LOCAL CONNECTIVITY OF THE MANDELBROT SET AT CERTAIN INFINITELY RENORMALIZABLE POINTS

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Abstract. We construct a subset of the Mandelbrot set which is dense on the boundary of the Mandelbrot set and which consists of only infinitely renormalizable points such that the Mandelbrot set is locally connected at every point of this subset. We prove the local connectivity by finding bases of connected neighborhoods directly.

Contents

1. Introduction

Consider a quadratic polynomial $P_c(z) = z^2 + c$, where c is a complex parameter. The filled-in Julia set K_c of P_c is the set of points z such that the orbit $O(z) = \{P_c^{\circ n}(z)\}_{n=0}^{\infty}$ is bounded. The Julia set J_c of P_c is the boundary of K_c. The Mandelbrot set $\mathcal{M}$ is the set of parameters c such that the critical orbit $O(c) = \{P_c^{\circ n}(0)\}_{n=0}^{\infty}$ is bounded (see Fig. 1). Equivalently, $\mathcal{M}$ is the set of parameters c such that J_c is connected. For a parameter c in the complement of the Mandelbrot set $\mathcal{M}$, the Julia set J_c is a Cantor set. Douady and Hubbard (see [2]) proved that the Mandelbrot set $\mathcal{M}$ is connected.

The Julia set of $P_0(z) = z^2$ is the round circle S^1. The map P_0 restricted to S^1 is an expanding self-map. From the Structure Stability Theorem in smooth dynamical systems (see [10, 12]), the dynamical system (P_c, J_c) for any $|c|$ small enough is topologically conjugate to (P_0, S^1). Thus J_c for any $|c|$ small enough is a Jordan curve. It is also known that J_c for any $|c|$ small enough is a quasi-circle (see [13]), and that the Hausdorff dimension $HD(J_c)$ of J_c is $1 + |c|/(4\log 2) + o(|c|)$ (see [11]).

1991 Mathematics Subject Classification, Primary : 58F23; Secondary: 30C10
Partially supported by NSF grant and PSC-CUNY award

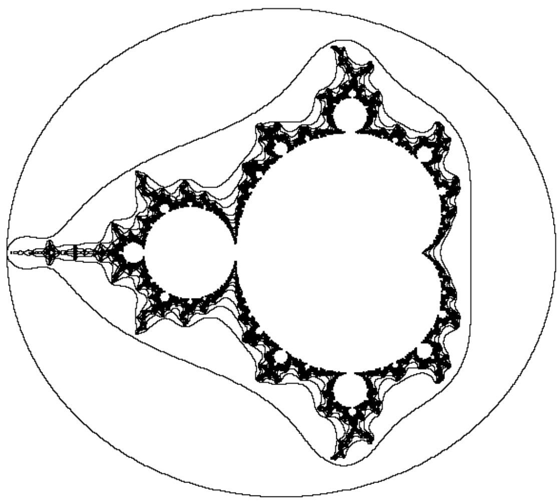

FIGURE 1. Computer generating picture of the boundary of the Mandelbrot set and some equipotential curves

The techniques in the study of the dynamics of P_c for any $|c|$ small enough can be used to the study of the dynamics of a hyperbolic quadratic polynomial which is a quadratic polynomial having an attractive or super-attractive periodic point in the complex plane. Let $\mathcal{H}P$ be the set of parameters c such that P_c is hyperbolic. The main conjecture in this direction is that

Conjecture 1. *The set $\mathcal{H}P$ is open and dense in $\mathcal{M}$.*

It is easy to verify that $\mathcal{H}P$ is open but to prove that $\mathcal{H}P$ is dense in $\mathcal{M}$ is a hard problem.

Douady and Hubbard [3] proved that this conjecture follows from

Conjecture 2. *The Mandelbrot set $\mathcal{M}$ is locally connected.*

In order to study the dynamics of quadratic polynomials more penetrating, Douady and Hubbard [4] introduced the concept of a quadratic-like map which is a proper, degree two, analytic branch cover $f : U \to V$ where U and V are open domains isomorphic to a disc and $\overline{U} \subset V$. A quadratic polynomial P_c is a quadratic-like map when restricted to any domain bounded by an equipotential curve (see [8]).

For a quadratic-like map $f : U \to V$, let 0 be the unique branch point of f. The filled-in Julia set K_f of f is defined as

$$K_f = \cap_{n=0}^{\infty} f^{-n}(U).$$

Suppose K_f is connected (otherwise it is a Cantor set). The map f is said to be once n_1-renormalizable if there are subdomains $0 \in U_1 \subset U$ and $V_1 \subset V$ isomorphic to a disc such that

$$f_1 = f^{\circ n_1} : U_1 \to V_1$$

is a quadratic-like map with connected filled-in Julia set K_{f_1}. Otherwise, f is called non-renormalizable. If $f_1 : U_1 \to V_1$ is also n_2-renormalizable, then we call f is twice (n_1, n_2)-renormalizable. So on one can define a *k-times* $(n_1, n_2, \cdots, n_k)$-renormalizable quadratic-like map as well as an infinitely renormalizable quadratic-like map.

Suppose $f : U \to V$ is a quadratic-like map. A point p is called a periodic point of period $n \geq 1$ of f if $f^{\circ n}(p) = p$ and $f^{\circ i}(p) \neq p$ for $1 \leq i < n$. The number $\lambda_p = (f^{\circ n})'(p)$ is called the multiplier of f at p. A periodic point p of period n of f is called parabolic if $\lambda_p = \exp(2\pi i p/q)$ where p and q are integers; irrationally indifferent if $\lambda_p = \exp(2\pi\theta i)$ where θ is irrational; repelling if $|\lambda_p| > 1$; attractive if $0 < |\lambda_p| < 1$; super-attractive if $\lambda_p = 0$. For a quadratic polynomial $P_c(z) = z^2 + c$, let $PCO(c) = \{P_c^{\circ n}(0)\}_{n=1}^{\infty}$ be its post-critical orbit. We classify the parameters c in $\mathcal{M} \setminus \mathcal{H}P$ as

$$\mathcal{P} = \{c \in \mathcal{M} \mid P_c \text{ has a parabolic periodic point }\},$$

$$\mathcal{I} = \{c \in \mathcal{M} \mid P_c \text{ has an irrationally indifferent periodic point }\},$$

$$\mathcal{N} = \{c \in \mathcal{M} \mid c \text{ is not in } \mathcal{P} \cup \mathcal{I} \text{ and } 0 \notin \overline{PCO(c)} \}.$$

$$\mathcal{F}R = \{c \in \mathcal{M} \mid c \text{ is not in } \mathcal{P} \cup \mathcal{I} \cup \mathcal{N} \text{ and } P_c \text{ is not infinitely renormalizable }\},$$

and

$$\mathcal{I}R = \{c \in \mathcal{M} \mid P_c \text{ is infinitely renormalizable }\}.$$

In particular, a parameter c in $\mathcal{N}$ is called Misiurewicz if $PCO(c)$ consists of only finitely many points. It is known that the set of all Misiurewicz points is dense on the boundary of the Mandelbrot set (see [1]). A point c in $\mathcal{I}R$ is called an infinitely renormalizable.

Recent work of Yoccoz (see [6]) shows that the Mandelbrot set $\mathcal{M}$ is locally connected at all points which are not infinitely renormalizable. (We would like to note that the local connectivity of the Mandelbrot set $\mathcal{M}$ at all points in $\mathcal{P}$ is assured by many experts [6, 15] but a proof is still needed). There remains the many infinitely renormalizable points for which to complete the proof of Conjecture 2. However, we prove the following theorem to further confirm Conjecture 2.

Main Theorem. *There is a subset Υ of the Mandelbrot set $\mathcal{M}$ which is dense on the boundary $\partial\mathcal{M}$ of the Mandelbrot set $\mathcal{M}$ and which consists of only infinitely renormalizable points such that the Mandelbrot set $\mathcal{M}$ is locally connected at every point c of Υ.*

The proof of Main Theorem is in the next section. The talks with Professors Tan Lei, C. Petersen, and A. Douady are very helpful. The article of J. Hubbard [6] and

the book of Carleson and Gamelin [1] are helpful during my understanding of the structure of the Mandelbrot set $\mathcal{M}$ and the work of Yoccoz. I would like to thank them very much. I also would like to thank Professor M. Shishikura for pointing out the work of Eckmann and Epstein [5]. Research is partially supported by NSF grant # DMS-9400974 and PSC-CUNY award # 6-65348. Research at MSRI is supported in part by NSF grant # DMS-9022140.

2. Construction of bases of connected neighborhoods

We give the proof of Main Theorem now. Let us start the construction of the subset Υ in Main Theorem around the simplest point $-2 \in \mathcal{M}$. Consider the polynomial $P(z) = z^2 - 2$ and its Julia set $J = [-2, 2]$. It has a non-separate fixed point $\beta = 2$ and a separate fixed point α, this menas that $J \setminus \{\beta\}$ is still connected and $J \setminus \{\alpha\}$ is disconnected. Let γ be a fixed equipotential curve of P. Two external rays land at α. Let Γ be the union of these two external rays. Let D be the closed domain bounded by γ. Let D_0 be the domain containing 0 and bounded by γ and $P^{-1}(\Gamma)$ (see Fig. 2). The restriction $P|(C \setminus (-\infty, -2])$ has two inverse branches

$$G_0 : C \setminus (-\infty, -2] \to LH = \{z = x + yi \in C \mid x < 0\}$$

and

$$G_1 : C \setminus (-\infty, -2] \to RH = \{z = x + yi \in C \mid x > 0\}.$$

Let $D_n = G_1^{\circ n}(D_0)$ and let $B_n = G_0(D_{n-1})$ for $n \geq 1$ (see Fig. 2). Since 2 is an expanding fixed point of P and $P(-2) = 2$, the diameter $\mathrm{diam}(B_n)$ tends to zero exponentially as n goes to infinity.

Fig. 2 can be constructed as well for c in a small neighborhood about -2 because β, α, and Γ are structurally stable. Let $0 \notin U_0$ be a small neighborhood about -2 with $\mathrm{diam}(U_0) \leq 1$ such that Fig. 2 is preserved for c in U_0. Let $\beta(c)$, $\alpha(c)$, $\Gamma(c)$, $D_n(c)$ for $0 \leq n < \infty$, and $B_n(c)$ for $1 \leq n < \infty$ be the corresponding points, sets, and domains for P_c, $c \in U_0$. Let $\overline{\beta}(c) \neq \beta(c)$ be another pre-image of $\beta(c)$ under P_c. Then the diameter $\mathrm{diam}(B_n)$ tends to 0 and the set $B_n(c)$ approaches to $\overline{\beta}(c)$ as n goes to infinity uniformly on U_0.

Let

$$W_n = \{c \mid c \in B_n(c)\}.$$

Because W_n is bounded by external rays and an equipotential curve of the Mandelbrot set $\mathcal{M}$,

$$\tilde{\mathcal{M}}_n = W_n \cap \mathcal{M}$$

is connected. Since the equation $P_c(0) - \overline{\beta}(c) = 0$ has a unique solution -2 in U_0, the Rouché Theorem assures that $P_c(0) - x = 0$ has a unique solution for any x in $B_n(c)$

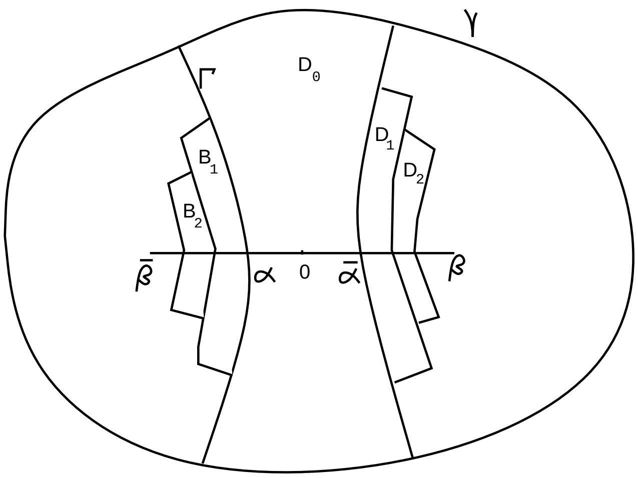
γ
Γ
D_0
D_1
D_2
B_1
B_2
$\bar{\beta}$
α
0
$\bar{\alpha}$
β

Fig. 2

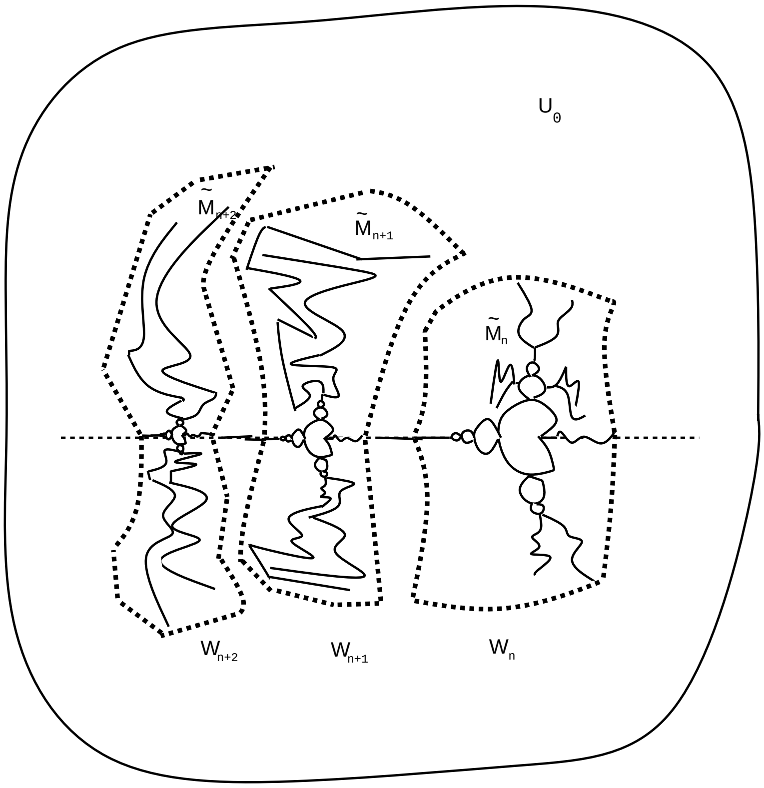
U_0
$\tilde{M}_{n+2}$
$\tilde{M}_{n+1}$
$\tilde{M}_n$
W_{n+2}
W_{n+1}
W_n

Fig. 3

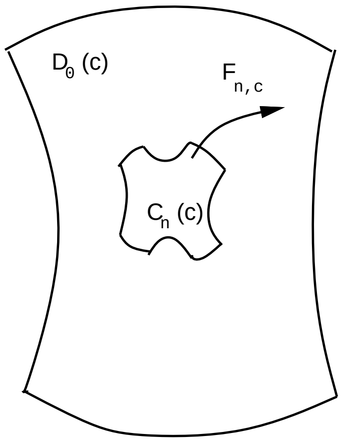

Fig. 4

and large n and the solution is close to -2. Therefore, there is an integer $N_0 > 0$ such that for $n \geq N_0$,

$$W_n \subset U_0.$$

That is,

$$\operatorname{diam}(W_n) \leq 1$$

for $n \geq N_0$ (see Fig. 3).

For each W_n where $n \geq N_0$, since $c \in B_n(c)$, $C_n(c) = P_c^{-1}(B_n(c))$ is a simply connected domain containing 0. It is also a subdomain of $D_0(c)$. Let

$$F_{n,c} = P_c^{\circ(n+1)} : \mathring{C}_n(c) \to \mathring{D}_0(c)$$

is a quadratic-like map (see Fig. 4). The family

$$\{F_{n,c} : \mathring{C}_n(c) \to \mathring{D}_0(c) \mid c \in W_n\}$$

is full (see [4]). So W_n contains a copy $\mathcal{M}_n$ of the Mandelbrot set $\mathcal{M}$ (see [4]). For $c \in \mathcal{M}_n$, the Julia set $J_{F_{n,c}}$ of $F_{n,c} : \mathring{C}_n(c) \to \mathring{D}_0(c)$ is connected. Therefore, for

$$c \in \Upsilon_1 = \cup_{n \geq N_0}^{\infty} \mathcal{M}_n,$$

P_c is once renormalizable.

For a fixed integer $i_0 \geq N_0$, consider W_{i_0} and $\mathcal{M}_{i_0}$; there is a parameter $c_{i_0} \in \mathcal{M}_{i_0}$ such that

$$F_{i_0} = F_{i_0, c_{i_0}} : C_{i_0} = C_{i_0}(c_{i_0}) \to D_{i_0} = D_0(c_{i_0})$$

is hypbrid equivalent (see [4]) to $P(z) = z^2 - 2$. The quadratic-like map $F_{i_0} : C_{i_0} \to D_{i_0}$ has the non-separate fixed point β_{i_0} and the separate fixed point α_{i_0}. Let $\overline{\beta}_{i_0}$ be another pre-image of β_{i_0} under F_{i_0}. Let Γ_{i_0} be the external rays of $P_{c_{i_0}}$ landing at α_{i_0}. Let $D_{i_0 0}$ be the domain containing 0 and bounded by ∂C_{i_0} and $F_{i_0}^{-1}(\Gamma_{i_0})$. Let

$\overline{\beta}_{i_0} \in E_{i_00}$ and $\beta_{i_0} \in E_{i_01}$ be the components of the closure of $C_{i_0} \setminus D_{i_00}$. Let G_{i_00} and G_{i_01} be the inverses of $F_{i_0}|E_{i_00}$ and $F_{i_0}|E_{i_01}$. Let

$$D_{i_0n} = G_{i_01}^{\circ n}(D_{i_00})$$

and let

$$B_{i_0n} = G_{i_00}(D_{i_0(n-1)})$$

for $n \geq 1$. Since β_{i_0}, α_{i_0}, and Γ_{i_0} are structurally stable, we can find a small neighborhood U_{i_0} about c_{i_0} with $\mathrm{diam}(U_{i_0}) \leq 1/2$ such that the corresponding domains $B_{i_0}(c)$ can be constructed for P_c, $c \in U_{i_0}$. Let

$$W_{i_0n} = \{c \in C \mid F_{i_0,c}(0) \in B_{i_0n}(c)\}.$$

The diameter $\mathrm{diam}(B_{i_0n}(c))$ tends to zero and the set $B_{i_0n}(c)$ approaches to $\overline{\beta_{i_0}}(c)$ uniformly on U_{i_0} as n goes to infinity. Since the equation $F_{i_0,c}(0) - \overline{\beta}_{i_0}(c) = 0$ has a unique solution c_{i_0}, the Rouché Theorem implies that $F_{i_0,c}(0) - x = 0$ has a unique solution near c_0 for any $x \in B_{i_0n}(c)$ and any n large. Thus, there is an integer $N_{i_0} \geq 0$ such that for $n \geq N_{i_0}$, $W_{i_0n} \subseteq U_{i_0}$. Thus $\mathrm{diam}(W_{i_0n}) \leq 1/2$. Since W_{i_0n} for $n \geq N_{i_0}$ is bounded by external rays and an equipotential curve of $\mathcal{M}$, $\tilde{\mathcal{M}}_{i_0n} = W_{i_0} \cap \mathcal{M}$ is connected. For each c in W_{i_0n}, $n \geq N_{i_0}$, let $C_{i_0n}(c) = F_{i_0,c}^{-1}(B_{i_0n}(c))$. Then

$$F_{i_0n,c} = F_{i_0,c}^{\circ(n+1)} : \mathring{C}_{i_0n}(c) \to \mathring{D}_{i_00}(c)$$

is a quadratic-like map. Moreover,

$$\{F_{i_0n,c} \mid c \in W_{i_0n}\}$$

is a full family. Thus W_{i_0n} contains a copy $\mathcal{M}_{i_0n}$ of the Mandelbrots set $\mathcal{M}$. For

$$c \in \Upsilon_2 = \cup_{i_0 \geq N_0} \cup_{i_1 \geq N_{i_0}} \mathcal{M}_{i_0i_1},$$

P_c is twice renormalizable.

We use the induction to complete the construction of the subset around -2. Suppose we have constructed W_w where $w = i_0i_1 \ldots i_{k-1}$ and $i_0 \geq N_0$, $i_1 \geq N_{i_1}$, $\ldots$, $i_{k-1} \geq N_{i_0i_1 \ldots i_{k-2}}$. Let $v = i_0 \ldots i_{k-2}$. There is a parameter $c_w \in \mathcal{M}_w$ such that

$$F_w = F_{w,c_w} : C_w = C_w(c_w) \to D_w = D_{v0}(c_w)$$

is hypbrid equivalent to $P(z) = z^2 - 2$. The quadratic-like map $F_w : C_w \to D_w$ has the non-separate fixed point β_w and the separate fixed point α_w. Let $\overline{\beta}_w$ be another pre-image of β_w under F_w. Let Γ_w be the external rays of P_{c_w} landing at α_w. Let D_{w0} be the domain containing 0 and bounded by ∂C_w and $F_w^{-1}(\Gamma_w)$. Let $\overline{\beta}_w \in E_{w0}$ and $\beta_w \in E_{w1}$ be the components of the closure of $C_w \setminus D_{w0}$. Let G_{w0} and G_{w1} be the inverses of $F_w|E_{w0}$ and $F_w|E_{w1}$. Let

$$D_{wn} = G_{w1}^{\circ n}(D_{w0})$$

and let

$$B_{wn} = G_{w0}(D_{w(n-1)})$$

for $n \geq 1$. Since β_w, α_w, and Γ_w are structurally stable, we can find a small neighborhood U_w about c_w with $\mathrm{diam}(U_w) \leq 1/2^k$ such that the corresponding domains $D_{wn}(c)$ and $B_{wn}(c)$ can be constructed for P_c, $c \in U_w$. Let

$$W_{wn} = \{c \in C \mid F_{w,c}(0) \in B_{wn}(c)\}.$$

The diameter $\mathrm{diam}(B_{wn}(c))$ tends to zero and the set $B_{wn}(c)$ approaches to $\overline{\beta_w}(c)$ uniformly on U_w as n goes to infinity. Since the equation $F_{w,c}(0) - \overline{\beta}_w(c) = 0$ has a unique solution c_w, the Rouché Theorem implies that $F_{w,c}(0) - x = 0$ for any x in $B_{wn}(c)$ and for n large has a unique solution which is close to c_w. Therefore, there is an integer $N_w \geq 0$ such that for $n \geq N_w$, $W_{wn} \subseteq U_w$. Thus $\mathrm{diam}(W_{wn}) \leq 1/2^k$. Because W_{wn} for $n \geq N_{i_1}$ is bounded by external rays and an equipotential curve of $\mathcal{M}$,

$$\tilde{\mathcal{M}}_{wn} = W_{wn} \cap \mathcal{M}$$

is connected.

For each c in W_{wn}, $n \geq N_w$, let $C_{wn}(c) = F_{w,c}^{-1}(B_{wn}(c))$. Then

$$F_{wn,c} = F_{w,c}^{\circ(n+1)} : \mathring{C}_{wn}(c) \to \mathring{D}_{w0}(c)$$

is a quadratic-like map. Moreover,

$$\{F_{wn,c} : \mathring{C}_{wn}(c) \to \mathring{D}_{w0}(c) \mid c \in W_{wn}\}$$

is a full family. Thus W_{wn} contains a copy $\mathcal{M}_{wn} \subset \tilde{\mathcal{M}}_{wn}$ of the Mandelbrot set $\mathcal{M}$. For

$$c \in \Upsilon_{k+1} = \cup_w \cup_{i_k \geq N_w} \mathcal{M}_{wi_k},$$

P_c is $(k+1)$-*times* renormalizable where $w = i_0 i_1 \ldots i_{k-1}$ runs over all sequences of integers of length k in the induction.

We have thus constructed a subset $\Upsilon(-2) = \cap_{k=1}^{\infty} \Upsilon_k$ such that for each $c \in \Upsilon(-2)$, P_c is infinitely renormalizable and such that -2 is a limit point of $\Upsilon(-2)$. For each $c \in \Upsilon(-2)$, there is a corresponding sequence $w_\infty = i_0 i_1 \ldots i_k \ldots$ of integers such that

$$\{c\} = \cap_{k=0}^{\infty} W_{i_0 \ldots i_k}.$$

Since

$$\tilde{\mathcal{M}}_{i_0 \ldots i_k} = W_{i_0 \ldots i_k} \cap \mathcal{M}$$

is connected,

$$\{W_{i_0 \ldots i_k}\}_{k=0}^{\infty}$$

is a basis of connected neighborhoods of the Mandelbrot $\mathcal{M}$ at c. In other words, the Mandelbrot set $\mathcal{M}$ is locally connected at c.

We prove further, after understood the above construction, that for any Misiurewicz point, there is a subset $\Upsilon(c_0)$ such that every c in $\Upsilon(c_0)$ is infinitely renormalizable, such that the Mandelbrot set $\mathcal{M}$ is locally connected at every point c in $\Upsilon(c_0)$, and such that c_0 is a limit point of $\Upsilon(c_0)$ as follows.

Suppose c_0 is a Misiurewicz point in $\mathcal{M}$. Then there is an integer $m > 1$ such that $p = P_{c_0}^{\circ m}(0)$ is a repelling periodic point of period $k \geq 1$. Let α be the separate fixed point of P_{c_0}. Without loss of generality, we assume that P_{c_0} is non-renormalizable. (If P_{c_0} is renormalizable, it must be finitely renormalizable. We would then take α as the separate fixed point of the last renormalization of P_{c_0} (see [7])). Let Γ be the union of a cycle of external rays landing at α. Let γ be a fixed equipotential curve of P_{c_0}. Using Γ and γ, we can construct the two-dimensional Yoccoz puzzle as follows (see [7] for the notation). Let C_{-1} be the domain bounded by γ. Then Γ cuts C_{-1} into a finite number of closed domains. Let η_0 denote the set of these domains. Let $\eta_n = P_{c_0}^{-n}(\eta_0)$. Let C_n be the member of η_n containing 0 for $n \geq 0$.

Let
$$p \in \cdots \subseteq D_n(p) \subseteq D_{n-1}(p) \subseteq \cdots \subseteq D_1(p) \subseteq D_0(p)$$
be a p-end, where $D_n(p) \in \eta_n$. Let
$$c_0 \in \cdots \subseteq E_n(c_0) \subseteq E_{n-1}(c_0) \subseteq \cdots \subseteq E_1(c_0) \subseteq E_0(c_0)$$
be a c_0-end, where $E_n(c_0) \in \eta_n$. We have $P_{c_0}^{\circ(m-1)}(E_{n+m-1}(c_0)) = D_n(p)$. Since the diameter $\operatorname{diam}(D_n(p))$ tends to zero as $n \to \infty$ and since p is a repelling periodic point, we can find an integer $l \geq m$ such that $|(P_{c_0}^{\circ k})'(x)| \geq \lambda > 1$ for all $x \in D_l(p)$ and such that
$$P_{c_0}^{\circ(m-1)} : E_{l+m-1}(c_0) \to D_l(p)$$
is a homeomorphism. Let $q \geq 0$ be the integer such that
$$f = P_{c_0}^{\circ q} : D_l(p) \to C_{r_0}$$
is a homeomorphism, where C_{r_0} is the domain containing 0 in η_{r_0}, $r_0 \geq 0$. There is an integer $r > r_0$ such that $r + q > l$ and such that $B_0 = f^{-q}(C_r)$ does not contain p. Thus
$$P_{c_0}^{\circ q} : B_0 \to C_r$$
is a homeomorphism. Define
$$B_n = \left(P_{c_0}^{\circ nk} | D_{l+nk}(p)\right)^{-1}(B_0) \subseteq D_{l+nk}(p)$$
for $n \geq 1$. Note that B_n is in η_{r+q+nk}. Then
$$P_{c_0}^{\circ(q+nk)} : B_n \to C_r$$
is a homeomorphism.

Since α, p, Γ, C_r, D_n, and B_n, for $n \geq 0$, are structurally stable, they can be constructed for P_c as long as c near c_0. Let U_0 be a neighborhood about c_0 with

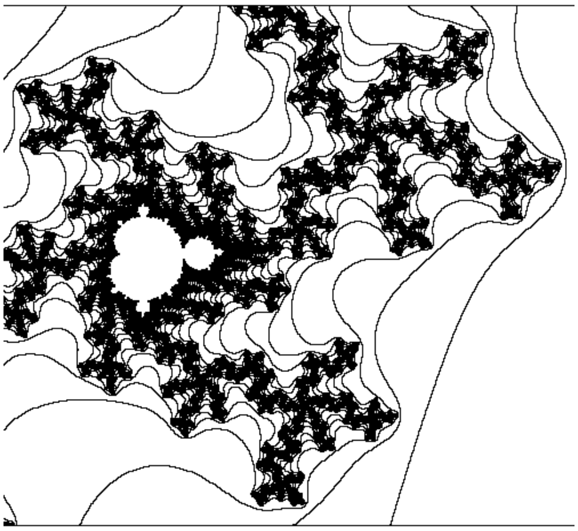

FIGURE 5. A small copy of the Mandelbrot set and hairs around it

$\mathrm{diam}(U_0) \leq 1$ such that the corresponding points $\alpha(c)$ and $p(c)$ and the corresponding sets $\Gamma(c)$, $C_r(c)$, $D_n(c)$, and $B_n(c)$, for $n \geq 0$, are all preserved for $c \in U_0$. Moreover, as n goes to infinity, the diameter $\mathrm{diam}(B_n(c))$ tends to zero and the set $B_n(c)$ approaches to $p(c)$ uniformly on U_0. Let

$$W_n = W_n(c_0) = \{c \in C \mid P_c^m(0) \in B_n(c)\}.$$

Since the equation $P_c^{\circ m}(0) - p(c) = 0$ has a unique solution c_0 in U_0 (see [3]), the Rouché Theorem implies that the equation $P_c^{\circ m}(0) - x = 0$ has a unique solution for any x in $B_n(c)$ and large n and that the solution is close to c_0. Therefore, there is an integer $N_0 = N_0(c_0) > 0$ such that $W_n \subset U_0$. Thus $\mathrm{diam}(W_n) \leq 1$ for all $n \geq N_0$. Since W_n is bounded by external rays and an equipotential curve of the Mandelbrot set $\mathcal{M}$, then

$$\tilde{\mathcal{M}}_n = \mathcal{M} \cap W_n$$

is connected (see Fig. 5).

For any $c \in W_n$, $n \geq N_0$, let $R_n(c)$ be the pre-image of $B_n(c)$ under the map

$$P_{c_0}^{\circ(m-1)} : E_{l+m-1}(c) \to D_l(p, c)$$

and let $C_{m+r+q+nk}(c) = P_c^{-1}(R_n(c))$. Then $C_{m+r+q+nk}(c)$ is the domain containing 0 in $\eta_{m+r+q+nk}$ and

$$F_{n,c} = P_c^{\circ(q+nk+m)} : \mathring{C}_{m+r+q+nk}(c) \to \mathring{C}_r(c)$$

is a quadratic-like map. Moreover,

$$\{F_{n,c} : \mathring{C}_{m+r+q+nk}(c) \to \mathring{C}_r(c) \mid c \in W_n\}$$

is a full family. Thus W_n contains a copy $\mathcal{M}_n = \mathcal{M}_n(c_0)$ of the Mandelbrot set $\mathcal{M}$ where $\mathcal{M}_n \subset \tilde{\mathcal{M}}_n$. For any $c \in \cup_{n \geq N_0} \mathcal{M}_n$, P_c is once renormalizable. Now repeat the induction we did for -2. We construct a subset $\Upsilon(c_0)$ such that every $c \in \Upsilon(c_0)$ is infinitely renormalizable, such that $\mathcal{M}$ is locally connected at every $c \in \Upsilon(c_0)$, and such that c_0 is a limit point of $\Upsilon(c_0)$.

Let $\Upsilon = \cup_{c_0} \Upsilon(c_0)$ where c_0 runs over all Misiurewicz points in $\mathcal{M}$. Then every $c \in \Upsilon$ is infinitely renormalizable and $\mathcal{M}$ is locally connected at every $c \in \Upsilon$. Since the set of Misiurewicz points is dense in $\partial\mathcal{M}$, the set Υ is dense in $\partial\mathcal{M}$. This completes the proof of the theorem.

Remark 1. Eckmann and Epstein [5] and Douady and Hubbard [4] estimated the size of $\mathcal{M}_n$ in the construction. Since $\tilde{\mathcal{M}}_n \setminus \mathcal{M}_n$ contains hairs (see Fig. 5) and may destroy the local connectivity of $\mathcal{M}$, we must estimate the size of $\tilde{\mathcal{M}}_n$.

Remark 2. For c in Υ, one can prove that the Julia set J_c is locally connected at the critical point 0 as well as at the grand critical orbit of P_c. But since we did not show that P_c is unbranched we could not conclude that J_c is locally connected from the result in [7]. However, we constructed a similar subset $\tilde{\Upsilon}$ of the Mandelbrot set $\mathcal{M}$ in [7] which is also dense on the boundary of the Mandelbrot set $\mathcal{M}$ such that P_c is unbranched infinitely renormalizable and has the complex a priori bounds for every point c of this subset. Thus the Julia set J_c of P_c for c in $\tilde{\Upsilon}$ is locally connected according to the result in [7].

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