COMPOSITIONAL ROOTS OF HÉNON MAPS

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Abstract. Let H denote a composition of complex Hénon maps in $\mathbb{C}^2$. In this paper we show that the only possible compositional roots of H are also compositions of Hénon maps, and that H can have compositional roots of only finitely many distinct orders.

1. Introduction

Following [BS], we say that a generalized Hénon map is a map of the form

$$H(z, w) = (w, p(w) - az),$$

where p is a monic polynomial of degree $d \geq 2$ and $a \in \mathbb{C} - \{0\}$, and we let $\mathcal{G}$ denote the space of finite compositions of such maps. From [FM], we know that any polynomial diffeomorphism of $\mathbb{C}^2$ is conjugate either to one of the maps in $\mathcal{G}$ or to an elementary map which preserves each line of the form $w = \text{const}$. In [BF], we classified, up to conjugacy, all polynomial diffeomorphisms which arise as the time-1 map of a holomorphic vector field. In particular, each of these maps is an elementary map and has compositional roots of all orders. Moreover, in some cases, these roots can be nonpolynomial. See [AF] for information about such cases.

In this paper we treat the question of the existence of compositional roots for the remaining cases. In particular, we show that any root of a map in $\mathcal{G}$ must be a polynomial map and that any map in $\mathcal{G}$ can have roots of only a finite number of distinct orders. For the remaining elementary maps which are not the time-1 map of a flow, we show that such maps have roots of arbitrarily high order and nonpolynomial roots, but that any root of such a map is conjugate to a polynomial elementary map.

2. Dynamical behavior and Green's functions

Fix $H \in \mathcal{G}$. Let $K^+(= K^+(H))$ and $K^-(= K^-(H))$ denote the set of points p in $\mathbb{C}^2$ such that the orbit of p under H is bounded with

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respect to forward or backward iteration, respectively. Also, for $R > 0$, define sets

$$V^- := \{(z,w) : |w| > R \text{ and } |w| > |z|\},$$
$$V^+ := \{(z,w) : |z| > R \text{ and } |w| < |z|\},$$
$$V := \{(z,w) : |z| \le R \text{ and } |w| \le R\}.$$

A simple argument shows that $K^\pm \subseteq V^\pm \cup V$ for R sufficiently large. Finally, we let d denote the degree of H as a polynomial map, and define functions

$$G^\pm(z,w) := \lim_{n\to\infty} \frac{1}{d^n} \log^+ \|H^{\pm n}(z,w)\|.$$

It is immediate that G^+ is continuous and plurisubharmonic (psh) on $\mathbb{C}^2$, identically 0 on K^+, and strictly positive and pluriharmonic on $\mathbb{C}^2 - K^+$. Analogous statements are true with G^- and K^- in place of G^+ and K^+. See, for example, [BS] for a more systematic discussion.

Note also that $G^\pm \circ H = d^{\pm 1} G^\pm$. Moreover, corollary 2.6 in [BS] implies the following.

Lemma 2.1. *There exist $R > 0$, $C > 0$ such that if $(z,w) \in \overline{V^- \cup V}$, then*

$$\log^+ |w| - C \le G^+(z,w) \le \log^+ |w| + C,$$

and if $(z,w) \in \overline{V^+ \cup V}$, then

$$\log^+ |z| - C \le G^-(z,w) \le \log^+ |z| + C.$$

Since $G^+(z_0, w)$ is subharmonic in w for each fixed z_0, we see that it is bounded from above in $\{|w| < |z_0|\}$ by its maximum on the boundary of this disk, which is contained in $\overline{V^- \cup V}$. Applying a similar argument to G^- gives the following.

Lemma 2.2. *There exists $C > 0$ such that for $(z,w) \in \mathbb{C}^2$,*

$$G^\pm(z,w) \le \max\{\log^+ |z|, \log^+ |w|\} + C.$$

3. Compositional roots and Green's functions

Fix $H \in \mathcal{G}$ and let d be the degree of H. In this section we show that if $F^n = H$ in the sense of composition, then $G^\pm \circ F = d^{\pm 1/n} G^\pm$. First a simple lemma.

Lemma 3.1. *Suppose F is an automorphism of $\mathbb{C}^2$ and $F^n = H$. Then K^+ and K^- are the same for F as for H, and F is a diffeomorphism of K^+ and of K^-.*

Proof: Take $p \in K^+(H)$ and let $\mathcal{O}_H^+(p)$ denote the forward orbit of p under H. Then $\cup_{j=0}^{n-1} F^j(\overline{\mathcal{O}_H^+(p)})$ is compact, and $\mathcal{O}_F^+(p)$ is contained in this set, hence is bounded. Thus $K^+(H) \subseteq K^+(F)$.

If $p \notin K^+(H)$, then the forward orbit is not bounded for $H = F^n$, hence is not bounded for F. Thus $K^+(H) = K^+(F)$, and a similar argument applies to K^-.

The fact that F is a diffeomorphism of $K^\pm = K^\pm(F)$ is clear from the definition of these sets. ■

Lemma 3.2. *Let F be as in the previous lemma. Then $G^\pm \circ F = d^{\pm 1/n} G^\pm$.*

Proof: Since F is holomorphic on $\mathbb{C}^2$ and preserves K^+, we see that $G^+ \circ F$ is 0 on K^+, plurisubharmonic and continuous on $\mathbb{C}^2$, and strictly positive and pluriharmonic on $\mathbb{C}^2 - K^+$.

Fix z_0 and define $g_{z_0}(w) := G^+ \circ F(z_0, w)$. Note that $K^+ \cap (\{z_0\} \times \mathbb{C})$ is a compact set and that g is harmonic on the complement of this set. Hence, outside a large disk, g_{z_0} has a harmonic conjugate in a neighborhood of any point. Using analytic continuation in the exterior of this disk, we obtain a harmonic conjugate with periods. Hence for some $r > 0$, some constant c_{z_0}, and a real harmonic function h_{z_0}, we get a function

$$\phi_{z_0}(w) = g_{z_0}(w) - c_{z_0} \log |w| + i h_{z_0}(w)$$

which is holomorphic for $|w| > r$.

Since $g_{z_0} \geq 0$, we have $|\exp(-\phi_{z_0}(w))| \leq |w|^{c_{z_0}}$. Hence $\exp(-\phi_{z_0}(w))$ has at most a pole at infinity, so we can write

$$\exp(-\phi_{z_0}(w)) = w^N \exp(f(w))$$

for some integer N and some f holomorphic in $|w| > r$ with a removable singularity at infinity.

Taking absolute value and log, we get $g_{z_0}(w) - c_{z_0} \log |w| = -N \log |w| - \mathrm{Re}(f(w))$. Hence $g_{z_0}(w) = b_{z_0} \log |w| + O(1)$ in $\{|w| > 2r\}$, for some b_{z_0}. Since $g_{z_0} \geq 0$, we have $g_{z_0}(w) = b_{z_0} \log^+ |w| + O(1)$ in $\mathbb{C}$.

We claim that b_z is independent of z. Note that $2\pi b_{z_0}$ is the period for the harmonic conjugate of g_{z_0} in $|w| > r$, and that $g_z(w)$ is pluriharmonic in (z, w) near (z_0, w_0) for any $|w_0| > r$.

Fix $|w_0| > r$. We can construct the harmonic conjugate for g in the bidisk $\Delta(z_0; r_0) \times \Delta(w_0; r_0)$ for some r_0 small. For $w \in \Delta(w_0; r_0)$, we can use analytic continuation as above to extend g_z around a circle in $\{z\} \times \{|w| > r\}$. Doing this for each such w gives a new harmonic conjugate for g in the bidisk, which must differ from the original by a constant. Thus $b_z = b_{z_0}$ for z near z_0.

Hence $g_{z_0}(w) = b\log^+|w| + O(1)$, and from lemma 2.1, we see $G^+(z_0, w) = \log^+|w| + O(1)$. Thus $g_{z_0}(w) - bG^+(z_0, w)$ is continuous for $w \in \mathbb{C}$ and harmonic for w such that $(z_0, w) \notin K^+$, has a removable singularity at ∞, and is 0 for w such that $(z_0, w) \in K^+$, which is a nonempty set. Hence $g_{z_0}(w) \equiv bG^+(z_0, w)$ for all $w \in \mathbb{C}$.

Similarly, $G^+ \circ F(z, w) - bG^+(z, w) \equiv 0$ for all $(z, w) \in \Delta(z_0; r_0) \times \mathbb{C}$. Since this difference is pluriharmonic in $\mathbb{C}^2 - K^+$, which is connected, it must be 0 throughout $\mathbb{C}^2 - K^+$, hence throughout $\mathbb{C}^2$ since both terms are 0 on K^+.

Finally, $G^+ \circ F^n(z, w) = G^+ \circ H(z, w) = dG^+(z, w)$, while induction with the above result shows that $G^+ \circ F^n(z, w) = b^n G^+(z, w)$ Hence $b^n = d$, and $b > 0$ since $G^+ \geq 0$. This gives the lemma for G^+, and the same proof applies to G^-. ∎

4. Polynomial roots

In the proof of the following theorem, we use the terminology and results of [FM]. In particular, we use the fact that the group of polynomial automorphisms of $\mathbb{C}^2$ is the amalgamated product of the group $\mathcal{A}$ of affine linear automorphisms and the group $\mathcal{E}$ of elementary automorphisms which preserve the set of lines of the form $w = const$. A *reduced word* is an automorphism of the form $g_1 \cdots g_k$, $k \geq 1$, where each g_k is in $\mathcal{A}$ or $\mathcal{E}$ but not in the intersection of these two groups and no two adjacent g_j's are in the same group. We say that k is the *length* of this reduced word. Also, we need to know that the identity cannot be written as a reduced word.

Theorem 4.1. *Suppose $H \in \mathcal{G}$ is a composition of generalized Hénon maps and F is an automorphism of $\mathbb{C}^2$ with $F^n = H$. Then $F \in \mathcal{G}$.*

Proof: Let $(z, w) \in \mathbb{C}^2$, and let $F = (F_1, F_2)$. If $F(z, w) \in \overline{V^- \cup V}$, then from lemmas 2.1, 3.2, and 2.2, we see

$$\begin{aligned}\log^+|F_2| - C_1 &\leq G^+(F(z, w)) \\ &= d^{1/n} G^+(z, w) \\ &\leq d^{1/n}(\max\{\log^+|z|, \log^+|w|\} + C_2).\end{aligned}$$

Exponentiating and using $|F_1| \leq |F_2| + R$, we obtain

$$|F(z, w)| \leq C \max\{(|z| + 1)^{d^{(1/n)}}, (|w| + 1)^{d^{(1/n)}}\}.$$

Similarly, if $F(z,w) \in V^+$, then

$$\begin{aligned}\log^+|F_1| - C_1 &\leq G^-(F(z,w))\\ &= 1/d^{(1/n)}G^-(z,w)\\ &\leq 1/d^{(1/n)}(\max\{\log^+|z|, \log^+|w|\} + C_2).\end{aligned}$$

Exponentiating and using $|F_2| \leq |F_1|$, we obtain

$$|F(z,w)| \leq C\max\{(|z|+1)^{1/d^{(1/n)}}, (|w|+1)^{1/d^{(1/n)}}\}.$$

Hence F has polynomial growth throughout $\mathbb{C}^2$, hence must be a polynomial.

We show next that $F \in \mathcal{G}$. Let $\tau(z,w) := (w,z)$. By [FM], we can write $H = \tau e_1 \cdots \tau e_m$, for some elementary maps e_j. Since $F^n = H$, F must be a reduced word with length at least 2. There are four possibilities for the form of F. The first is

$$F = a_1 e_1' \cdots a_l e_l'$$

for some affine, non-elementary maps a_j and some elementary, non-affine maps e_j'. By [FM] or [AR], each a_j can be written $a_j = a_j^1 \tau a_j^2$, where a_j^1 and a_j^2 are affine and elementary, and a_1^1 has the form

$$a_1^1(z,w) = (bz + cw, w).$$

Now, since a_j^k is elementary, F^n has the form $a_1^1 \circ (p,q)$, where p and q are polynomials and $(p,q) = \tau e_1'' \cdots \tau e_{nl}''$. By [FM], we have $\deg(q) > \deg(p)$, and likewise the degree of the second coordinate function of $H = F^n$ is larger than the degree of the first coodinate function. This implies that $c = 0$, so $a_1^1(z,w) = (bz,w)$. Replacing a_2^1 by σa_2^1, where $\sigma(z,w) = (z,bw)$, we obtain

$$F = \tau E_1 \cdots \tau E_l.$$

Hence $F \in \mathcal{G}$.

The second case is

$$F = e_1' a_2 \cdots a_l e_l'.$$

In this case, we can replace each a_j by $a_j^1 \tau a_j^2$ as before, and hence relabeling, we may assume $F = e_1' \tau \cdots \tau e_l'$. But then $H = F^n = e_1' \tau \cdots \tau e_{nl}'$, which implies that the degree of the first coordinate of H is larger than the degree of the second coordinate, which is impossible. Hence F cannot have this form.

The third case is

$$F = e_1' a_2 \cdots e_{l-1}' a_l.$$

As before, we may relabel to assume that $F = e_1'\tau \cdots e_{l-1}'\tau a_l^2$. But then $H = F^n = e_1'\tau \cdots e_k'\tau a_l^2$, but also $H = \tau e_1 \cdots \tau e_m$. Hence

$$I = (F^n)^{-1}H = (a_l^2)^{-1}(\tau(e_k')^{-1} \cdots (e_1')^{-1})(\tau e_1 \cdots \tau e_m).$$

But then I has been written as a reduced word, which is impossible from [FM]. Thus F cannot have this form.

In the final case, we have

$$F = a_1 e_1' \cdots e_{l-1}' a_l.$$

Again we may relabel and collect terms and assume $H = F^n = a_1^1 \tau e_1' \cdots e_{k-1}' \tau a_l^2$. Since a_l^2 is linear, we can use an argument like that in the first case to relabel and replace a_1^1 by the identity. Since a_l^2 is elementary, we can write $a_l^2(z,w) = (az+bw+c, dw+e)$ with $a \neq 0$. Applying $\tau e_1' \cdots e_{k-1}'\tau$ to this, we see that the homogeneous polynomial of highest degree in F^n depends on z. But a simple inductive argument shows that the corresponding polynomial for H is independent of z. Hence F cannot have this form.

Thus $F \in \mathcal{G}$. ■

Remark 4.2. *In general, a map H can have distinct roots of a given order. For example, the map H given by squaring $F(z,w) = (w, z+w^2)$ has three square roots. This is true because $(F \circ s)^2 = H$ for $s(z,w) = (\omega^2 z, \omega w)$, where $\omega^3 = 1$. In fact, one can check that these are the only possible square roots of H.*

5. Roots of elementary maps

In [BF], we showed that no Hénon map can be the time-1 map of the flow of a holomorphic vector field and gave a precise classification of those maps which can be the time-1 map of such a flow. In [AF] and [AFV], it was shown that any flow of a holomorphic vector field whose time-1 map is an elementary map is in fact conjugate to a flow which is polynomial for all time.

In this section, we consider the set of elementary maps which are not the time-1 map of any holomorphic flow and show that such maps have roots of arbitrarily high order but that any root is conjugate to a polynomial map.

The elementary maps which cannot be the time-1 map of a flow have the form

$$F(z,w) = (\beta^\mu(z + w^\mu q(w^r)), \beta w),$$

where β is a primitive rth root of unity, $q(w) = w^k + q_{k-1}w^k + \cdots q_1 w + q_0$, and $k \geq 1$. A simple check shows that if we replace $w^\mu q(w^r)$ by

$(w^\mu q(w^r))/(lr+1)$ for any $l \in \mathbb{Z}^+$, then the resulting map is an $(lr+1)$st root of F.

In general, maps of this form can have nonpolynomial roots. For instance, let $F(z,w) = (-(z + w(w^4+1)), -w)$ and let k be any entire function of one variable. Define $\phi(z,w) = (i(z + w(w^4+1)/2 + w^3 k(w^4)), iw)$. A simple check shows that $\phi^2 = F$, and ϕ is nonpolynomial whenever k is transcendental.

We claim that any root of F is conjugate to a polynomial automorphism. Suppose that ϕ is an automorphism of $\mathbb{C}^2$ with $\phi^n = F$. Then $\phi F^r \phi^{-1} = F^r$, so an argument like that in [FM, theorem 6.10] shows that $\phi(z,w) = (e^{g(w)}z + h(w), aw + b)$ for some entire g, h, and some $a, b \in \mathbb{C}$, $a \neq 0$. Since $\phi^n = F$, we see that $a^n = \beta$ and $b(a^n - 1)/(a-1) = 0$, so that $b = 0$.

Using this form for ϕ and the fact that $\phi F^r = F^r \phi$, it follows that $e^{g(w)}$ is a nonzero rational function, hence is a constant, $c \neq 0$. Moreover, since $\phi F = F\phi$, we see that $c = a^\mu$. Thus $\phi(z,w) = (a^\mu z + h(w), aw)$.

Now, since $\phi^{rn} = F^r$, it follows that $\sum_{j=0}^{rn-1} (a^\mu)^{-j} h(a^j w) = r w^\mu q(w^r)$. Write $h = h_1 + h_2$, where $h_2 = w^{\mu + kr}\tilde{h}_2$ for some entire $\tilde{h}_2$. Then the sum just given is valid with h_2 in place of h and 0 in place of $rw^\mu q(w^r)$. Hence by [AF], there exists f entire such that $f(aw) - a^\mu f(w) = h_2(w)$.

A simple check shows that if $\psi(z,w) = (z + f(w), w)$, then $\psi^{-1}\phi\psi$ is a polynomial, and in fact, $\psi^{-1}\phi^n\psi = F$. Thus any root of F is conjugate to a polynomial automorphism.

Given any elementary map F and a root $\phi^n = F$, one can ask if ϕ is conjugate to a polynomial automorphism. There are a few cases such as the above where this result is relatively straightforward, but in general, this seems to be a hard question. For some results along these lines in the case $F = I$, see [AR].

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