ON COMBINATORIAL DESCRIPTIONS OF HOMOTOPY GROUPS OF $\Sigma K(\pi, 1)$

JIE WU

ABSTRACT. We give a combinatorial description of homotopy groups of $\Sigma K(\pi, 1)$. In particular, all of the homotopy groups of the 3-sphere are combinatorially given.

1. INTRODUCTION

In this article, we study homotopy groups and some related problems by using simplicial homotopy theory. The point of view here is that combinatorial aspects of group theory provide further information about homotopy groups.

The homotopy groups of the 3-sphere, the suspension of $K(\pi, 1)$ and wedges of 2-spheres are shown to be the centers of certain groups with specific generators and specific relations. We list two group theoretical descriptions of $\pi_*(S^3)$ as follows.

Definition 1.1. ***A bracket arrangement of weight*** *n in a group is a set of elements defined recursively as follows:*

Let G be a group and let $a_1, a_2, \cdots, a_n$ be a finite sequence of elements of G. Let

$$\beta^1(a_1) = a_1$$

and if $n > 1$, then let

$$\beta^n(a_1, \cdots, a_n) = [\beta^k(a_1, \cdots, a_k), \beta^l(a_{k+1}, \cdots, a_n)]$$

for some k and $l > 0$.

Theorem 1.2. *For $n \geq 1$, $\pi_{n+2}(S^3)$ is isomorphic to the center of the group with generators $y_0, \ldots, y_n$ and relations*

$$[y_{i_1}^{\epsilon_1}, y_{i_2}^{\epsilon_2}, \ldots, y_{i_t}^{\epsilon_t}]$$

with $\{i_1, \ldots, i_t\} = \{-1, 0, \ldots, n\}$ as sets in which the indices i_j can be repeated, where $\epsilon_j = \pm 1, y_{-1} = (y_0 \ldots y_n)^{-1}$ and the commutator bracket $[\ldots]$ runs over all bracket arrangements of weight t for each t.

Research at MSRI is supported in part by NSF grant DMS-9022140

Notations 1.3. *Let G be a group and let S be a subset of G. Let $< S >$ denote the normal subgroup generated by S. Let H_j be a sequence of subgroups of G for $1 \leq j \leq k$. Let $[[H_1, \ldots, H_k]]$ denote the subgroup of G generated by all of the commutators $[h^{(1)}_{i_1}, \ldots, h^{(t)}_{i_t}]$ with $\{i_1, \ldots, i_t\} = \{1, \ldots, k\}$ as sets in which the indices i_j can be repeated, where $h^{(s)}_j \in H_j$ and the commutator bracket $[\ldots]$ runs over all of the bracket arrangements of weight t for each t.*

Theorem 1.4. *Let $n \geq 1$. In the free group $F(y_0, \ldots, y_n)$ freely generated by $y_0, \ldots, y_n$ we have*

$$([[< y_0 >, < y_1 >, \ldots, < y_n >]] \cap < y_{-1} >)/[[< y_{-1} >, < y_0 >, \ldots, < y_n >]] \cong \pi_{n+2}(S^3)$$

where $y_{-1} = (y_0 \ldots y_n)^{-1}$.

The method of the proofs of these theorems is to study the Moore chain complex of Milnor's construction $F(S^1)$ for the 1-sphere S^1. A group theoretical description of the homotopy groups $\pi_*(\Sigma K(\pi, 1))$ is as follows.

Theorem 1.5. *Let π be any group and let $\{x^{(\alpha)} | \alpha \in J\}$ be a set of generators for π. Then, for $n \neq 1$, $\pi_{n+2}(\Sigma K(\pi, 1))$ is isomorphic to the center of the quotient group of the free product*

$$\coprod^{groups}_{0 \leq j \leq n} (\pi)_j$$

modulo the relations

$$[y^{(\alpha_1)\epsilon_1}_{i_1}, y^{(\alpha_2)\epsilon_2}_{i_2}, \cdots, y^{(\alpha_t)\epsilon_t}_{i_t}]$$

with $\{i_1, i_2, \cdots, i_t\} = \{-1, 0, 1, \cdots, n\}$ as sets, where $(\pi)_j$ is a copy of π with generators $\{x^{(\alpha)}_j | \alpha \in J\}$, $\epsilon_j = \pm 1$, $y^{(\alpha)}_{-1} = x^{(\alpha)}_0{}^{-1}$, $y_j = x^{(\alpha)}_j x^{(\alpha)}_{j+1}{}^{-1}$ for $1 \leq j \leq n-1$, $y^{(\alpha)}_n = x^{(\alpha)}_m$ and the commutator bracket $[\cdots]$ runs over all of the commutator bracket arrangment of weight t for each t.

In particular, for $\pi = Z/m$, we have

Corollary 1.6. *$\pi_{n+2}(\Sigma K(Z/m, 1))$ is isomorphic to the center of the group with generators $x_0, \ldots, x_n$ and relations: (1). x^m_j for $0 \leq j \leq n$ and (2).*

$$[y^{\epsilon_1}_{i_1}, y^{\epsilon_2}_{i_2}, \ldots, y^{\epsilon_t}_{i_t}]$$

with $\{i_1, \ldots, i_t\} = \{-1, 0, \ldots, n\}$ as sets in which the indices i_j can be repeated, where $\epsilon_j = \pm 1$, $y_{-1} = x^{-1}_0$, $y_j = x_{j-1} x^{-1}_j$ for $-1 \leq j \leq n-1$, $y_n = x_n$ and the commutator bracket $[\ldots]$ runs over all bracket arrangements of weight t for each t.

One of the features of Theorems 1.2 and 1.5 is that homotopy groups embed in certain 'enveloping groups'. These 'enveloping groups' have systematic and uniform structure. The centers of these groups are of course more complicated to analyze. Theorem 1.4 is very similar to the combinatorial decription of J. H. C. Whitehead's conjecture [see, Bo2, pp.317]. These decriptions give a combinatorial question how to give a computable way to understand the quotient groups $R \cap S/[R, S]$ for certain subgroups R and S of a finite generated free group. Few informations about this question are known [see, Bo1]. The article is organized as follows.

In Section 2, we study some general properties of simplicial groups. Central extensions in the Moore-Postnikov systems will be considered. In Section 3, we study the intersection of certain subgroups in free groups. The proofs of Theorems 1.2 and 1.4 is given in Section 4, where Theorem 4.5 is Theorem 1.2 and Theorem 4.10 is Theorem 1.4. The proof of Theorem 1.5 is given in Section 5, where Theorem 5.9 is Theorem 1.5. In Section 6, we give some applications of our descriptions. One example is to compute the homotopy groups of the Cohen's construction on the 1-sphere. A direct corollary is to give a short proof of the Milnor's counter example for the minimal simplicial groups.

The author would like to thank F. Cohen, B. Gray, J. Harper, D. Kan, M. Mahowald, J. Moore, P. May, S. Priddy and many other mathematicians for their kindly encouragement and helpful suggestions. The author is indebted to helpful discussions with J. C. Moore and F. R. Cohen.

2. Central Extensions in simplicial group theory

In this section, we study some general properties of simplicial groups. A simplicial set K is called a Kan complex if it satisfies the *extension condition*,i.e, for any simplicial map $f : \Lambda^k[n] \to K$ has an extension $g : \Delta[n] \to K$, where $\Delta[n]$ is the standard n simplex, $\Lambda^k[n]$ is the subcomplex of $\Delta[n]$ generated by all $d_i(\sigma_n)$ for $i \neq k$ and σ_n is the nondegenerate n simplex in $\Delta[n]$,[K1, C2]. Recall that any simplicial group is a Kan complex [Mo1]. Given a simplicial group G, the Moore chain complex (NG, d_0) is defined by $NG_n = \bigcap_{j\neq 0} Kerd_j$ with $d_0 : NG_n \to NG_{n-1}$. The classical Moore Theorem is that $\pi_n(G) \cong H_n(NG)$ [Mo1, Theorem 4;C2, Theorem 3.7 or K2, Proposition 5.4]. Now let $\mathcal{Z}_n = \mathcal{Z}_n(G) = \bigcap_j Kerd_j$ denote the cycles and let $\mathcal{B}_n = \mathcal{B}_n(G) = d_0(NG_{n+1})$ denote the boundaries. It is easy to check that $\mathcal{B}_n$ is a normal subgroup of G_n for any simplicial group G.

Lemma 2.1. *Let G be a simplicial group. Then the homotopy group $\pi_n(G)$ is contained in the center of $G_n/\mathcal{B}G_n$ for $n \geq 1$.*

Proof. Notice that $\pi_n(G) \cong \mathcal{Z}G_n/\mathcal{B}G_n$. It suffices to show that the commutator $[x, y] \in \mathcal{B}G_n$ for any $x \in \mathcal{Z}G_n$ and $y \in G_n$.

Notice that x is a cycle. There is a simplicial map $f_x : S^n \to G$ such that $f_x(\sigma_n) = x$, where S^n is the standard n-sphere with a nondegenerate n-simplex σ_n. Now let the simplicial map $f_y : \Delta[n] \to G$ be the representative of y, i.e, $f_y(\tau_n) = y$ for the nondegenerate n-simplex τ_n. Let ϕ be the composite

$$S^n \xrightarrow{j} S^n \wedge \Delta[n] \xrightarrow{[f_x,f_y]} G,$$

where $j(\sigma_n) = \sigma_n \wedge \tau_n$ and $[f_x, f_y](a \wedge b) = [f_x(a), f_y(b)]$, the commutator of $f_x(a)$ and $f_y(b)$. Notice that $\phi(\sigma_n) = [x, y]$ and $S^n \wedge \Delta[n]$ is contactible. Thus $[x, y] \in \mathcal{B}_n$. The assertion follows.

Definitions 2.2. *Let G be a simplicial group. The subsimplicial group R_nG is defined by setting*

$$(R_nG)_q = \{x \in G_q | f_x(\Delta[q])^{[n]} = 1\},$$

where f_x is the representative of x and $X^{[n]}$ is the n-skeleton of the simplicial set X. The subsimplicial group $\overline{R}_nG$ is defined by setting

$$(\overline{R}_nG)_q = \{x \in G_q | f_x((\Delta[q])_n) \subseteq \mathcal{B}G_n\}.$$

Let P_nG denote G/R_nG and let $\overline{P}_nG$ denote $G/\overline{R}_nG$.

It is easy to check that both R_nG and $\overline{R}_nG$ are normal subsimplicial group of G. Thus both P_nG and $\overline{P}_nG$ are quotient simplicial group of G. Notice that

$$R_nG \subseteq \overline{R}_nG \subseteq R_{n-1}G.$$

The quotient simplicial group $\overline{P}_nG$ is between P_nG and $P_{n-1}G$. By checking the definition of Moore-Postnikov systems of a simplicial set [Mo1;C2], we have

Lemma 2.3. *The quotient simplicial group P_nG is the standard n-th Moore-Postnikov system of the simplicial group G.*

The quotient simplicial group $\overline{P}_nG$ has the same homotopy type of P_nG.

Proposition 2.4. *The quotient simplicial homomorphism $q_n : P_nG \to \overline{P}_nG$ is a homotopy equivalence for each n.*

Proof: The Moore chain complex of P_nG is as follows:

$$N(P_nG)_q = \begin{cases} 1 & \text{for}\, q > n+1, \\ NG_{n+1}/\mathcal{Z}G_{n+1} & \text{for}\, q = n+1, \\ NG_q & \text{for}\, q < n+1. \end{cases}$$

The Moore chain complex of $\overline{P}_nG$ is as follows:

$$N(\overline{P}_nG)_q = \begin{cases} 1 & \text{for}\, q > n, \\ NG_n/\mathcal{B}G_n & \text{for}\;\, q = n, \\ NG_q & \text{for}\;\, q < n. \end{cases}$$

Thus $(q_n)_*\colon \pi_*(P_nG)\to\pi_*(\overline{P}_nG)$ is an isomorphism and the assertion follows.

Let $\overline{F}_n$ denote the kernel of the quotient simplicial homomorphism $r_n\colon \overline{P}_nG\to P_{n-1}G$ for $n>0$.

Proposition 2.5. *The simplicial group $\overline{F}_n$ is the minimal simplicial group $K(\pi_nG,n)$ for each $n>0$.*

Proof: It is directly to check that

$$N(\overline{F}_nG)_q=\begin{cases}1 & \text{for}\, q\neq n,\\ \pi_nG & \text{for }\ q=n.\end{cases}$$

The assertion follows.

Proposition 2.6. *The short exact sequence of simplicial groups*

$$0\to\overline{F}_nG\to\overline{P}_nG\to P_{n-1}G\to 0$$

is a central extension for each $n>0$.

Proof: Consider the relative commutator sussimplicial group $[\overline{F}_nG,\overline{P}_nG]$. By Lemma 2.1, $[\overline{F}_nG,\overline{P}_nG]_n=1$. Notice that $[\overline{F}_nG,\overline{P}_nG]$ is a subsimplicial group of minimal simplicial group $\overline{F}_nG\cong K(\pi_nG,n)$. Thus $[\overline{F}_nG,\overline{P}_nG]=1$ and the assertion follows.

Definition 2.7. *A simplicial group is said to be r-centerless if the center $Z(G_n)=\{1\}$ for $n\geq r$.*

Proposition 2.8. *Let G be a reduced r-centerless simplicial group. Then $\pi_n(G)\cong Z(G_n/\mathcal{B}_n)$ for $n\geq r+1$*

Proof. By Lemma 2.1, $\mathcal{Z}_n/\mathcal{B}_n\subseteq Z(G_n/\mathcal{B}_n)$. It suffices to show that $Z(G_n/\mathcal{B}_n)\subseteq\mathcal{Z}_n/\mathcal{B}_n$ for $n\geq r+1$. Now let $\tilde{x}\in Z(G_n/\mathcal{B}_n)$. Choose $x\in G_n$ with $p(x)=\tilde{x}$, where $p\colon G_n\to G_n/\mathcal{B}_n$ is the quotient homomorphism. To check $\tilde{x}\in\mathcal{Z}_n/\mathcal{B}_n$, it suffices to show that $x\in\mathcal{Z}_n$ or $d_jx=1$ for all j. Since $Z(G_{n-1})=\{1\}$, $d_jx=1$ if and only if $[d_jx,y]=1$ for all $y\in G_{n-1}$. Now $[d_jx,y]=d_j[x,s_{j-1}y]$ for $j>0$ and $[d_0x,y]=d_0[x,s_0y]$. Since $\tilde{x}\in Z(G_n/\mathcal{B}_n)$, $[x,z]\in\mathcal{B}_n\subseteq\mathcal{Z}_n$ for all $z\in G_n$ and therefore $[d_jx,y]=1$ for all $y\in G_{n-1}$. The assertion follows.

By inspecting the proof, we also have

Proposition 2.9. *Let G be a reduced r-centerless simplicial group. Then*

$$Z(G_n/\mathcal{Z}_n)=\{1\}$$

for $n\geq r+1$.

Lemma 2.10. *Let G be a reduced simplicial group so that G_n is cyclic or centerless for each n. Then there exists a unique integer $\gamma_G>0$ so that $G_n=\{1\}$ for $n<\gamma_G$ and $Z(G_n)=\{1\}$ for $n>\gamma_G$.*

Proof. Let $\gamma_G = max\{\gamma | G_n = \{1\} \; for \; n < \gamma\}$. Then $\gamma_G > 0$. If $\gamma_G < \infty$. Then $G_{\gamma_G} \neq \{1\}$. We show that $G_{\gamma_G + q}$ is centerless for each $q > 0$. Notice that $d_0^q \circ s_0^q : G_n \to G_{n+q} \to G_n$ and $d_1^q \circ s_1^q : G_n \to G_{n+q} \to G_n$ are identities for $n > 0$ Thus $s_0^q(G_{\gamma_G})$ and $s_1^q(G_{\gamma_G})$ are nontrivial summands of G_{γ_G+q}. Now let $x = s_0^q y = s_1^q z \in s_0^q(G_{\gamma_G}) \cap s_1^q(G_{\gamma_G})$. Then $d_{q+1}x = d_{q+1}s_0^q y = s_0^q d_1 y = 1 = d_{q+1}s_1^q z = s_1^{q-1} d_2 s_1 z = s_1^{q-1} z$. Thus $x = s_1^q z = 1$. And therefore $s_0^q(G_{\gamma_G}) \cap s_1^q(G_{\gamma_G}) = \{1\}$. The assertion follows.

Corollary 2.11. *Let G be a reduced simplicial group such that G_n is cyclic or centerless for each n. Then $\pi_n(G) \cong Z(G_n/\mathcal{B}_n)$ for $n \neq \gamma_G + 1$, where γ_G is defined as above.*

Notice that, for any free group F, $rank(F) \geq 2 \Leftrightarrow Z(F) = \{1\}$ and $F \neq \{1\}$. We have

Lemma 2.12. *Let G be a reduced simplicial group such that G_n is a free group for each n. Then there exits a unique integer $\gamma_G > 0$ so that $G_n = \{1\}$ for $n < \gamma_G$ and $rank(G_n) \geq 2$ for $n > \gamma_G$.*

Example 2.13. *The 1-stem is determined in this example.*

Let $G = F(S^n)$, Milnor's F-construction on the standard n-sphere for $n \geq 1$. Then $G_n \cong F(\sigma) \cong Z(\sigma)$, the free abelian group generated by σ,

$$G_{n+1} \cong F(s_0\sigma, s_1\sigma, \ldots, s_n\sigma)$$

and $G_{n+2} \cong F(s_i s_j \sigma | 0 \leq j < i \leq n)$. It is easy to check that $\Gamma^2 G_{n+1} = \mathcal{Z}_{n+1}$, where $\Gamma^q G$ is the q-th term in the lower central series of a group G starting with $\Gamma^1 G = G$. By Lemma 2.1,

$$\Gamma^3 G_{n+1} = [\mathcal{Z}_{n+1}, G_{n+1}] \subseteq \mathcal{B}_{n+1}.$$

If $n = 1$, then it will be shown that $\Gamma^3 G_{n+1} = \mathcal{B}_{n+1}$ in Section 4 and therefore $\pi_3(S^2) \cong \pi_2(F(S^1)) \cong Z$, which is generated by $[s_0\sigma, s_1\sigma]$.

Suppose that $n > 1$. Consider the following equations

$$d_k([s_{j-1}s_i\sigma, s_{j+1}s_j\sigma]) = \begin{cases} 1 & \text{for } k \neq j, \\ [s_i\sigma, s_j\sigma] & \text{for k=j}, \end{cases}$$

for $i + 1 < j \leq n$,

$$d_k[s_{i+2}s_{i+1}\sigma, s_{i+3}s_i\sigma] = \begin{cases} [s_{i+1}\sigma, s_{i+2}\sigma] & \text{for k=i+1}, \\ [s_{i+1}\sigma, s_i\sigma] & \text{for k=i+3}, \\ 1 & \text{otherwise}, \end{cases}$$

and

$$d_k[s_{i+2}s_i\sigma, s_{i+3}s_{i+1}\sigma] = \begin{cases} [s_{i+1}\sigma, s_{i+2}\sigma] & \text{for k=i+1}, \\ [s_i\sigma, s_{i+2}\sigma] & \text{for k=i+2}, \\ [s_i\sigma, s_{i+1}\sigma] & \text{for k=i+3}, \\ 1 & \text{otherwise}. \end{cases}$$

By the Homotopy Addition Theorem [C2, Theorem 2.4], $[s_i\sigma, s_j\sigma]\in\mathcal{B}_{n+1}$ for $i+1<j$, $[s_{i+1}\sigma, s_{i+2}\sigma]\equiv[s_i\sigma,s_{i+1}\sigma] \mod \mathcal{B}_{n+1}$ and $0\equiv[s_i\sigma,s_{i+2}\sigma]\equiv[s_{i+1}\sigma,s_{i+2}\sigma]\equiv 2[s_i\sigma,s_{i+1}\sigma]$ if $i+2\leq n$. Notice that $[s_0\sigma,s_1\sigma]\neq 0$ in $\pi_{n+1}(\Gamma^2G/\Gamma^3G)$. Thus $[s_0\sigma,s_1\sigma]\notin\mathcal{B}_{n+1}$ and, by the relations above,

$$\pi_{n+2}(S^{n+1})\cong\pi_{n+1}(G)\cong\mathcal{Z}_{n+1}/\mathcal{B}_{n+1}\cong Z/2$$

for $n\geq 2$, which can be represented by $[s_i\sigma,s_{i+1}\sigma]$ for $0\leq i\leq n-1$.

3. Intersections of certain subgroups in free groups

In this section, we give some group theory preliminary. The intersections of certain subgroups in free groups are considered in this section. We will use these informations to determine the Moore chain complex of certain simplicial groups in the next sections.

Definition 3.1. *let S be a set and let $T\subseteq S$ a subset. The* ***projection homomorphism***

$$\pi: F(S)\to F(T)$$

is defined by

$$\pi(x)=\begin{cases} x & x\in T,\\ 1 & x\notin T.\end{cases}$$

Now let $\pi: F(S)\to F(T)$ be a projection homomorphism and let R equal the kernel of π. Define the subsets of the free group $F(S)$ as follows.

$$\mathcal{A}_T(k)=\{[[x,y_1^{\epsilon_1}]\cdots],y_t^{\epsilon_t}]|0\leq t\leq k,\epsilon_j=\pm1, y=y_1^{\epsilon_1}\dots y_t^{\epsilon_t}\in F(T), x\in S-T\},$$

where $y=y_1^{\epsilon_1}\cdots y_t^{\epsilon_t}\in F(T)$ runs over reduced words in $F(T)$ with $t\leq k$ and $y_j\in T$. Furthermore define $[[x,y_1^{\epsilon_1}]\dots],y_t^{\epsilon_t}]=x$ for $t=0$. Define

$$\mathcal{B}_T(k)=\{\phi^{-1}x\phi|\phi\in F(T)\, a\, reduced\, word\, with\, lenth\, l(\phi)\leq k, x\in S-T\},$$

$$\mathcal{A}_T=\cup_{k\geq0}\mathcal{A}_T(k)$$

and

$$\mathcal{B}_T=\cup_{k\geq0}\mathcal{B}_T(k).$$

By the classical Kurosch-Schreier theorem (see [MKS, pp.243, K2, Theorem 18.1]), we have

Proposition 3.2. *The subgroup R is a free group freely generated by $\mathcal{B}_T$.*

We will show that $\mathcal{A}_T$ is also a set of free generators for R. We need a lemma.

Lemma 3.3. *Let $\phi: F_1\to F_2$ be a homomorphism of free groups. Suppose that $\phi^{ab}: F_1^{ab}\to F_2^{ab}$ is an isomorphism, where F^{ab} is the abelianlizer of the group F. Then $\phi: F_1\to F_2$ is a monomorphism.*

Proof. Notice that $\phi_*: H_*(F_1) \to H_*(F_2)$ is an isomorphism, where $H_*(G)$ is the homology of the group G. Thus $F_1/\Gamma^r F_1 \to F_2/\Gamma^r F_2$ is an isomorphism for each r, where $\Gamma^r G$ is the r-th term in the lower central seris of the group G starting with $\Gamma^1 G = G$ and so

$$lim_r F_1/\Gamma^r F_1 \to lim_r F_2/\Gamma^r F_2$$

is an isomorphism. Notice that $\cap_r \Gamma^r F = 1$ for any free group F. Thus $F \to lim_r F/\Gamma^r F$ is a monomorphism. The assertion follows.

Proposition 3.4. *The subgroup R is a free group freely generated by $\mathcal{A}_T$.*

Proof. First we assume that both S and T are finite sets. Denote by $i_k: \mathcal{A}_T(k) \to R$ and $j_k: \mathcal{B}_T(k) \to R$ the natural inclusions. Notice that

$$R = F(\mathcal{B}_T) = colim_k F(\mathcal{B}_T(k)).$$

We set up the following steps.

Step1. $\mathcal{A}_T(k) \subseteq F(\mathcal{B}_T(k))$.

The proof of this statement is given by induction on k starting with $\mathcal{A}_T(0) = \mathcal{B}_T(0) = S-T$. Suppose that $\mathcal{A}_T(k-1) \subseteq F(\mathcal{B}_T(k-1))$ and let $w = [[x, y_1^{\epsilon_1}, \cdots, y_t^{\epsilon_t}] \in \mathcal{A}_T(k)$. If $t < k$, then $w \in \mathcal{A}_T(k-1) \subseteq F(\mathcal{B}_T(k-1)) \subseteq F(\mathcal{B}_T(k))$, by induction. Now

$$[[x, y_1^{\epsilon_1}]\cdots], y_k^{\epsilon_k}] = [[x, y_1^{\epsilon_1}]\cdots], y_{k-1}^{\epsilon_{k-1}}]^{-1} \cdot y_k^{-\epsilon_k}[[x, y_1^{\epsilon_1}]\cdots], y_{k-1}^{\epsilon_{k-1}}]y_k^{\epsilon_k}$$

Now since $[[x, y_1^{\epsilon_1}]\cdots], y_{k-1}^{\epsilon_{k-1}}] \in F(\mathcal{B}_T(k-1))$, $[[x, y_1^{\epsilon_1}]\cdots], y_{k-1}^{\epsilon_{k-1}}] = \prod_{j=1}^s (\phi_j^{-1} x_j \phi_j))^{\eta_j}$ with $\phi_j^{-1} x_j \phi_j \in \mathcal{B}_T(k-1)$ and $\eta_j = \pm 1$. Thus

$$w = (\prod_{j=1}^s (\phi_j^{-1} x_j \phi_j))^{\eta_j})^{-1} \cdot \prod_{j=1}^s (y_k^{-\epsilon_k} \phi_j^{-1} x_j \phi_j y_k^{\epsilon_k}))^{\eta_j} \in F(\mathcal{B}_T(k))$$

The induction is finished.

Step 2. $\tilde{i}_k: F(\mathcal{A}_T(k)) \to F(\mathcal{B}_T(k))$ is an epimorphism, where the homomorphism $\tilde{i}_k$ is induced by the inclusion $i_k: \mathcal{A}_T(k) \to F(\mathcal{B}_T(k))$

The proof of this step is given induction on k starting with $F(\mathcal{A}_T(0)) = F(\mathcal{B}_T(0)) = F(S-T)$. Suppose that $F(\mathcal{A}_T(k-1)) \to F(\mathcal{B}_T(k-1))$ is an epimorphism and consider $\tilde{i}_k: F(\mathcal{A}_T(k)) \to F(\mathcal{B}_T(k))$. Let $\phi^{-1} x \phi \in \mathcal{B}_T(k)$, where $\phi = y_1^{\epsilon_1} \cdots y_t^{\epsilon_t}$ is a reduced word with $t \leq k$. If $t \leq k-1$, then $\phi^{-1} x \phi \in Im \varphi_k$ by induction. Let $\phi = y_1^{\epsilon_1} \cdots y_k^{\epsilon_k}$ be a reduced word and let z denote the word $(y_1^{\epsilon_1} \cdots y_{k-1}^{\epsilon_{k-1}})^{-1} x y_1^{\epsilon_1} \cdots y_{k-1}^{\epsilon_{k-1}}$. Then $\phi^{-1} x \phi = z \cdot [z, y_k^{\epsilon_k}]$. Notice that $z \in Im(\tilde{i}_k)$ by induction. It suffices to show that

$[w,y^{\epsilon}]\in F(\mathcal{A}_T(k))$ for $w\in\mathcal{A}_T(k-1)$ for all $w\in\mathcal{A}_T(k-1)$, $y\in T$ and $\epsilon=\pm1$ by the Witt-Hall identity that

$$[ab,c]=[a,c]\cdot[[a,c],b]\cdot[b,c].$$

We show this by second induction starting with

$$\mathcal{A}_T(1)=\{[x,y^{\epsilon}]|y\in T,x\in S-T,\epsilon=\pm1\}.$$

Let $w=[[x,y_1^{\epsilon_1}],\cdots],y_t^{\epsilon_t}]$ be in $\mathcal{A}_T(k-1)$ with $k>1$, where $y_1^{\epsilon_1}\cdots y_t^{\epsilon_t}$ is a reduced word. Let $y\in T$ and let $\epsilon=\pm1$. If $y_1^{\epsilon_1}\cdots y_t^{\epsilon_t}y^{\epsilon}$ is a reduced word, then $[w,y^{\epsilon}]\in F(\mathcal{A}_T(k))$ by definition. Suppose that $y_1^{\epsilon_1}\cdots y_t^{\epsilon_t}y^{\epsilon}$ is not a reduced word. Then $t>0$, $y=y_t$ and $\epsilon=-\epsilon_t$. Let w' denote $[[x,y_1^{\epsilon_1}],\cdots,],y_{t-1}^{\epsilon_{t-1}}]\in\mathcal{A}_T(k-2)$. Then $w=[w',y_t^{\epsilon_t}]$. By the Witt-Hall identities, there is an equation

$$1=[w',y_t^{-\epsilon_t}]\cdot w\cdot[w,y_t^{-\epsilon_t}].$$

By induction, $[w',y_t^{-\epsilon_t}]\in F(\mathcal{A}_T(k-1))\subseteq F(\mathcal{A}_T(k))$ and so

$$[w,y_t^{-\epsilon_t}]=w^{-1}[w',y_t^{-\epsilon_t}]^{-1}\in F(\mathcal{A}_T(k-1))\subseteq F(\mathcal{A}_T(k)).$$

The second induction is finished and so the first induction is finished. The assertion follows.

Step 3. $\tilde{i}_k:F(\mathcal{A}_T(k))\to F(\mathcal{B}_T(k))$ is an isomorphism.

By Step 2, $Z(\mathcal{A}_T(k))\to Z(\mathcal{B}_T(k))$ is an epimorphism. Notice that

$$rank(F(\mathcal{B}_T(k)))=|\mathcal{B}_T(k)|=|\mathcal{A}_T(k)|=rank(F(\mathcal{A}_T(k))).$$

Thus $\tilde{i}_k:Z(\mathcal{A}_T(k))\to Z(\mathcal{B}_T(k))$ is an isomorphism and so $\tilde{i}_k:F(\mathcal{A}_T(k))\to F(\mathcal{B}_T(k))$ is a monomorphism. Thus $\tilde{i}_k$ is an isomorphism.

Step 4. Since $F(\mathcal{A}_T(k))\to F(\mathcal{B}_T(k))$ is an isomorphism for each k, $F(\mathcal{A}_T)=colim_kF(\mathcal{A}_T(k))\to F(\mathcal{B}_T)=colim_kF(\mathcal{B}_T(k))$ is an isomorphism.

Now consider the general case. By Lemma 3.6, it suffices to show that $\tilde{i}:F(\mathcal{A}_T)\to F(\mathcal{B}_T)$ is an isomorphism. To check that $F(\mathcal{A}_T)\to F(\mathcal{B}_T)$ is an epimorphism. Let $w\in\mathcal{B}_T$, there exist finite subsets S' and T' of S and T, respectively, so that $w\in\mathcal{B}_{T'}$. By the special case as above, $\tilde{i}|_{F(\mathcal{A}_{T'})}:F(\mathcal{A}_{T'})\to F(\mathcal{B}_{T'})$ is an isomorphism and $w\in Im\tilde{i}|_{F(\mathcal{A}_{T'})}$. Thus $\tilde{i}$ is an epimorphism. To check that $F(\mathcal{A}_T)\to F(\mathcal{B}_T)$ is a monomorphism. Let $w\in Ker\tilde{i}$, there exist finite subsets S' and T' of S and T, respectively, so that $w\in F(\mathcal{A}_{T'})$. Notice that $\tilde{i}|_{F(\mathcal{A}_{T'})}$ is an isomorphism. Thus $w=1$ and the assertion follows.

Now let's consider the intersection of kernels of projection homomorphisms. Let S be a set and let T_j be a subset of S for $1 \leq j \leq k$. Let $\pi_j : F(S) \to F(T_j)$ be the projection homomorphism for $1 \leq j \leq k$. We construct a subset $\mathcal{A}(T_1, \cdots, T_k)$ of $F(S)$ by induction on k as follows.

$$\mathcal{A}(T_1) = \mathcal{A}_{T_1},$$

where $\mathcal{A}_T$ is defined in Definition3.1. Let

$$T_2^{(2)} = \{w \in \mathcal{A}(T_1) | w = [[x, y_1^{\epsilon_1}, \cdots, y_t^{\epsilon_t}] \, with \, x, y_j \in T_2 \, for \, all \, j\}$$

and define

$$\mathcal{A}(T_1, T_2) = \mathcal{A}(T_1)_{T_2^{(2)}}.$$

Suppose that $\mathcal{A}(T_1, T_2, \cdots, T_{k-1})$ is well defined so that all of the elements in $\mathcal{A}(T_1, T_2, \cdots, T_{k-1})$ are written down as certain commutators in $F(S)$ in terms of elements in S. Let $T_k^{(k)}$ be the subset of $\mathcal{A}(T_1, T_2, \cdots, T_{k-1})$ defined by

$$T_k^{(k)} = \{w \in \mathcal{A}(T_1, T_2, \cdots, T_{k-1}) | w = [x_1^{\epsilon_1}, \cdots, x_l^{\epsilon_l}] \, with \, x_j \in T_k \, for \, all \, j\},$$

where $[x_1^{\epsilon_1}, \cdots, x_l^{\epsilon_l}]$ are the elements in $\mathcal{A}(T_1, T_2, \cdots, T_{k-1})$ which are written down as commutators. Then define

$$\mathcal{A}(T_1, T_2, \cdots, T_k) = \mathcal{A}(T_1, T_2, \cdots, T_{k-1})_{T_k^{(k)}}$$

Theorem 3.5. *Let S be a set and let T_j be a subset of S for $1 \leq j \leq k$. Let $\pi_j : F(S) \to F(T_j)$ be the projection homomorphism for $1 \leq j \leq k$. Then the intersection $\bigcap_{j=1}^k Ker\pi_j$ is a free group freely generated by $\mathcal{A}(T_1, T_2, \cdots, T_k)$.*

Proof. The proof is given by induction on k. If $k = 1$, the assertion follows from the above lemma. Suppose that $\bigcap_{j=1}^{k-1} Ker\pi_j = F(\mathcal{A}(T_1, T_2, \cdots, T_{k-1}))$ and consider $\pi_k : F(S) \to F(T_k)$. Then

$$\bigcap_{j=1}^{k} Ker\pi_j = Ker(\bar{\pi}_k : F(\mathcal{A}(T_1, T_2, \cdots, T_{k-1})) \to F(T_k))$$

, where $\bar{\pi}_k$ is π_k restricted to the subgroup $F(\mathcal{A}(T_1, T_2, \cdots, T_{k-1}))$. Let

$$w = [x_1^{\epsilon_1}, \cdots, x_l^{\epsilon_l}] \in \mathcal{A}(T_1, T_2, \cdots, T_{k-1}).$$

If $w \notin T_k^{(k)}$, then $x_j \notin T_k$ for some j and $\bar{\pi}_k(w) = 1$. Thus $\bar{\pi}_k$ factors through $F(T_k^{(k)})$, i.e, there is a homomorphism $j : F(T_k^{(k)}) \to F(T_k)$ so that $\bar{\pi}_k = j \circ \pi$, where $\pi : F(\mathcal{A}(T_1, T_2, \cdots, T_{k-1})) \to F(T_k^{(k)})$ is the projection homomorphism. We claim that $j : F(T_k^{(k)}) \to F(T_k)$ is a monomorphism. Consider the commutative diagram

$$\begin{array}{ccccc} F(\mathcal{A}(T_1,T_2,\cdots,T_{k-1})) & \stackrel{\bar\pi_k}{\to} & F(T_k) & \hookrightarrow & F(S)\\ \uparrow & & \uparrow j & & \uparrow\\ F(T_k^{(k)}) & \stackrel{=}{\to} & F(T_k^{(k)}) & \hookrightarrow & F(\mathcal{A}(T_1,T_2,\cdots,T_{k-1})), \end{array}$$

where $F(T_k^{(k)}) \to F(\mathcal{A}(T_1,T_2,\cdots,T_{k-1}))$ and $F(\mathcal{A}(T_1,T_2,\cdots,T_{k-1})) \to F(S)$ are inclusions of subgroups. Thus $j: F(T_k^{(k)}) \to F(T_k)$ is a monomorphism and

$$Ker\bar\pi_k = Ker(F(\mathcal{A}(T_1,T_2,\cdots,T_{k-1})) \to F(T_k^{(k)}) = F(\mathcal{A}(T_1,T_2,\cdots,T_k)).$$

The assertion follows.

Corollary 3.6. *Let π_j be the projection homomorphisms as in Theorem 3.5. Then the intersection subgroup $\bigcap_{j=1}^k Ker\pi_j$ equals the commutator subgroup $[[<T_1>,\cdots,<T_k>]]$ which is defined in Notations 1.3.*

4. On the Homotopy Groups of the 3-sphere

In this section, we study the Moore chain complex of the Milnor's construction $F(S^1)$. The proofs Theorems 1.2 and 1.4 are given in this section, where Theorem 4.5 is Theorem 1.2 and Theorem 4.10 is Theorem 1.4. Recall that the simplicial 1-sphere S^1 is a free simplicial set generated by a 1-simplex σ. Thus $S^1_0=\{*\}$, $S^1_1=\{\sigma,*\}$ and $S^1_{n+1}=\{*, s_n\cdots s_{i+1}s_{i-1}\cdots s_0\sigma|0\le j\le n\}$. Let x_i denote $s_n\cdots s_{i+1}s_{i-1}\cdots s_0\sigma$. Then $F(S^1)_{n+1}=F(x_0,x_1,\cdots,x_n)$ the free group freely generated by $x_0,\cdots,x_n$. Let y_i denote $x_{i-1}x_i^{-1}$ for $-1\le i\le n$, where we put $x_{-1}=x_{n+1}=1$ in $F(S^1)_{n+1}=F(x_0,\cdots,x_n)$. By direct calculation, we have

Lemma 4.1. *$F(S^1)_{n+1}=F(y_0,\cdots,y_n)$ with*

$$d_jy_k=\begin{cases} y_{k-1} & j\le k,\\ 1 & j=k+1,\\ y_k & j>k+1,\end{cases}$$

and

$$s_jy_k=\begin{cases} y_{k+1} & j\le k,\\ y_ky_{k+1} & j=k+1, and\\ y_k & j>k+1\end{cases}$$

for $0\le j\le n+1$, where $y_{-1}=(y_0\cdots y_{n-1})^{-1}$ in $F(S^1)_n$.

Now let C_{n+1} denote the subgroup of $F(y_0,\cdots,y_n)$ generated by all of the commutators $[y_{i_1}^{\epsilon_1},\cdots,y_{i_t}^{\epsilon_t}]$ with $\{i_1,\cdots,i_t\}=\{0,1,\cdots,n\}$ as sets, i.e. each j $(0\le j\le n)$ appears in the index set $\{i_1,\cdots,i_t\}$ at least one time, where $\epsilon_j=\pm1$ and the commutator $[y_{i_1}^{\epsilon_1},\cdots,y_{i_t}^{\epsilon_t}]$ runs over all of the commutator bracket arrangements of weight t for $y_{i_1}^{\epsilon_1},\cdots,y_{i_t}^{\epsilon_t}$.

Lemma 4.2. *The group C_{n+1} is a subgroup of $NF(S^1)_{n+1}$,i.e $C_{n+1}\subseteq\cap_{j\neq0}Ker(d_j)$.*

Proof. Notice that $d_j y_{j-1} = 1$ for $1 \leq j \leq n+1$. Since $\{y_{i_1}, \cdots, y_{i_t}\} = \{y_0, y_1, \cdots, y_n\}$, $d_j[y_{i_1}^{\epsilon_1}, \cdots, y_{i_t}^{\epsilon_t}] = 1$ for each $j > 0$. The assertion follows.

Theorem 4.3. $NF(S^1)_{n+1} = C_{n+1}$.

Proof. For $1 \leq j \leq n+1$, let $S = \{y_0, y_1, \cdots, y_n\}$ and let $T_j = \{y_0, \cdots, \hat{y}_j \cdots, y_n\}$. By Lemma 4.1, there is a commutative diagram

$$\begin{array}{ccc} F(S) & \stackrel{\pi_j}{\to} & F(T_{j-1}) \\ \downarrow d_j & & \bar{d}_j \downarrow\cong \\ F(y_0, \cdots, y_{n-1}) & \stackrel{=}{\to} & F(y_0, \cdots, y_{n-1}), \end{array}$$

where

$$\bar{d}_j(y_k) = \begin{cases} y_k & 0 \leq k \leq j-2, \\ y_{k-1} & j \leq k \leq n. \end{cases}$$

Thus $Ker d_j = Ker \pi_j$ and $NF(S^1)_{n+1} = \bigcap_{j=1}^{n+1} Ker d_j = \bigcap_{j=1}^{n+1} Ker \pi_j$. By Theorem 3.5, $NF(S^1)_{n+1} = F(\mathcal{A}(T_0, T_1, \cdots, T_n))$, where the notation $\mathcal{A}(T_0, T_1, \cdots, T_n)$ is given in Section 3. To check that

$$F(\mathcal{A}(T_0, T_1, \cdots, T_n)) \subseteq C_{n+1},$$

it suffices to show that $\mathcal{A}(T_0, T_1, \cdots, T_n) \subseteq C_{n+1}$. This will follow from the following statement.

Statement. For each $0 \leq j \leq n$ and $w = [y_{i_1}^{\epsilon_1}, y_{i_2}^{\epsilon_2}, \cdots, y_{i_t}^{\epsilon_t}] \in \mathcal{A}(T_0, T_1, \cdots, T_j)$, $\{y_0, y_1, \cdots, y_j\} \subseteq \{y_{i_1}, y_{i_2}, \cdots, y_{i_t}\}$.

We show this statement by induction on j. Note that $F(T_0) = F(y_1, \cdots, y_n)$. For $j = 0$,

$$\mathcal{A}(T_0) = \{[[y_0, y_{i_1}^{\epsilon_1}], \cdots,], y_{i_t}^{\epsilon_t}] | i_s > 0, y_{i_1}^{\epsilon_1} \cdots y_{i_t}^{\epsilon_t}\, a\, reduced\, word\, in\, F(T_0)\}.$$

Thus the assertion holds for $j = 0$. Suppose that the assertion holds for $j-1$ with $j \leq n$. Notice that $T_j = \{y_0, \cdots, \hat{y}_j, \cdots, y_n\}$. Thus

$$T_j^{(j)} = \{w \in \mathcal{A}(T_0, T_1, \cdots, T_{j-1}) | w = [y_{i_1}^{\epsilon_1}, \cdots, y_{i_t}^{\epsilon_t}]\, with\, y_j \notin \{y_{i_1}, \cdots, y_{i_t}\}\}.$$

and so $y_j \in \{y_{i_1}, \cdots, y_{i_t}\}$ for $w = [y_{i_1}^{\epsilon_1}, \cdots, y_{i_t}^{\epsilon_t}] \in \mathcal{A}(T_0, T_1, \cdots, T_{j-1}) - T_j^{(j)}$. Hence, by induction,

$$\{y_0, y_1, \cdots, y_j\} \subseteq \{y_{i_1}, y_{i_2}, \cdots, y_{i_t}\}$$

for $w = [y_{i_1}^{\epsilon_1}, \cdots, y_{i_t}^{\epsilon_t}] \in \mathcal{A}(T_0, T_1, \cdots, T_{j-1}) - T_j^{(j)}$. Notice that

$$\mathcal{A}(T_0, T_1, \cdots, T_j) = \mathcal{A}(T_0, T_1, \cdots, T_{j-1})_{T_j^{(j)}}.$$

Thus

$$\{y_0, y_1, \cdots, y_j\} \subseteq \{y_{i_1}, y_{i_2}, \cdots, y_{i_t}\}$$

for each $w = [y_{i_1}^{\epsilon_1}, \cdots, y_{i_t}^{\epsilon_t}] \in \mathcal{A}(T_0, T_1, \cdots, T_j)$. The induction is finished and the theorem follows.

Corollary 4.4. *$\mathcal{A}(T_0, T_1, \cdots, T_n)$ is a set of free generators for $NF(S^1)_{n+1}$.*

Theorem 4.5 (Theorem 1.2). *For $n \geq 1$, $\pi_{n+2}(S^3)$ is isomorphic to the center of the group with generators $y_0, y_1, \cdots, y_n$ and relations $[y_{i_1}^{\epsilon_1}, y_{i_2}^{\epsilon_2}, \cdots, y_{i_t}^{\epsilon_t}]$ with*

$$\{i_1, i_2, \cdots, i_t\} = \{-1, 0, 1, \cdots, n\}$$

as sets, where $\epsilon_j = \pm 1$,$y_{-1} = (y_0 y_1 \cdots y_n)^{-1}$ and the commutator bracket $[\cdots]$ runs over all bracket arrangements of weight t for each t.

Proof. Notice that $\pi_{n+2}(S^3) \cong \pi_{n+1}F(S^1)$ for $n \geq 1$ and $\mathcal{B}_{n+1} = d_0(NF(S^1)_{n+2})$. By Lemma 4.1 and Theorem 4.3, $\mathcal{B}_{n+1}$ is generated by $[y_{i_1}^{\epsilon_1}, y_{i_2}^{\epsilon_2}, \cdots, y_{i_t}^{\epsilon_t}]$ so that $\{y_{i_1}, y_{i_2}, \cdots, y_{i_t}\} = \{y_0, y_1, \cdots, y_n\}$, where $\epsilon_j = \pm 1$. By Proposition 2.8, it suffices to check that $\pi_2(F(S^1)) \cong Z(F(S^1)_2/\mathcal{B}_2) = Z(F(y_0, y_1)/\mathcal{B}_2)$. By Example 2.13, $\mathcal{Z}_2 = \Gamma^2 F(S^1)_2 = \Gamma^2 F(y_0, y_1)$ and $\Gamma^3 F(y_0, y_1) \subseteq \mathcal{B}_2$. Now , by the construction of $\mathcal{B}_2$, $\mathcal{B}_2 \subseteq \Gamma^3 F(y_0, y_1)$ and therefore

$$\pi_2(F(S^1)) \cong \mathcal{Z}_2/\mathcal{B}_2 = \Gamma^2 F(y_0, y_1)/\Gamma^3 F(y_0, y_1) = Z(F(y_0, y_1)/\mathcal{B}_2).$$

The assertion follows.

Remark 4.6. *The relations in above theorem are not minimal, i.e, many of them can be cancelled out.*

Proposition 4.7. *Let ${C'}_{n+1}$ be the subgroup of $F(S^1)_{n+1}$ generated by all commutators given by $[\cdots[y_{i_1}^{\epsilon_1}, y_{i_2}^{\epsilon_2}], \cdots,], y_{i_t}^{\epsilon_t}]$ with $\{i_1, \cdots, i_t\} = \{0, 1, \cdots, n\}$ and $\epsilon_j = \pm 1$. Then*

$$NF(S^1)_{n+1}/\Gamma^s \cap NF(S^1)_{n+1} \cong C'_{n+1}/\Gamma^s \cap C'_{n+1}$$

for each s, where $\Gamma^s = \Gamma^s F(S^1)_{n+1}$ is the s-term in the lower central series of $F(S^1)_{n+1}$.

Proof. Notice that $C'_{n+1} \subseteq NF(S^1)_{n+1}$. The induced homomorphism $f_s : C'_{n+1}/\Gamma^s \cap C'_{n+1} \to NF(S^1)_{n+1}/\Gamma^s \cap NF(S^1)_{n+1}$ is a monomorphism. We check that f is an epimorphism. It suffices to show that, for each $w \in NF(S^1)$, there exists a sequence of elements $\{x_j\}$ so that $x_j \in \Gamma^j \cap C'_{n+1}$ and $wx_1x_2 \cdots x_s \in \Gamma^{s+1}$ for each s. In fact, if this statement holds, $wx_1x_2 \cdots x_{s-1} \equiv 1 \ mod\ \Gamma^s \cap NF(S^1)_{n+1}$ for each s and $w = (wx_1 \cdots x_{s-1}) \cdot (x_1x_2 \cdots x_{j-1})^{-1} \in C'_{n+1} \ mod\, \Gamma^s$. Now we construct x_j by induction, which depends on w. Notice that, for $n \geq 1$, $NF(S^1)_{n+1} \in \Gamma^{n+1} \subseteq \Gamma^2$. Choose $x_j = 1$ for $j \leq n$. Suppose that there are $x_1, \cdots, x_{s-1}$ so that $x_j \in \Gamma^j \cap C'_{n+1}$ and $wx_1 \cdots x_{s-1} \in \Gamma^s$. Since $C'_{n+1} \subseteq NF(S^1)_{n+1}$, $wx_1 \cdots x_{s-1} \in \Gamma^s \cap NF(S^1)_{n+1}$

and $d_j(wx_1\cdots x_{s-1})=1$ for $j>1$. Let $\pi:\Gamma^s\to\Gamma^s/\Gamma^{s+1}$ be the quotient homomorphism. $\pi(wx_1\cdots x_{s-1})$ is a linear combination of basic Lie products. We claim that $\pi(wx_1\cdots x_{s-1})$ is a linear combination of basic Lie products in which each y_j appears in the Lie product for $0\leq j\leq n$. If not, there exists j so that $\pi(wx_1\cdots x_{s-1})=b+c$, where b is a nontrivial linear combination of basic Lie products in which y_j does not appear and c is a linear combination of basic Lie products in which y_j appears. Now the face homomorphism $d_{j+1}:F(y_0,\cdots,y_n)\to F(y_0,\cdots,y_{n-1})$ induces a homomorphism

$$\bar{d}_{j+1}:\Gamma^sF(y_0,\cdots,y_n)/\Gamma^{s+1}F(y_0,\cdots,y_n)\to\Gamma^sF(y_0,\cdots,y_{n-1})/\Gamma^{s+1}F(y_0,\cdots,y_{n-1})$$

and

$$1=\bar{d}_{j+1}\pi(wx_1\cdots x_{s-1})=\bar{d}_{j+1}(b)+\bar{d}_{j+1}(c)=\bar{d}_{j+1}(b).$$

Notice that

$$d_{j+1}|_{F(y_0,\cdots,y_{j-1},y_{j+1},\cdots,y_n)}:F(y_0,\cdots,y_{j-1},y_{j+1},\cdots,y_n)\to F(y_0,\cdots,y_{n-1})$$

is an isomorphism. Thus $b=1$. This contradicts to that b is a nontrivial linear combination of a basis. Thus $\pi(wx_1\cdots x_{s-1})$ is a linear combination of basic Lie products in which all of the y_j appear. By Theorem 5.12 in [MKS,pp.337], there exists x_s in C'_{n+1} so that $\pi(wx_1\cdots x_s)=1$, or $wx_1\cdots x_s\in\Gamma^{s+1}$. The induction is finished now.

By Corollary 3.6, we have

Theorem 4.8. *In the free group $F(y_0,\cdots,y_n)$, $NF(S^1)_{n+1}=[[<y_0>,\cdots,<y_n>]]$ and $\mathcal{B}F(S^1)_{n+1}=[[<y_{-1}>,<y_0>,\cdots,<y_n>]]$.*

Thus Theorem 4.5 can be rewritten as follows.

Theorem 4.9. *In the free group $F(y_0,\cdots,y_n)$ for $n\geq 1$, the center*

$$Z(F(y_0,\cdots,y_n)/[[<y_{-1}>,<y_0>,\cdots,<y_n>]])\cong\pi_{n+2}(S^3)$$

By Lemma 4.1 and Proposition 3.5, $Kerd_0=<x_0>=<y_{-1}>$ and therefore

$$\mathcal{Z}_{n+1}=[[<y_0>,\cdots,<y_n>]]\cap<y_{-1}>.$$

Thus we have

Theorem 4.10 (Theorem 1.4). *In the free group $F(y_0,\cdots,y_n)$ with $n\geq 1$,*

$$[[<y_0>,\cdots,<y_n>]]\cap<y_{-1}>]]/[[<y_{-1}>,<y_0>,\cdots,<y_n>]]\cong\pi_{n+2}(S^3).$$

5. On the Homotopy Groups of $\Sigma K(\pi,1)$

In this section, we give group theoretical descriptions for $\pi_*(\Sigma K(\pi,1))$ for any group π. The proof of Theorem 1.5 is given in this section, where Theorem 5.9 is Theorem 1.5. We will use the notations defined in Section 3. First we extend our description for $\pi_*(S^2)$ to the case $\pi_*(\vee_{\alpha\in J}S^2)$. Recall that $(\vee_{\alpha\in J}S^1)_0=*$, $(\vee_{\alpha\in J}S^1)_1=\{\sigma_\alpha,*|\alpha\in J\}$ and $(\vee_{\alpha\in J}S^1)_{n+1}=\{s_n\cdots\hat{s}_i\cdots s_0\sigma_\alpha,*|\alpha\in J,0\le i\le n\}$. Let $x_i^{(\alpha)}$ denote $s_n\cdots\hat{s}_i\cdots s_0\sigma_\alpha$. Then

$$F(\vee_{\alpha\in J}S^1)_{n+1}=F(x_0^{(\alpha)},x_1^{(\alpha)},\cdots,x_n^{(\alpha)}|\alpha\in J).$$

Let $y_j^{(\alpha)}=x_j^{(\alpha)}\cdot x_{j+1}^{(\alpha)}$ for $0\le j\le n-1$ and $y_n^{(\alpha)}=x_n^{(\alpha)}$. By Lemma 4.1, we have

Lemma 5.1. *$F(\vee_{\alpha\in J}S^1)_{n+1}=F(y_j^{(\alpha)}|0\le j\le n,\alpha\in J)$ with*

$$d_j(y_k^{(\alpha)})=\begin{cases}y_{k-1}^{(\alpha)} & j\le k,\\ 1 & j=k+1,\\ y_k^{(\alpha)} & j>k+1,\end{cases}$$

and

$$s_j(y_k^{(\alpha)})=\begin{cases}y_{k+1}^{(\alpha)} & j\le k,\\ y_k^{(\alpha)}\cdot y_{k+1}^{(\alpha)} & j=k+1,\\ y_k^{(\alpha)} & j>k+1,\end{cases}$$

for $0\le j\le n+1$, where $y_{-1}^{(\alpha)}=(y_0^{(\alpha)}\cdots y_{n-1}^{(\alpha)})^{-1}$ in $F(\vee_{\alpha\in J}S^1)$

Let C_{n+1}^J denote the subgroup of $F(\vee_{\alpha\in J}S^1)_{n+1}$ generated by all of the commutators $[y_{i_1}^{(\alpha_1)\epsilon_1},\cdots,y_{i_t}^{(\alpha_t)\epsilon_t}]$ with $\{i_1,\cdots,i_t\}=\{0,1,\cdots,n\}$ as sets, where $\epsilon_j=\pm1$, $\alpha_j\in J$ and the commutator bracket $[\cdots]$ runs over all of the commutator bracket arrangements of weight t for each t.

Lemma 5.2. $C_{n+1}^J\subseteq NF(\vee_{\alpha\in J}S^1)_{n+1}$.

Proof: For each $1\le j\le n+1$, there exists some $i_s=j-1$. Thus $d_j(y_{i_s}^{(\alpha_s)\epsilon_s})=1$ for some i_s and therefore

$$d_j([y_{i_1}^{(\alpha_1)\epsilon_1},\cdots,y_{i_t}^{(\alpha_t)\epsilon_t}])=1$$

for each $j>0$. The assertion follows.

Lemma 5.3. *For each $1\le j\le n+1$,*

$$Kerd_j\cap F(\vee_{\alpha\in J}S^1)_{n+1}=<y_{j-1}^{(\alpha)}|\alpha\in J>,$$

the normal subgroup generated by $y_{j-1}^{(\alpha)}$ with $\alpha\in J$.

Proof: By the definition of d_j, there is a commutative diagram

$$\begin{array}{ccc} F(y_j^{(\alpha)}|0\leq j\leq n, \alpha\in J) & \xrightarrow{p} & F(y_0^{(\alpha)}\cdots \hat{y}_{j-1}^{(\alpha)}\cdots y_n^{(\alpha)}|\alpha\in J) \\ d_j\downarrow & & \cong\downarrow \bar{d}_j \\ F(y_j^{(\alpha)}|0\leq j\leq n-1, \alpha\in J) & \xrightarrow{=} & F(y_j^{(\alpha)}|0\leq j\leq n-1, \alpha\in J) \end{array}$$

where p is the projection and

$$\bar{d}_j y_k^{(\alpha)} = \begin{cases} y_{k-1}^{(\alpha)} & j\leq k, \\ y_k^{(\alpha)} & j>k+1. \end{cases}$$

The assertion follows.

Theorem 5.4. *Let C_{n+1}^J be defined as above. Then*

$$NF(\vee_{\alpha\in J}S^1)_{n+1}=C_{n+1}^J$$

Proof: By lemma 5.1, each d_j with $j>0$ is a projection homomorphism. Thus, by Theorem 3.5,

$$NF(\vee_{\alpha\in J}S^1)_{n+1}=\bigcap_{j=1}^{n+1} Kerd_j = F(\mathcal{A}(T_0,T_1,\cdots,T_n))$$

where $T_j=\{y_0^{(\alpha)},\cdots,\hat{y}_j^{(\alpha)},\cdots,y_n^{(\alpha)}|\alpha\in J\}$. It suffices to show that

$$\mathcal{A}(T_0,T_1,\cdots,T_n)\subseteq C_{n+1}^J.$$

This follows from the next lemma.

Lemma 5.5. *For $0\leq j\leq n$, let $W=[y_{i_1}^{(\alpha_1)\epsilon_1},\cdots,y_{i_t}^{(\alpha_t)\epsilon_t}]\in\mathcal{A}(T_0,T_1,\cdots,T_j)$. Then*

$$\{0,1,\cdots,j\}\subseteq\{i_1,\cdots,i_t\}.$$

Proof: The proof is given by induction on j for $0\leq j\leq n$. Notice that, by construction, each element in $\mathcal{A}(T_0,T_1,\cdots,T_j)$ is written as a certain commutator. If $j=0$, then

$$\mathcal{A}(T_0)=\{y_0^{(\alpha)},[[y_0^{(\alpha)},y_{i_1}^{(\alpha_1)\epsilon_1}],\cdots,],y_{i_t}^{(\alpha_t)\epsilon_t}]\}$$

where $\alpha,\alpha_j\in J$, $\epsilon_j=\pm1$ and $y_{i_1}^{(\alpha_1)\epsilon_1}\cdots y_{i_t}^{(\alpha_t)\epsilon_t}$ runs over all of the reduced words $\neq 1$ in $F(y_j^{(\alpha)}|\alpha\in J, 1\leq j\leq n)$. Thus the assertion holds for $j=0$. Suppose that the the assertion holds for $j-1$ with $j\leq n$. Recall that

$$T_j^{(j)}=\{W\in\mathcal{A}(T_0,T_1,\cdots,T_{j-1}|W=[y_{i_1}^{(\alpha_1)\epsilon_1},y_{i_2}^{(\alpha_2)\epsilon_2},\cdots,y_{i_t}^{(\alpha_t)\epsilon_t}],\ with\, y_{i_s}^{(\alpha_s)}\in T_j\}.$$

Notice that $y_{i_s}^{(\alpha_s)}\in T_j \iff i_s\neq j$. Thus, for $W=[y_{i_1}^{(\alpha_1)\epsilon_1},y_{i_2}^{(\alpha_2)\epsilon_2},\cdots,y_{i_t}^{(\alpha_t)\epsilon_t}]\in\mathcal{A}(T_0,\cdots,T_{j-1})-T_j^{(j)}$, $j\in\{i_1,\cdots,i_t\}$. By induction, $\{0,\cdots,j-1\}\subseteq\{i_1,\cdots,i_t\}$. Hence

$$\{0,1,\cdots,j\}\subseteq\{i_1,i_2,\cdots,i_t\}$$

for any $W = [y_{i_1}^{(\alpha_1)\epsilon_1}, y_{i_2}^{(\alpha_2)\epsilon_2}, \cdots, y_{i_t}^{(\alpha_t)\epsilon_t}] \in \mathcal{A}(T_0, \cdots, T_{j-1} - T_j^{(j)}$. Recall that, by construction,

$$\mathcal{A}(T_0, \cdots, T_j) = \mathcal{A}_{T_j^{(j)}}.$$

$$\{0, 1, \cdots, j\} \subseteq \{i_1, i_2, \cdots, i_t\}$$

for any $W = [y_{i_1}^{(\alpha_1)\epsilon_1}, y_{i_2}^{(\alpha_2)\epsilon_2}, \cdots, y_{i_t}^{(\alpha_t)\epsilon_t}] \in \mathcal{A}(T_0, \cdots, T_j)$. This completes the proof.

Theorem 5.6. *$\pi_{n+2}(\vee_{\alpha\in J} S^2)$ is isomorphic to the center of the group with generators*

$$y_0^{(\alpha)}, y_1^{(\alpha)}, \cdots, y_n^{(\alpha)}$$

for $\alpha \in J$. and relations

$$[y_{i_1}^{(\alpha_1)\epsilon_1}, y_{i_2}^{(\alpha_2)\epsilon_2}, \cdots, y_{i_t}^{(\alpha_t)\epsilon_t}]$$

with $\{i_1, i_2, \cdots, i_t\} = \{-1, 0, 1, \cdots, n\}$ as sets, where the indices i_j can be repeated, $\epsilon_j = \pm 1$, $\alpha_j \in J$ and the commutator bracket runs over all of the commutator bracket arrangements of weight t for each t.

Proof: Notice that $\pi_{n+2}(\vee_{\alpha\in J} S^2) \cong \pi_{n+1}(F(\vee_{\alpha\in J} S^1))$. By the above theorem, $\mathcal{B}_{n+1}$ is generated by

$$[y_{i_1}^{(\alpha_1)\epsilon_1}, y_{i_2}^{(\alpha_2)\epsilon_2}, \cdots, y_{i_t}^{(\alpha_t)\epsilon_t}]$$

By Theorem 2.8, the assertion holds for $n \geq 1$. For $n = 0$, $\mathcal{B}_1 = \Gamma^2(F(y_0^{(\alpha)} | \alpha \in J))$ and

$$F(y_0^{(\alpha)} | \alpha \in J)/\mathcal{B}_1 \cong \oplus_{\alpha\in J} Z\pi_2(\vee_{\alpha\in J} S^2).$$

The assertion holds for all of n.

For the general case, we need a simplicial group construction.

Definition 5.7. *Let G be a simplicial group and let X be a pointed simplicial set with a point $*$. The simplicial group $F^G(X)$ is defined by setting*

$$F^G(X)_n = \coprod_{x\in X_n} (G_n)_x,$$

the free product, modulo the relations $(G_n)_*$, where $(G_n)_x$ is a copy of G_n. The faces and degeneracies homomorphisms in $F^G(X)$ is given in the canonical way by the universal property of the coproduct in the category of groups and group homomorphisms.

Lemma 5.8 (Ca, Theorem 9, pp.88). *Let G be a simplicial group and let X be a pointed simplicial set. Then the goemetric realization $|F^G(X)|$ is homotopy equivalent to $\Omega(B|G| \wedge |X|)$.*

A generalization of this lemma by using fibrewise simplicial groups is given in [Wu1].

Theorem 5.9 (Theorem 1.5). *Let π be any group and let $\{x^{(\alpha)}|\alpha\in J\}$ be a set of generators for π. Then, for $n\neq 1$, $\pi_{n+2}(\Sigma K(\pi,1))$ is isomorphic to the center of the quotient group of the free product*

$$\coprod^{groups}_{0\leq j\leq n}(\pi)_j$$

modulo the relations

$$[y_{i_1}^{(\alpha_1)\epsilon_1},y_{i_2}^{(\alpha_2)\epsilon_2},\cdots,y_{i_t}^{(\alpha_t)\epsilon_t}]$$

with $\{i_1,i_2,\cdots,i_t\}=\{-1,0,1,\cdots,n\}$ as sets, where $(\pi)_j$ is a copy of π with generators $\{x_j^{(\alpha)}|\alpha\in J\}$, $\epsilon_j=\pm 1$, $y_{-1}^{(\alpha)}=x_0^{(\alpha)^{-1}}$, $y_j=x_j^{(\alpha)}x_{j+1}^{(\alpha)^{-1}}$ for $1\leq j\leq n-1$ and $y_n^{(\alpha)}=x_n^{(\alpha)}$, the commutator bracket $[\cdots]$ runs over all of the commutator bracket arrangment of weight t for each t.

Proof: Since $\{x^{(\alpha)}|\alpha\in J\}$ is a set of generators for π, $F(x^{(\alpha)}|\alpha\in J)\to\pi$ is an epimorphism. Thus

$$F(\vee_{\alpha\in J}S^1)\cong F^{F(x^{(\alpha)}|\alpha\in J)}(S^1)\to F^{\pi}(S^1)$$

is an epimorphism. Hence

$$NF(\vee_{\alpha\in J}S^1)\to NF^{\pi}(S^1)$$

and

$$\mathcal{B}F(\vee_{\alpha\in J}S^1)\to\mathcal{B}F^{\pi}(S^1)$$

are epimorphisms. The assertion follows from Theorem 5.6, Lemma 5.7 and Proposition 2.8.

6. Applications

In this section, we consider Cohen's K-construction. Our descriptions for the homotopy groups of 3-sphere gives a calculation for the K-construction of 1-sphere.

Definition 6.1. *Let X be set. The group $K(X)$ is defined to the the quotient group of the free group $F(X)$ modulo the normal subgroup generated by all of the commutators $[[x_1,x_2],\cdots,],x_t]$ with $x_i\in X$ and $x_i=x_j$ for some $1\leq i<j\leq t$ Now let S be a pointed simplicial set. The simplicial group $K(S)$ is defined to be the quotient simplicial group of $F(S)$ modulo the normal simplicial subgroup generated by all of the commutators $[[x_1,x_2],\cdots,],x_t]$ with $x_i\in S$ and $x_i=x_j$ for some $1\leq i<j\leq t$, where $F(S)$ is the Milnor's $F(K)$-construction for the simplicial set S.*

Definition 6.2. *The group $Lie(n)$ is the elments of weight n in the Lie algebra $Lie(x_1,x_2,\cdots,x_n)$ which is the quotient Lie algebra of the free Lie algebra*

$$L(x_1,x_2,\cdots,x_n)$$

over Z modulo the two sided Lie ideal generated by the Lie elements $[[x_{i_1},x_{i_2}],\cdots,],x_{i_t}]$ with $i_l=i_k$ for some $1\leq l<k\leq t$.

The following lemmas are due to Fred Cohen.

Lemma 6.3 (Co). *$\Gamma^q K(x_1,x_2,\cdots,x_n)=\{1\}$ for $q\geq n$ and*

$$\Gamma^n K(x_1,x_2,\cdots,x_n)\cong Lie(n),$$

where $\Gamma^q G$ is the q-th term in the lower central series for the group G starting with $\Gamma^1 G=G$.

Lemma 6.4 (Co). *In the group $K(x_1,x_2,\cdots,x_n)$, the normal subgroup grnerated by x_j is abelian for each $1\leq j\leq n$.*

Lemma 6.5 (Co). *The set $\{[x_1,x_{\sigma(2)},\cdots,x_{\sigma(n)}]|\sigma\in\Sigma_{n-1}\}$ is a Z-basis for $Lie(n)$, where Σ_{n-1} acts on $\{2,3,\cdots,n\}$ by permutation.*

Recall that the simplicial 1-sphere S^1 is a free simplicial set generated by a 1-simplex σ. More precisely, $S^1_0=\{*\}$, $S^1_1=\{\sigma,*\}$ and $S^1_{n+1}=\{*,s_n\cdots\hat{s}_j\cdots s_0\sigma|0\leq j\leq n\}$. Let x_i denote $s_n\cdots\hat{s}_j\cdots s_0\sigma$. Then

Lemma 6.6. *The face functions $d_i:S^1_{n+1}\to S^1_n$ and the degenarate functions $s_i: S^1_{n+1}\to S^1_{n+2}$ are as follows:*

$$d_i x_j=\begin{cases}x_j & j<i\\ x_{j-1} & j\geq i\end{cases}$$

and

$$s_i x_j=\begin{cases}x_j & j<i\\ x_{j+1} & j\geq i\end{cases},$$

where we put $x_{-1}=*$ and $x_n=*$ in S^1_n.

Theorem 6.7. *$\pi_n(K(S^1))$ is isomorphic to $Lie(n)$*

Proof: Let $\pi:F(S^1)\to K(S^1)$ be the quotient homomorphism. Then $NF(S^1)\to NK(S^1)$ is an epimorphism. Recall that $NF(S^1)_{n+1}$ is generated by all of the commutators

$$[y_{i_1},y_{i_2},\cdots,y_{i_t}]$$

so that

$$\{i_1,i_2,\cdots,i_t\}=\{0,1,\cdots,n\}$$

by Theorem 4.3. Thus $NF(S^1)_{n+1}\subseteq\Gamma^{n+1}F(S^1)_{n+1}$ and therefore

$$NK(S^1)_{n+1}\subseteq\Gamma^{n+1}K(S^1)_{n+1}.$$

Notice that $K(S^1)_{n+1}\cong K(x_0,x_1,\cdots,x_n)$. Thus $\Gamma^{n+1}K(S^1)_{n+1}\cong Lie(n+1)$ and $\Gamma^{n+1}K(S^1)_n=\{1\}$. Thus

$$d_j|_{\Gamma^{n+1}K(S^1)_{n+1}}:\Gamma^{n+1}K(S^1)_{n+1}\to K(S^1)_n$$

is trivial for each $j \geq 0$. And therefore

$$NKS^1)_{n+1} = \Gamma^{n+1} K(S^1)_{n+1} \cong Lie(n+1)$$

with $d_0 : NK(S^1)_{n+1} \to NK(S^1)_n$ is trivial. the assertion follows.

Corollary 6.8. *Let $\pi : F(S^1) \to K(S^1)$ be the quotient simplicial homomorphism. Then*

$$\pi_* : \pi_n(F(S^1)) \to \pi_n(K(S^1))$$

is an isomorphism for $n = 1, 2$ and zero for $n > 2$.

Now we consider the Samelson product in $\pi_*(K(S^1))$. Let x_j denote $s_n \cdots \hat{s}_j \cdots s_0\sigma$ in S^1_{n+1}. The following lemma follows directly from Lemma 6.6.

Lemma 6.9. *Let $I = (i_1, i_2, \cdots, i_m)$ be a sequence with $i_1 < i_2 < \cdots < i_m$. Then $s_I : S^1_{n+1} - * \to S^1_{n+m+1} - *$ is the composite*

$$\{x_0, x_1, \cdots, x_n\} \xrightarrow{\bar{s}_I} \{x_0, x_1, \cdots, \hat{x}_{i_1}, \cdots, \hat{x}_{i_2}, \cdots, \hat{x}_{i_m}, \cdots, x_{n+m}\}$$
$$\hookrightarrow \{x_0, x_1, \cdots, x_{n+m}\}.$$

where $s_I = s_{i_m} \cdots s_{i_1}$ and $\bar{s}_I$ is the order preserving isomorphism.

Recall that, for $\sigma \in \pi_n(G)$ and $\tau \in \pi_m(G)$, the Samelson product [C1] is defined to be

$$< \sigma, \tau >= \prod_{(a,b)} [s_b\sigma, s_a\tau]^{sign(a,b)},$$

where G is a simplicial group, $(a, b) = (a_1, \cdots, a_n, b_1, \cdots, b_m)$ runs over all shuffles of $(0, 1, \cdots, m+n-1)$,i.e. all permutations, so that $a_1 < a_2 < \cdots < a_n$, $b_1 < b_2 < \cdots < b_m$, $sign(a, b)$ is the sign of the permutation (a, b), the order of the product $\prod$ is right lexicographic on a and $s_a = s_{a_n} \cdots s_{a_1}$.

Proposition 6.10. *The Samelson product in $\pi_*(K(S^1))$ is as follows:*

$$< [x_{\sigma(0)}, x_{\sigma(1)}, \cdots, x_{\sigma(n)}], [x_{\tau(0)}, x_{\tau(1)}, \cdots, x_{\tau(m)}] >$$
$$= \sum_{(I,J)} sign(I, J)[[x_{i_{\sigma(0)}}, \cdots, x_{i_{\sigma(n)}}], [x_{j_{\tau(0)}}, \cdots, x_{j_{\tau(m)}}]]$$

for the commutators

$$[x_{\sigma(0)}, x_{\sigma(1)}, \cdots, x_{\sigma(n)}] \in \pi_{n+1}(KS^1)) \cong Lie(n+1)$$

and

$$[x_{\tau(0)}, x_{\tau(1)}, \cdots, x_{\tau(m)}] \in \pi_{m+1}(K(S^1)) \cong Lie(m+1)$$

where

$$(I, J) = (i_0, i_1, \cdots, i_n, j_0, j_1, \cdots, j_m)$$

runs over all shuffles of $(0,1,\cdots,m+n+1)$ *so that* $i_0<i_1<\cdots<i_n$, $j_0<j_1<\cdots<j_m$, *sign*(I,J) *is the sign of the permutation* (I,J), $\sigma\in\Sigma_{n+1}$ *acts on* $\{0,1,\cdots,n\}$ *and* $\tau\in\Sigma_{m+1}$ *acts on* $\{0,1,\cdots,m\}$

Proof: Notice that $\{x_0,\cdots,\hat{x}_{j_0},\cdots,\hat{x}_{j_m},\cdots,x_{n+m+1}\}=\{x_{i_0},\cdots,x_{i_n}\}$ and $\bar{s}_J:\{x_0,\cdots,x_n\}\to\{x_{i_0},\cdots,x_{i_n}\}$ is an ordered isomorphism.

$$s_J([x_{\sigma(0)},x_{\sigma(1)},\cdots,x_{\sigma(n)}])=[x_{i_{\sigma(0)}},\cdots,x_{i_{\sigma(n)}}]$$

and

$$s_I([x_{\tau(0)},x_{\tau(1)},\cdots,x_{\tau(m)}])=[x_{j_{\tau(0)}},\cdots,x_{j_{\tau(m)}}].$$

The assertion follows.

Definition 6.11. *A simplicial group is* ***minimal*** *if it is also a minimal simplicial set.*

Recall that a simplicial group G is minimal if and only if the Moore chain complex NG is minimal [C2].

Theorem 6.12. *The simplicial group* $K(S^1)$ *is the universal minimal simplicial quotient simplicial group of* $F(S^1)$ *in the following sense:*
(1). $K(S^1)$ *is a minimal simplicial group.*
(2). Let G *be a minimal simplicial group. Then every simplicial homomorphism* $f:F(S^1)\to G$ *factors through* $K(S^1)$*.*

Proof: By inspecting the proof of Theorem 6.7, $K(S^1)$ is a minimal simplicial group. The assertion (2) follows from the following statement.

Statement: $K(S^1)$ is the quotient simplicial group of $F(S^1)$ modulo the normal subsimplicial group generated by the boundaries.

Let H denote the kernel of the quotient map $p:F(S^1)\to K(S^1)$ and let $\overline{B}$ denote the normal subsimplicial group of $F(S^1)$ generated by the boundaries $\mathcal{B}F(S^1)$. Notice that $K(S^1)$ is a minimal simplicial group. Thus $\overline{B}$ is contained in H. Let Q denote the quotient simplicial group $F(S^1)/\overline{B}$. Then Q is a minimal simplicial group. By Proposition 2.6, there is a central extension

$$0\to K(\pi_{n+1}Q,n+1)\to P_{n+1}Q\to P_nQ\to 0,$$

where P_nQ is the n-th Moore-Postnikov system of Q. Notice that $P_1Q=K(\pi_Q,1)=K(Z,1)$. Thus $\Gamma^{n+2}P_{n+1}Q=1$ by induction on n. Notice that $Q_{n+1}\cong(P_{n+1}Q)_{n+1}$. Thus $\Gamma^{n+2}Q_{n+1}=1$. Now we show that H is contained in $\overline{B}$ by induction on the dimension starting with $H_1=\overline{B}_1=1$. Suppose that $H_n\subseteq\overline{B}_n$ with $n>0$. Notice that $F(S^1)_{n+1} = F(x_0, \cdots, x_n)$ and $K(S^1)_{n+1} = K(x_0, \cdots, x_n)$. Thus H_{n+1} is a normal subgroup of $F(x_0, \cdots, x_n)$ generated by the commutators

$$[[x_{i_1}, x_{i_2}], \cdots, x_{i_t}], x_{i_t}]$$

such that $i_p \neq i_q$ for $p < q$. Now consider $W = [[x_{i_1}, x_{i_2}], \cdots, x_{i_t}], x_{i_t}]$. If $t \geq n+1$, then $W \in \Gamma^{n+2} F(x_0, \cdots, x_n)$. Thus $W \in \overline{B}_{n+1}$ since $\Gamma^{n+2} Q_{n+1} = 1$.

If $t < n+1$, then there exists an index $j \in \{0, 1, \cdots, n\} - \{i_1, \cdots, i_t\}$. Recall that

$$s_i x_k = \begin{cases} x_k & k < i, \\ x_{k+1} & k \geq i, \end{cases}$$

for $x_k = s_{n-1} \cdots \hat{s}_k \cdots s_0 \sigma \in S^1_n$. Thus

$$s_j[[x_{i'_1}, x_{i'_2}], \cdots, x_{i'_t}], x_{i'_t}] = [[x_{i_1}, x_{i_2}], \cdots, x_{i_t}], x_{i_t}],$$

where $i'_k = i_k$ if $i_k < j$ and $i'_k = i_k - 1$ if $i_k > j$. By induction, $[[x_{i'_1}, x_{i'_2}], \cdots, x_{i'_t}], x_{i'_t}] \in \overline{B}$. Thus $W = [[x_{i_1}, x_{i_2}], \cdots, x_{i_t}], x_{i_t}] \in \overline{B}$. The induction is finished and the assertion follows.

The simplicial group $K(S^1)$ is homotopy eqivalent to a product of the Eilenberg-Maclane spaces with a different product structure.

Proposition 6.13. *$\Omega K(S^1)$ is an abelian simplicial group. Therefore $K(S^1)$ is homotopy equivalent to a product of the Eilenberg-MacLane spaces as a simplicial set.*

Proof: Consider

$$d_0 : K(x_0, x_1, \cdots, x_n) \to K(x_0, x_1, \cdots, x_{n-1})$$

$d_0(x_0) = 1$ and $d_0(x_j) = x_{j-1}$. Thus $Kerd_0 \cap K_{n+1}(S^1) \cong < x_0 >$ is the normal subgroup generated by x_0 which is abelian by Lemma 6.4. The assertion follows.

In the end of this section, we give some applications of $K(S^1)$ to minimal simplicial groups.

Proposition 6.14. *Let G be a minimal simplicial group such that the abelianlizer G^{ab} is a minimal simplicial group $K(\pi, 1)$ for a cyclic group π. Then G is homotopy equivalent to a product of Eilenberg-Maclane spaces.*

Proof: Notice that $G_1 = \pi$. Let x be a generator for the cyclic group π and let $f_x : S^1 \to G$ be a representive map of x, i.e, $f_x(\sigma) = x$. Let $g : F(S^1) \to G$ be the simplicial homomorphism induced by f_x. We need a lemma.

Lemma 6.15. *The simplicial homomorphism $g : F(S^1) \to G$ is simplicial surjection.*

Proof: It suffices to show that the subsimplicial group, denote by H, of G generated by G_1 is G itself. This is given by induction on the dimensions starting with $H_1 = G_1$. Suppose that $H_{n-1} = G_{n-1}$ with $n > 1$. By a result of Condule [see, e.g, Po, Proposition 1, pp.6], G_n is generated by the degenerate images of lower order Moore chain complex terms and NG_n. Thus G_n is genereted by NG_n and H_n by induction. Notice that G is a minimal simplicial group. Thus $NG_n = \mathcal{Z}G_n$, the cycles. By Proposition 2.1, $\mathcal{Z}G_n$ is contained in the center of G_n. Thus H_n is a normal subgroup of G_n and the composite $\phi : \pi_n G \cong \mathcal{Z}G_n \to G_n \to G_n/H_n$ is an epimorphism. Thus G_n/H_n is an abelian group and so the quotient homomorphism $G_n \to G_n/H_n$ factors through G_n^{ab}. Notice that $G^{ab} = K(\pi, 1)$. Thus $G_n^{ab} \cong G_n^{ab}/\mathcal{Z}G_n$ and so $\phi : \pi_n G \to G_n/H_n$ is trivial. Thus G_n/H_n is trivial and the assertion follows.

Continuation of Proof of Proposition 6.14: Notice that G is minimal. The simplicial epimorphism $g : F(S^1) \to G$ factors through $K(S^1)$ by Proposition 6.12. By Proposition 6.13, $\Omega K(S^1)$ is an abelian simplicial group. Thus ΩG is also an abelian simplicial group. Thus G is homotopy equivalent to a product of Eilenberg-Maclane spaces, which is the assertion.

The following counter-example for minimal simplicial groups is due to J. W. Milnor (unpublished).

Proposition 6.16. $\Omega(S^{n+1}[n+1, n+2, n+3])$ *does not have a homotopy type of a minimal simplicial group for* $n > 0$, *where* $S^{n+1}[n+1, n+2, n+3]$ *is the 3-stage Postnikov system by taking the first three nontrivial homotopy groups of* S^{n+1}.

Proof: Suppose that G is a minimal simplicial group such that $G \simeq \Omega(S^{n+1}[n+1, n+2, n+3])$. Let $f : F(S^1) \to \Omega^{n-1}G$ be a simplicial homomorphism such that $f(\sigma)$ is a generator of $(\Omega^{n-1}G)_1 \cong G_n \cong Z$. Then $f_* : \pi_j(F(S^1)) \to \pi_j(\Omega^{n-1}G)$ is an isomorphism for $j \leq 3$. Notice that $\Omega^{n-1}G$ is also a minimal simplicial group. The simplicial homomorphism $f : F(S^1) \to \Omega^{n-1}G$ factors through $K(S^1)$. Notice that $\pi_3(F(S^1)) \cong \pi_3\Omega^{n-1}G) \cong Z/2$ and $\pi_3(K(S^1)) \cong Lie(3) \cong Z \oplus Z$. There is a contradiction and the assertion follows.

More examples and counter-examples for minimal simplicial groups will be given in [Wu2]. It was know that there are still many counter-examples of two-stage Postnikov systems for minimal simplicial groups [Wu2].

References

[A] J. F. Adams, *On products in minimal complexes*, Trans. Amer. Math. Soc. 82 (1956), 180-189.

[AM] M. Artin and B. Mazur, *On the van Kampen theorem*, Topology 5(1966), 179-189.

[BK] A. K. Bousfield, D. Kan, *Homotopy limits, completions and localizations*, Springer Lecture Notes in Math. 304, 1972.

[Bo1] W. A. Bogley, *An embedding for π_2 of a subcomplex of a finite contractible two-complex,* Glasgow Math. J. 33(1991), 365-371.

[Bo2] W. A. Bogley, *J. H. C. Whitehead's Asphericity Question,* London Math. Soc. Lect. Note Series 197 (1993), 309-334.

[C1] E. B. Curtis, *Some relations between homotopy and homology*, Ann. of Math. 83 (1965), 386–413.

[C2] E. B. Curtis, *Simplicial homotopy theory*, Advances in Math, 107-209 (1971).

[Car] G. Carlsson, *A simplicial group construction for balanced products* , Topology, vol.23, no.1 (1985) 85–89.

[Co] F. Cohen, *On the combinatorial methods in homotopy theory*, To appear.

[Cond] D. Conduche, *Modules croises generalises de longeur 2*, J. Pure. Appl. Algebra 34 (1984), 155-178.

[K1] D. Kan, *On c.s.s. complexes*, Amer. J. Math. 79 (1957), 449-476.

[K2] D. Kan, *A combinatorial definition of homotopy groups*, Ann. of Math. 67 (1958), 288–312.

[Mi1] J. Milnor, *On the construction $F(K)$*, Algebraic Topology- A Student Guide, by J.F. Adams, 119-136, Cambridge Univ. Press.

[Mi2] J. Milnor, *The geometrical realization of a semi-simplicial complex*, Ann. of Math.65(1957) 357-362.

[Fi] Z. Fiedorowicz, *Classification spaces of topological manoids and category*, Amer. J. Math. 106 (1984), 301-350.

[KT] D. Kan and W. Thurston, *Every connected spaces has the homology of a $K(\pi, 1)$*, Topology 15 (1976), 253-258.

[Mo1] J. C. Moore, *Homotopie des complexes monöideaux*, Seminaire Henri Cartan, 1954-55.

[Mo2] J. C. Moore, *Algebraic homotopy theory*, mimeographed notes, Princeton,1956.

[Mo3] J.C. Moore, *Semi-simplicial complexes and Postnikov systems*, Symposium Internationale de Topologica Algebraica, pp.232-247(1958)

[Po] T. Porter, *N-types of simplicial groups and crossed N-cubes*, Topology, Vol.32, No.1, (1993), 5–24.

[Qu1] D.G. Quillen, *The geometrical realization of a Kan fibration is a Serre fibration*, Proc. Amer. Math. Soc. 19 (1968),1499-1500.

[S] C. R. Stover, *A ven Kampen spectral sequence for higher homotopy groups*, Topology 29(1990), 9-26.

[Wh1] J.H.C. Whitehead, *On the asphericity of regions in a 3-sphere*, Fund. Math., 32(1939), 149-166.

[MKS] W. Magnus, A. Karrass and D. Solitar, *Combinatorial Group Theory*, Interscience Publisher, New York, London, Sydney,1966.

[Wi] C. W. Wilkerson, *Applications of minimal simplicial groups*, Topology 15(1976), 111-130.

[Wu1] Jie Wu, *On fibrewise simplicial groups and a generalization of Milnor's $F(K)$ construction*, to appear.

[Wu2] Jie Wu, *On minimal simplicial groups*, In preparing.

MSRI, 1000 Centennial Drive, Berkeley, CA 94720-5070