ON THE MUMFORD-TATE CONJECTURE FOR ABELIAN VARIETIES WITH REDUCTION CONDITIONS

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ABSTRACT. We study monodromy action on abelian varieties satisfying certain bad reduction conditions. These conditions allow us to get some control over the Galois image. As a consequence we verify the Mumford-Tate conjecture for such abelian varieties.

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Introduction

There are two long outstanding conjectures due to Hodge and Tate related to the structure of the ring of algebraic cycles modulo homological equivalence. The Mumford-Tate conjecture implies that for abelian varieties the two are equivalent. The focus of this work is on the Mumford-Tate conjecture for special classes of abelian varieties.

Let X be a smooth projective algebraic variety over $\mathbb{C}$. Then its r-th cohomology group admits the Hodge decomposition: $H^r(X,\mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(X)$. The *Hodge*

1991 *Mathematics Subject Classification.* 11G10, 14C25, 14C30.

Key words and phrases. abelian variety, algebraic cycles, Galois representations, monodromy action, Mumford-Tate conjecture, Tate conjecture, Hodge conjecture.

Research at MSRI supported in part by N.S.F. grant #DMS 9022140.

cycles are those rational cohomology classes, i.e., elements of $H^{\cdot}(X, \mathbb{Q})$ that sit in the components $H^{p,p}(X)$ via the canonical embedding $H^{\cdot}(X,\mathbb{Q}) \hookrightarrow H^{\cdot}(X,\mathbb{C})$. Denote $\mathcal{H}^p(X) := H^{p,p}(X) \cap H^{2p}(X,\mathbb{Q})$ the group of codimension p Hodge cycles. Then that $\mathcal{H}(X) := \oplus_p \mathcal{H}^p(X)$ has a ring structure with respect to the cup-product. It is immediate that rational linear combinations of the cohomology classes of algebraic subvarieties in X (=: *algebraic cycles*) are Hodge. The *Hodge conjecture* claims that the converse is also true, viz., *all the Hodge cycles are algebraic.*

The only general result in this direction is the Lefschetz (1,1)-theorem asserting algebraicity of all codimension 1 Hodge cycles (= rational (1,1) cohomology classes, hence the name). Denote $\mathcal{D}(X)$ the subring of $\mathcal{H}(X)$ generated by $\mathcal{H}^1$, $\mathcal{D}^p(X) := \mathcal{D}(X) \cap \mathcal{H}^p(X)$ is the group of codimension p cycles which are linear combinations of cup-products of divisors. If $\mathcal{D}(X) = \mathcal{H}(X)$, then the (1,1)-theorem implies the Hodge conjecture.

On the other hand, for an algebraic variety defined over an algebraic number field, say $K \subset \overline{\mathbb{Q}}$, one can consider ℓ-adic étale cohomology $H^{\cdot}_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$. The Galois group $\mathcal{G}al(\overline{\mathbb{Q}}/K)$ acts continuously on $H^{\cdot}_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$. If F is a finite extension of K, then the open subgroup $\mathcal{G}al(\overline{\mathbb{Q}}/F)$ of $\mathcal{G}al(\overline{\mathbb{Q}}/K)$ acts by the p^{th} power of the inverse of the cyclotomic character χ_ℓ on the cohomology classes of F-rational codimension p algebraic cycles. If for an arbitrary ℓ-adic Galois representation W and an integer $n \in \mathbb{Z}$ $W(n) := W \otimes \chi_\ell^n$ denotes the n^{th} Tate twist of W, then the these cohomology classes are in $H^{2p}_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(p)^{\mathcal{G}al(\overline{\mathbb{Q}}/F)}$. By a *codimension p Tate cycle* we mean a cohomology class in $H^{2p}_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(p)$ fixed by an open subgroup of $\mathcal{G}al(\overline{\mathbb{Q}}/K)$. Since $H^{\cdot}_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ is a finite dimensional ℓ-adic representation, we can find the largest open subgroup of the Galois group fixing all the Tate cycles (hence these cycles are all defined over a large enough number field). Let $\mathfrak{g}$ be the Lie algebra of the image of the Galois group in $\operatorname{End}_{\mathbb{Q}_l}(H^{\cdot}_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell))$. Then codimension p Tate cycles $\mathcal{T}^p_\ell(X)$ are, by definition, the $\mathfrak{g}$-invariants $H^{2p}_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(p)^{\mathfrak{g}}$. Tate [T 0] conjectured that *all the Tate cycles are algebraic.* We denote $\mathcal{T}_\ell := \bigoplus\limits_p \mathcal{T}^p_\ell$ the ring (under the cup-product) of the Tate cycles.

From now on, we restrict ourselves to the case of *abelian varieties* defined over number fields. On the one hand, this case is more concrete, and some progress has been made; on the other, it has important arithmetical applications.

Although not known in general, the analog of the (1,1)-theorem for the Tate cycles of codimension 1 for abelian varieties has been proved by Faltings [F]. Hence, as above, we can conclude that the Tate conjecture holds for an abelian variety A satisfying $\mathcal{T}_\ell(A) = \mathcal{D}_\ell(A)$, where $\mathcal{D}_\ell(A)$ is the ring of étale cohomology classes generated by divisors.

It is known that *generically*, but not always, the Hodge (resp. the Tate) cycles are all generated by divisors [Ma], [Ab 1].

The first (counter)example due to Mumford (cf. [Po]) features a CM abelian 4-fold. Weil [W] has shown that the essential feature of Mumford's example causing $\mathcal{H} = \mathcal{D}$ to fail is an action in a special way of a quadratic imaginary field $\Bbbk$ on an abelian variety. Namely, consider a family of abelian varieties of even dimension, say $2d$, whose endomorphism algebra contains such a field $\Bbbk$ with the signature of the $\Bbbk$-action (d, d). Generically for such a family, the ring of Hodge cycles is generated by divisors together with an exceptional (non-divisorial) cycle of codimension d.

Recently, C. Schoen proved the Hodge conjecture for one family of abelian 4-folds of Weil type (with an action of $\mathbb{Q}(\mu_3)$).

Our *initial* motivation was to answer a question of Tate on whether the Tate conjecture holds for this family (cf. [T 2], p. 82). The affirmative answer was obtained independently by Moonen-Zarhin (cf. [MZ]).

In general, both conjectures seem to be very difficult in codimensions > 1.

The existence of the comparison isomorphisms between the ℓ-adic and singular cohomology theories carrying algebraic cycles in one theory to another suggests that the Hodge and the Tate conjectures describe essentially the same object. So, it is natural to ask if the two conjectures are equivalent in some sense. The precise statement in the case of abelian varieties constitutes the ***Mumford-Tate conjecture***, which we denote by **MT**. It asserts that ***the Hodge and the Tate conjectures are equivalent for an abelian variety and all its self-products***.

Concretely, for an abelian variety A over a number field K, there exists a connected reductive algebraic subgroup $\mathcal{H}g(A)$ of $\mathrm{GL}(V)$ defined over $\mathbb{Q}$, $V = H^1(A_{\mathbb{C}}, \mathbb{Q})$ (resp. a connected reductive algebraic subgroup $G_\ell(A)$ of $\mathrm{GL}(V_\ell)$ defined over $\mathbb{Q}_\ell$, $V_\ell = H^1_{\text{ét}}(A_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ for some prime number $\ell \in \mathbb{Z}$), such that the Hodge (resp. the Tate) cycles of codimension p are obtained as invariants in $H^{2p}(A, \mathbb{Q}) \cong \bigwedge^{2p} V$ (resp. in $H^{2p}_{\text{ét}}(A_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \cong \bigwedge^{2p} V_\ell$) of $\mathfrak{h} = \mathcal{L}ie(\mathcal{H}g(A))$ (resp. $\mathfrak{g}_\ell = \mathcal{L}ie(G_\ell(A))$). Because of the comparison isomorphism $V_\ell \cong V \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ between the two cohomology theories, $\mathcal{H}g_\ell(A) := \mathcal{H}g(A) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ acts on V_ℓ. Let $\mathfrak{h}_\ell := \mathcal{L}ie(\mathcal{H}g_\ell) = \mathfrak{h} \otimes \mathbb{Q}_\ell$. MT asserts that $\mathfrak{g}_\ell = \mathfrak{h}_\ell$. (Note that $\mathfrak{g}_\ell(A)$ is *not* the Lie algebra of the image of $\mathcal{G}al(\overline{\mathbb{Q}}/K)$ in $\mathrm{GL}(V_\ell)$, but the intersection of the Lie algebra of this image with $\mathfrak{sl}(V_\ell)$. That is why we do not Tate-twist the étale cohomology group.)

Deligne, Piatetskii-Shapiro and Borovoi proved a "half" of MT, viz., $\mathfrak{h}_\ell(A) \supseteq \mathfrak{g}_\ell(A)$. Hence the Tate conjecture implies the Hodge conjecture.

MT for abelian varieties of CM-type is a consequence of the results of Shimura and Taniyama (cf. [ShT], [Po]). This must have been the motivating factor behind

Mumford-Tate.

MT has been proved in a few (non-CM) cases by imposing restrictions on the size of the endomorphism algebra and adding some divisibility conditions on the dimension of abelian varieties, cf. [S 0], [C 0]. In [Z 4], [Z 5], [LZ] MT was verified for abelian varieties satisfying certain conditions on the Galois action at a prime of good reduction. In all these cases the Tate cycles are generated by divisors, hence the Tate conjecture holds. The result now follows from the "known half" of MT.

The main thrust of this work is to show that under suitable bad reduction conditions we can control the image of Galois; in particular, MT holds for a class of abelian varieties, including some Weil-type abelian varieties (for which the Tate conjecture is *not* known).

Note that if A is an absolutely simple abelian variety, $e = (\mathrm{End}^\circ(A) : \mathbb{Q})$ the degree over $\mathbb{Q}$ of its endomorphism algebra, and A has bad reduction at some prime $\wp$, then e divides the dimension of the toric part of the reduction.

The following is the main result of this work (see Theorem 6.4)

Main Theorem. *Let A be an absolutely simple abelian variety, $\mathrm{End}^\circ(A) = \Bbbk$: imaginary quadratic field, $g = \dim(A)$. Assume A has bad semi-stable reduction at some prime $\wp$, with the dimension of the toric part of the reduction equal to $2r$, and $\gcd(r, g) = 1$, and $(r, g) \neq (15, 56)$ or $(m-1, \frac{m(m+1)}{2})$. Then MT holds.*

Roughly speaking, the idea is the following. If A is an abelian variety with bad semi-stable reduction at some prime $\wp$ (of its field of definition), then the action of the inertia at $\wp$ on on the ℓ-adic ($\wp \nmid \ell$) Tate module of A is unipotent of "rank" equal to the dimension of the toric part of the identity component of the the special fiber of the Nèron model of A at $\wp$ (=: toric rank). If A satisfies the conditions on the "size" of the endomorphism algebra and the toric rank imposed above, then the rank of the unipotents (in the inertia image) is prime to the dimension of the Galois representation, which is a sufficiently restrictive condition, given our knowledge of the possible Galois representations arising in this situation.

Note that people have looked at the special elements in the monodromy action before (cf. 6.8).

As mentioned above, MT is known to hold for CM abelian varieties. It is also known (e.g., [ST]) that such abelian varieties have good reduction at all primes, after possibly a finite base change. But the set of abelian varieties with (potentially) good reduction *everywhere* is "small" in a corresponding moduli space, i.e., it is a very rare occasion that an abelian variety has everywhere (potentially) good reduction. Indeed, such abelian varieties correspond to "integral" points of the moduli space (cf. *loc. cit.*, Remark (1), p. 498) and as "sparse" as integers in a number field.

So, "most" of the abelian varieties do have bad reduction "somewhere." We have reasons to believe that abelian varieties with *minimal* bad reduction (e.g., the case $r = 1$ of Theorem A) are the "most typical" (cf. [L]).

Along the way we established various other results. They are:

- *If A is an abelian variety with* $\mathrm{End}^\circ(A) = \mathbb{Q}$ *and the dimension of the toric part of its reduction is either* 2 *or prime to* $2\dim(A)$*, then MT holds (Theorems 6.5, 6.6).*
- *MT holds for some abelian 4-folds A with* $\mathrm{End}^\circ(A) = \mathbb{Q}$ *(Theorem 5.2).*
- *For some abelian varieties, either MT or the Hodge conjecture holds (Theorem 7.1, Remark 7.2(4)).*

The question of existence of abelian varieties considered here and the "size" of the set of such varieties in the corresponding moduli spaces is addressed in [L].

This work started with 4-dimensional case, part of which was independently obtained by Moonen-Zarhin. Since this has been published ([MZ]), we do not treat this case here (see, however, section 2).

Acknowledgment. I greatly benefited from conversations with many people over the period of time I was working on this paper. I would like to thank my advisor, Dinakar Ramakrishnan, for encouragement and guidance throughout the research and preparation of my PhD thesis (at Caltech) in which most of these results appeared. I am especially grateful to Don Blasius, Haruzo Hida and Michael Aschbacher for numerous helpful remarks and enlightening comments.

This work could not have been possible without generous help of Fedor Bogomolov. I can hardly thank him enough.

I. Preliminaries, first applications

0. Basics and Notation

0.0. Let A be a *simple* (:=absolutely simple) abelian variety defined over some number field, say, $K \hookrightarrow \overline{\mathbb{Q}}$ with a fixed embedding, $D := \mathrm{End}^\circ(A) := \mathrm{End}^\circ_{\overline{\mathbb{Q}}}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$, $V := H_1(A(\mathbb{C}), \mathbb{Q})$. Then $D \hookrightarrow \mathrm{End}_{\mathbb{Q}}(V)$.

We recall the Albert classification of the possible *types* of the endomorphism algebras D (cf. [MAV, §20]):

I : D is a totally real field;

II : D is a *indefinite* quaternion algebra over a totally real field F, i.e., $D \otimes_F \mathbb{R}$ is a sum of $f := (F : \mathbb{Q})$ copies of $\mathrm{M}_2(\mathbb{R})$;

III : D is an *definite* quaternion algebra over a totally real field F, i.e., $D \otimes_F \mathbb{R}$ is a sum of $f := (F : \mathbb{Q})$ copies of the Hamiltonian quaternions $\mathbb{H}$;

IV : D is a division algebra over a CM-field;

By abuse of language we say that A is "of type I (II, ...)" if its D is of this type.

0.1. Recall that $V_{\mathbb{R}} := V \otimes_{\mathbb{Q}} \mathbb{R}$ is given a complex structure induced by the natural isomorphism between $V_{\mathbb{R}}$ and the universal covering space of $A(\mathbb{C})$ (cf. [MAV]). Therefore we obtain a homomorphism of algebraic groups,

$$\varphi : T \to \mathrm{GL}(V),$$

defined over $\mathbb{R}$, where T is the compact one-dimensional torus over $\mathbb{R}$, i.e., $T_{\mathbb{R}} = \{z \in \mathbb{C} \mid |z| = 1\}$, by the formula

$$\varphi(e^{i\theta}) = \text{the element of } \mathrm{GL}(V)\text{, which is multiplication by } e^{i\theta}$$
$$\text{in the complex structure on } V_{\mathbb{R}}.$$

Note that there is a non-degenerate skew symmetric (Riemann) form $\Theta : V \times V \to \mathbb{Q}$ and that φ satisfies the Riemann conditions (cf. [M 1]):

1. $\varphi(T) \subseteq \mathrm{Sp}(V, \Theta)$,
2. $\Theta(v, \varphi(i) \cdot v) > 0, \ \forall v \in V, \ x \neq 0.$

Definition. The *Hodge group* $\mathcal{H}g(A)$ of A is the smallest algebraic subgroup of $\mathrm{Sp}(V) := \mathrm{Sp}(V, \Theta)$ defined over $\mathbb{Q}$ which after extension of scalars to $\mathbb{R}$ contains the image of φ.

0.1.1. For the purpose of completeness and further reference, we give the following reformulation of the above construction and definition. The reference for what follows is [D 3, § 3].

Since $A(\mathbb{C})$ is a compact smooth Kähler manifold, $V_{\mathbb{C}} := H_1(A(\mathbb{C}), \mathbb{C})$ admits a Hodge decomposition

$$H_1(A(\mathbb{C}), \mathbb{C}) = H_{-1,0}(A) \oplus H_{0,-1}(A).$$

Thus we obtain a homomorphism

$$\mu : \mathbb{G}_{m,\mathbb{C}} \to \mathrm{GL}(V)_{\mathbb{C}}$$

by defining $\mu(z)$, $\forall z \in \mathbb{C}^\times$, to be the automorphism of $V_{\mathbb{C}}$ which is multiplication by z on $H_{-1,0}(A)$ and by the identity on $H_{0,-1}(A)$.

Definition. The *Mumford-Tate group* $\mathcal{M}(A)$ of A is the smallest algebraic subgroup of $\mathrm{GL}(V)$ defined over $\mathbb{Q}$ which after extension of scalars to $\mathbb{C}$ contains the image of μ.

Clearly, over $\mathbb{C}$, $\mathcal{M}(A)$ is the subgroup of $\mathrm{GL}(V)_{\mathbb{C}}$ generated by the conjugates ${}^{\sigma}\mu$, $\forall \sigma \in \mathrm{Aut}(\mathbb{C})$.

Definition. The *Hodge group* $\mathcal{H}g(A)$ of A (or the *special* Mumford-Tate group of A) is the connected component of the identity of the intersection $\mathcal{M}(A) \cap \mathrm{SL}(V)$ in $\mathrm{GL}(V)$.

Remarks. 1. The construction of $\mathcal{M}(A)$ furnishes it with a canonical character $\nu : \mathcal{M}(A) \to \mathbb{G}_m$ defined over $\mathbb{Q}$ and characterized by the condition $\nu \circ \mu = \mathrm{id}_{\mathbb{G}_m}$. Then $\mathcal{H}g(A) = \mathrm{Ker}(\nu)$. This is the reason why we use the Hodge group instead of the Mumford-Tate group (cf. also 0.3).

2. One can easily show that the two definitions of $\mathcal{H}g(A)$ are equivalent.

0.1.2. The following theorem lists the properties of $\mathcal{H}g(A)$ (cf. [M 1, § 2, Theorem (i)], [Ta, Lemma 1.4]).

Theorem.

1. $\mathcal{H}g(A)$ *is a connected reductive group.*
2. D $(= \mathrm{End}^{\circ}(A)) = \mathrm{End}_{\mathcal{H}g(A)}(V) = \mathrm{End}_{\mathfrak{h}}(A)$, *where* $\mathfrak{h} := \mathcal{L}ie(\mathcal{H}g(A))$.
3. $\mathcal{H}g(A)$ *is semi-simple for an abelian variety* A *of type I, II or III.*
4. $\mathcal{H}g(A^a \times B^b) \cong \mathcal{H}g(A \times B)$ *for any abelian varieties* A, B *and* $a, b \in \mathbb{N}$.

Remarks. 1. Part 2 of the theorem implies that A is simple if and only if V is $\mathfrak{h}$-simple, if and only if (Schur's lemma) D is a division algebra.

2. One can refine part 3, cf. 0.7.1.

0.1.3. Recall that the *Hodge classes* of A are classes of type (p,p) in the Hodge decomposition of homology of A.

The *Hodge conjecture* states that all the Hodge classes are algebraic.

0.1.4. By the Künneth formula $H_{\cdot}(A) = \overset{\cdot}{\bigwedge} H_1(A)$ (cf. [MAV]), hence $\mathcal{H}g(A)$ acts on $H_{\cdot}(A)$. One can show (cf. [M 1]) that the Hodge classes of A are exactly those classes in $H_{\cdot}(A)$ that are fixed by $\mathcal{H}g(A)$. In fact, the Hodge group has the following *characteristic* property (*loc. cit.*, § 2, Corollary).

Theorem. *The Hodge group* $\mathcal{H}g(A)$ *is the largest (reductive) subgroup of* $\mathrm{GL}(V)$ *fixing all the Hodge classes of* A^s, $s \geq 1$.

By the Künneth formula $H_2(A^s) = \bigoplus_{i=1}^{s} H_2(A)$. Hence, in the view of the previous theorem, the Lefschetz (1,1)-theorem for abelian varieties takes the following form.

Theorem. *Let $s \in \mathbb{N}$, $sV := V \oplus ... \oplus V$ (s times), then the $\mathfrak{h}$-invariants $(\bigwedge\limits_{\mathbb{Q}}^{2} sV)^{\mathfrak{h}}$ is exactly the ($\mathbb{Q}$-span of homological classes of) divisors on $A^s = A \times ... \times A$ (s times).*

0.1.5. Following Ribet and Murty (cf. [Ri], [Mu]) we make the following

Definition. The *Lefschetz group* $L(A)$ of A is the connected component of the identity of the centralizer of $\mathrm{End}^{\circ}(A)$ in $\mathrm{Sp}(V, \Theta)$ (inside $\mathrm{End}_{\mathbb{Q}}(V)$, Θ is a polarization, cf. 0.1).

The following are the main results about the Lefschetz group (cf. [Mu], also [Ri], [H 1]).

Theorem.

0.(i) *$L(A)$ is a connected reductive algebraic group defined over $\mathbb{Q}$.*
(ii) *$\mathcal{H}g(A) \subseteq L(A)$.*
(iii) *$L(A)$ is semi-simple for A of type I, II or III; moreover, it is symplectic for A of type I and II, orthogonal for type III.*
(iv) *$L(A_1^{n_1} \times ... \times A_s^{n_s}) = L(A_1) \times ... \times L(A_s)$.*
1. *All the Hodge classes on A^s are divisorial if and only if $\mathcal{H}g(A) = L(A)$ and A is not of type III.*
2. *If A is of type III, then it has a non-divisorial Hodge class.*

Let $\mathfrak{l} := \mathcal{L}ie(L(A))$, then

0.(ii)$'$ $\quad \mathfrak{h} \hookrightarrow \mathfrak{l} \hookrightarrow \mathfrak{sp}(V),\ \mathfrak{h}^{ss} \hookrightarrow \mathfrak{l}^{ss},\ C_{\mathfrak{h}} \hookrightarrow C_{\mathfrak{l}}.$

Here $C_?$ is the center of ?, and $?^{ss}$ is the semi-simple part of ?.

0.2. Let $\Bbbk \hookrightarrow D$ be an imaginary quadratic field, $\mathcal{G}al(\Bbbk/\mathbb{Q}) = \{\sigma, \rho\}, \rho\ (= \sigma^2)$ is the fixed (identity) embedding $\Bbbk \hookrightarrow \overline{\mathbb{Q}}$. In this case $V_{\mathbb{R}} := V \otimes \mathbb{R}$ has two complex structures. One is given by the isomorphism $V_{\mathbb{R}} = \mathcal{L}ie(A(\mathbb{C}))$, (cf. 0.1), and the other by the action of $\Bbbk \otimes_{\mathbb{Q}} \mathbb{R} (\simeq \mathbb{C})$. Hence the splitting $V_{\mathbb{R}} = V^{\sigma} \oplus V^{\rho}$ ($\Bbbk$ acts by $\sigma(\Bbbk)$ on V^{σ} and by $\rho(\Bbbk)$ on V^{ρ}). The two complex structures coincide on one of the subspaces, say V^{ρ} , and conjugate on the other, V^{σ}. If $m_{\sigma} = \dim_{\mathbb{C}}(V^{\sigma})$, $m_{\rho} = \dim_{\mathbb{C}}(V^{\rho})$, then (m_{σ}, m_{ρ}) is the *signature* of the $\Bbbk$-action; $m_{\sigma} + m_{\rho} = g = \dim_{\mathbb{C}}(V_{\mathbb{R}}) = \dim(A)$.

0.2.1. Recall that the *Rosati involution* is the involution on $D = \mathrm{End}^{\circ}(A)$ induced by the Riemann form (cf. 0.1). The Rosati involution is *positive*, consequently, the field it fixes is totally real (cf. [MAV]).

In the case $\mathbb{k} \subseteq D$ we always assume that the *Rosati involution preserves* $\mathbb{k}$. The positivity of the involution implies that it acts on $\mathbb{k}$ non-trivially. Hence this action coincides with (the complex conjugation) σ.

0.2.2. Since $\mathfrak{h}$ and $\mathfrak{l}$ centralize D (cf. 0.1.2 and 0.1.5)

$$\mathfrak{h} \hookrightarrow \mathfrak{l} \hookrightarrow \mathfrak{sp}(V)^{\mathbb{k}} \hookrightarrow \mathfrak{sp}(V),$$

where $\mathfrak{sp}(V)^{\mathbb{k}}$ is the centralizer of $\mathbb{k}$ in $\mathfrak{sp}(V)$. By [D 3, Lemma 4.6]

$$\mathfrak{sp}(V)^{\mathbb{k}} = \mathfrak{u}(V),$$

the Lie algebra of the unitary group of a $\mathbb{k}$-Hermitian form on V viewed as the $\mathbb{k}$-vector space (cf. 0.2). Extending scalars to $\mathbb{k}$ we get

$$\mathfrak{h}_{\mathbb{k}} \hookrightarrow \mathfrak{l}_{\mathbb{k}} \hookrightarrow \mathfrak{u}(V) \times \mathfrak{u}(V)^{\sigma} \hookrightarrow \mathfrak{sp}(V_{\mathbb{k}}), \tag{0.2.2.1}$$

where $\mathfrak{h}_{\mathbb{k}} := \mathfrak{h} \otimes_{\mathbb{Q}} \mathbb{k}$, $\mathfrak{l}_{\mathbb{k}} := \mathfrak{l} \otimes_{\mathbb{Q}} \mathbb{k}$, $V_{\mathbb{k}} := V \otimes_{\mathbb{Q}} \mathbb{k} = V \oplus U$, U is the same V, but with the conjugate $\mathbb{k}$-vector space structure.

The $\mathfrak{h}$-invariant $\mathbb{k}$-Hermitian form referred to above is a non-degenerate element of $\check{V} \otimes \check{U}$ (cf. [D 3, Lemma 4.6]), hence the isomorphism

$$U \cong \check{V}$$

of $\mathfrak{h}$-modules. Clearly the projection of $\mathfrak{h}_{\mathbb{k}}$ to $\mathfrak{u}(V)$ is $\mathfrak{h}$, thus we can rewrite (0.2.2.1) as

$$\mathfrak{h} \hookrightarrow \mathfrak{l} \hookrightarrow \mathfrak{u}(V) \overset{\Delta}{\hookrightarrow} \mathfrak{sp}(V \oplus \check{V}),\ g \overset{\Delta}{\mapsto} \begin{pmatrix} g & 0 \\ 0 & {}^t g^{-1} \end{pmatrix}. \tag{0.2.2.2}$$

(*Note.* The embeddings above are considered over $\mathbb{k}$.)

0.2.3. From this we get:

$$\overline{\mathfrak{h}} \hookrightarrow \overline{\mathfrak{l}} \hookrightarrow \mathfrak{gl}(W) \overset{\Delta}{\hookrightarrow} \mathfrak{sp}(W \oplus \check{W}),$$

where $\overline{\mathfrak{h}} := \mathfrak{h} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$, $\overline{\mathfrak{l}} := \mathfrak{l} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$, $W := V \otimes_{\mathbb{k},\rho} \overline{\mathbb{Q}}$, $\check{W}$: dual of W $(= W \otimes_{\mathbb{k},\sigma} \overline{\mathbb{Q}})$.

0.3. Let $V_\ell := T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong V \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, where $T_\ell(A)$ is the ℓ-adic Tate module $(= H_1^{\text{ét}}(A \times_K \overline{\mathbb{Q}}, \mathbb{Z}_\ell) = \varprojlim_n {}_{\ell^n}A(\overline{\mathbb{Q}})$, where ${}_{\ell^n}A(\overline{\mathbb{Q}})$ = kernel of multiplication by $\ell^n : A(\overline{\mathbb{Q}}) \to A(\overline{\mathbb{Q}}))$, $V = H_1(A(\mathbb{C}), \mathbb{Q})$ as above. By abuse of language we call V_ℓ Tate module too. Let G_ℓ be the image of $\mathcal{G}al(\overline{\mathbb{Q}}/K)$ in $\operatorname{End}_{\mathbb{Q}_\ell}(V_\ell)$, where K is the base field of A, $\mathfrak{G}_\ell := \mathcal{L}ie(G_\ell)$. It is known (cf. [Bo]) that $\mathfrak{G}_\ell$ is algebraic and $\mathbb{Q}_\ell \cdot 1_{V_\ell} \subseteq \mathfrak{G}_\ell \subset \mathfrak{gsp}(V_\ell)$. Let $\mathfrak{g}_\ell := \mathfrak{G}_\ell \cap \mathfrak{sl}(V_\ell) \subset \mathfrak{sp}(V_\ell)$. Then $C_{\mathfrak{G}_\ell} = C_{\mathfrak{g}_\ell} \oplus \mathbb{Q}_\ell \cdot 1_{V_\ell}$, $\mathfrak{g}_\ell^{ss} = \mathfrak{G}_\ell^{ss}$.

Remarks. 1. $\mathfrak{g}_\ell$ does not depend on finite extensions of K (cf. [S 2]).

2. $V_\ell \cong V_\ell(r)$, $\forall r \in \mathbb{Z}$, as $\mathfrak{g}_\ell$-modules, but not as $\mathfrak{G}_\ell$-modules.

0.3.0. The *Tate conjecture* states that the *Tate cycles*, i.e., the Galois invariants $H^{\text{ét}}_{\cdot}(A_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)^{\mathfrak{g}_\ell} = (\overset{\bullet}{\bigwedge} H^{\text{ét}}_1(A_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell))^{\mathfrak{g}_\ell}$, are algebraic.

0.3.1. Faltings (cf. [F]) has proved the analogs of Mumford's Theorem 0.1.2(1,2) and the (1,1)-theorem (a special case of the Tate conjecture) for abelian varieties.

Theorem.

1. *Let* $s \in \mathbb{N}$, $(\overset{2}{\underset{\mathbb{Q}_\ell}{\bigwedge}} sV_\ell)^{\mathfrak{g}_\ell}$ *is exactly the* ($\mathbb{Q}_\ell$*-span of homological classes of*) *divisors of* $A^s = A \times ... \times A$ (s *times*).
2. $\mathrm{End}_{\mathfrak{g}_\ell}(V_\ell) = D \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$.
3. $\mathfrak{g}_\ell$ *is reductive.*

0.3.2. $\mathfrak{h}_\ell := \mathfrak{h} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \hookrightarrow \mathrm{End}_{\mathbb{Q}_\ell}(V_\ell)$. The known relation between $\mathfrak{g}_\ell$ and $\mathfrak{h}_\ell$ is given by the following theorem.

Theorem (Deligne [D 3], Piatetskii-Shapiro [P-Sh], Borovoi [Bor]).
$\mathfrak{g}_\ell \subseteq \mathfrak{h}_\ell \subset \mathrm{End}_{\mathbb{Q}_\ell}(V_\ell)$.

0.4. The *Mumford-Tate conjecture* (=: MT) states $\mathfrak{g}_\ell = \mathfrak{h}_\ell$. Since $\mathfrak{g}_\ell$ and $\mathfrak{h}_\ell$ are reductive, it is the same as equivalence of the Hodge and the Tate conjectures for an abelian variety and all its self-products.

0.4.1. In order to prove MT it is enough to establish the conjecture for one ℓ ([LP, Theorem 4.3]).

0.4.2. Moreover, it is enough to show $\overline{\mathfrak{g}}_\ell = \overline{\mathfrak{h}}_\ell$, where $\overline{\mathfrak{g}}_\ell := \mathfrak{g}_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$, $\overline{\mathfrak{h}}_\ell := \mathfrak{h}_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$ ([Z 2, §5, Key Lemma]).

0.4.3. The (1,1)-theorems imply $(\overset{2}{\underset{\mathbb{Q}_\ell}{\bigwedge}} sV_\ell)^{\mathfrak{g}_\ell} = (\overset{2}{\underset{\mathbb{Q}_\ell}{\bigwedge}} sV_\ell)^{\mathfrak{h}_\ell}$.

0.4.4. Theorems 0.3.2, 0.1.2(2) and 0.3.1(2) imply $\mathfrak{g}_\ell^{ss} \subset \mathfrak{h}_\ell^{ss}$, $C_{\mathfrak{g}_\ell} \subset C_{\mathfrak{h}_\ell}$.

In fact, $C_{\mathfrak{g}_\ell} = C_{\mathfrak{h}_\ell}$. This can be shown in a way similar to the proof of MT for CM abelian varieties (cf. [ShT], and also [D 1]).

So, ***in order to prove MT one must show that*** $\overline{\mathfrak{g}}_\ell^{ss} = \overline{\mathfrak{h}}_\ell^{ss}$.

To simplify notations we write sometimes $\widetilde{\mathfrak{g}}$ for $\overline{\mathfrak{g}}_\ell^{ss}$ and $\widetilde{\mathfrak{h}}$ for $\overline{\mathfrak{h}}_\ell^{ss}$.

0.5. Let $\mathfrak{a}$ be a semi-simple Lie algebra over an algebraically closed field of characteristic 0, and $\mathfrak{a} = \mathfrak{a}_1 \times ... \times \mathfrak{a}_n$ be the decomposition of $\mathfrak{a}$ into the product of its simple ideals. For any faithful irreducible representation U of $\mathfrak{a}$, U decomposes as a tensor product of irreducible representations U_i of $\mathfrak{a}_i$. Since U is faithful, none of the U_i's is trivial. Moreover, if the representation U admits a non-degenerate invariant bilinear form, then so does each U_i.

We say that the representation is *minuscule* if the highest weight of each U_i is minuscule, see [B, Ch.VIII, §7.3]. The following is the list of minuscule weights, [B, Ch.VIII, §7.3 and Table 2]:

type $\mathbf{A_m}$ $(m \geq 1):$ $\varpi_1,\ \varpi_2,\ \ldots\ ,\ \varpi_m$; $\dim(\varpi_s) = \binom{m+1}{s}$;
type $\mathbf{B_m}$ $(m \geq 2):$ ϖ_1; $\dim(\varpi_1) = 2m+1$;
type $\mathbf{C_m}$ $(m \geq 2):$ ϖ_1; $\dim(\varpi_1) = 2m$;
type $\mathbf{D_m}$ $(m \geq 3):$ $\varpi_1, \varpi_{m-1}, \varpi_m$; $\dim(\varpi_1) = 2m$,
$\dim(\varpi_{m-1}) = \dim(\varpi_m) = 2^{m-1}$;
type $\mathbf{E_6}$: $\varpi_1,\ \varpi_6$; $\dim(\varpi_1) = \dim(\varpi_6) = 27$;
type $\mathbf{E_7}$: ϖ_7; $\dim(\varpi_7) = 56$;
there are *no* minuscule representations for the types $\mathbf{E_8},\ \mathbf{F_4},\ \mathbf{G_2}$.

0.5.1. It is known that the representations of $\widetilde{\mathfrak{g}},\ \widetilde{\mathfrak{h}}$ are minuscule (cf. [S $*$], [D 2]). It is also known that $\widetilde{\mathfrak{g}}$ is not exceptional, see [S $*$, Theorem 7] (for the corresponding result for $\widetilde{\mathfrak{h}}$ see [D 2, Remarque 1.3.10(i)]).

0.6.1. Let again $\Bbbk \hookrightarrow D$ and $\Bbbk_\ell := \Bbbk \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. Then, as in 0.2.2,

$$\mathfrak{g}_\ell \subset \mathfrak{h}_\ell \subset \mathfrak{sp}(V_\ell)^{\Bbbk_\ell} \subset \mathfrak{sp}(V_\ell).$$

If l splits in $\Bbbk$, λ, λ' being the primes of $\Bbbk$ over l, $\lambda' = \lambda^\sigma$, then $\Bbbk_\ell \cong \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell, V_\ell = V_\lambda \oplus V_{\lambda'}$, where $V_\lambda, V_{\lambda'}$ are vector spaces over $\Bbbk_\lambda \cong \mathbb{Q}_\ell$, $\Bbbk_{\lambda'} \cong \mathbb{Q}_\ell$ respectively, and

$$\mathfrak{g}_\ell \subset \mathfrak{h}_\ell \subset \mathfrak{gl}(V_\lambda) \oplus \mathfrak{gl}(V_{\lambda'}) \overset{\Delta}{\subset} \mathfrak{sp}(V_\lambda \oplus V_{\lambda'}).$$

Since $\lambda' = \lambda^\sigma$, as in 0.2.2 we conclude $V_{\lambda'} \cong \check{V}_\lambda$ and can rewrite the above sequence as

$$\mathfrak{g}_\ell \subset \mathfrak{h}_\ell \subset \mathfrak{gl}(V_\lambda) \overset{\Delta}{\hookrightarrow} \mathfrak{sp}(V_\lambda \oplus \check{V}_\lambda).$$

Remark. If ℓ does *not* split in $\Bbbk$, then $\Bbbk_\ell$ is a field, $(\Bbbk_\ell : \mathbb{Q}_\ell) = 2$, and the rest is identical to 0.2.2.

0.6.2. As in 0.2.2, by extending scalars to $\overline{\mathbb{Q}}_\ell$ we get

$$\overline{\mathfrak{g}}_\ell \subset \overline{\mathfrak{h}}_\ell \subset \mathfrak{gl}(W_\lambda) \overset{\Delta}{\hookrightarrow} \mathfrak{sp}(W_\lambda \oplus \check{W}_\lambda),$$

where $W_\lambda := V_\ell \otimes_{\Bbbk_\ell, \rho_\lambda} \overline{\mathbb{Q}}_\ell$, $\rho_\lambda : \Bbbk_\ell \to \Bbbk_\lambda$ is the projection.

Remark. For ℓ non-split in $\Bbbk$, the same holds (cf. 0.2.2, 0.2.3).

0.7. We will need the following simple facts. We assume that $D = \Bbbk$.

0.7.1. Proposition. *The representations of $\overline{\mathfrak{g}}_\ell$ and $\overline{\mathfrak{h}}_\ell$ are non-self-dual.* $\square$

Remarks. 1. This is true for any irreducible subrepresentation of W_λ for any type IV abelian variety (e.g., [Mu], [H]).

2. If the abelian variety is of type I (respectively II, respectively III), then the irreducible components are symplectic (respectively symplectic, respectively orthogonal) (*loc. cit.*).

0.7.2. Proposition. *$\overline{\mathfrak{g}}_\ell$ and $\overline{\mathfrak{h}}_\ell$ are semi-simple if and only if the signature of the $\Bbbk$-action is (m,m). Further, if this is not the case, the centers $C_{\overline{\mathfrak{g}}_\ell}, C_{\overline{\mathfrak{h}}_\ell}$ are 1-dimensional.*

Proof. This is essentially proved in [D 3], [W]. Let us, however, briefly explain why this holds and fix notations.

Let $\mu : \mathbb{G}_{m,\mathbb{C}} \to \mathrm{GL}(V_\mathbb{R})$ be the cocharacter defining the Hodge structure on V, then the map $h : \mathbb{S} = R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to \mathrm{GL}(V_\mathbb{C})$ is given by $h(z) = \mu(z)$ on $V_\mathbb{R}$ (cf. 0.1.1), $h(z) = \overline{\mu}(z)$ on $U_\mathbb{R}$, where $U_\mathbb{R}$ is the "same" $V_\mathbb{R}$ but with the conjugate $\Bbbk \otimes \mathbb{R}$-action, $V_\mathbb{C} = V_\mathbb{R} \oplus U_\mathbb{R}$ (cf. 0.2). If the $\Bbbk$-signature is (m_σ, m_ρ) then $V_\mathbb{R} = V_\mathbb{R}^\sigma \oplus V_\mathbb{R}^\rho$, $\dim_\mathbb{C}(V_\mathbb{R}^\sigma) = m_\sigma$, $\dim_\mathbb{C}(V_\mathbb{R}^\rho) = m_\rho$ ($\Bbbk$ acts by $\sigma(\Bbbk)$ on $V_\mathbb{R}^\sigma$ and by $\rho(\Bbbk)$ on $V_\mathbb{R}^\rho$). But $V_\mathbb{R} = H_{-1,0}$, hence the power of z by which $\mu(z)$ acts on $V_\mathbb{R}^\sigma$, $V_\mathbb{R}^\rho$ is 1. We will call this power the *μ-weight*. Similarly, $V_\mathbb{C} = H_1(A(\mathbb{C}), \mathbb{C}) = V_\mathbb{C}^\sigma \oplus V_\mathbb{C}^\rho$. But also $V_\mathbb{C} = H_{-1,0} \oplus H_{0,-1} (= V_\mathbb{R} \oplus U_\mathbb{R})$ and these two decompositions commute, since the former is determined by $\Bbbk \subseteq D$ and the Hodge group centralizes D in $\mathrm{End}_\mathbb{Q}(V)$. Hence we can write

$$\begin{aligned} V_\mathbb{C} &= V_\mathbb{R} \oplus U_\mathbb{R} \\ &= (V_\mathbb{R}^\sigma \oplus V_\mathbb{R}^\rho) \oplus (U_\mathbb{R}^\sigma \oplus U_\mathbb{R}^\rho) \\ &= (V_\mathbb{R}^\sigma \oplus U_\mathbb{R}^\sigma) \oplus (V_\mathbb{R}^\rho \oplus U_\mathbb{R}^\rho) \\ &= V_\mathbb{C}^\sigma \oplus V_\mathbb{C}^\rho, \end{aligned}$$

where $U_\mathbb{R}^\sigma$ (respectively $U_\mathbb{R}^\rho$) is the conjugate of $V_\mathbb{R}^\rho$ (respectively $V_\mathbb{R}^\sigma$). Thus $\dim_\mathbb{C}(U_\mathbb{R}^\sigma) = m_\rho$, $\dim_\mathbb{C}(U_\mathbb{R}^\rho) = m_\sigma$, so $\dim_\mathbb{C}(V_\mathbb{C}^\sigma) = m_\rho + m_\sigma = g = \dim_\mathbb{C}(V_\mathbb{C}^\rho)$.

The μ-weight of $U_\mathbb{R}$ is 0, hence the decomposition

$$V_\mathbb{C}^\sigma = V_\mathbb{R}^\sigma \oplus U_\mathbb{R}^\sigma$$

is according to μ-weights 1, 0. (This exactly corresponds to the Hodge-Tate decomposition of the λ-adic representationbelow, cf. (1.0.$*$ $\sim$1.1.$*$).) Now the μ-weight (respectively the Hodge type) of

$$\bigwedge_\mathbb{C}^g V_\mathbb{C}^\sigma = \bigwedge_\mathbb{C}^{m_\sigma} V_\mathbb{R}^\sigma \otimes \bigwedge_\mathbb{C}^{m_\rho} U_\mathbb{R}^\sigma \subseteq \bigwedge V_\mathbb{C}^\sigma$$

is m_σ (respectively $(-m_\sigma, -m_\rho)$). Hence $\bigwedge\limits_{\Bbbk}^{g} V$ is a Hodge cycle (i.e., of Hodge type $(-\frac{g}{2}, -\frac{g}{2})$) if and only if $m_\sigma = m_\rho$. Hence it is fixed by $\mathfrak{h}$ if and only if $m_\sigma = m_\rho$, i.e., $\mathfrak{h} \subset \mathfrak{sl}(V) \cap \mathfrak{u}(V)$ only in this case (here V is considered as a $\Bbbk$-vector space). In other words, the center $C_{\mathfrak{h}}$ of $\mathfrak{h}$ kills the determinant $\bigwedge\limits_{\Bbbk}^{g} V$ (and hence $\neq \{0\}$) if and only if $m_\sigma \neq m_\rho$. Now, since $\overline{V} := V \otimes \overline{\mathbb{Q}} = W \oplus \check{W}$, W is irreducible, $\overline{\mathfrak{h}} \subset \mathfrak{gl}(W) \overset{\Delta}{\to} \mathfrak{sp}(\overline{V})$. Summarizing,

$$\overline{\mathfrak{h}} = \overline{\mathfrak{h}}^{ss} \subset \mathfrak{sl}(W) \overset{\Delta}{\to} \mathfrak{sp}(W \oplus \check{W}) \ \text{ if } m_\sigma = m_\rho,$$
$$\overline{\mathfrak{h}} = \overline{\mathfrak{h}}^{ss} \oplus \overline{\mathbb{Q}} \subset \mathfrak{gl}(W) \overset{\Delta}{\to} \mathfrak{sp}(W \oplus \check{W}) \ \text{ if } m_\sigma \neq m_\rho.$$

Using 0.4.4 we conclude

$$\begin{aligned} &\overline{\mathfrak{g}}_\ell \subset \overline{\mathfrak{h}}_\ell \subset \mathfrak{sl}(W_\lambda) \overset{\Delta}{\to} \mathfrak{sp}(W_\lambda \oplus \check{W}_\lambda), \\ &\overline{\mathfrak{g}}_\ell = \overline{\mathfrak{g}}_\ell^{ss}, \overline{\mathfrak{h}}_\ell = \overline{\mathfrak{h}}_\ell^{ss}, \ \text{if } m_\sigma = m_\rho, \end{aligned}$$

$$\begin{aligned} &\overline{\mathfrak{g}}_\ell \subset \overline{\mathfrak{h}}_\ell \subset \mathfrak{gl}(W_\lambda) \overset{\Delta}{\to} \mathfrak{sp}(W_\lambda \oplus \check{W}_\lambda), \\ &C_{\overline{\mathfrak{g}}_\ell} = C_{\overline{\mathfrak{h}}_\ell} = \overline{\mathbb{Q}}_\ell, \ \text{if } m_\sigma \neq m_\rho. \quad \square \end{aligned}$$

0.7.3. As one can see from the proof, if $\Bbbk \subseteq \mathrm{End}^\circ(A)$ (not necessarily equal), and $m_\sigma \neq m_\rho$, the center $C_{\mathfrak{h}}$ of $\mathfrak{h}$ must kill the determinant $\bigwedge\limits_{\Bbbk}^{g} V$, hence the center is non-trivial. However, if $\Bbbk \neq \mathrm{End}^\circ(A)$, then the center can be non-trivial even if $m_\sigma = m_\rho$ (e.g., CM case).

But even if $\Bbbk \subsetneq \mathrm{End}^\circ(A)$, as follows from the proof, $\overline{\mathfrak{g}}_\ell \subset \overline{\mathfrak{h}}_\ell \subset \mathfrak{sl}(W_\lambda)$ if and only if $m_\sigma = m_\rho$. In this form the result (using [D 3, Proposition 4.4]) can be generalized to the case of an arbitrary CM-field $E \hookrightarrow D$ (cf. [MZ, Lemma 2.8]).

Note that since $\mathfrak{g}_\ell$ and $\mathfrak{h}_\ell$ are semi-simple for abelian varieties of types II or III (cf. 0.1.2(3)), if $\Bbbk \hookrightarrow \mathrm{End}^\circ(A)$ is Rosati-stable, A of type II or III, then the signature of the $\Bbbk$-action is necessarily (m, m).

Definition. If the signature of the $\Bbbk$-action on an abelian variety is (m, m), we call such an abelian variety a *Weil type* abelian variety (cf. [W]).

1. On the Hodge-Tate decomposition

1.0. We recall here certain basic facts on the Hodge-Tate decomposition and then give some applications. The classical/standard reference is [T 1].

1.0.1. According to Tate and Raynaud, $\overline{V}_\ell := V_\ell \otimes \mathbb{C}_\ell = H_1((A \otimes_K \overline{K})_{\text{ét}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathbb{C}_\ell,$ $\mathbb{C}_\ell$ is a completion of $\overline{\mathbb{Q}}_\ell$, admits a decomposition

$$\overline{V}_\ell = \overline{V}_\ell(0) \oplus \overline{V}_\ell(1),$$

where $\overline{V}_\ell(i) := \overline{V}_\ell^{(i)} \otimes_{\mathbb{Q}_\ell} \mathbb{C}_\ell,\ i = 1, 2$. The $\mathbb{Q}_\ell$-subspaces (but *not* $\mathbb{C}_\ell$-subspaces) $\overline{V}_\ell^{(i)}$'s of $\overline{V}_\ell$ are defined as follows:

$$\overline{V}_\ell^{(i)} := \{v \in \overline{V}_\ell \mid v^\sigma = \chi_\ell(\sigma)^i \cdot v,\ \forall \sigma \in \mathcal{I}\},\ i = 1, 2,$$

where $\mathcal{I}$ is the absolute inertia group at a prime of K (=base field of A) over ℓ (cf. 0.3 for why we take the inertia instead of the whole decomposition group), χ_ℓ is the cyclotomic character. Recall that the Galois action is continous and semi-linear on $\overline{V}_\ell$ (see [S 2, 1.2]), and, clearly, the $\overline{V}_\ell^{(i)}$'s are Galois submodules of $\overline{V}_\ell$. The Galois action on $\overline{V}_\ell(i)$ is by the formula

$$(v \otimes c)^\sigma := v^\sigma \otimes c^\sigma,\ \forall v \in \overline{V}_\ell^{(i)},\ \forall c \in \mathbb{C}_\ell,\ \forall \sigma \in \mathcal{I},$$

extended by linearity.

1.0.2. According to S. Sen ([Se, Section 4, Theorem 1]), to the the Hodge-Tate decomposition on $\overline{V}_\ell$ one can associate a cocharacter

$$\phi : \mathbb{G}_{m,\mathbb{C}_\ell} \to \mathrm{GL}(V_\ell)_{\mathbb{C}_\ell},$$

by defining $\phi(z),\ \forall z \in \mathbb{C}_\ell^\times$, to be the automorphism of $\overline{V}_\ell$ which is multiplication by z on $\overline{V}_\ell(1)$ and by the identity on $\overline{V}_\ell(0)$. This association is made in such a manner that the algebraic envelope $\tilde{G}_\ell$ of the Galois image (cf. 0.3) turns out to be the smallest algebraic group defined over $\mathbb{Q}_\ell$ which after extension of scalars to $\mathbb{C}_\ell$ contains the image of ϕ.

Remark. This cocharacter ϕ is completely analogous to the cocharacter μ associated to the Hodge decomposition on $V_\mathbb{C}$, $\tilde{G}_\ell$ is the analog of the Mumford-Tate group $M(A)$ and $\mathfrak{g}_\ell$ is the analog of $\mathfrak{h}$, see 0.1.1.

1.0.3. Before proceeding, recall that for a $\mathcal{G}al(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$-module X, the *Tate twist* $X(1)$ of X is defined to be $X \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(1)$ with the Galois structure of a tensor product of Galois modules (as in 1.0.1). Here $\mathbb{Q}_\ell(1)$ is the *Tate module:*

$$\mathbb{Q}_\ell(1) := (\varprojlim_n \mu_{\ell^n}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell,\ \mu_{\ell^n} = \{\zeta \in \overline{\mathbb{Q}}_\ell \mid \zeta^{\ell^n} = 1\},$$

with the natural $\mathcal{G}al(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$-action by χ_ℓ:

$$\zeta^\sigma = \zeta^{\chi_\ell(\sigma)},\ \forall \sigma \in \mathcal{G}al(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell),\ \zeta \in \mu_{\ell^n},\ \text{for some } n.$$

1.0.4. The Hodge-Tate decomposition of $\overline{V}_\ell$ can be rewritten in the following explicit form ([T 1, § 4, Corollary 2], see also the Remark following that Corollary):

$$\overline{V}_\ell = \mathcal{L}ie(A^\vee_{\mathbb{C}_\ell})^\vee \oplus \mathcal{L}ie(A_{\mathbb{C}_\ell})(1),$$

where $\mathcal{L}ie(A^\vee_{\mathbb{C}_\ell})^\vee$ is the cotangent space of the dual abelian variety $A^\vee_{\mathbb{C}_\ell}$ at its origin and $\mathcal{L}ie(A_{\mathbb{C}_\ell})(1)$ is the tangent space of $A_{\mathbb{C}_\ell}$ at its origin Tate-twisted by χ_ℓ.

1.0.5. On the other hand, we have the Hodge decomposition on $V_\mathbb{C} = H_1(A(\mathbb{C}), \mathbb{C})$:

$$\begin{aligned} H_1(A(\mathbb{C}), \mathbb{C}) &= H_1(A_\mathbb{C}, \mathcal{O}_{A_\mathbb{C}}) \oplus H_0(A_\mathbb{C}, \Omega^1_{A_\mathbb{C}}) \\ &= \mathcal{L}ie(A^\vee_\mathbb{C})^\vee \oplus \mathcal{L}ie(A_\mathbb{C}), \end{aligned}$$

or, in our notation,

$$V_\mathbb{C} = U_\mathbb{R} \oplus V_\mathbb{R},$$

see the proof of 0.7.2.

1.0.6. Fix an isomorphism $\mathbb{C}_\ell \cong \mathbb{C}$. Then the comparison isomorphism

$$c : H_1^{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathbb{C}_\ell \cong H_1(A(\mathbb{C}), \mathbb{C})$$

or, in our notation,

$$c : V_\ell \otimes_{\mathbb{Q}_\ell} \mathbb{C}_\ell \cong V \otimes_\mathbb{Q} \mathbb{C},$$

provides isomorphisms

$$c : V_\mathbb{R} \to \overline{V}_\ell(1),\ \ c : U_\mathbb{R} \to \overline{V}_\ell(0).$$

Note, that $\dim_\mathbb{C}(V_\mathbb{R}) = g = \dim_{\mathbb{C}_\ell}(\overline{V}_\ell(1)),\ \dim_\mathbb{C}(U_\mathbb{R}) = g = \dim_{\mathbb{C}_\ell}(\overline{V}_\ell(0))$.

1.0.7. On the other hand, both homology groups admit decompositions according to the action of $\mathbb{k} \subseteq \mathrm{End}^\circ(A)$:

$$\overline{V}_\ell := \overline{V}_\lambda \oplus \overline{V}_{\lambda'}$$

where $\overline{V}_\lambda := W_\lambda \otimes_{\overline{\mathbb{Q}}_\ell} \mathbb{C}_\ell = V_\ell \otimes_{\mathbb{k}_\ell, \rho_\lambda} \mathbb{C}_\ell$, $\overline{V}_{\lambda'} := W_{\lambda'} \otimes_{\overline{\mathbb{Q}}_\ell} \mathbb{C}_\ell = V_\ell \otimes_{\mathbb{k}_\ell, \rho_{\lambda'}} \mathbb{C}_\ell$, (cf. 0.6.2) and

$$V_\mathbb{C} := V^\sigma_\mathbb{C} \oplus V^\rho_\mathbb{C}.$$

These two types of splittings commute, and consequently $V^\sigma_\mathbb{C}, V^\rho_\mathbb{C}$ admit the Hodge decomposition, and $\overline{V}_\lambda$, $\overline{V}_{\lambda'}$ admit the Hodge-Tate decomposition. The map c respects these splittings, hence maps either $V^\sigma_\mathbb{C}$ to $\overline{V}_\lambda$, or $V^\rho_\mathbb{C}$ to $\overline{V}_\lambda$.

1.0.8. Let's *assume* that

$$c : V_{\mathbb{C}}^{\sigma} \xrightarrow{\sim} \overline{V}_{\lambda},$$

hence $\dim_{\mathbb{C}}(V_{\mathbb{C}}^{\sigma}) = g = \dim_{\mathbb{C}_\ell}(\overline{V}_{\lambda})$.

As a result we conclude

$$c : V_{\mathbb{R}}^{\sigma} \xrightarrow{\sim} \overline{V}_{\lambda}(1),$$

hence $\dim_{\mathbb{C}_\ell}(\overline{V}_{\lambda}(1)) = m_\sigma,\ \dim_{\mathbb{C}_\ell}(\overline{V}_{\lambda}(0)) = m_\rho$.

Remark. If $c : V_{\mathbb{C}}^{\rho} \to \overline{V}_{\lambda}$, then m_σ and m_ρ exchange roles. This does not affect our results.

1.1. For abelian varieties of types I, II and III $\mathfrak{g}_\ell,\ \mathfrak{h}_\ell$ are semi-simple and have the same invariants on $V_\ell \otimes_{\mathbb{Q}_\ell} V_\ell$ (cf. 0.4.3 and Remark 0.7.1(2)). In the case of type IV, the Lie algebras can be non-semi-simple. Since $\mathfrak{g}_\ell^{ss} \subseteq \mathfrak{h}_\ell^{ss}$, *a priori* $\mathfrak{g}_\ell^{ss}$ can have more invariants than $\mathfrak{h}_\ell^{ss}$ in $V_\ell^{\otimes 2}$. However, in our special case the following is true.

Theorem. *If $D = \mathbb{k}$, then $\mathfrak{g}_\ell^{ss}$ and $\mathfrak{h}_\ell^{ss}$ are non-self-dual.*

Remark. If $\mathfrak{g}_\ell = \mathfrak{g}_\ell^{ss},\ \mathfrak{h}_\ell = \mathfrak{h}_\ell^{ss}$ we get nothing new (cf. 0.7.1).

Proof. **(0).** If $\overline{\mathfrak{g}}_\ell^{ss}$ is symplectic or orthogonal, then so is $\overline{\mathfrak{g}}_\ell^{ss} \otimes_{\overline{\mathbb{Q}}_\ell} \mathbb{C}_\ell$. If $\overline{\mathfrak{h}}_\ell^{ss} \otimes_{\overline{\mathbb{Q}}_\ell} \mathbb{C}_\ell$ fixes a bilinear form on $\overline{V}_\ell$ coming from $V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$, then $\overline{\mathfrak{h}}_\ell^{ss}$ fixes the form. So, we can extend scalars to $\mathbb{C}_\ell$ and will use the same notation $\overline{\mathfrak{h}}_\ell,\ \overline{\mathfrak{g}}_\ell$ for the corresponding extentions of the Lie algebras.

(i). Let us consider first the symplectic case.

If $\overline{\mathfrak{g}}_\ell^{ss}$ *fixes* 1-dimensional subspace $\chi \subseteq \bigwedge^2 \overline{V}_\lambda$, then χ is a 1-dimensional $\overline{\mathfrak{g}}_\ell$-subrepresentation. The Hodge-Tate decomposition implies

$$\bigwedge^2 \overline{V}_\lambda = \bigwedge^2 \overline{V}_\lambda(0) \oplus (\overline{V}_\lambda(1) \otimes \overline{V}_\lambda(0)) \oplus \bigwedge^2 \overline{V}_\lambda(1).$$

The Hodge-Tate weight of the terms on the right is 0, 1 and 2 respectively.

Since $\dim_{\mathbb{C}_\ell}(\chi) = 1$, it is of Hodge-Tate weight 0, 1 or 2. $\chi^{g/2} = \det(\overline{V}_\lambda) := \bigwedge^g \overline{V}_\lambda = \bigwedge^{m_\sigma} \overline{V}_\lambda(1) \otimes \bigwedge^{m_\rho} \overline{V}_\lambda(0)$, the Hodge-Tate weight of the RHS is m_σ. Hence the Hodge- Tate weight of $\chi^{g/2}$ is m_σ. On the other hand, the weight of $\chi^{g/2}$ is $g/2$-times the weight of χ, hence is equal to 0, $g/2$ or g. So $m_\sigma = 0, g/2$ or g. The cases $m_\sigma = 0$ or g correspond to the $\mathbb{k}$-signature $(0, g)$ or $(g, 0)$. In either case the abelian variety is isogenous to a product of CM elliptic curves ([Sh 1, Proposition 14]). In the case $m_\sigma = g/2$ we have $m_\sigma = m_\rho$ and the Lie algebra $\overline{\mathfrak{g}}_\ell = \overline{\mathfrak{g}}_\ell^{ss}$ (as well as $\overline{\mathfrak{h}}_\ell = \overline{\mathfrak{h}}_\ell^{ss}$) is non-self-dual (cf. 0.7.1), hence this χ does not exist!

(ii). The orthogonal case is a direct consequence of the symplectic and 0.4.3 (take $s = 2 :\ \mathrm{Sym}^2(\overline{V}_\lambda) \hookrightarrow \bigwedge^2(2\overline{V}_\lambda)$). $\square$

Remark. Another way to conclude that the Hodge-Tate weight of χ is 1 is to use a result of Raynaud that the Galois action on 1-dimensional subrepresentations of (co)homology is by (powers of) the cyclotomic character χ_ℓ. Either way, the result is a consequence of the existence of the Hodge-Tate decomposition.

1.2. We can apply this consideration of the Hodge-Tate decomposition to abelian varieties with $D = \Bbbk$ and $\Bbbk$-signature (m_σ, m_ρ) such that $\gcd(m_\sigma, m_\rho) = 1$ (we call them *Ribet-type* abelian varieties, cf. [Ri]), then an argument of Serre ([S 1, §4]) implies the Tate conjecture in that case. Indeed, Ribet's proof of *loc. cit.*, Theorem 3, *verbatim* provides the Tate cycles are generated by divisors and hence the following theorem.

Theorem. *If A is a Ribet-type abelian variety, then the Tate cycles (on the abelian variety, and all its self-products) are generated by divisors and hence the Tate, the Hodge and the Mumford-Tate conjectures hold.* □

Remark. Applicability of this argument to the Tate cycles was undoubtly known to Ribet and was also noticed in [C 1] and [MZ].

2. Abelian 4-folds

2.0. The first non-trivial case of the Hodge, the Tate and the Mumford-Tate conjectures is that of abelian *fourfolds.* In that case $1 \leq (D : \mathbb{Q}) \leq 8$ and the dimensions of the irreducible subrepresentations of $\mathfrak{h}$, $\mathfrak{g}_\ell$ are ≤ 8.

The case of a simple 4-fold with $(D : \mathbb{Q}) \geq 2$ was studied in [MZ]. In this section we survey the 4-dimensional case just indicating the ideas involved. For details (in particular, proof of Theorem 2.4 below) see [MZ].

2.1. If $(D : \mathbb{Q}) \geq 2$, then the dimensions ≤ 4 and the restrictions imposed on $\mathfrak{g}_\ell$, $\mathfrak{h}$ (cf. 0.1.2, 0.3.1, 0.5) force the representations to be *unique*, hence abelian varieties in these cases verify MT.

Moreover, in "most cases" these Lie algebras are the "largest possible," viz., coincide with the Lie algebra of the Lefschetz group. Thus for any such abelian variety (excluding type III, cf. 0.1.5) the Hodge and the Tate conjectures hold.

2.2. The case $D = \mathbb{Q}$ is slightly more subtle: the choice for $\mathfrak{g}_\ell$, $\mathfrak{h}$ is *not* unique anymore. In fact, both possibilities, $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$ and $\mathfrak{sp}_8$, do occur.

The first one, viz., $\overline{\mathfrak{g}}_\ell = \overline{\mathfrak{h}}_\ell \cong \mathfrak{sp}_8$, is the generic case (cf. [Ab 1], [Ma]). Abelian varieties with $\overline{\mathfrak{h}} \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$, were constructed by Mumford in [M 2].

However, imposing some *extra conditions* on a simple 4-fold with $D = \mathbb{Q}$ we still can conclude MT (cf. 5.2).

2.2.1. In the generic case $\overline{\mathfrak{g}}_\ell = \overline{\mathfrak{h}}_\ell = \bar{\mathfrak{l}}_\ell$, where $\bar{\mathfrak{l}}_\ell := \mathcal{L}ie(L(A)) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell$, the Lie algebra of the Lefschetz group, and all the conjectures follow from 0.1.5.

2.2.2. In the Mumford case, $\overline{\mathfrak{h}} \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$, again $\overline{\mathfrak{g}}_\ell = \overline{\mathfrak{h}}_\ell$ and MT holds. However, $\bar{\mathfrak{l}} \cong \mathfrak{sp}_8$ and *not* all the Hodge/Tate cycles (on *self-products*!) are divisorial.

2.2.3. Meanwhile, whatever the case, $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$ or $\mathfrak{sp}_8$, [Ta, Lemma 4.10] implies that all the Hodge and the Tate cycles on the abelian variety *itself* are divisorial. However, calculations show (cf. [H 1, Lemma 5.2, (5.2.2)]) that on the "square" of the Mumford 4-fold not all the Hodge cycles are divisorial.

2.3.1. If $D = \mathbb{Q}$, then either $\overline{\mathfrak{g}}_\ell = \overline{\mathfrak{h}}_\ell$ and MT holds, or $\overline{\mathfrak{g}}_\ell \neq \overline{\mathfrak{h}}_\ell$ and then $\overline{\mathfrak{g}}_\ell \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$, $\overline{\mathfrak{h}}_\ell \cong \mathfrak{sp}_8$. In the latter case the (Lie algebra of the) Hodge group is equal to the (Lie algebra of the) Lefschetz group, thus all the Hodge cycles on the self-products of the abelian variety are divisorial, see also §7.

2.3.2. For the Weil case (cf. 0.7.3) *generically* the ring of Hodge cycles is not generated by divisors ([W], also [MZ]). So, if there are any doubts about the Hodge conjecture (and hence the Tate conjecture), the Weil abelian varieties are the ones to look at. Recently, C. Schoen ([Sc], also [vG]) succeded in proving the Hodge conjecture for one family of Weil 4-folds admitting an action of $\mathbb{Q}(\mu_3)$.

2.3.3. 2.1 answers a question of Tate (cf. [T 2], p. 82) on whether the Tate conjecture is true for the Schoen family.

2.4. We summarize the above discussion in the following theorem.

Theorem. *1. If A is any 4-dimensional abelian variety, then the rings of the Tate cycles and the Hodge cycles coincide (hence, the Hodge and the Tate conjectures for this variety are equivalent).*

2. If, additionally, $\mathrm{End}^\circ(A) \neq \mathbb{Q}$, *then MT holds.*

Remarks. 1. Later (Theorem 5.2) we will see that even when $\mathrm{End}^\circ(A) = \mathbb{Q}$, MT holds under some reduction conditions.

2. Recall (0.4) that MT implies that the Hodge and the Tate conjectures are equivalent for an abelian variety and *all its self-products.*

3. If A is non-simple, then the second hypothesis of the theorem is satisfied.

Proof. The only "real" case to consider is that of A *simple*. The proof can be found in [MZ] (in fact, Moonen-Zarhin considered a deeper problem of "when and why" a simple 4-fold has an exceptional Weil class).

If A is a *non*-simple abelian 4-fold, say A is isogenous to $A_1 \times A_2$, then $\dim(A_i) \leq 3$. Hence, by the (1,1)-theorems and duality, all the Hodge cycles on A_i are divisorial. The embeddings $\mathfrak{g}_\ell \subset \mathfrak{h}_\ell \hookrightarrow \mathfrak{sp}(V_\ell)$ factor through the sub-representations corresponding to the simple components of A. The dimensions of the sub-representations are ≤ 6, and there is not "enough room" for $\mathfrak{g}_\ell$ and $\mathfrak{h}_\ell$ to be different, i.e., $\mathfrak{g}_\ell = \mathfrak{h}_\ell$, hence MT holds. □

2.5. Let us indicate what is the situation regarding the Hodge and the Tate conjectures for *non-simple* abelian 4-folds. As above, let A be isogenous to $A_1 \times A_2$. Then all the Hodge and the Tate cycles on the A_i's are divisorial.

2.5.1. We can also say that all the Hodge cycles (and hence the Tate cycles) on A and all its self-products are generated by divisors in the following cases:

1. Neither of the A_i's is of type IV ([H 2, Theorem 0.1]).
2. A_1 is not of type IV, A_2 is of CM-type (*loc. cit.*, Proposition 3.1).
3. If the A_i's are *non*-CM, type IV abelian surfaces, then according to [Sh 1, Theorem 5, Propositions 17, 19], the A_i's are products of CM elliptic curves. Hence so is $A = A_1 \times A_2$ and for such abelian varieties the result stated above is known ([Im]; [H 1, Theorem 2.7]).
4. If the A_i's are *isogenous* CM surfaces, then by remark 0.1.2(1) and 0.1.5, $\mathcal{H}g(A) = \mathcal{H}g(A_1)$, $L(A) = L(A_1)$. By 0.1.5(1) $L(A_1) = \mathcal{H}g(A_1)$, hence $L(A) = \mathcal{H}g(A)$, and applying 0.1.5(1) once again we conclude the result.

2.5.2. For the remaining case, viz., both the A_i's are *non-isogenous* CM abelian varieties, let us just mention that Shioda constructed an example of a product of a simple CM 3-fold, A_1, with a CM elliptic curve, A_2, such that on $A = A_1 \times A_2$ there exist exceptional, non-divisorial, Hodge cycles, [Shi, Example 6.1]. In this example, however, the Hodge (hence the Tate) conjecture holds.

II. Abelian varieties with reduction conditions

3. Bad reduction and monodromy action

3.0. Let A be an abelian variety defined over a number field K. Assume A has bad reduction at a prime $\wp$ of $\mathcal{O}_K$. Let $\tilde{A}$ be the identity component of the special fiber of the Néron model of A. Then $\tilde{A}$ is semi-abelian:

$$0 \to H \to \tilde{A} \to B \to 0,$$

where H is the affine subgroup of $\tilde{A}$, B is the abelian quotient.

3.0.1. Since we are concerned with the Lie algebra of the (image of) Galois, we may pass to a finite extension of K (cf. Remark 1 in 0.3). So, according to the semi-stable reduction theorem ([G, Théorèm 3.6]), by extending the base field if necessary, we may assume that the reduction is *semi-stable* (i.e., H is a torus) and *split* (i.e., H is split: $H \cong \mathbb{G}_m^r$).

The dimension r of H we call the *toric rank* of (the reduction of) A.

3.0.2. $D = \mathrm{End}^\circ(A)$ as before, there is a homomorphism $D \to \mathrm{End}^\circ(H)$, $1_A \mapsto 1_H$. But $\mathrm{End}^\circ(\mathbb{G}_m^r) = M_r(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$, hence $(D : \mathbb{Q})|r$.

3.1. Consider the corresponding "specialization sequence"

$$0 \to V_\ell^{\mathcal{I}} \to V_\ell(A) \to U \to 0,$$

where $V_\ell(A)$ is the Tate module of A, $\mathcal{I} := \mathcal{I}(\wp)$ is the inertia group at $\wp$, $V_\ell^{\mathcal{I}} := V_\ell(A)^{\mathcal{I}}$ is the submodule of $\mathcal{I}$-invariants and U is a trivial $\mathcal{I}$-module (cf. [G, Proposition 3.5]). We have $\dim_{\mathbb{Q}_\ell}(V_\ell(A)) = 2g,\ \dim_{\mathbb{Q}_\ell}(V_\ell^{\mathcal{I}}) = 2g - r,\ \dim_{\mathbb{Q}_\ell}(U) = r$.

3.2. The above sequence is a sequence of of $\mathcal{I}$-modules. The $\mathcal{I}$-action is called the *local monodromy* action.

3.3. The reduction map at $\wp$ induces an isomorphism $V_\ell^{\mathcal{I}} \cong V_\ell(\tilde{A})$, the Tate module of $\tilde{A}$, and takes a submodule $W \subseteq V_\ell^{\mathcal{I}}$ to the Tate module $V_\ell(H) \subseteq V_\ell(\tilde{A})$ of the toric part H of $\tilde{A}$ (cf. [ST, Lemma 2], [G, 2.3], [I], [O]). In fact, according to the "Igusa-Grothendieck Orthogonality Theorem," $W = (V^{\mathcal{I}})^\perp$ with respect to the Weil pairing on V (cf. [I, Theorem 1], [G, Théorème 2.4], also [O, Theorem (3.1)]).

3.4. The monodromy action on $V_\ell(A)$ is, in general, quasi-unipotent (e.g., [G], [ST], [O]). However, since (we assumed that) the reduction of $A$is *semi-stable* and *split*, this action is, in fact, *unipotent* (cf. [G, Corollaire 3.8]).

3.4.0. Picking a vector subspace T of $V_\ell(A)$ specializing to U, we get the matrix form of the monodromy action:

$$\begin{matrix} V_\ell^{\mathcal{I}}\left\{\begin{matrix} W\{ \\ \\ \end{matrix}\right. \\ T\{ \end{matrix} \begin{pmatrix} 1_r & 0 & *_r \\ 0 & 1_{2g-2r} & 0 \\ 0 & 0 & 1_r \end{pmatrix}.$$

3.4.1. Passing to the Lie algebra $\mathfrak{i} := \mathcal{L}ie(\mathcal{I})$, we conclude the existence of nilpotents, $\tau \in \mathfrak{i} \subset \mathfrak{g}_\ell$, of order 2, i.e., $\tau^2 = 0$, and rank (with respect to V_ℓ) $\mathrm{rk}_{V_\ell}(\tau) \leq r$, where $\mathrm{rk}_{V_\ell}(\tau) := \dim_{\mathbb{Q}_\ell}(\tau V_\ell) =$ rank of the matrix of $\tau \in \mathfrak{gl}(V_\ell)$.

3.4.2. The Neron-Ogg-Shafarevich criterion ensures that $\exists\, \tau \neq 0$, since A has bad reduction.

3.4.3. Moreover, if N is given by the above matrix, then $\tau = N - 1_{2g}$ is the logarithm of the monodromy corresponding to the *monodromy filtration* (cf. [G, 4.1 and also Corollaire 4.4]; also [Il, 2.6])

$$(0) \subset W \subset V_\ell^{\mathcal{I}} \subset V_\ell,$$

and τ maps $V_\ell \to W,\ V_\ell^{\mathcal{I}} \to 0$, inducing an isomorphism of the quotients $V_\ell / V_\ell^{\mathcal{I}} \stackrel{\sim}{\to} W$, or, in our notation,

$$\tau : T \stackrel{\sim}{\to} W$$

(cf. [G, 4.1.2], [Il, (2.6.3)]; see also our Remark in 0.3 for why we omit the Tate twist in this formula). In particular, $\mathrm{rk}_{V_\ell}(\tau) = r$.

3.5. By extending scalars to $\overline{\mathbb{Q}}_\ell$ we get the corresponding nilpotents (of the same order) in each irreducible component of $V_\ell \otimes \overline{\mathbb{Q}}_\ell$ with the sum of the ranks with respect to each of the components being equal to the rank with respect to V_ℓ.

4. Minimal reduction

4.0. We say that an abelian variety A over a number field has *minimal* bad reduction at a prime $\wp$ of this field (or, just minimal reduction, for short) if the reduction is bad and the rank of the toric part H of $\tilde{A}$ (cf. 3.0) is the minimal possible.

4.1. Let us go back to the case $D = \Bbbk$, in which $\overline{\mathfrak{g}}_\ell \subseteq \overline{\mathfrak{h}}_\ell \subseteq \mathfrak{gl}(W_\lambda) \stackrel{\Delta}{\to} \mathfrak{sp}(W_\lambda \oplus \check{W}_\lambda)$ (cf. 0.6.2). The toric rank should be even (cf. 3.0.2), say, $2r$. If $\tau' = \Delta(\tau) \in \Delta(\overline{\mathfrak{g}}_\ell)$ is a nilpotent of rank $\mathrm{rk}_{V_\ell \otimes \overline{\mathbb{Q}}_\ell}(\tau') = 2r$ (cf. 3.4.3), then $\tau^2 = 0,\ \mathrm{rk}_{W_\lambda}(\tau) = r$.

4.1.1. 3.0.2 and 3.4.3 imply that in the case $D = \Bbbk$ the minimal toric rank is 2.

4.1.2. Hence in the minimal reduction case $\exists\, \tau \in \overline{\mathfrak{g}}_\ell \subseteq \overline{\mathfrak{h}}_\ell \subseteq \mathfrak{gl}(W_\lambda)$ such that $\tau^2 = 0,\ \mathrm{rk}_{W_\lambda}(\tau) = 1$. The same, clearly, holds if we replace the Lie algebras with their semi-simple components, since all nilpotents live in these components.

Let $U := W_\lambda$, and (as in 0.4.4) $\widetilde{\mathfrak{g}} = \overline{\mathfrak{g}}_\ell^{ss},\ \widetilde{\mathfrak{h}} = \overline{\mathfrak{h}}_\ell^{ss}$, and rewrite the above as:

$$\tau \in \widetilde{\mathfrak{g}} \subseteq \widetilde{\mathfrak{h}} \subseteq \mathfrak{sl}(U),\ \tau^2 = 0,\ \mathrm{rk}_U(\tau) = 1,$$
$$\widetilde{\mathfrak{g}},\ \widetilde{\mathfrak{h}}\ :\ \text{semi-simple irreducible representations.}$$

Such an element τ of rank 1 and order 2 is called a *transvection*.

4.2. It is a very restrictive condition for an irreducible representation of a semi-simple Lie algebra to contain a transvection.

Lemma. *If* $\mathfrak{a} \hookrightarrow \mathfrak{sl}(U)$ *is a semi-simple faithful irreducible representation,* $\tau \in \mathfrak{a}$, $\tau^2 = 0$, $\mathrm{rk}_U(\tau) = 1$, *then* $\mathfrak{a}$ *is simple and, moreover, it is either* $\mathfrak{sp}(U)$ *or* $\mathfrak{sl}(U)$.

Proof. This is proved in [McL] (cf. also [PS]). □

4.3. We will also need the following simple fact.

Lemma. *If* $\mathrm{rk}_U(\tau)$ *is prime to* $\dim(U)$, *then* $\mathfrak{a}$ *is simple.* □

5. Applications of minimal reduction

5.1. An immediate application of 4.1.2 (existence of rank 1 quadratic nilpotents in $\mathfrak{g}$) and Lemma 4.2 is the following theorem.

Theorem. *If* A *is a simple abelian variety with* $D \subseteq \Bbbk$, *having minimal reduction, then MT holds. Moreover, if* $D = \mathbb{Q}$, *then all the Hodge and the Tate cycles are divisorial, hence the Hodge and the Tate conjectures hold.*

Proof. 1. If $D = \mathbb{Q}$, then the Tate module $V_\ell(A)$ is absolutely irreducible and symplectic. The minimality of reduction implies that the rank of a correspondent nilpotent is 1. The result now follows from 4.2.

2. If $D = \Bbbk$, the result follows from 4.1.2, 4.2 and 1.1 (cf. 0.1.2). □

Remarks. 1. Such abelian varieties exist and, moreover, form a subset dense in the complex topology in the corresponding moduli space (cf. [L]).

2. The importance of the Weil type abelian varieties is not limited to the fact that they (may) have non-divisorial Weil cycles (cf. 4.4). Proving the algebraicity of the Weil cycles is a critical ingredient in proving the Tate conjecture (cf. [D 3, §§ 4∼6]; also [An], [Ab 2, § 6]).

5.2. Using the same method we can now extend Theorem 2.4 in the following way.

Theorem. *If* A *is a simple abelian 4-fold with* $D = \mathbb{Q}$ *admitting bad but not purely multiplicative reduction, then all the Hodge and the Tate cycles are divisorial, hence the Hodge conjecture, the Tate conjecture and MT hold.*

Proof. The possible values of the toric rank in this case are 1, 2, 3 (4 corresponds to the purely multiplicative reduction).

First recall (2.2) that the only choices for $\widetilde{\mathfrak{g}}$ and $\widetilde{\mathfrak{h}}$ are $\mathfrak{sp}_8$ or $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$. But $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$ does not contain quadratic nilpotents of rank 1, 2 or 3. So, $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{h}} \cong \mathfrak{sp}_8$ and 0.1.5 finishes the proof. □

Remarks. 1. Applying results/methods of [Z 5], [LZ] one can also verify all the conjectures for a simple abelian 4-fold admitting certain types of *good* reductions, further restricting the class of 4-folds for which the conjectures are not yet known.

2. G. Mustafin [Mus] has proved a "geometric analog" of MT for (families of) abelian varieties with purely multiplicative reduction (i.e., the algebraic envelope of the image of the monodromy coincides with the Hodge group for a "sufficiently general" abelian variety in the family), cf. also [H 3]. It appears, though, that his methods cannot be transplanted to the arithmetic situation.

6. Another type of bad reduction

6.0. Now we want to establish a result analogous to Theorem 5.1 for another type of bad reduction. Namely, consider an abelian variety A admitting bad (semi-stable, split) reduction of toric rank r such that $r/(D:\mathbb{Q})$ is prime to $2\dim(A)/(D:\mathbb{Q})$, where as before $D = \mathrm{End}^\circ(A)$. Assume also D is *commutative*. In this case $\widetilde{\mathfrak{g}}$, $\widetilde{\mathfrak{h}}$ (notation as in 4.1.2) are simple (cf. 4.3) and contain nilpotents of rank prime to the dimension of the representations (cf. 3.4.3). In place of Lemma 4.2 we use the results of Premet-Suprunenko [PS] on classification of quadratic elements (= nilpotents of order 2) and quadratic modules (= representations containing non-trivial quadratic elements) of simple Lie algebras.[†]

We use the fact that the representations of $\widetilde{\mathfrak{g}}$, $\widetilde{\mathfrak{h}}$ are minuscule and $\widetilde{\mathfrak{g}}$ is not exceptional (0.5.1).

We may assume that the dimensions of the representations is > 4.

6.1. One of our key tools replacing Lemma 4.2 is the following result.

Theorem. *If* $\mathfrak{a} \subset \mathfrak{b} \hookrightarrow \mathfrak{sl}(U)$, $\mathfrak{a} \neq \mathfrak{b}$, *both Lie algebras are simple and the representations are (faithful) irreducible and minuscule, then* $\mathfrak{b}$ *is classical and (its highest weight is)* ϖ_1.

Proof. Since any minuscule representation is quadratic (cf. [B, Ch VIII, §7.3]), we can apply [PS, Theorem 3], and exclude non-minuscule cases. □

6.2. We will be interested in 2 cases: $D = \Bbbk$ and $D = \mathbb{Q}$. In the former case we know that $\widetilde{\mathfrak{g}}$ and $\widetilde{\mathfrak{h}}$ are both non-self-dual (cf. Theorem 1.1) and if they satisfy the conditions of the theorem, then

- $\widetilde{\mathfrak{h}}$ is classical, ϖ_1 and non-self-dual, hence $\widetilde{\mathfrak{h}} = (A_n, \varpi_1)$
- $\widetilde{\mathfrak{g}}$ is classical, minuscule and non-self-dual, hence $\widetilde{\mathfrak{g}} = (A_m, \varpi_s)$, $m \neq 2s$.

If $D = \mathbb{Q}$, we know that $\widetilde{\mathfrak{g}}$, $\widetilde{\mathfrak{h}}$ are symplectic and we again have a unique possibility for $\widetilde{\mathfrak{h}}$, viz., $\widetilde{\mathfrak{h}} = (C_n, \varpi_1)$. However, there are several *a priori* possible choices for $\widetilde{\mathfrak{g}}$.

[†]This terminology is apparently standard in the finite groups theory, cf. [Th].

To eliminate (as many as we can) possibilities of $\widetilde{\mathfrak{g}} \neq \widetilde{\mathfrak{h}}$ we use the existence in the representations of a quadratic nilpotent of *rank prime to the dimension* of the representation.

So, we consider a slightly more general situation. As above

- $\widetilde{\mathfrak{g}} \subsetneq \widetilde{\mathfrak{h}} \subset \mathfrak{sl}(U)$, $\widetilde{\mathfrak{g}}$, $\widetilde{\mathfrak{h}}$ are classical simple Lie algebras,
- $\widetilde{\mathfrak{g}}$ is minuscule, $\widetilde{\mathfrak{h}}$ is ϖ_1,
- there exists $\tau \in \widetilde{\mathfrak{g}} \subset \widetilde{\mathfrak{h}} \subset \mathfrak{sl}(U)$ with $\tau^2(U) = 0$, $\mathrm{rk}_U(\tau) = r$, $\dim(U) = n$ and $\gcd(r, n) = 1$.

We add the following condition wich is satisfied in both our cases:

- $\widetilde{\mathfrak{g}}$, $\widetilde{\mathfrak{h}}$ are non-self-dual, or orthogonal, or symplectic *simultaneously.*

6.3. First we exclude the cases $\widetilde{\mathfrak{g}} = (D_m, \varpi_{m-1})$, (D_m, ϖ_m) for $m > 4$.

Lemma. *If $\tau \in \widetilde{\mathfrak{g}} \hookrightarrow \mathfrak{sl}(U)$, $\widetilde{\mathfrak{g}} = (D_m, \varpi_{m-1})$ or (D_m, ϖ_m), τ is a quadratic element, then* $\gcd(r, n) > 2$.

Proof. [PS, Lemma 21, Note 2, Lemma 17] imply $r = 2^{m-3}$ or 2^{m-2}, while $n = 2^{m-1}$. ☐

$\widetilde{\mathfrak{g}} \neq (D_4, \varpi_3)$, (D_4, ϖ_4) either. It is enough to show this for ϖ_4, since they are (graph)isomorphic.

Proposition. $(D_4, \varpi_4) \not\hookrightarrow$ *(classical,* ϖ_1*).*

Proof. Note that $n = 8$ in here. We do this case by case:

1. $(D_4, \varpi_4) \not\hookrightarrow (B_\bullet, \varpi_1)$, since the dimension of the RHS is odd.
2. $(D_4, \varpi_4) \not\hookrightarrow (C_\bullet, \varpi_1)$, since the LHS is orthogonal while the RHS is symplectic (cf. [B, table 1]).
3. $(D_4, \varpi_4) \not\hookrightarrow (D_4, \varpi_1)$ (e.g., [Z 2, §5, Key lemma], although this is overkill).
4. $(D_4, \varpi_4) \not\hookrightarrow (A_\bullet, \varpi_1)$, since the LHS is orthogonal, the RHS is not. ☐

Proposition. *If* $\widetilde{\mathfrak{g}} = (D_m, \varpi_1)$*, then* $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{h}}$.

Proof. $\widetilde{\mathfrak{h}} = A_\bullet$, $C_\bullet$ are excluded: $\widetilde{\mathfrak{g}}$ is orthogonal, $\widetilde{\mathfrak{h}}$ is not; $\widetilde{\mathfrak{h}} = B_\bullet$ is excluded by a dimensional reason. ☐

Proposition. *If* $\widetilde{\mathfrak{g}} = (C_m, \varpi_1)$*, then* $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{h}}$.

Proof. $\widetilde{\mathfrak{h}} = A_\bullet$, $B_\bullet$, $D_\bullet$ are not symplectic ... ☐

Proposition. *If* $\widetilde{\mathfrak{g}} = (B_m, \varpi_1)$*, then* $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{h}}$.

Proof. $\widetilde{\mathfrak{h}} = A_\bullet$, $C_\bullet$ are not orthogonal... $(D_\bullet, \varpi_1)$ is even-dimensional... ☐

Proposition. *If* $\widetilde{\mathfrak{g}} = (A_m, \varpi_s)$, *then* $\widetilde{\mathfrak{h}}$ *must be* (A_{n-1}, ϖ_1).

Proof. First note that the only self-dual representation of A_m is ϖ_s with $s = \frac{m+1}{2}$ (there is no such a representation if m is even). So, if $\widetilde{\mathfrak{h}}$ is self-dual, then so is $\widetilde{\mathfrak{g}}$ (cf. 6.2) and we may assume $s = \frac{m+1}{2}$, m : odd. But then $\dim(A_m, \varpi_s) = \binom{2s}{s}$: even, thus $\widetilde{\mathfrak{h}} \neq B_\bullet$. To exclude the other cases (i.e., $C_\bullet$, $D_\bullet$) we use the fact that $r = \mathrm{rk}(\tau) = \binom{m-1}{s-1}$ ([PS, §2 & Lemma 18]):

$$r = \binom{2(s-1)}{s-1}, \quad n = \binom{2s}{s} = \binom{2(s-1)}{s-1}\frac{(2s-1)2s}{s^2} = r\frac{2(2s-1)}{s};$$

$\gcd(n, r) = 1 \Rightarrow r|s$, which is not true: $n > 4 \Rightarrow s > 3 \Rightarrow \binom{2(s-1)}{(s-1)} > s$. □

Remark. If the $\Bbbk$-signature of the abelian variety is (m_σ, m_ρ) with $m_\sigma \neq m_\rho$, then, even if $\widetilde{\mathfrak{g}}$ is not simple, simple components of $\widetilde{\mathfrak{g}}$ are of type A (cf. [Y]).

So, the only possibility for $\widetilde{\mathfrak{g}} \subsetneq \widetilde{\mathfrak{h}}$ is $\widetilde{\mathfrak{g}} = (A_m, \varpi_s)$ for some s, $\widetilde{\mathfrak{h}} = (A_{n-1}, \varpi_1)$. In this case we can say the following.

Proposition. *Let* $\widetilde{\mathfrak{g}} = (A_m, \varpi_s) \hookrightarrow \widetilde{\mathfrak{h}} = (A_{n-1}, \varpi_1) \cong \mathfrak{sl}(U)$ *(fix the isomorphism),* $2 \leq s < \frac{m+1}{2}$, $\tau \in \widetilde{\mathfrak{g}}$, $r = \mathrm{rk}_U(\tau)$, $\gcd(n, r) = 1$. *Then either* $s = 3$, $m = 7$, *or* $s = 2$.

Proof. $r = \binom{m-1}{s-1}$ ([PS, §2 & Lemma 18]), $n = \binom{m+1}{s} = \binom{m-1}{s-1}\frac{m(m+1)}{s(m+1-s)}$, $\gcd(r, n) = 1 \Rightarrow r \mid s(m+1-s)$ and the result follows from the following simple observation:
$\binom{m-1}{s-1}|s(m+1-s)$ if and only if $(m, s) = (7, 3)$ or $s = 2$. □

Remark. If $s = 2$, then $r = \binom{m-1}{s-1} = m-1$, $n = \binom{m+1}{s} = \frac{m(m+1)}{2}$. Since $\gcd(m, m-1) = 1$ and $\gcd(m-1, m+1) = 1$ or 2, $\gcd(r, n) = 1$ if and only if m is even or $m \equiv 1 \pmod 4$ (i.e., $m \not\equiv 3 \pmod 4$).

6.4. So, if $D = \Bbbk$ we have the following result.

Theorem. *If A is a simple abelian variety with $D = \Bbbk$, $g = \dim(A)$ having bad reduction with the toric rank $2r$ and r is prime to g, then MT holds if (g, r) is neither (56, 15) nor of the form* $(\frac{m(m+1)}{2}, m-1)$. □

6.5. If $D = \mathbb{Q}$, then $\widetilde{\mathfrak{g}}$, $\widetilde{\mathfrak{h}}$ are symplectic, hence the theorem holds with no exceptions.

Theorem. *If A is an abelian variety with $D = \mathbb{Q}$, $g = \dim(A)$ having bad reduction with the toric rank r prime to $2g$, then MT holds.* □

6.6. Using the same methods one can handle the case of quadratic elements of rank 2. Namely, the following result holds.

Theorem. *Let A be a simple abelian variety with $D = \mathbb{Q}$. If A has bad reduction with toric rank 2, then MT holds.*

Proof. (Sketch) Let $\dim(A) = g$. We can assume $g \geq 4$.

One can check (cf. [PS, §2, Lemma 18]) that if $\mathfrak{b} \hookrightarrow \mathfrak{sl}(U)$ is a semi-simple irreducible classical Lie algebra, $\dim(U) \geq 8$ and $\exists\, \tau \in \mathfrak{b},\ \tau^2 = 0,\ \mathrm{rk}_U(\tau) = 2$, then either $\mathfrak{b}$ is simple, and hence $\mathfrak{b} = \mathfrak{sl}(U),\ \mathfrak{sp}(U)$ or $\mathfrak{so}(U)$ (since $\dim(U) \geq 8,\ \mathfrak{b} \ncong (A_3, \varpi_2)$), or $\mathfrak{b} = \mathfrak{a} \times \mathfrak{sl}_2$, where $\mathfrak{a} \cong \mathfrak{sl}_g$ or $\mathfrak{sp}_g$. Since $\widetilde{\mathfrak{g}},\ \widetilde{\mathfrak{h}}$ are *symplectic*, the only possibility is $\mathfrak{sp}(U)$! $\square$

6.7. All the varieties considered in §6 exist and dense in the (complex topology in the) corresponding moduli spaces (cf. [L]).

6.8. 1. The idea of using special element(s) in the representation of the Hodge group has been used before. However, to our knowledge, in those earlier cases the element was semi-simple of low rank (e.g., [Z 1]) and the results then follow from a theorem of Kostant [Ko] (cf. also [Z 3]).

2. Katz used special unipotent elements to show that certain monodromy groups are large. However, the unipotents he considered were of the maximal possible rank, i.e., having only *one* Jordan block ([Ka 1, Ch. 7]).*

Katz was also using semi-simple elements for similar purposes, [Ka 2].

7. A curious result

7.1. Let us mention another application of Theorem 6.1.

Theorem. *If A is a simple abelian variety with (semi-simple parts of) $\mathfrak{h}$, $\mathfrak{g}_\ell$ simple, satisfying one of the following conditions:*

- *the variety is of type I or II,*
- *$D = \mathbb{k}$, the variety is of non-Weil type (i.e., the $\mathbb{k}$-signature is (m_σ, m_ρ) with $m_\sigma \neq m_\rho$),*

then one of the following must hold:

- *MT holds for A,*
- *all the Hodge cycles (on A and all its self-products) are generated by divisor classes (hence the Hodge conjecture holds).*

Proof. As we mentioned in 6.0, the representations of $\mathfrak{h}_\ell$, $\mathfrak{g}_\ell$ are minuscule, hence quadratic (cf. [B, Ch VIII, §7.3, Proposition 7]), then so is $\mathfrak{l}_\ell := \mathfrak{l} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, where $\mathfrak{l}$ is

*As an application of Katz's classification of representations containing such unipotents (cf. *loc. cit.*, 11.5∼11.7) one can find modular curves for which the image of Galois in the corresponding ℓ-adic representation is large.

the Lie algebra of the Lefschetz group (cf. 0.1.5), and $\widetilde{\mathfrak{g}} \subset \widetilde{\mathfrak{h}} \subset \widetilde{\mathfrak{l}} \subseteq \mathfrak{sl}(U)$ (cf. 0.2.3; here $\widetilde{\mathfrak{l}} := \overline{\mathfrak{l}}_\ell^{ss}$, $\overline{\mathfrak{l}}_\ell = \mathfrak{l}_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$). If $\widetilde{\mathfrak{g}} \subsetneqq \widetilde{\mathfrak{h}}$ (i.e., MT does *not* hold), then by Theorem 6.1 $\widetilde{\mathfrak{h}} \hookrightarrow \mathfrak{sl}(U)$ is classical and ϖ_1. Thus $\widetilde{\mathfrak{l}}$ is also simple, classical and ϖ_1. We want to show that in this case $\overline{\mathfrak{h}}_\ell = \overline{\mathfrak{l}}_\ell$ and the theorem then follows from 0.1.5.

Consider first the case of an abelian variety of type I or II. The Lie algebras $\overline{\mathfrak{h}}_\ell$ and $\overline{\mathfrak{l}}_\ell$ are both symplectic, simple, classical and ϖ_1. Hence $\overline{\mathfrak{h}}_\ell = \mathfrak{sp}(U)$ (resp. $\mathfrak{so}(U)$) $= \overline{\mathfrak{l}}_\ell$.

If $D = \Bbbk$, then $C_{\overline{\mathfrak{h}}_\ell}$ is 1-dimensional (0.7.2), hence $C_{\overline{\mathfrak{h}}_\ell} = C_{\overline{\mathfrak{l}}_\ell}$ and $\widetilde{\mathfrak{h}} = \widetilde{\mathfrak{l}} = \mathfrak{sl}(U)$ (cf. 1.1; also [Mu, Lemma 2.3]). The theorem follows. $\square$

7.2. Remarks. 1. As one can see from the proof, if there is a way to assure that $\mathfrak{h}$ is simple and ϖ_1, then $\mathfrak{h} = \mathfrak{l}$, hence (for types I or II) the Hodge conjecture holds (e.g., 2.3).

2. Abelian varieties with bad reduction as in 6.4∼6.5 have simple (semi-simple parts of) Hodge and Galois groups (cf. 4.3).

3. For the Weil type varieties, i.e., $m_\sigma = m_\rho$, $C_{\overline{\mathfrak{h}}_\ell} = \{0\} \neq C_{\overline{\mathfrak{l}}_\ell}$ *generically*. This is (a restatement of) the main result of [W].

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