules,

(1) *If $\dim D \geq 4$ then:*

$$\mathcal{C}_c^\infty(AA) = |\det|^s \otimes \operatorname{ind}_{Q_2}^{H_D}(\mathcal{C}_c^\infty(A))$$

(2) *If $\dim D = 2$ then:*

$$\mathcal{C}_c^\infty(AA) = |\det|^s \otimes \operatorname{ind}_{Q^+}^{SL_3}(\mathcal{C}_c^\infty(A^+)) + |\det|^s \otimes \operatorname{ind}_{Q^-}^{SL_3}(\mathcal{C}_c^\infty(A^-))$$

(3)

$$\mathcal{C}_c^\infty(BB) = |\det|^s \otimes \operatorname{ind}_{QQ \times Q_1}^{GL_2(F) \times H_D}(\mathcal{C}_c^\infty(B))$$

(4)

$$V_{N_D} \cong |\det|^t \otimes V_M + |\det|^s \otimes 1$$

where V_M is the minimal representation of M_D (center acting trivially).

In the above formulas det *is the usual determinant of* 2×2 *matrices, and* s *and* t *are given by the following table:*

G	s	t
E_6	2	3/2
E_7	3	2
E_8	5	3

Remark: In this section we have assumed that G_D, and hence H_D are simply connected. In fact, the isogeny class of H_D determines the isogeny class of G_D and vice-versa. Note that the action of H_D on $\mathcal{C}_c^\infty(\overline{\Omega\Omega})$ is geometric. It comes from the (algebraic) action of H_D on J_D^0 which extends uniquely to the group of adjoint type. Therefore we have a canonical way of extending the representation V to groups of adjoint type.

8. Θ-lifts from $G_2(F)$

In this section we compute Θ-lifts of spherical tempered representations of $G_2(F)$ in all three cases. In the process we also compute the normalizing factors (i.e. coefficients s) in Theorems 7.1 and 7.4. We assume that the residual characteristic of F is odd.

We study the dual pair $G_2(F)\times F_4(F)$ in a simple group G of type E_8 first. Let $\tilde{Y}$ be a spherical tempered representation of $G_2(F)$. Then $\tilde{Y}=\mathrm{Ind}_{P_2}^{G_2}(\tilde{X})$ where $\tilde{X}$ is a spherical representation of $GL_2(F)$ with the parameter $(\chi^{-1}|\cdot|^{3/2},\mu^{-1}|\cdot|^{3/2})$. As before, χ and μ must be unitary characters. Let $Q_2=L_2V_2$ be the maximal parabolic subgroup of $F_4(F)$ stabilizing an amber projective line. In Section 6 we have shown that the action of L_2 on the corresponding amber line gives an exact sequence

$$1\to SL_3\to L_2\to GL_2\to 1.$$

One checks that $\rho_{V_2}=|\det|^7$, here det is the usual determinant on GL_2. Let X be a spherical representation of GL_2 with the parameter $(\chi|\cdot|^{7/2},\mu|\cdot|^{7/2})$. Pull X back to L_2. Let

$$W=\mathrm{Ind}_{Q_2}^{F_4}X.$$

Note that W is a unitarizable representation of $F_4(F)$. It is quite possible that W is always irreducible but we do not know.

Theorem 8.1. *Let* $\tilde{Y}$ *be a spherical tempered representation of* $G_2(F)$ *and* W *the representation of* $F_4(F)$ *defined above. Then* $\Theta(\tilde{Y})\neq\{\}$ *and if* $W'\in\Theta(\tilde{Y})$ *then* W' *is a summand of* W. *Let* $s\in G_2(\mathbb{C})$ *be the Satake parameter of* $\tilde{Y}$. *Then the Satake parameter of* W *is* $\Psi(s\times\rho)$ *where* $\Psi: G_2(\mathbb{C})\times SO_3(\mathbb{C})\to F_4(\mathbb{C})$ *and* $\rho\in SO_3(\mathbb{C})$ *is the Satake parameter of the trivial representation of* $SL_2(F)$.

Proof. Let $P=MN$ be the maximal parabolic subalgebra of G studied in the last two sections. We described an embedding of the dual pair $G_2(F)\times F_4(F)$ in G such that $G_2(F)\cap P$ is the Heisenberg maximal parabolic subgroup P_2.

Yet another embedding of the dual pair $G_2(F)\times F_4(F)$ is given by the inclusion of Jordan algebras $J_F\to J$ (use the description of the exceptional Lie algebras given in Section 6). In this case, $G_2(F)\subset M$ and $F_4(F)\cap P=Q_4=L_4V_4$, the Heisenberg maximal parabolic subgroup of $F_4(F)$. The Levi component L_4 is isomorphic to $GSp_6(F)$. Note that the inclusion $GSp_6(F) \to M$ induces an isomorphism $GSp_6(F)/Sp_6(F) \cong M/[M, M]$. This is easily seen by considering the action of $GSp_6(F)$ and M on Z, the center of both, V_4 and N.

Let V be the minimal representation of G. As an $G_2(F) \times GSp_6(F)$-module

$$V_N \cong V_M \otimes |\det|^3 + 1 \otimes |\det|^5$$

where det denotes the usual determinant on $GSp_6(F)$ (Prop. 4.1 [S]). Let $\tilde{Y} = \mathrm{Ind}_{P_2}^{G_2}(\tilde{X})$ be a spherical tempered representation of $G_2(F)$ and let Y' be the Langlands lift of $\tilde{Y}$ to $PGSp_6(F)$. By Theorem 5.4 $\tilde{Y} \otimes Y'$ is a quotient of V_M. It follows from the Frobenius reciprocity that $\tilde{Y} \otimes W'$ is a quotient of V for some W', a subquotient of

$$\mathrm{Ind}_{Q_4}^{F_4}(Y' \otimes |\det|^3).$$

On the other hand, $\tilde{X} \otimes W'$ must be a quotient of V_{U_2}. By Theorem 7.4

$$\mathcal{C}_c^\infty(AA) = \delta(\det) \otimes \mathrm{ind}_{Q_2}^{H_D}(\mathcal{C}_c^\infty(A))$$

for a certain character δ which we shall now determine. Although we do not know δ, for a generic choice of χ and μ, $\tilde{X} \otimes W'$ will be a quotient of $\mathcal{C}_c^\infty(AA)$. Since $\mathcal{C}_c^\infty(A)$ is a regular representation of $GL_2(F)$ twisted by δ, $\tilde{X} \otimes W'$ must be a quotient of

$$\tilde{X} \otimes \mathrm{Ind}_{Q_2}^{F_4} X$$

where X is a representation of L_2 pulled back from a representation of $GL_2(F)$ with a parameter $(\chi\delta|\cdot|^{-3/2}, \mu\delta|\cdot|^{-3/2})$. We get that W' is a subquotient of both, $\mathrm{Ind}_{Q_4}^{F_4}(Y' \otimes |\det|^3)$ and $\mathrm{Ind}_{Q_2}^{F_4} X$. This immediately implies that

$$\delta(\det) = |\det|^5.$$

The knowledge of δ implies that any $\tilde{X}$ is a quotient of $\mathcal{C}_c^\infty(AA)$ only and, therefore Θ-lifts of $\tilde{Y}$ are the summands of W.

It remains to check the statement about Satake parameters. The L-group of $F_4(F)$ is $F_4(\mathbb{C})$. Let $Q_3 = L_3V_3$ be the maximal parabolic subgroup of F_4 such that $[L_3, L_3] = SL_2 \times SL_3$ where SL_2 is spanned by long roots and SL_3 by short roots. Since $L_3(\mathbb{C})$ is the L-group of L_2, it fits into exact sequence

$$1 \to GL_2 \to L_3 \to PGL_3 \to 1.$$

Let $s \in GL_2(\mathbb{C})$ be the parameter (χ, μ). Obviously, the Satake parameter of W is $s \times \rho \in GL_2 \times SO(3) \subset L_3$. Let G_2 be the centralizer of $SO(3)$ in F_4. Then $L_3 \cap G_2 = GL_2$ and s is precisely the Satake parameter of $\tilde{Y}$ as in Section 4.

Theorem 7.4 can be used, in a similar way, to prove converses of Theorems 4.4 and 5.4. We state results without giving details of proofs.

Theorem 8.2. *Let $\Phi : SL_3(\mathbb{C}) \to G_2(\mathbb{C})$ be the inclusion of the L-groups of $PGL_3(F)$ and $G_2(F)$. Let $\tilde{Y}$ be a tempered spherical representation of $G_2(F)$. The Satake parameter of $\tilde{Y}$ is $\Phi(s)$ for some s, a Satake parameter of a tempered spherical representation $\tilde{Y}$ of $G_2(F)$. Note that $\Phi(s) = \Phi(s^*)$ where s^* is a Satake parameter of Y^*, the dual of Y. Then*

$$\Theta(\tilde{Y}) = \{Y, Y^*\}.$$

Theorem 8.3. *Let $\Phi : G_2(\mathbb{C}) \to Spin_7(\mathbb{C})$ be the inclusion of the L-groups of $G_2(F)$ and $PGSp_6(F)$. Let $\tilde{Y}$ be a tempered spherical representation of $G_2(F)$. Then*

$$\Theta(\tilde{Y}) = \{Y'\}$$

where Y' is the spherical tempered representation of $PGSp_6(F)$ whose Satake parameter is $\Phi(s)$.

Acknowledgments

It is a pleasure to thank B. Gross for initiating this project and on many discussions during the course of this work.

A part of this work was written in the Spring of 1995. During that time the second author enjoyed hospitality of the Mathematical Sciences Research Institute and was generously supported by an NSF Postdoctoral Fellowship and a Sloan Research Fellowship.

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